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On Modules with Trivial Self-Extensions

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Throughout this note, all rings will be self-basic connected artin algebras over a fixed commutative local artin ring k and all modules will be finitely generated. Homomorphisms will be written on the opposite side of the scalars. By mod-A (resp. A-mod), we denote the category of all right (resp. left) A-modules for an artin algebra A. The ordinary duality functor is denoted by D: mod-$A \rightleftarrows A$-mod, i.e., $D = \mathrm{Hom}_k(?, I)$ with the minimal injective cogenerator I over k.

We are interested in the bimodules ${}_BT_A$ with the following properties:

(A1) $B = \mathrm{End}(T_A)$ and $\mathrm{End}({}_BT) = A$,

(A2) $\mathrm{Ext}^i_B(T, T) = 0 = \mathrm{Ext}^i_A(T, T)$ for all integers $i \geqq 1$.

It is well known that a tilting module T_A with $B = \mathrm{End}(T_A)$ has the above properties. Another example of such a module appears in the study of generalized Nakayama conjecture: Let

$$0 \to A \to I_0 \to I_1 \to I_2 \to \cdots$$

be the minimal injective resolution of the module A_A. Let T_A be the direct sum of all representatives of indecomposable injective A-modules which appear in the above resolution as direct summands of some term and put $B = \mathrm{End}(T_A)$. Then the bimodule ${}_BT_A$ has the properties. This fact was already used by Tachikawa [3] in the study of Nakayama's conjecture.

By $n(X_A)$ we denote the number of nonisomorphic indecomposable direct summands of a module X_A.

For a tilting module T_A with $B = \mathrm{End}(T_A)$, it is proved by Happel and Ringel [2] that ${}_BT$ becomes again a tilting module and $n({}_BT) = n(T_A)$.

Auslander and Reiten [1] conjectured that the injective module T_A

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defined as the second example above is, in fact, an injective cogenerator. They called it the generalized Nakayama conjecture. It is obvious that the validity of their conjecture follows from the conditions $\mathrm{id}({}_BT)=\mathrm{id}(T_A)$ or $n({}_BT)=n(T_A)$.

Therefore, it is natural to consider the following problems on the bimodule ${}_BT_A$ which has the properties (A1) and (A2):

Problem 1. $pd({}_BT)=pd(T_A)$? ($\mathrm{id}({}_BT)=\mathrm{id}(T_A)$?)

Problem 2. $n({}_BT)=n(T_A)$?

In this note, we consider the above problems and prove the following

THEOREM. (1) *If* $pd({}_BT), pd(T_A)<\infty$ *then* $pd({}_BT)=pd(T_A)$.

(2) *If* $({}_BT), \mathrm{id}(T_A)<\infty$ *then* $\mathrm{id}({}_BT)=\mathrm{id}(T_A)$.

(3) *If* $pd({}_BT)$, $pd(T_A)<\infty$ *or* $\mathrm{id}({}_BT)$, $\mathrm{id}(T_A)<\infty$ *then* $n({}_BT)=n(T_A)$.

We start by characterizing the one-sided module T_A for which the bimodule ${}_BT_A$ with $B=\mathrm{End}(T_A)$ has the properties (A1) and (A2).

PROPOSITION 1. *Let T_A be a module with $B=\mathrm{End}(T_A)$ and X_A another module. Assume that there is an infinite exact sequence*

$$0\to X\to T_0\xrightarrow{f_0}T_1\xrightarrow{f_1}T_2\xrightarrow{f_2}\cdots,$$

such that $T_k\in\mathrm{add}(T_A)$ and $\mathrm{Ext}^1_A(\mathrm{Ker}\, f_{k+1}, T)=0$ for any $k\geqq 0$. Then the canonical morphism $d_X\colon X_A\to\mathrm{Hom}_B(\mathrm{Hom}_A(X,T),T)_A$ is an isomorphism and further $\mathrm{Ext}^i_B(\mathrm{Hom}_A(X,T),T)=0$ for any $i\geqq 1$.

Proof. We denote the contravariant functors $\mathrm{Hom}(?,{}_BT_A)$ and $\mathrm{Ext}^i(?,{}_BT_A)$ by $F=F_0$ and F_i, respectively. From the short exact sequences

$$0\to\mathrm{Ker}\, f_k\to T_k\to\mathrm{Ker}\, f_{k+1}\to 0\qquad (k\geqq 0,\ \mathrm{Ker}\, f_0=X),$$

we have the following exact sequences

$$0\to F(\mathrm{Ker}\, f_{k+1})\to F(T_k)\to F(\mathrm{Ker}\, f_k)\to F_1(\mathrm{Ker}\, f_{k+1})=0.$$

Hence we obtain a projective resolution of the left B-module $F(X)$,

$$\cdots\xrightarrow{F(f_2)}F(T_2)\xrightarrow{F(f_1)}F(T_1)\xrightarrow{F(f_0)}F(T_0)\to F(X)\to 0,$$

where $\mathrm{Cok}\, F(f_k)=F(\mathrm{Ker}\, f_k)$ for each $k\geqq 0$.

Now consider the following commutative diagrams with exact rows:

$$\begin{array}{ccccccccc} 0 \to & \operatorname{Ker} f_k & \to & T_k & \to & \operatorname{Ker} f_{k+1} & \to 0 & & \\ & \downarrow{\scriptstyle d_{\operatorname{Ker} f_k}} & & \downarrow{\scriptstyle d_{T_k}} & & \downarrow{\scriptstyle d_{\operatorname{Ker} f_{k+1}}} & & & \\ 0 \to & F^2(\operatorname{Ker} f_k) & \to & F^2(T_k) & \to & F^2(\operatorname{Ker} f_{k+1}) & \to & F_1F(\operatorname{Ker} f_k) & \to 0. \end{array}$$

Since $\operatorname{Ker} f_k$ are cogenerated by T_k, all the maps $d_{\operatorname{Ker} f_k}$ are monomorphisms and all the maps d_{T_k} are isomorphisms by our assumption. Then, by the snake lemma, we know that all the maps $d_{\operatorname{Ker} f_k}$ are isomorphisms and $F_iF(X) \cong F_1F(\operatorname{Ker} f_{i-1}) = 0$ for all $i \geqq 1$.

This completes the proof.

COROLLARY 2. *Let $B = \operatorname{End}(T_A)$ and assume that there is an infinite exact sequence*

$$0 \to A \to T_0 \xrightarrow{f_0} T_1 \xrightarrow{f_1} T_2 \xrightarrow{f_2} \cdots,$$

such that $T_k \in \operatorname{add}(T_A)$ and $\operatorname{Ext}^1_A(\operatorname{Ker} f_k, T) = 0$ for all $k \geqq 0$. Then the bimodule ${}_BT_A$ has the property (A1) *and satisfies the condition $\operatorname{Ext}^i_B(T, T) = 0$ for all $i \geqq 1$.*

PROPOSITION 3. *Assume that the bimodule ${}_BT_A$ has the property* (A1) *and satisfies the condition $\operatorname{Ext}^1_A(T, T) = 0$. Suppose that a module ${}_BY$ is T-reflexive (i.e., $Y \cong F^2(Y)$) and satisfies the condition $\operatorname{Ext}^i_B(Y, T) = 0$ for all $i \geqq 1$. Then there is an infinite exact sequence*

$$0 \to F(Y)_A \to T_0 \xrightarrow{f_0} T_1 \xrightarrow{f_1} T_2 \xrightarrow{f_2} \cdots,$$

such that $T_k \in \operatorname{add}(T_A)$ and $\operatorname{Ext}^1_A(\operatorname{Ker} f_{k+1}, T) = 0$ for all $k \geqq 0$.

Proof. Let $\cdots \to^{g_2} P_2 \to^{g_1} P_1 \to^{g_0} P_0 \to Y \to 0$ be a projective resolution of the left B-module Y. From the exact sequences

$$0 \to \operatorname{Cok} g_{k+1} \to P_k \to \operatorname{Cok} g_k \to 0 \qquad (k \geqq 0, \operatorname{Cok} g_0 = Y),$$

we have the sequences

$$0 \to F(\operatorname{Cok} g_k) \to F(P_k) \to F(\operatorname{Cok} g_{k+1}) \to F_1(\operatorname{Cok} g_k) \cong F_{k+1}(Y) = 0.$$

Hence we obtain an infinite exact sequence

$$0 \to F(Y) \to F(P_0) \xrightarrow{F(g_0)} F(P_1) \xrightarrow{F(g_1)} F(P_2) \xrightarrow{F(g_2)} \cdots,$$

where $\operatorname{Ker} F(g_k) = F(\operatorname{Cok} g_k)$ and $F(P_k) \in \operatorname{add}(T_A)$ for all $k \geqq 0$.

We have to prove $F_1F(\text{Cok } g_k)=0$ for each $k \geqq 1$. By the condition $\text{Ext}_A^1(T, T)=0$, we have the following commutative diagrams with exact rows:

$$\begin{array}{ccccccccc} 0 \to & \text{Cok } g_{k+1} & \to & P_k & \to & \text{Cok } g_k & \to 0 & & \\ & \downarrow{\scriptstyle d_{\text{Cok } g_{k+1}}} & & \downarrow{\scriptstyle d_{P_k}} & & \downarrow{\scriptstyle d_{\text{Cok } g_k}} & & & \\ 0 \to & F^2(\text{Cok } g_{k+1}) & \to & F^2(P_k) & \to & F^2(\text{Cok } g_k) & \to & F_1F(\text{Cok } g_{k+1}) & \to 0. \end{array}$$

From those diagrams, by using the snake lemma, we see that $\text{Ker } d_{\text{Cok } g_{k+1}} = 0$, $\text{Cok } d_{\text{Cok } g_{k+1}} \cong \text{Ker } d_{\text{Cok } g_k}$ and $\text{Cok } d_{\text{Cok } g_k} \cong F_1F(\text{Cok } g_{k+1})$ for each $k \geqq 0$. For $k=0$, by our assumption, the map $d_{\text{Cok } g_k}$ is an isomorphism. Therefore, by induction on k, we know that all the maps $d_{\text{Cok } g_k}$ are isomorphisms and $F_1F(\text{Cok } g_{k+1}) \cong \text{Cok } d_{\text{Cok } g_k} = 0$.
This finishes the proof.

COROLLARY 4. *Assume that the bimodule* ${}_BT_A$ *has the property* (A1) *and satisfies the conditions* $\text{Ext}_A^1(T, T)=0$ *and* $\text{Ext}_B^i(T, T)=0$ *for any* $i \geqq 1$. *Then there is an infinite exact sequence*

$$0 \to A \to T_0 \xrightarrow{f_0} T_1 \xrightarrow{f_1} T_2 \xrightarrow{f_2} \cdots,$$

such that $T_k \in \text{add}(T_A)$ *and* $\text{Ext}_A^1(\text{Ker } f_k, T)=0$ *for any* $k \geqq 0$.

Combining Corollary 2 with Corollary 4, we get a characterization of bimodules ${}_BT_A$ possessing the properties (A1) and (A2).

PROPOSITION 5. *For a module* T_A *with* $B=\text{End}(T_A)$, *the following two assertions are equivalent*:

(1) ${}_BT_A$ *has the properties* (A1) *and* (A2).

(2) (i) $\text{Ext}_A^i(T, T)=0$ *for any* $i \geqq 1$;

(ii) *there is an infinite exact sequence*

$$0 \to A \to T_0 \xrightarrow{f_0} T_1 \xrightarrow{f_1} T_2 \xrightarrow{f_2} \cdots,$$

such that $T_k \in \text{add}(T_A)$ *and* $\text{Ext}_A^1(\text{Ker } f_k, T)=0$ *for any* $k \geqq 0$.

From now on, we assume that the bimodule ${}_BT_A$ has the properties (A1) and (A2).

We call an exact sequence $0 \to X \to T_0 \to^{f_0} T_1 \to^{f_1} T_2 \to^{f_2} \cdots$, a T-sequence of X_A if $T_k \in \text{add}(T_A)$ and $\text{Ext}_A^1(\text{Ker } f_{k+1}, T)=0$ for any

$k \geqq 0$. Further, we define for a module X_A which has a T-sequence its T-dimension, T-dim(X_A), to be the natural number or the symbol ∞:

inf$\{n \mid$There is a T-sequence of X such that $T_n \neq 0$ and $T_k = 0$ for any $k \geqq n+1\}$.

By the proofs of Proposition 1 and Proposition 3, we know that there is a bijection between the set of all projective resolutions of ${}_BF(X)$ and the set of all T-sequences of X_A. By this bijection, we have

COROLLARY 6. *The equalities* $T\text{-dim}(X_A) = pd({}_BF(X))$ *and* $pd(F(Y)_A) = T\text{-dim}({}_BY)$ *hold for any modules* X_A *and* ${}_BY$ *which have their T-sequences.*

By using this corollary, we can prove the statements (1) and (2) in the theorem.

PROPOSITION 7. (1) $pd({}_BT), pd(T_A) < \infty \Rightarrow pd({}_BT) = pd(T_A)$.

(2) $\mathrm{id}({}_BT), \mathrm{id}(T_A) < \infty \Rightarrow \mathrm{id}({}_BT) = \mathrm{id}(T_A)$.

Proof. (1) By the above corollary, we may assume $pd(T_A) \leqq pd({}_BT)$. Suppose that $pd(T_A) = n$, $pd({}_BT) = T\text{-dim}(A_A) = n+m$ and $m \geqq 1$. Let

$$0 \to A \to T_0 \xrightarrow{f_0} T_1 \xrightarrow{f_1} \cdots \longrightarrow T_{n+m-1} \xrightarrow{f_{n+m-1}} T_{n+m} \to 0$$

be a T-sequence of A_A. From the exact sequences $0 \to \mathrm{Ker}\, f_k \to T_k \to \mathrm{Ker}\, f_{k+1} \to 0$ $(0 \leqq k \leqq n+m-1)$, we have $\mathrm{Ext}_A^i(T, \mathrm{Ker}\, f_{k+1}) \cong \mathrm{Ext}_A^{i+1}(T, \mathrm{Ker}\, f_k)$ for any $i \geqq 1$. Therefore, we have the isomorphisms

$$\begin{aligned} 0 = \mathrm{Ext}_A^{n+1}(T, \mathrm{Ker}\, f_{m-1}) &\cong \mathrm{Ext}_A^n(T, \mathrm{Ker}\, f_m) \\ &\cong \cdots \cong \mathrm{Ext}_A^1(T, \mathrm{Ker}\, f_{n+m-1}). \end{aligned}$$

$\mathrm{Ext}_A^1(T, \mathrm{Ker}\, f_{n+m-1}) = 0$ implies that the short exact sequence

$$0 \to \mathrm{Ker}\, f_{n+m-1} \to T_{n+m-1} \to T_{n+m} \to 0$$

is splitting. Obviously, this contradicts $T\text{-dim}(A_A) = n+m$.

The assertion (2) is now obvious, since ${}_BT_A$ has the properties (A1) and (A2) if and only if so does the dual bimodule ${}_ADT_B$.

In order to prove Proposition 9, we need the following.

LEMMA 8. *Consider the commutative diagram with exact rows*:

$$\begin{array}{ccccccccc} X_1 & \to & X_2 & \to & X_3 & \to & X_4 & \to & X_5 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\ Y_1 & \to & Y_2 & \to & Y_3 & \to & Y_4 & \to & Y_5. \end{array}$$

Denote by K_i, C_i, *and* I_i *the modules* Ker f_i, Cok f_i, *and* Im f_i, *respectively. Then the isomorphisms*

$$H(K_3 \to K_4 \to K_5) \cong H(I_2 \to I_3 \to I_4) \cong H(C_1 \to C_2 \to C_3)$$

hold, where $H(X \to^f Y \to^g Z)$ *denotes the homology group* (Ker g)/(Im f) *for a sequence* $X \to^f Y \to^g Z$ *with* $gf = 0$.

Proof. An exercise in homological algebra.

PROPOSITION 9. *Let* $0 \to X_1 \to^u P_0 \to^p X_A \to 0$ *be an exact sequence with* P_0 *projective. Put* $X^* = \text{Cok } F(p)$. *Denote by* v *the inclusion map* $X^* \to F(X_1)$ *and by* q *the projection map* $F(X_1) \to F_1(X)$. *By using these notations, the following statements hold*:

(1) $F_{i+1}(X^*) \cong F_i F(X)$ *for any* $i \geqq 1$.

(2) $F_1(X^*) \cong \text{Cok } d_X$.

(3) *The following infinite sequence is exact*:

$$0 \to FF_1(X) \xrightarrow{\text{Cok } d_{X_1} \cdot F(q)} \text{Cok } d_{X_1} \to \text{Ker } d_X$$

$$\to F_1F_1(X) \xrightarrow{F_1(q)} F_1F(X_1) \xrightarrow{F_1(v)} F_1(X^*)$$

$$\to F_2F_1(X) \xrightarrow{F_2(q)} F_2F(X_1) \xrightarrow{F_2(v)} F_2(X^*)$$

$$\to \cdots.$$

Proof. From the exact sequence $0 \to F(X) \to^{F(p)} F(P_0) \to X^* \to 0$, we have $0 \to F(X^*) \to F^2(P_0) \to^{F^2(p)} F^2(X) \to F_1(X^*) \to 0$ and $F_{i+1}(X^*) \cong F_iF(X)$ for any $i \geqq 1$. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 \to & X_1 & \xrightarrow{u} & P_0 & \xrightarrow{p} & X & \to 0 \\
 & \downarrow{\scriptstyle d_{X_1}} & & & & & \\
 & F^2(X_1) & & \downarrow{\scriptstyle d_{P_0}} & & \downarrow{\scriptstyle d_X} & \\
 & \downarrow{\scriptstyle F(v)} & & & & & \\
0 \to & F(X^*) & \to & F^2(P_0) & \xrightarrow[F^2(p)]{} & F^2(X) & \to F_1(X^*) \to 0.
\end{array}$$

By the snake lemma, we see that $\text{Cok } d_X \cong F_1(X^*)$, $\text{Ker}(F(v) \cdot d_{X_1}) = 0$ and $\text{Cok}(F(v) \cdot d_{X_1}) \cong \text{Ker } d_X$.

From the exact sequence $0 \to X^* \to^{v} F(X_1) \to^{q} F_1(X) \to 0$, we have the long exact sequence

$$0 \to FF_1(X) \xrightarrow{F(q)} F^2(X_1) \xrightarrow{F(v)} F(X^*) \to F_1^2(X) \xrightarrow{F_1(q)} F_1F(X_1) \xrightarrow{F_1(v)} F_1(X^*) \to \cdots.$$

Next consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_1 & \longrightarrow & F(X^*) & \to & \operatorname{Cok} d_X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d_{X_1}} & & \| & & \downarrow{\scriptstyle m} & & \\ 0 \to FF_1(X) & \xrightarrow[F(q)]{} & F^2(X_1) & \xrightarrow[F(v)]{} & F(X^*) & \to & F_1F_1(X) & \xrightarrow[F_1(q)]{} & F_1F(X_1), \end{array}$$

where the map m is the canonical one induced from the commutativity of the left-hand square. Applying the previous lemma to the above diagram, we have the isomorphisms

$$\operatorname{Ker} m \cong H(0 \to \operatorname{Ker} m \to 0) \cong H(FF_1(X) \to \operatorname{Cok} d_{X_1} \to 0)$$

and

$$0 \cong H(\operatorname{Ker} d_{X_1} \to 0 \to \operatorname{Ker} m) \cong H(0 \to FF_1(X) \to \operatorname{Cok} d_{X_1}).$$

Those isomorphisms imply the exactness of the sequence:

$$0 \to FF_1(X) \to \operatorname{Cok} d_{X_1} \to \operatorname{Ker} m \to 0.$$

It is easy to see that $\operatorname{Im}(m)$ is isomorphic to $\operatorname{Im}(F(X^*) \to F_1^2(X))$. Hence we have the desired long exact sequence. This completes the proof.

Using the long exact sequence in the above proposition, we can prove statement (3) in the theorem.

THEOREM 10. *Assume* $\operatorname{id}({}_BT)$, $\operatorname{id}(T_A) < \infty$. *Then, for any module* X, *the following equality holds*:

$$\mathbf{dim}\, X = \sum_{i,j \geq 0} (-1)^{i+j} \mathbf{dim}\, F_iF_j(X) \qquad (F_0 = F);$$

that is, $K_0(A) \cong K_0(B)$ *by the correspondence*: $\mathbf{dim}\, X \mapsto \sum_{i \geq 0} (-1)^i \mathbf{dim}\, F_i(X)$. *Here,* $K_0(A)$ *and* $K_0(B)$ *stand for the Grothendieck groups of* A *and* B, *respectively.*

Proof. Let $\cdots \to^{g_2} P_2 \to^{g_1} P_1 \to^{g_0} P_0 \to X \to 0$ be a projective resolution of X. Put $X_k = \mathrm{Cok}\, g_k$ $(k \geqq 0)$. Then, by the previous proposition, we have

$$\begin{aligned} 0 &\to FF_1(X) \to \mathrm{Cok}\, d_{X_1} \to \mathrm{Ker}\, d_X \\ &\to F_1F_1(X) \to F_1F(X_1) \to \mathrm{Cok}\, d_X \to F_2F_1(X) \\ &\to F_2F(X_1) \to F_1F(X) \to F_3F_1(X) \\ &\to F_3F(X_1) \to F_2F(X) \to \cdots . \end{aligned} \tag{E$_0$}$$

and, for any $i \geqq 1$,

$$\mathrm{Ker}\, d_{X_i} = 0, \qquad FF_1(X_i) \cong \mathrm{Cok}\, d_{X_{i+1}}$$

and

$$\begin{aligned} 0 &\to F_1F_1(X_i) \to F_1F(X_{i+1}) \to \mathrm{Cok}\, d_{X_i} \to F_2F_1(X_i) \\ &\to F_2F(X_{i+1}) \to F_1F(X_i) \to F_3F_1(X_i) \to F_3F(X_{i+1}) \\ &\to F_2F(X_i) \to \cdots . \end{aligned} \tag{E$_i$}$$

Here it should be noted that $F_1(X_i) \cong F_{i+1}(X)$ for any $i \geqq 1$.

By our assumption, we can consider the sums

$$M_i = \sum_{k \geqq 1} (-1)^{k+i}\, \mathbf{dim}\, F_kF_i(X) \qquad \text{for} \quad i \geqq 0,$$

and

$$N_j = \sum_{k \geqq 1} (-1)^k\, \mathbf{dim}\, F_kF(X_j) \qquad \text{for} \quad j \geqq 1.$$

Then we have

$$\begin{aligned} M_0 + M_1 &- \mathbf{dim}\, FF_1(X) \\ &= \mathbf{dim}\, \mathrm{Ker}\, d_X - \mathbf{dim}\, \mathrm{Cok}\, d_X - \mathbf{dim}\, \mathrm{Cok}\, d_{X_1} - N_1, \\ &\qquad M_2 = \mathbf{dim}\, \mathrm{Cok}\, d_{X_1} + N_1 + N_2 \end{aligned}$$

and

$$M_{i+1} + (-1)^i\, \mathbf{dim}\, FF_i(X) = (-1)^{i+1}(N_i + N_{i+1}),$$

from (E_0), (E_1), and (E_i) $(i \geqq 2)$, respectively.

For a large number k, the terms M_k, N_k, and $\mathbf{dim}\, FF_k(X)$ are all zero (for M_k and $\mathbf{dim}\, FF_k(X)$ this is obvious and for N_k it is proved as follows: We have the isomorphism $F_iF(X_k) \cong F_{i+1}(X_k^*)$ and we may assume

$F_1(X_k) \cong F_{k+1}(X) = 0$; therefore, by definition, $X_k^* \cong F(X_{k+1})$ and $F_{i+1}(X_k^*) \cong F_{i+1}F(X_{k+1}) \cong F_{i+2}(X_{k+2}^*)$. Thus we have the isomorphisms

$$F_iF(X_k) \cong F_{i+1}F(X_{k+1}) \cong \cdots \cong F_{i+r}F(X_{k+r}) \cong F_{i+r+1}(X_{k+r}^*) = 0$$

for some $r \geqq k-(i+1)$). Hence, we get the equalities

$$\begin{aligned}\left(\sum_{i \geqq 0} M_i\right) + \left(\sum_{i \geqq 1} (-1)^i \,\mathbf{dim}\, FF_i(X)\right) &= \mathbf{dim}\,\mathrm{Ker}\, d_X - \mathbf{dim}\,\mathrm{Cok}\, d_X \\ &= \mathbf{dim}\, X - \mathbf{dim}\, F^2(X)\end{aligned}$$

and

$$\mathbf{dim}\, X = \sum_{i,j \geqq 0} (-1)^{i+j}\,\mathbf{dim}\, F_iF_j(X).$$

This completes the proof.

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