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# Differentiation of Fuzzy Continuous Mappings on Fuzzy Topological Vector Spaces

MARIO FERRARO* AND DAVID H. FOSTER

*Department of Communication and Neuroscience, University of Keele, Keele, Staffordshire ST5 5BG, England*

*Submitted by L. A. Zadeh*

## 1. INTRODUCTION

In a classic paper [1], Zadeh introduced the notion of fuzzy sets and fuzzy set operations. Chang [2], Wong [3], Lowen [4], and others developed a theory of fuzzy topological spaces and Rosenfeld [5] initiated a theory of fuzzy groups. These were brought together by Foster [6] to form the elements of a theory of fuzzy topological groups.

Starting with a vector space $E$, a structure for fuzzy vector spaces and fuzzy topological vector spaces was proposed by Katsaras and Liu [7]. In this paper, we develop the theory of fuzzy topological vector spaces further and introduce the notion of the differentiability of fuzzy continuous mappings defined on fuzzy topological vector spaces. The properties of derivatives and formal rules of derivation are also briefly discussed. We point out that our approach does not depend upon the imposition of a norm on the space $E$. In particular, the derivative defined here should be distinguished from the differential of a "fuzzy function" described by Puri and Ralescu [8] which relates to mappings from an open subset of a normed space into a subset of fuzzy sets defined on a reflexive Banach space.

## 2. PRELIMINARIES

Definitions and notation for fuzzy sets follow Zadeh [1], and those for fuzzy points and neighbourhoods follow Pu and Liu [9].

Let $X$ be a set and $I$ the unit interval [0, 1]. A fuzzy set $A$ in $X$ is characterized by a membership function $\mu_A$ which associates with each point $x \in X$ its "grade of membership" $\mu_A(x) \in I$.

* On leave from Istituto di Fisiologia e Chimica Biologica, Università di Torino, Italy.



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DEFINITION 2.1. Let $A$ and $B$ be fuzzy sets in $X$. Then

$$A = B \Leftrightarrow \mu_A(x) = \mu_B(x), \qquad x \in X;$$
$$A \subset B \Leftrightarrow \mu_A(x) \leqslant \mu_B(x), \qquad x \in X;$$
$$C = A \cup B \Leftrightarrow \mu_C(x) = \max\{\mu_A(x), \mu_B(x)\}, \qquad x \in X;$$
$$D = A \cap B \Leftrightarrow \mu_D(x) = \min\{\mu_A(x), \mu_B(x)\}, \qquad x \in X;$$
$$E = C_X A \Leftrightarrow \mu_E(x) = 1 - \mu_A(x), \qquad x \in X.$$

More generally for a family $\{A_j\}$, $j \in J$, of fuzzy sets the *union* $C = \bigcup_{j \in J} A_j$ and the *intersection* $D = \bigcap_{j \in J} A_j$ are defined by

$$\mu_C(x) = \sup_{j \in J} \mu_{A_j}(x), \qquad x \in X,$$
$$\mu_D(x) = \inf_{j \in J} \mu_{A_j}(x), \qquad x \in X.$$

We denote by $k_c$ the fuzzy set in $X$ with membership function $\mu_{k_c}(x) = c$, $c \in I$, $x \in X$. The fuzzy set $k_1$ corresponds to the set $X$ and the fuzzy set $k_0$ to the empty set $\varnothing$.

DEFINITION 2.2. A *fuzzy point* in $X$ is a fuzzy set with membership function $\mu_{y_\lambda}(x)$, $x \in X$, defined by

$$\mu_{y_\lambda}(x) = \lambda, \qquad \text{for } x = y,$$
$$= 0, \qquad \text{otherwise},$$

where $0 < \lambda \leqslant 1$. The point $y$ is called the *support* of $y_\lambda$ and $\lambda$ its *value* (compare Goguen [10], Pu and Liu [9], Sarkar [11]). The fuzzy point $y_\lambda$ is said to be *contained in*, or to *belong to*, a fuzzy set $A$, written $y_\lambda \in A$, iff $\lambda \leqslant \mu_A(y)$.

DEFINITION 2.3. Let $f$ be a mapping from a set $X$ to a set $Y$. Let $B$ be a fuzzy set in $Y$, with membership function $\mu_B$. Then the *inverse image* of $B$, written $f^{-1}[B]$, is the fuzzy set in $X$ with membership function defined by

$$\mu_{f^{-1}[B]}(x) = \mu_B(f(x)), \qquad x \in X.$$

Conversely, let $A$ be a fuzzy set in $X$, with membership function $\mu_A$. Then the *image* of $A$, written $f[A]$, is the fuzzy set in $Y$ with membership function $\mu_{f[A]}(y)$, $y \in Y$, defined by

$$\mu_{f[A]}(y) = \sup_{z \in f^{-1}(y)} \mu_A(z), \qquad \text{if } f^{-1}(y) \text{ is nonempty,}$$
$$= 0, \qquad \text{otherwise,}$$

where $f^{-1}(y) = \{x \mid f(x) = y\}$.

LEMMA 2.1. *Let $f$ be a mapping from a set $X$ to a set $Y$, let $A_1, A_2$ be fuzzy sets in $X$ and let $B_1, B_2$ be fuzzy sets in $Y$. Then*

(i) $A_1 \subset A_2 \Rightarrow f[A_1] \subset f[A_2]$,

(ii) $B_1 \subset B_2 \Rightarrow f^{-1}[B_1] \subset f^{-1}[B_2]$.

*Proof.* See [2].

## 3. Fuzzy Topological Spaces

The following definition of a fuzzy topological space is due to Lowen [4].

DEFINITION 3.1. A *fuzzy topology* on a set $X$ is a family $\mathscr{T}$ of fuzzy sets in $X$ which satisfies the following conditions.

(i) For all $c \in I$, $k_c \in \mathscr{T}$.

(ii) If $A, B \in \mathscr{T}$, then $A \cap B \in \mathscr{T}$.

(iii) If $A_j \in \mathscr{T}$ for all $j \in J$, then $\bigcup_{j \in J} A_j \in \mathscr{T}$.

The pair $(X, \mathscr{T})$ is called a *fuzzy topological space*, or fts for short, and the members of $\mathscr{T}$ are called *$\mathscr{T}$-open fuzzy sets*, or simply *open fuzzy sets*.

In the definition of a fuzzy topology by Chang [2], the condition (i) is just

(i)′ $k_0, k_1 \in \mathscr{T}$.

The inclusion in $\mathscr{T}$ of all fuzzy sets with constant membership function is required for the fuzzy continuity of the constant functions.

DEFINITION 3.2. A fuzzy set in $X$ is said to be *$\mathscr{T}$-closed*, or *closed* for short, iff its complement is an open fuzzy set.

DEFINITION 3.3. Let $(X, \mathscr{T})$ be a fts. A fuzzy set $A$ in $X$ is called a *neighbourhood* of a fuzzy point $y_\lambda$ in $X$ iff there exists an open fuzzy set $B \in \mathscr{T}$ such that $y_\lambda \in B \subset A$.

LEMMA 3.1. *Let* $(X, \mathscr{T})$ *be a fts and let* $\mathscr{G}(y_\lambda)$ *denote the set of all neighbourhoods of a fuzzy point* $y_\lambda$ *in* $X$. *Then every member of* $\mathscr{G}(y_\lambda)$ *has the following properties.*

(i) *Every fuzzy set which contains a fuzzy set belonging to* $\mathscr{G}(y_\lambda)$ *itself belongs to* $\mathscr{G}(y_\lambda)$.

(ii) *Every finite intersection of sets of* $\mathscr{G}(y_\lambda)$ *belongs to* $\mathscr{G}(y_\lambda)$.

*Proof.* Straightforward.

DEFINITION 3.4. In a fts $(X, \mathscr{T})$ a *fundamental system of neighbourhoods* of a fuzzy point $y_\lambda$ is a set $\mathscr{B}(y_\lambda)$ of neighbourhoods of $y_\lambda$ such that for each neighbourhood $A$ of $y_\lambda$ there is a $B \in \mathscr{B}(y_\lambda)$ such that $B \subset A$.

DEFINITION 3.5. Let $(X, \mathscr{T})$, $(Y, \mathscr{V})$ be two fts's. A mapping $f$ of $(X, \mathscr{T})$ into $(Y, \mathscr{V})$ is *fuzzy continuous* iff for each open fuzzy set $V$ in $\mathscr{V}$ the inverse image $f^{-1}[V]$ is in $\mathscr{T}$. Conversely, $f$ is *fuzzy open* iff for each open fuzzy set $U$ in $\mathscr{T}$, the image $f[U]$ is in $\mathscr{V}$. The mapping $f$ is *fuzzy continuous at a point* $x \in X$ iff for each open fuzzy set $V$ in $\mathscr{V}$ containing the fuzzy point $y_\delta = (f(x))_\delta$, $0 < \delta \leqslant 1$, the inverse image $f^{-1}[V]$ is an open fuzzy set in $\mathscr{T}$ containing $x_\lambda$, $0 < \lambda \leqslant \delta$.

LEMMA 3.2. *If* $(X, \mathscr{T})$, $(Y, \mathscr{V})$ *are fts's and* $f$ *is a mapping of* $(X, \mathscr{T})$ *into* $(Y, \mathscr{V})$ *the following assertions are equivalent.*

(i) *The mapping* $f$ *is fuzzy continuous.*

(ii) *For each fuzzy set* $A$ *in* $X$ *and each neighbourhood* $V$ *of* $f[A]$, *there is a neighbourhood* $U$ *of* $A$ *such that* $f[U] \subset V$.

*Proof.* See [2].

DEFINITION 3.6. Let $(X, \mathscr{T})$, $(Y, \mathscr{V})$ be two fts's. A bijective mapping of $(X, \mathscr{T})$ onto $(Y, \mathscr{V})$ is a *fuzzy homeomorphism* iff both $f$ and $f^{-1}$ are fuzzy continuous.

DEFINITION 3.7. Given a family $\{(X_j, \mathscr{T}_j)\}$, $j \in J$, of fts's, we define their product $\prod_{j \in J} (X_j, \mathscr{T}_j)$ to be the fts $(X, \mathscr{T})$, where $X = \prod_{j \in J} X_j$ is the usual set product and $\mathscr{T}$ is the coarsest topology on $X$ for which the projections $p_j$ of $X$ onto $X_j$ are fuzzy continuous for each $j \in J$. The fuzzy topology $\mathscr{T}$ is called the *product fuzzy topology* on $X$, and $(X, \mathscr{T})$ a *product* fts.

LEMMA 3.3. *Let $\{(X_j, \mathscr{T}_j)\}$, $j \in J$, be a family of fts's and $(X, \mathscr{T})$ the product fts. The product fuzzy topology $\mathscr{T}$ on $X$ has as a base the set of finite intersections of fuzzy sets of the form $p_j^{-1}[U_j]$, where $U_j \in \mathscr{T}_j$, $j \in J$.*

*Proof.* See [3].

Let $\{X_j\}$, $j = 1, 2, \ldots, n$, be a finite family of sets and for each $j$ let $A_j$ be a fuzzy set in $X_j$. We define the *product* $A = \prod_{j=1}^{n} A_j$ of the family $\{A_j\}$ as the fuzzy set in $X = \prod_{j=1}^{n} X_j$ that has membership function given by

$$\mu_A(x_1, \ldots, x_n) = \min\{\mu_{A_1}(x_1), \ldots, \mu_{A_n}(x_n)\}, \qquad (x_1, \ldots, x_n) \in X.$$

It follows from the above that if $X_j$ has fuzzy topology $\mathscr{T}_j$, $j = 1, 2, \ldots, n$, the product fuzzy topology $\mathscr{T}$ on $X$ has as a base the set of product fuzzy sets of the form $\prod_{j=1}^{n} U_j$, where $U_j \in \mathscr{T}_j$, $j = 1, 2, \ldots, n$.

We make use of the following separation property.

DEFINITION 3.8. A fts $(X, \mathscr{T})$ is called a *fuzzy $T_1$ space* iff every fuzzy point is a closed fuzzy set.

LEMMA 3.4. *A fts $(X, \mathscr{T})$ is a fuzzy $T_1$ space iff for each $x \in X$ and each $\lambda \in I$ there exists $B \in \mathscr{T}$ such that $\mu_B(x) = 1 - \lambda$, $\mu_B(y) = 1$, for all $y \neq x$.*

*Proof.* See [9].

LEMMA 3.5. *If $\{(X_j, \mathscr{T}_j)\}$, $j = 1, 2, \ldots, n$, is a finite family of fts's, each of which is a fuzzy $T_1$ space, then the product fts $(X, \mathscr{T})$ is a fuzzy $T_1$ space.*

*Proof.* Every fuzzy point $y_\lambda$, $0 < \lambda \leqslant 1$, in $X$ can be thought of as the product of fuzzy points $(y_j)_\lambda$, $j = 1, 2, \ldots, n$, each with suport $y_j$ and the same value $\lambda$. By hypothesis each $(y_j)_\lambda$ is closed, whence the product itself is closed (see [12]). ∎

## 4. FUZZY TOPOLOGICAL VECTOR SPACES

The first part of this section follows Katsaras and Liu [7]. Let $E$ denote a vector space over the field $K$ of real or complex numbers.

DEFINITION 4.1. Let $\{A_j\}$, $j = 1, 2, \ldots, n$, be a finite family of fuzzy sets in a vector space $E$. The *sum* $A = A_1 + A_2 + \cdots + A_n$ of the family $\{A_j\}$, $j = 1, 2, \ldots, n$, is the fuzzy set in $E$ whose membership function is given by

$$\mu_A(x) = \sup_{x_1 + \cdots + x_n = x} \min\{\mu_{A_1}(x_1), \ldots, \mu_{A_n}(x_n)\}, \qquad x \in E.$$

The *scalar product* $\alpha A$ of $\alpha \in K$ and $A$ a fuzzy set in $E$ is the fuzzy set in $E$ that has membership function $\mu_{\alpha A}(x)$, $x \in E$, given by

$$\begin{aligned} \mu_{\alpha A}(x) &= \mu_A(x/\alpha) \qquad \text{for } \alpha \neq 0, \\ &= \mu_{0_\lambda}(x) \qquad \text{for } \alpha = 0, \end{aligned}$$

where $\lambda = \sup_{y \in E} \mu_A(y)$.

LEMMA 4.1. *Let $E$, $F$ be vector spaces over $K$, and let $f$ be a linear mapping from $E$ into $F$. Then, for all fuzzy sets $A$, $B$ in $E$ and all scalars $\alpha$,*

(i) $f[A+B] = f[A] + f[B]$,

(ii) $f[\alpha A] = \alpha f[A]$.

*Proof.* See [7].

COROLLARY. $\alpha[A+B] = \alpha A + \alpha B$ *for all fuzzy sets $A$, $B$ in $E$ and all scalars $\alpha$.*

LEMMA 4.2. *If $A$, $B$ are two fuzzy sets in $E$ and $\alpha \in K$, $\alpha \neq 0$, then*

$$\alpha A \subset B \Rightarrow A \subset 1/\alpha B.$$

*Proof.* Obvious.

DEFINITION 4.2. A fuzzy set $A$ in a vector space $E$ is said to be *balanced* if $\alpha A \subset A$ for all $\alpha \in K$, $|\alpha| \leqslant 1$.

LEMMA 4.3. *Let $A$, $B$ be fuzzy sets in $E$. If $A$, $B$ are balanced, then the sum $A+B$ and the scalar product $\alpha A$, $\alpha \in K$, are balanced.*

*Proof.* See [7].

LEMMA 4.4. *If $A$ is a balanced fuzzy set in $E$, then*

$$\mu_A(x) = \mu_A(-x), \qquad x \in E.$$

*Proof.* Obvious.

LEMMA 4.5. *If $A$ is a balanced fuzzy set in $E$ then*

(i) $\mu_A(x) \geqslant \mu_A(\xi x)$, $x \in E$, *for all* $|\xi| \geqslant 1$,

(ii) $\mu_A(0) \geqslant \mu_A(x)$, $x \in E$.

*Proof.* Since $A$ is balanced $\mu_A(x) \geqslant \mu_{\alpha A}(x)$, $x \in E$, for all $\alpha$, $|\alpha| \leqslant 1$.

(i) Choose $\alpha \neq 0$ and set $\xi = 1/\alpha$. Then $\mu_A(x) \geqslant \mu_{1/\xi A}(x) = \mu_A(\xi x)$ for $|\xi| \geqslant 1$.

(ii) Choose $\alpha = 0$. Then by Definition 4.1 $\mu_{0A}(x) = \mu_{0_\lambda}(x)$, $x \in E$, where $\lambda = \sup_{y \in E} \mu_A(y)$. Thus $\mu_{0A}(0) = \lambda \geqslant \mu_A(x)$, $x \in E$, and, since $A$ is a balanced fuzzy set, $\mu_A(0) = \mu_{0A}(0) \geqslant \mu_A(x)$, $x \in E$. ▮

DEFINITION 4.3. Let $A$ be a fuzzy set in a vector space $E$. The *balanced hull* of $A$ is the intersection of all balanced fuzzy sets in $E$ which contain $A$.

LEMMA 4.6. *Let $A$ be a fuzzy set in $E$. Then the balanced hull of $A$ is the fuzzy set* $\bigcup_{|\alpha| \leqslant 1} \alpha A$.

*Proof.* See [7].

DEFINITION 4.4. A *fuzzy topological vector space*, or ftvs for short, is a vector space $E$ over a field $K$, equipped with a fuzzy topology $\mathscr{T}$ such that the two mappings

(i) $\sigma\colon (x, y) \to x + y$ of $(E, \mathscr{T}) \times (E, \mathscr{T})$ into $(E, \mathscr{T})$,

(ii) $\pi\colon (\alpha, x) \to \alpha x$ of $(K, \mathscr{K}) \times (E, \mathscr{T})$ into $(E, \mathscr{T})$,

where $\mathscr{K}$ is the usual topology on $K$, are fuzzy continuous.

In the sequel $(E, \mathscr{T})$, or $E$ for short, denotes a ftvs with scalar field $K$.

PROPOSITION 4.1. *For each $a \in E$ and each $\alpha \in K$, $\alpha \neq 0$, the mapping $x \to \alpha x + a$ of $(E, \mathscr{T})$ into $(E, \mathscr{T})$ is a fuzzy homeomorphism.*

*Proof.* The mappings $x \to (\alpha, x)$ and $x \to (x, a)$ are fuzzy continuous (see [6]) and the mappings $(\alpha, x) \to \alpha x$ and $(x, a) \to x + a$ are fuzzy continuous by Definition 4.4. ▮

The structure of a ftvs places constraints on which fuzzy sets can be neighbourhoods, as the following lemma shows.

LEMMA 4.7. *Let $0_\lambda$ be a fuzzy point in $E$. Let $V$ be a fuzzy set in $E$ containing $0_\lambda$. If there is a point $a \in E$ such that $\mu_V(ka) = 0$, for all nonzero $k \in K$, then $V$ is not a neighbourhood of $0_\lambda$.*

*Proof.* Suppose that $V$ is a neighbourhood of $0_\lambda$, and without loss in generality that $V$ is open. Consider the function $\pi\colon (k, a) \to ka$ and let $a_\delta$, $0 < \delta \leqslant \lambda$, be a fuzzy point. For $k = 0$ the point $ka_\delta \in V$. Since $\pi$ is fuzzy continuous there is a neighbourhood of $(0, a_\delta)$ such that $\mu_{\varepsilon a_\delta}(x) \leqslant \mu_V(x)$, $x \in E$, for $\varepsilon$ a nonzero scalar. Set $x = \varepsilon a$. Then $\mu_{a_\delta}(a) \leqslant \mu_V(\varepsilon a)$. But this contradicts the definition of $V$. ▮

Let $0_\lambda$ be a fuzzy point and $A$, $B$ be fuzzy sets in a ftvs $E$. The following lemmas are needed for Proposition 4.2.

LEMMA 4.8. *If $A$ and $B$ are neighbourhoods of $0_\lambda$ then the sum $A+B$ and the scalar product $\alpha A$, $\alpha \in K$, $\alpha \neq 0$, are neighbourhoods of $0_\lambda$.*

*Proof.* If $A$, $B$ are neighbourhoods of $0_\lambda$ there exist two open fuzzy sets $U$, $V$ in $E$ such that $0_\lambda \in U$, $0_\lambda \in V$ and $U \subset A$, $V \subset B$. The sum $U+V$ can be represented as the union $\bigcup_{x \in E}(x_v + V)$, where $v = \mu_V(x)$. But $x_v + V$ is an open fuzzy set in $E$ since its membership function is given by $\mu_{x_v+V}(y) = \sup_{x_1+x_2=y} \min\{\mu_{x_v}(x_1), \mu_V(x_2)\} = \min\{v, \mu_V(y-x)\} = \min\{\mu_{k_v}(y), \mu_{f_x[V]}(y)\}$, $y \in E$, where $k_v$ is the fuzzy set with constant membership function and $f_x$: $y \to y+x$, i.e., $x_v + V$ is the intersection of two open fuzzy sets. Since $U+V$ is the union of open fuzzy sets, it is itself open. Obviously $U+V \subset A+B$. The membership function of $U+V$ is given by

$$\mu_{U+V}(x) = \sup_{x_1+x_2=x} \min\{\mu_U(x_1), \mu_V(x_2)\}, \qquad x \in E.$$

If $x_1 = x_2 = 0$, then $\min\{\mu_U(x_1), \mu_V(x_2)\} \geqslant \lambda$, and, a fortiori,

$$\mu_{U+V}(0) = \sup_{x_1+x_2=0} \min\{\mu_U(x_1), \mu_V(x_2)\} \geqslant \lambda.$$

The second statement of the Lemma is obvious. ∎

LEMMA 4.9. *If $A$ is a neighbourhood of $0_\lambda$ then there exists a neighbourhood $B$ of $0_\lambda$ such that $B+B \subset A$.*

*Proof.* By fuzzy continuity of the sum, for every neighbourhood $A$ of $0_\lambda$ there exist neighbourhoods $B_1$, $B_2$ of $0_\lambda$ such that $\mu_{B_1+B_2}(x) \leqslant \mu_A(x)$, $x \in E$. Let $B = B_1 \cap B_2$. Then $\mu_{B+B}(x) \leqslant \mu_{B_1+B_2}(x)$, for all $x \in E$. ∎

LEMMA 4.10. *If $A$ is a neighbourhood of $0_\lambda$ then there exists a neighbourhood $B$ of $0_\lambda$ such that $\alpha B \subset A$ for every $\alpha \in K$, $|\alpha| \leqslant 1$.*

*Proof.* Let $A$ be a neighbourhood of $0_\lambda$. Since the scalar product is continuous there exist an $\varepsilon > 0$ and a neighbourhood $U$ of $0_\lambda$ such that for $\xi \in K$, $|\xi| < \varepsilon$, $\mu_{\xi U}(x) \leqslant \mu_A(x)$, $x \in E$. By hypothesis $|\alpha| \leqslant 1$. Hence $|\alpha\xi| < \varepsilon$ and $\mu_{\alpha\xi U}(x) \leqslant \mu_A(x)$. Set $\xi U = B$, and the result follows. ∎

PROPOSITION 4.2. *Let $E$ be a ftvs. For every fuzzy point $0_\lambda$, $0 < \lambda \leqslant 1$, there exists a fundamental system of neighbourhoods $\mathscr{B}(0_\lambda)$ in $E$ for which the following results hold.*

(i) *For each $U \in \mathscr{B}(0_\lambda)$ there is a $V \in \mathscr{B}(0_\lambda)$ with $V+V \subset U$.*

(ii) *For each* $U \in \mathscr{B}(0_\lambda)$ *there is a* $V \in \mathscr{B}(0_\lambda)$ *for which* $\alpha V \subset U$ *for all* $\alpha \in K$, $|\alpha| \leqslant 1$.

(iii) *Every* $U \in \mathscr{B}(0_\lambda)$ *is balanced.*

*Proof.* Let $\mathscr{B}(0_\lambda)$ be any fundamental system of neighbourhoods. Then (i) and (ii) are true by Lemmas 4.9 and 4.10, respectively. For (iii) we first show that the set of balanced hulls of fuzzy sets in $\mathscr{B}(0_\lambda)$ is itself a fundamental system of neighbourhoods of $0_\lambda$. Let $W$ be a neighbourhood of $0_\lambda$. Then there exists a fuzzy set $V \in \mathscr{B}(0_\lambda)$ such that $\mu_{\alpha V}(x) \leqslant \mu_W(x)$, $x \in E$, $|\alpha| \leqslant 1$. The membership function of the balanced hull $U$ of $V$ is $\mu_U(x) = \sup_{|\alpha| \leqslant 1} \mu_{\alpha V}(x)$, $x \in E$. Hence $\mu_W(x) \geqslant \mu_U(x)$, $x \in E$. We show next that every $U$ is balanced. Let $\varepsilon$ be a scalar such that $|\varepsilon| \leqslant 1$. Suppose that $\mu_U(x) = \mu_{\alpha V}(x)$, $x \in E$, for $\alpha = 0$. Then $\mu_{\varepsilon U}(x) = \mu_U(x)$. Next suppose that it is not true that $\mu_U(x) = \mu_{0V}(x)$, $x \in E$. Let $\varepsilon \neq 0$. Then $\mu_{\varepsilon U}(x) = \mu_U(x/\varepsilon) = \sup_{|\alpha| \leqslant 1} \mu_{\alpha V}(x/\varepsilon) = \sup_{|\alpha| \leqslant 1} \mu_{\alpha\varepsilon V}(x)$. Defining $\alpha\varepsilon = \xi$, we obtain $\mu_{\varepsilon U}(x) = \sup_{|\xi| \leqslant \varepsilon} \mu_{\xi V}(x) \leqslant \sup_{|\xi| \leqslant 1} \mu_{\xi V}(x) = \mu_U(x)$. If $\varepsilon = 0$, then $\mu_{0U}(x)$ is nonzero only for $x = 0$. Suppose that $\mu_{0U}(0) > \mu_U(0)$, i.e., there is an $x \neq 0$ such that $\mu_U(x) > \mu_U(0)$. Since $\mu_U(x) = \sup_{|\alpha| \leqslant 1} \mu_{\alpha V}(x)$, there must exist a nonzero $\alpha_0$ such that $\mu_V(y) > \mu_U(0)$ with $y = x/\alpha_0$. But, by the definition of balanced hull, $\mu_U(0) \geqslant \mu_{0V}(0) = \sup_{x \in E} \mu_V(x)$, i.e., $\mu_U(0) \geqslant \mu_V(y)$, which contradicts the initial supposition. ∎

## 5. Fuzzy Differentiation

The treatment here is based on the definition given by Lang [13] of the derivative of a mapping from one topological vector space to another. Let $E$, $F$ be two fuzzy topological vector spaces and let $\phi$ be a mapping from $E$ into $F$. Let $o(t)$ denote any function of a real variable $t$ such that $\lim_{t \to 0} o(t)/t = 0$.

Definition 5.1. The mapping $\phi$ is said to be *tangent* to 0 if given a neighbourhood $W$ of $0_\delta$, $0 < \delta \leqslant 1$, in $F$ there exists a neighbourhood $V$ of $0_\lambda$, $0 < \lambda \leqslant \delta$, in $E$ such that

$$\phi[tV] \subset o(t)\, W,$$

for some function $o(t)$.

Lemma 5.1. *If the mapping* $\phi$ *is tangent to* 0, *then* $\phi$ *is fuzzy continuous at* $0 \in E$.

*Proof.* By Lemma 4.10, for every neighbourhood $W$ of $0_\delta$, $0 < \delta \leqslant 1$, in $F$ there exists a neighbourhood $W'$ such that $\mu_{o(t)W'}(y) \leqslant \mu_W(y)$, $y \in F$,

$|o(t)| \leqslant 1$. By Definition 5.1, for each $W'$ there exist neighbourhoods $V$, $V'$, $V = tV'$, of $0_\lambda$, $0 < \lambda \leqslant \delta$, in $E$ such that $\mu_{\phi[V]}(y) = \mu_{\phi[tV']}(y) \leqslant \mu_{o(t)W'}(y)$, $y \in F$. ∎

LEMMA 5.2. *If $\phi$ and $\psi$ are two mappings tangent to* 0 *then their sum $\phi + \psi$ is a mapping tangent to* 0.

*Proof.* For every neighbourhood $W$ of $0_\delta$, $0 < \delta \leqslant 1$, in $F$ there exists, by Lemma 4.9, a neighbourhood $W'$ such that $W' + W' \subset W$. Hence, $\mu_{o(t)W' + o(t)W'}(y) = \mu_{o(t)[W' + W']}(y) \leqslant \mu_{o(t)W}(y)$, $y \in F$, by Corollary to Lemma 4.1, and Lemma 2.1. By Definition 5.1, there exist two neighbourhoods $V_1$ and $V_2$ of $0_\lambda$, $0 < \lambda \leqslant \delta$, in $E$ such that $\mu_{\phi[tV_1]}(y) \leqslant \mu_{o(t)W'}(y)$, $\mu_{\psi[tV_2]}(y) \leqslant \mu_{o(t)W'}(y)$. Set $V = V_1 \cap V_2$. Then $\mu_{\phi[tV]}(y) \leqslant \mu_{o(t)W'}(y)$, $\mu_{\psi[tV]}(y) \leqslant \mu_{o(t)W'}(y)$. Whereupon $\mu_{\phi[tV] + \psi[tV]}(y) \leqslant \mu_{o(t)W}(y)$. ∎

LEMMA 5.3. *Let $E$, $F$, $G$ be ftvs's. If $\phi$ is a mapping of $E$ into $F$ tangent to* 0 *and $f$ is a linear mapping of $F$ into $G$ that is fuzzy continuous at $0 \in F$, then $f \circ \phi$ is tangent to* 0. *Conversely if $f$ is a linear mapping of $E$ into $F$, fuzzy continuous at $0 \in E$, and $\phi$ is a mapping of $F$ into $G$ tangent to* 0, *then $\phi \circ f$ is tangent to* 0.

*Proof.* By fuzzy continuity of $f$, for every neighbourhood $W$ of $0_\nu$, $0 < \nu \leqslant 1$, in $G$ there is a neighbourhood $V$ of $0_\delta$, $0 < \delta \leqslant \nu$, in $F$ such that $\mu_{f[V]}(z) \leqslant \mu_W(z)$, $z \in G$. For every such $V$ there exists a neighbourhood $U$ of $0_\lambda$, $0 < \lambda \leqslant \delta$, in $E$ such that $\mu_{\phi[tU]}(y) \leqslant \mu_{o(t)V}(y)$, $y \in F$. By Lemmas 2.1 and 4.1, $\mu_{f[\phi[tU]]}(z) \leqslant \mu_{f[o(t)V]}(z) = \mu_{o(t)f[V]}(z) \leqslant \mu_{o(t)W}(z)$, $z \in G$. The proof of the second part of the Lemma proceeds in a similar way. ∎

COROLLARY. *If $\phi$ is a mapping tangent to* 0 *then the scalar product $\alpha\phi$ is a mapping tangent to* 0.

DEFINITION 5.2. Let $E$, $F$ be two ftvs's, each endowed with a $T_1$ fuzzy topology. Let $f\colon E \to F$ be a fuzzy continuous mapping. We say that $f$ is *fuzzy differentiable* at a point $x \in E$ if there exists a linear fuzzy continuous mapping $u$ of $E$ into $F$ such that we can write

$$f(x + y) = f(x) + u(y) + \phi(y), \qquad y \in E,$$

where $\phi$ is tangent to 0. The mapping $u$ is called the *fuzzy derivative* of $f$ at $x$. We denote the fuzzy derivative by $f'(x)$; it is an element of $L(E, F)$, the set of all linear fuzzy continuous mappings of $E$ into $F$.

From this point on we shall suppose that each ftvs is equiped with a $T_1$ fuzzy topology.

PROPOSITION 5.1. *The fuzzy derivative $f'(x)$ of a mapping $f$ of $E$ into $F$ at a point $x \in E$ is unique.*

*Proof.* Suppose that the derivative is not unique. Then there exist two linear fuzzy continuous mappings $u_1, u_2$ such that $u_1(y) + \phi(y) = u_2(y) + \psi(y)$, $y \in E$, where $\phi$ and $\psi$ are each tangent to 0. Set $n(y) = u_1(y) - u_2(y)$, $y \in E$. Then $n(y) = \psi(y) - \phi(y)$, and, by Lemma 5.2, $n(y)$ must be tangent to 0. By hypothesis, $n$ is not zero. Let $a \in E$ be such that $n(a) = r \neq 0$. By Lemma 3.4, for each $r \neq 0$, $r \in F$, there exists an open fuzzy set $B$ in $F$ such that $\mu_B(0) = 1$ and $\mu_B(r) = 0$. If $\mathscr{B}(0_\delta)$ is a fundamental system of balanced neighbourhoods of $0_\delta$, $0 < \delta \leqslant 1$, in $F$, there is a $W \in \mathscr{B}(0_\delta)$ with membership function $\mu_W(z) \leqslant \mu_B(z)$, $z \in F$, with $\mu_{\varepsilon W}(z) \leqslant \mu_W(z)$ for all $|\varepsilon| \leqslant 1$. If $\xi = 1/\varepsilon$ for $\varepsilon \neq 0$, then $\mu_W(\xi r) \leqslant \mu_W(r) = 0$, $|\xi| \geqslant 1$. It follows that for every $r'$ of the form $kr$, $k \in K$, $k \neq 0$, there exists $\xi$ such that $\mu_W(\xi r') = 0$. Since $n$ is tangent to 0, there must be a neighbourhood $V$ of $0_\lambda$, $0 < \lambda \leqslant \delta$, in $E$ such that $\mu_{n[tV]}(z) \leqslant \mu_{o(t)W}(z)$, $z \in F$, whence $\mu_{n[V]}(z) \leqslant \mu_{(o(t)/t)W}(z)$ by linearity of $n$ and Lemma 4.2. In particular, setting $z = r'$ and $t/o(t) = \xi$, we have $\mu_{n[V]}(r') = \sup_{x \in n^{-1}(kr)} \mu_V(x) = 0$, which implies $\mu_V(ka) = 0$. But, by Lemma 4.7, a fuzzy set $V$ with a membership function $\mu_V(ka) = 0$ for all $k \neq 0$ is not a neighbourhood of $0_\lambda$. Hence $n$ must be zero. The fuzzy derivative is thus unique. ▌

PROPOSITION 5.2. *A constant function from a ftvs E into a ftvs F is fuzzy differentiable at every point of E.*

*Proof.* Straightforward.

PROPOSITION 5.3. *The fuzzy derivative of a linear fuzzy continuous mapping u of a ftvs E into a ftvs F exists at every point $x \in E$.*

*Proof.* Straightforward.

PROPOSITION 5.4. *Suppose that $F = \prod_{j=1}^{n} F_j$ is the product ftvs of a finite family of ftvs's $F_j$, $j = 1, 2, \ldots, n$, and that $f$ is a fuzzy continuous mapping of $E$ into $F$. In order for $f$ to be fuzzy differentiable at $x \in E$, a necessary and sufficient condition is that each $p_j \circ f$ be fuzzy differentiable at $x$.*

*Proof.* ($\Rightarrow$) By linearity of the projections $p_j$ we can write, for every $j$, $p_j(f(x+y) - f(x)) = p_j(f'(x)(y)) + p_j(\phi(y))$, $y \in E$. By Definition 3.7, $p_j \circ f'(x)$ is fuzzy continuous and linear, and, by Lemma 5.3, $p_j \circ \phi$ is tangent to 0. Since $f'(x)$ is unique $p_j \circ f'(x)$ is unique.

($\Leftarrow$) For every $j$ we can write $p_j(f(x+y)) - p_j(f(x)) = u_j(y) + \phi_j(y)$, where $u_j$ is a linear fuzzy continuous mapping and $\phi_j$ is tangent to 0. Let $W$ be a neighbourhood of $0_\delta$, $0 < \delta \leqslant 1$, in $F$. By the remark following

Lemma 3.3, $W$ can be expressed without loss in generality as the product of neighbourhoods $W_j$ of $0_\delta$ in $F_j$, $j=1, 2, \ldots, n$. By hypothesis, for every $W_j$ there is a neighbourhood $V_j$ of $0_\lambda$, $0<\lambda\leqslant\delta$, in $E$ such that $\mu_{\phi_j[tV_j]}(z_j)\leqslant \mu_{o(t)W_j}(z_j)$, $z_j\in F_j$. Setting $V=\bigcap_j V_j$, we have $\mu_{\phi_j[tV]}(z_j)\leqslant\mu_{\phi_j[tV_j]}(z_j)\leqslant \mu_{o(t)W_j}(z_j)$, $z_j\in F_j$, for all $j$. But $\mu_{o(t)W}(z)=\min\{\mu_{o(t)W_j}(z_j)\mid j=1,2,\ldots,n\}$. Set $\phi=\prod_{j=1}^n \phi_j$. Then $\mu_{\phi[tV]}(z)\leqslant\min\{\mu_{\phi_j[tV]}(z_j)\}\leqslant\mu_{o(t)W}(z)$, i.e., $\phi$ is tangent to 0. Define $f'(x)=\prod_{j=1}^n u_j$. This mapping is linear and fuzzy continuous by the fuzzy continuity of the $u_j$ (see [6]). The uniqueness of $f'(x)$ follows by the uniqueness of the $u_j$. ∎

PROPOSITION 5.5. *Let E, F, G be ftvs's, f a fuzzy continuous mapping of E into F, and g a fuzzy continuous mapping of F into G. Let* $x\in E$ *and* $y=f(x)$. *If f is fuzzy differentiable at x and g is fuzzy differentiable at y, then the composition* $h=g\circ f$ *is fuzzy differentiable at x.*

*Proof.* By hypothesis $f$ and $g$ are fuzzy differentiable. Hence we can write

$$f(x+r)-f(x)=f'(x)(r)+\phi(r), \qquad r\in E,$$

$$g(y+s)-g(y)=g'(y)(s)+\psi(s), \qquad s\in F,$$

where $\phi$ and $\psi$ are each tangent to 0. Defining $h=g\circ f$, we obtain, after substitution, $h(x+r)-h(x)=g'(y)(f'(x)(r))+g'(y)(\phi(r))+\psi(f'(x)(r)+\phi(r))$, $r\in E$. By Lemma 5.3, $g'(y)\circ\phi$ is tangent to 0. Consider the mapping $\psi\circ(f'(x)+\phi)$. For every neighbourhood $W$ of $0_\nu$, $0<\nu\leqslant 1$, in $G$ there is a neighbourhood $V$ of $0_\delta$, $0<\delta\leqslant\nu$, in $F$ such that $\mu_{\psi[tV]}(z)\leqslant\mu_{o(t)W}(z)$, $z\in G$. Given $V$ there exists a neighbourhood $V'$ of $0_\delta$ such that $V'+V'\subset V$. We can suppose, without loss in generality, that both $V$ and $V'$ belong to a fundamental system of balanced neighbourhoods $\mathscr{B}(0_\delta)$. By the fuzzy continuity of $f'(x)$ there is a neighbourhood $A$ of $0_\lambda$, $0<\lambda\leqslant\delta$, in $E$ such that $\mu_{f'(x)[A]}(y)\leqslant\mu_{V'}(y)$, which implies that $\mu_{tf'(x)[A]}(y)\leqslant\mu_{tV'}(y)$, i.e., $\mu_{f'(x)[tA]}(y)\leqslant\mu_{tV'}(y)$, $y\in F$. For every $V'$ there exists a neighbourhood $B$ of $0_\lambda$ in $E$ for which $\mu_{\phi[tB]}(y)\leqslant\mu_{o(t)V'}(y)$ and, for $|o(t)/t|\leqslant 1$, $\mu_{o(t)V'}(y)\leqslant\mu_{tV'}(y)$, $y\in F$. Setting $U=A\cap B$ and using Lemma 2.1, we obtain $\mu_{\phi[tU]+f'(x)[tU]}(y)\leqslant\mu_{tV}(y)$, which implies that $\psi[\phi[tU]+f'(x)[tU]]\subset\psi[tV]\subset o(t)W$, i.e., the mapping $\psi\circ(f'(x)+\phi)$ from $E$ to $G$ is tangent to 0. Thus at last we can write $h(x+r)-h(x)=g'(y)\circ f'(x)(r)+\chi(r)$, $r\in E$, where $g'(y)\circ f'(x)$ is linear and fuzzy continuous, and $\chi$, the sum of two mappings tangent to 0, is tangent to 0. ∎

PROPOSITION 5.6. *Let f, g be two fuzzy continuous mappings of E into F. If f and g are fuzzy differentiable at x, so are* $f+g$ *and* $\alpha f$, $\alpha\in K$.

*Proof.* The mapping $f+g$ is composed of $x\to(f(x), g(x))$ from $E$ into

$F \times F$ and of $(u, v) \to u + v$ from $F \times F$ into $F$. The first is fuzzy differentiable by Proposition 5.4 and the second by definition of the sum; the result follows from Proposition 5.5. For $\alpha f$ it is sufficient to note that the mapping $u \to \alpha u$ of $F$ into itself is fuzzy differentiable by Proposition 5.3. ∎

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