FILTRATION THROUGH ELASTIC-PLASTIC POROUS MEDIA: 
THE CASE OF SMALL ELASTIC RECOVERY

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(Received and accepted February 1993)

Abstract—We study the self-similar source solution of the Barenblatt equation for elasto-plastic filtration through porous rock, in the limit of large compressibility and small elastic recovery of the rock.

Consider the flow of an elastic fluid through an elasto-plastic porous medium. Then the pressure $p$, the density $\rho$, the porosity $\varphi$ and the velocity $v$ are governed by the following physical laws:

\[(\varphi \rho)_t + \text{div}(\rho v) = 0 \quad \text{(conservation of mass)}\]
\[v = -\frac{k}{\mu} \text{grad} p \quad \text{(Darcy's law)}\]
\[\frac{\rho}{\rho_0} = 1 + A(p - p_0) \quad \text{(equation of state of the fluid)}\]
\[\varphi_t = \varphi_0 B(p_t)p_t \quad \text{(equation of state of the medium)},\]

where $k$ denotes the permeability of the medium, $\mu$ the viscosity of the fluid and $A$ and $B$ coefficients of compressibility. Because the medium is assumed to be elasto plastic the compressibility of the medium depends on the sign of $p_t$: we, therefore, write $B(p_t) = B_+$ if $p_t > 0$ and $B(p_t) = B_-$ if $p_t < 0$, where $B_+$ and $B_-$ are positive numbers. If we eliminate $p$, $\varphi$ and $v$, we obtain to first order

\[p_t = K A p, \quad (1)\]

where $K = k/\{\mu \rho_0(A + B_+)\}$. For further details about the derivation of this equation, we refer to [1-4] and about its mathematical analysis to [5-7].

If we set

\[\gamma = \frac{\kappa_- - \kappa_+}{\kappa_+ + \kappa_-} \quad \text{and} \quad t = \frac{1}{2} \left( \frac{1}{\kappa_+} + \frac{1}{\kappa_-} \right) t^*,\]

and omit the asterisk again, we can write (1) in the canonical form

\[p_t + \gamma |p_t| = \Delta p. \quad (2)\]

In this note, we focus on the limiting situation in which the compressibility of the rocks is much greater than that of the water and the rocks exhibit very little elastic recovery. It is well-known that these conditions often pertain in rocks close to the surface. Thus, we consider the limit $A = 0$ and $B_- \ll B_+$, so that $\kappa_- \gg \kappa_+$ and $\gamma \approx 1$.

It has been shown [1,7,8] that equation (2) has a self-similar solution of the form

\[p(x,t) = t^{-\alpha/2} f(\eta), \quad \eta = \frac{|x|}{t^{1/2}}, \quad \alpha > 0, \quad (3)\]

which is singular at the origin and vanishes elsewhere as $t \to 0$:

\[p(x,t) \to 0 \quad \text{as} \quad t \to 0^+ \text{ and } x \neq 0\]
The exponent $\alpha$ in (3) appears as an eigenvalue in a nonlinear eigenvalue problem involving a two-point boundary value problem for $f$. This eigenvalue problem has recently been studied in some detail and the following properties of $\alpha$ have been established.

(A) For each $\gamma \in (-1, 1)$ there exists a unique exponent $\alpha$, which we denote by $\alpha(\gamma)$.

(B) The function $\alpha(\gamma)$ is analytic.

(C) $\alpha(\gamma)$ is strictly increasing and

\[
\begin{cases}
\alpha(\gamma) \rightarrow (N - 2) & \text{as } \gamma \rightarrow -1, \\
\alpha(\gamma) \rightarrow \infty & \text{as } \gamma \rightarrow +1,
\end{cases}
\]

where $(z)_+ = \max\{z, 0\}$.

(D)

\[
\alpha(0) = N \text{ and } \frac{\partial \alpha}{\partial \gamma}|_{\gamma=0} = \frac{4N^{N/2}}{\pi^{N/2} \Gamma\left(\frac{N}{2}\right)}.
\]

The properties (A) and (C) were proved in [7], (B) was proved in [9], and in (D) the value of $\alpha'(0)$ was found for $N = 1$ in [10] and for arbitrary $N$ in [9].

In this letter, we announce further asymptotic results result about the exponent $\alpha(\gamma)$ and the scaled pressure profile $f(q; \gamma)$. The first result refines the limiting behaviour of $\alpha(\gamma)$ given in (C).

**THEOREM 1.** We have

\[
\lim_{\gamma \rightarrow 1} (1 - \gamma) \alpha^2(\gamma) = \frac{1}{2} \rho_\nu^2,
\]

where

\[
\nu = \frac{N}{2} - 1
\]

and $\rho_\nu$ denotes the first zero of the Bessel function $J_\nu$.

Let $p(x, t)$ be the self-similar solution defined by (3). Then one finds that for every $t > 0$, there exists exactly one radius $R(t; \gamma)$ at which $p_t = 0$ and the character of the equation changes. In the following theorem, we give the asymptotic behaviour of $R$ as $\gamma \rightarrow 1$.

**THEOREM 2.** For every fixed $t > 0$, we have

\[
t^{-1/2}R(t; \gamma) \sim \left(2^{3/2} \rho_\nu\right)^{1/2}(1 - \gamma)^{-1/4} \quad \text{as } \gamma \rightarrow 1.
\]

For the function $f$ we find:

**THEOREM 3.** Let $f(\eta; \gamma)$ be the solution of the nonlinear eigenvalue problem obtained by substituting (3) into (2). Then

\[
\lim_{\gamma \rightarrow 1} \left(\sqrt{\alpha(\gamma)}; \gamma\right) \rightarrow \hat{f}(\zeta) \quad \text{as } \gamma \rightarrow 1,
\]

where

\[
\hat{f}(\zeta) = \begin{cases}
\Gamma(\nu + 1) \left(\frac{1}{2} \zeta \rho_\nu\right)^{-\nu} J_\nu\left(\frac{1}{2} \zeta \rho_\nu\right) & \text{if } 0 < \zeta < 2 \\
0 & \text{if } \zeta \geq 2,
\end{cases}
\]

uniformly with respect to $0 \leq \zeta \leq 2$.

The proofs of these limits are based on a careful analysis of the Riccati equation obtained after transforming the variables $\eta$ and $f$ to

\[
t = \frac{1}{4} \eta^2 \quad \text{and} \quad y(t) = -\frac{\eta f'}{f}.
\]

It reads

\[
y' = -F(y - \alpha) + \frac{y}{2t} \left(y - (N - 2)\right), \quad t > 0.
\]

The details of this analysis will appear in [11].
REFERENCES