Small subsets of groups

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ABSTRACT

Given an infinite group $G$ and an infinite cardinal $\kappa \leq |G|$, we say that a subset $A$ of $G$ is $\kappa$-large ($\kappa$-small) if there exists $F \in [G]^{<\kappa}$ such that $G = FA$ ($G \setminus FA$ is $\kappa$-large for each $F \in [G]^{<\kappa}$). The subject of the paper is the family $S_\kappa$ of all $\kappa$-small subsets. We describe the left ideal of the right topological semigroup $\beta G$ determined by $S_\kappa$. We study interrelations between $\kappa$-small and other ($P_\kappa$-small and $\kappa$-thin) subsets of groups, and prove that $G$ can be generated by some 2-thin subsets. We partition $G$ in countable many subsets which are $\kappa$-small for each $\kappa \geq \omega$. We show that $[G]^{<\kappa}$ is dual to $S_\kappa$ provided that either $\kappa$ is regular and $\kappa = |G|$, or $G$ is Abelian and $\kappa$ is a limit cardinal, or $G$ is a divisible Abelian group.

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1. Introduction

Let $G$ be an infinite group, $\kappa$ be an infinite cardinal such that $\kappa \leq |G|$. A subset $A$ of $G$ is called:

- $\kappa$-large if there exists $F \in [G]^{<\kappa}$ such that $G = FA$;
- $\kappa$-small if $G \setminus FA$ is $\kappa$-large for every $F \in [G]^{<\kappa}$;
- $\kappa$-thick if, for every $F \in [G]^{<\kappa}$, there exists $a \in A$ such that $Fa \subseteq A$;
- piecewise $\kappa$-large if there exists $F \in [G]^{<\kappa}$ such that $FA$ is $\kappa$-thick;
- $\kappa$-extralarge if $A \cap L$ is $\kappa$-large for every $\kappa$-large subset $L$ of $G$.

To be more precise, we should add to each of above types the adjective “left” because each of them has the “right” counterpart, for example, a subset $A$ is right $\kappa$-large if $G = AF$ for some $F \in [G]^{<\kappa}$. But in this paper we deal with only left-sided variants, so we omit the adjective left.

The $\omega$-large and piecewise $\omega$-large subsets are known in topological dynamic as syndedec and piecewise syndedec subsets. A lot of their usages can be find in [8]. The names large and small subsets for $\omega$-large and $\omega$-small subsets were suggested in [1]. Every infinite group $G$ can be partitioned in $|G|$ $\omega$-thick subsets. This statement was proved in [10] to show that every infinite totally bounded topological group can be partitioned in $|G|$ dense subset. For generalization of this statement see [2,14].
We note that all above types of subsets are not of the specific group nature, but can be defined (see [12,13] or Section 2 below) for some general structures, namely the balleans, which are the counterparts of the uniform topological spaces.

The subject of this paper is the family $S_k$ of all $\kappa$-small subsets of a group $G$. In Section 2, using the ballean approach, we show that $S_k$ is a translation invariant ideal in the Boolean algebra of all subsets of $G$ and, for a subset $A$ of $G$, the following statements are equivalent: $A$ is $\kappa$-small, $A$ is not piecewise $\kappa$-large, $G \setminus A$ is $\kappa$-extralarge.

Let $G$ be an infinite group, $\kappa$, $\kappa'$ be infinite cardinals such that $\kappa < \kappa' \leq |G|$. It follows directly from corresponding definitions that every $\kappa$-large subset of $G$ is $\kappa'$-large and every $\kappa'$-thick subset is $\kappa$-thick. In contrast, the families $S_k$ and $S_{\omega_1}$, could be incomparable (Theorems 3.1 and 4.2).

For every discrete group $G$, the Stone–Čech compactification $\beta G$ has a natural structure of compact right topological semigroup. We identify $\beta G$ with the set of all ultrafilters on $G$. In Section 3, we consider the set

$$\hat{S}_k = \{ p \in \beta G : G \setminus A \in p \text{ for all } A \in S_k \},$$

which is a closed left ideal of semigroup $\beta G$. We show that $\hat{S}_\omega$ is the minimal closed ideal of $\beta G$ but, for $\kappa > \omega$, $\hat{S}_k$ needs not to be a right ideal.

Given a group $G$ and a cardinal $\kappa \leq |G|$, we say that a subset $A$ of $G$ is

- $P_\kappa$-small if there exists an injective $\kappa$-sequence $(x_\alpha)_{\alpha < \kappa}$ such that the subsets $\{x_\alpha A : \alpha < \kappa\}$ are pairwise disjoint;
- $\kappa$-thin if $|gA \cap A| < \kappa$ for every element $g \in G$, $g \neq e$, where $e$ is the identity of $G$.

For Abelian groups, $P_\omega$-small subsets were introduced by I. Prodanov [11]. C. Chou [3] applied 3-thin subsets to show that every infinite amenable group $G$ admits $2^{2^{[G]}}$ distinct left invariant Banach measures. $\omega$-thin subsets are widely used also in $\beta$-theory. For example, let $G$ be an infinite group and $A$ be a $\omega$-thin subset of $G$ such that $|A| = |G|$. Then the principal left ideals $\{\beta Gp : p \in A, \ p \in \beta G \setminus G\}$ are pairwise disjoint. This implies van Douwen Theorem [8, Theorem 6.53] stating that the semigroup $\beta G$ has $2^{2^{[G]}}$ pairwise disjoint closed left ideals. For modifications of $P_\omega$-small subsets see [4], for $\kappa$-thin subsets, where $\kappa$ is a natural number, see [9].

In Section 4, we study interrelations between $\kappa$-small, $P_\kappa$-small and $\kappa$-thin subsets of a group. By [12, Theorem 13.1], every infinite group $G$ can be generated by some $\omega$-small subset. We strengthen this statement showing that $G$ can be generated by some 2-thin subset.

In Section 5, we prove that every infinite group $G$ can be partitioned in $\omega$ subsets which are $\kappa$-small for each $\kappa$ such that $\omega \leq \kappa \leq |G|$. Given a family $\mathcal{F}$ of subsets of a group $G$, we put

$$\mathcal{F}^* = \{ X \subset G : X^{-1}A \neq G \text{ for every } A \in \mathcal{F} \}.$$  

In [16], W. Serediński asked if there exists (in ZFC) a family $\mathcal{F}$ of subsets of $\mathbb{R}$ such that $\mathcal{F}^* = \{\text{countable subsets of } \mathbb{R}\}$. In [17], S. Solecki answered this question in the affirmative proving that, for every infinite Abelian group $G$ and every infinite regular cardinal $\kappa$ with $\kappa \leq |G|$, there exists a translation invariant ideal $I$ of subsets of $G$ such that $I^* = |G|^{<\kappa}$. This was done as a byproduct of some complicated construction and $I$ has some complementary properties.

In Section 6, we prove that $|G|^{<\kappa} = S_k^*$ provided that either $\kappa$ is regular and $\kappa \leq |G|$, or $G$ is Abelian and $\kappa$ is a limit cardinal, or $G$ is a divisible Abelian group. On the other hand, we show that, for every infinite group of regular cardinality, there is not a family $\mathcal{F}$ of subsets of $G$ such that $S_k = \mathcal{F}^*$.

2. Ballean context

A ball structure is a triple $B = (X, P, B)$, where $X, P$ are non-empty sets and, for every $x \in X$ and $\alpha \in P$, $B(x, \alpha)$ is a function from $X \times P$ to $\mathcal{P}(X)$. It is supposed that $x \in B(x, \alpha)$ for all $x \in X$ and $\alpha \in P$. The set $X$ is called the support of $B$ and $P$ is called the set of radii.

Given any $x \in X$ and $A \subset X$, $\alpha \in P$, we put

$$B^*(x, \alpha) = \{ y \in X : x \in B(y, \alpha) \} \quad \text{and} \quad B(A, \alpha) = \bigcup_{\alpha \in A} B(a, \alpha).$$

A ball structure $B$ is called:

- lower symmetric if for every $\alpha, \beta \in P$, there exist $\alpha', \beta' \in P$ such that, for every $x \in X$,

$$B^*(x, \alpha') \subset B^*(x, \alpha) \quad \text{and} \quad B^*(x, \beta') \subset B^*(x, \beta);$$

- upper symmetric if, for any $\alpha, \beta \in P$, there exist $\alpha', \beta' \in P$ such that, for every $x \in X$,

$$B(x, \alpha) \subset B^*(x, \alpha') \quad \text{and} \quad B(x, \beta) \subset B^*(x, \beta').$$
• **lower multiplicative** if, for any \( \alpha, \beta \in P \), there exists \( y \in P \) such that, for every \( x \in X \),

\[
B(B(x, y), y) \subseteq B(x, \alpha) \cap B(x, \beta);
\]

• **upper multiplicative** if, for any \( \alpha, \beta \in P \), there exists \( y \in P \) such that, for every \( x \in X \),

\[
B(B(x, \alpha), \beta) \subseteq B(x, y).
\]

Let \( \mathcal{B} = (X, P, B) \) be a lower symmetric and lower multiplicative ball structure. Then the family

\[
\left\{ \bigcup_{\alpha \in P} B(x, \alpha) \times B(x, \alpha) : \alpha \in P \right\}
\]

is a base of entourages for some (uniquely determined) uniformity on \( X \). On the other hand, if \( \mathcal{U} \subseteq X \times X \) is a uniformity on \( X \), then the ball structure \((X, \mathcal{U}, B)\) is lower symmetric and lower multiplicative, where \( B(x, U) = \{ y \in X : (x, y) \in U \} \). Thus, the lower symmetric and lower multiplicative ball structures can be identified with uniform topological spaces.

We say that a ball structure \( \mathcal{B} \) is a **balllean** if \( \mathcal{B} \) is upper symmetric and upper multiplicative. A structure on \( X \), equivalent to a balllean, can be also be defined in terminology of entourages. In this case it is called a coarse structure [15]. In this paper we follow the terminology from [13]. We note also that the concept of balllean is originated in asymptotic topology (see [5,7]).

Let \( B_1 = (X_1, P_1, B_1), B_2 = (X_2, P_2, B_2) \) be balleans. We say that a mapping \( f : X_1 \to X_2 \) is a \( \preceq \)-mapping if, for every \( x \in X_1 \),

\[
f(B_1(x, \alpha)) \subseteq B_2(f(x), \beta).
\]

A bijection \( f : X_1 \to X_2 \) is called an asymorphism if \( f \) and \( f^{-1} \) are \( \preceq \)-mappings.

Let \( \mathcal{B} = (X, P, B) \) be a balllean. We say that a subset \( A \) of \( X \) is

• **large** if there exists \( \alpha \in P \) such that \( X = B(A, \alpha) \);
• **small** if \( X \setminus B(A, \alpha) \) is large for every \( \alpha \in P \);
• **thick** if, for every \( \alpha \in P \), there exists \( a \in A \) such that \( B(a, \alpha) \subseteq A \);
• **piecewise large** if, there exists \( \beta \in P \) such that \( B(A, \beta) \) is thick;
• **extralarge** if \( A \cap L \) is large for every large subset \( L \) of \( X \).

We note (see [13, Chapter 9]) that large, small, thick subsets of a balllean can be considered as the counterparts of dense, nowhere dense, open subsets respectively of a uniform space.

**Proposition 2.1.** Let \( \mathcal{B} = (X, P, B) \) be a balllean and \( S \subseteq X \). Then the following statements are equivalent:

1. \( S \) is small;
2. \( S \) is not piecewise large;
3. \( X \setminus S \) is extralarge.

**Proof.** [12, Theorem 11.1].  \( \Box \)

**Proposition 2.2.** Let \( \mathcal{B} = (X, P, B) \) be a balllean. If the subsets \( S_1, \ldots, S_n \) of \( X \) are small, then \( S_1 \cup \cdots \cup S_n \) is small. If the subsets \( X_1, \ldots, X_n \) are extralarge, then \( X_1 \cap \cdots \cap X_n \) is extralarge.

**Proof.** [12, Theorem 11.2].  \( \Box \)

Let \( G \) be an infinite group with the identity \( e \), \( \kappa \) be an infinite cardinal with \( \kappa \leq |G| \). We consider the ball structure \((G, \mathcal{B}) = (G, [G]^{< \kappa}, B)\), where \( B(g, F) = (F \cup \{e\})g \), and note that \( B(G, \kappa) \) is a balllean. Then a subset \( A \) of \( G \) is large (small, thick, piecewise large, extralarge) in \( B(G, \kappa) \) if and only if \( A \) is \( \kappa \)-large (\( \kappa \)-small, \( \kappa \)-thick, piecewise \( \kappa \)-large, \( \kappa \)-extralarge).

Applying Propositions 2.1 and 2.2 to \( B(G, \kappa) \) we get the following statements.

**Proposition 2.3.** Let \( G \) be an infinite group, \( \kappa \) be an infinite cardinal such that \( \kappa \leq |G| \) and \( S \subseteq G \). Then the following statements are equivalent:

1. \( S \) is \( \kappa \)-small;
2. \( S \) is not piecewise \( \kappa \)-large;
3. \( G \setminus S \) is \( \kappa \)-extralarge.
Proposition 2.4. Let \( G \) be an infinite group, \( \kappa \) be an infinite cardinal such that \( \kappa \leq |G| \). If the subsets \( S_1, \ldots, S_n \) of \( X \) are \( \kappa \)-small, then \( S_1 \cup \cdots \cup S_n \) is \( \kappa \)-small. If the subsets \( X_1, \ldots, X_n \) are \( \kappa \)-extralarge, then \( X_1 \cap \cdots \cap X_n \) is \( \kappa \)-extralarge.

By Proposition 2.4, \( S_\kappa \) is an ideal in the Boolean algebra of all subsets of \( G \). It follows directly from definition of \( \kappa \)-small subset that the \( S_\kappa \) is left and right translation invariant, i.e. \( gS \in S_\kappa, \ 5g \in S_\kappa \) for all \( S \in S_\kappa \) and \( g \in G \).

Proposition 2.4 has also the following simple combinatorial application. Let \( G = A_1 \cup \cdots \cup A_n \) be a partition of \( G \). Since \( G \) is not \( \kappa \)-small, at least one cell \( A_i \) of the partition is piecewise \( \kappa \)-large. Hence, there exists \( F \in [G]^\kappa \) such that, for every \( H \in [G]^{<\kappa}, \ aH \subseteq FA_i \) for some \( a \in A_i \).

Remark 2.1. Let \( G \) be a finitely generated group and let \( S \) be a finite system of generators of \( G \) such that \( S = S^{-1} \) and \( e \notin S \). The Cayley graph \( \text{Cay}(G, S) \) is a graph with the set of vertices \( G \) and the set of edges \( \{(x, y): \ x, y \in G \text{ and } xy^{-1} \in S\} \). If we take another system \( S' \) of generators of \( G \), the graphs \( \text{Cay}(G, S) \) and \( \text{Cay}(G, S') \) need not to be isomorphic. But the graphs \( \text{Cay}(G, S) \) and \( \text{Cay}(G, S') \) provided with the path metrics are asymorphic as balleans, and this is a departure point of the asymptotic geometry of groups [7]. In fact, the metric ballean \( \text{Cay}(G, S) \) is isomorphic to the ballean \( B(G, \omega) \). Thus, the ballean \( B(G, \omega) \) plays the part of the Cayley graph in the case of non-finitely generated group \( G \).

3. Connection with ultrafilters

Given a discrete space \( X \), the Stone–Čech extension \( \beta X \) of \( X \) can be identified with the set of all ultrafilters on \( X \). The topology of \( \beta X \) can be defined by stating that the sets of the form \( \bar{A} = \{p \in \beta X: \ A \in p\} \), where \( A \) is a subset of \( X \), are a base for the open sets. We shall use the universal property of \( \beta X \) stating that every mapping \( f: X \rightarrow Y \), where \( Y \) is a compact Hausdorff space, can be extended to the continuous mapping \( f^\beta: \beta X \rightarrow Y \).

Now let \( G \) be a discrete group. Using the universal property of the space \( \beta G \), we can extend the group multiplication from \( G \) to \( \beta G \) in two steps. Given \( g \in G \), the continuous mapping \( x \mapsto gx: G \rightarrow \beta G \) extends to the continuous mapping \( q \mapsto gq: \beta G \rightarrow \beta G \). Then, for each \( q \in \beta G \), we extend the mapping \( g \mapsto gq \) defined from \( G \) into \( \beta G \) to the continuous mapping \( p \mapsto pq: \beta G \rightarrow \beta G \).

The product \( pq \) of the ultrafilters \( p, q \) can also be defined by the rule: given a subset \( A \subseteq G \),

\[
A \in pq \iff \{g \in G: \ g^{-1}A \in q\} \in p.
\]

For the structure of compact right topological semigroup \( \beta G \) and its combinatorial applications see [8]. We note that \( \mathcal{S}_\kappa \) is a closed left ideal of \( \beta G \).

Proposition 3.1. Given \( p \in \beta G \), \( p \in \mathcal{S}_\kappa \) if and only if every member of \( p \) is a piecewise \( \kappa \)-large subset of \( G \).

Proof. Apply the equivalence (i) \( \iff \) (ii) from Proposition 2.3.

For every group \( G \), the semigroup \( \beta G \) has the minimal ideal \( K(\beta G) \) and its closure \( M(\beta G) \) is the minimal closed ideal [8, Section 4.4].

Proposition 3.2. For every infinite group \( G \), \( M(\beta G) = \mathcal{S}_\omega \).

Proof. Apply Proposition 3.1 and Corollary 4.41 from [8].

Theorem 3.1. Let \( G \) be an infinite group of regular cardinality \( \kappa, \ k > \omega \). Then the following statements hold:

(i) \( \mathcal{S}_\kappa \setminus \mathcal{S}_\omega \neq \emptyset \);
(ii) \( \mathcal{S}_\omega \setminus \mathcal{S}_\kappa \neq \emptyset \);
(iii) \( \mathcal{S}_\kappa \) is not right ideal of \( \beta G \).

Proof. (i) We enumerate \( [G]^{<\omega} = \{G_\alpha: \ \alpha < \kappa\} \) and choose inductively an increasing family \( \{G_\alpha: \ \alpha < \kappa\} \) of subgroups of \( G \) and a \( \kappa \)-sequence \( (x_\alpha)_{\alpha < \kappa} \) in \( G \) such that, for each \( \alpha < \kappa \),
(1) \(|G_0| < |G|\) and \(G = \bigcup_{\alpha < \kappa} G_\alpha\);
(2) \(\{G_{\alpha+1} : \alpha \} \) is infinite;
(3) \(\{\bigcup_{\gamma < \alpha} G_\alpha F_\gamma X_\gamma \cap G_\alpha F_\alpha x_\alpha = \emptyset\); 
(4) \((G_{\alpha+1}) \setminus \bigcup_{\gamma < \alpha} G_\alpha F_\gamma X_\gamma \neq \emptyset\) for each \(x \in \bigcup_{\gamma < \alpha} G_\alpha F_\gamma X_\gamma\).

We put \(X = \bigcup_{\alpha < \kappa} F_\alpha x_\alpha\). Since \(F_\alpha x_\alpha \subseteq X\) for each \(\alpha < \kappa\), \(X\) is \(\omega\)-thick so \(X\) is piecewise \(\omega\)-large and \(X\) is not \(\omega\)-small.

To show that \(X\) is \(\kappa\)-small, we assume the contrary and choose \(Y \in [G]^{<\kappa}\) such that \(XY\) is \(\kappa\)-thick. Since \(\kappa\) is regular, by (1), there exists \(\alpha < \kappa\) such that \(Y \subseteq G_\alpha\) so \(G_\alpha X\) is \(\kappa\)-thick. We pick \(x \in X\) such that \(G_\alpha x \subseteq G_\alpha X\). If \(x \in \bigcup_{\gamma < \alpha} G_\alpha F_\gamma X_\gamma\), by (3), \(G_{\alpha+1} x \subseteq \bigcup_{\gamma < \alpha} G_\alpha F_\gamma X_\gamma\) contradicting (4). Thus, \(x \in K_\alpha x_\alpha\) for some \(\lambda > \alpha\). By (3), \(G_{\alpha+1} x \subseteq G_\alpha K_\lambda x_\lambda\) contradicting (2).

(ii) We pick \(x \in \mathcal{S}_\kappa \setminus \mathcal{S}_\omega\). Since \(X\) is piecewise \(\omega\)-large, by Propositions 2.4, 3.1 and Theorem 3.11 from [8], there exists \(p \in \mathcal{S}_\kappa\) such that \(x \in p\). On the other hand, \(p \notin \mathcal{S}_\kappa\). Thus, \(\mathcal{S}_\omega \setminus \mathcal{S}_\kappa \neq \emptyset\).

(iii) Apply (i) and Proposition 3.2. \(\square\)

4. \(P_\kappa\)-small and \(k\text{-thin}\) subsets

**Theorem 4.1.** Let \(G\) be an infinite Abelian group and let \(\kappa\) be a limit cardinal such that \(\kappa \leq |G|\). Then every \(P_\kappa\)-small subsets of \(G\) is \(\kappa\)-small.

**Proof.** We suppose the contrary and fix a \(P_\kappa\)-small subset \(A\) which is not \(\kappa\)-small. Then there exists \(K \in [G]^{<\kappa}\) such that \(G \setminus (K + A)\) is not \(\kappa\)-large. For each \(F \in [G]^{<\kappa}\), we pick \(x_F \in G\) such that \(x_F \notin F + (G \setminus (K + A))\) so \(x_F \in K + A\). Since \(\kappa\) is a limit cardinal and \(A\) is \(P_\kappa\)-small, we can choose \(F^\prime \in [G]^{<\kappa}\) such that \(|F^\prime| > |K|\) and \((F^\prime - F) \cap (A) = \emptyset\). Since \(x_{F^\prime} - y \in K + A\) and \(|F^\prime| > |K|\), there exists \(y \in \kappa\) such that \(|x_{F^\prime} - y \cap (g + A)| > 1\). We pick distinct elements \(y, z \in F^\prime\) such that \(x_{F^\prime} - y \in g + A\) and \(x_{F^\prime} - z \in g + A\). Then \(z - y \in A - A\) contradicting \((F^\prime - F^\prime) \cap (A - A) = \emptyset\). \(\square\)

**Theorem 4.2.** Let \(G\) be an uncountable Abelian group and let \(\kappa\) be a cardinal such that \(\omega < \kappa \leq |G|\). Then the following statements hold:

(i) \(\mathcal{S}_\omega \setminus \mathcal{S}_\kappa \neq \emptyset\);
(ii) \(\mathcal{S}_\kappa \setminus \mathcal{S}_\omega \neq \emptyset\).

**Proof.** (i) We fix some countable subgroup \(H\) of \(G\), decompose \(G\) in cosets by \(H\) and choose some system \(A\) of representatives. Clearly, the subsets \(\{g + A : g \in H\}\) are pairwise disjoint so \(A\) is \(P_\omega\)-small. By Theorem 4.1, \(A\) is \(\omega\)-small. Since \(G = H + A\) and \(\kappa > \omega\), \(A\) is \(\kappa\)-large so \(A \notin \mathcal{S}_\kappa\). Thus, \(A \in \mathcal{S}_\omega \setminus \mathcal{S}_\kappa\).

(ii) We fix a subset \(A\) from (i). Since \(A\) is piecewise \(\kappa\)-large, by Propositions 2.4, 3.1 and Theorem 3.11 from [6], there exists \(p \in \mathcal{S}_\kappa\) such that \(A \in p\). On the other hand, \(p \notin \mathcal{S}_\kappa\). Thus, \(\mathcal{S}_\omega \setminus \mathcal{S}_\kappa \neq \emptyset\). \(\square\)

**Theorem 4.3.** For every infinite group \(G\) of regular cardinality \(\kappa\), there exists a \(\kappa\)-small subset \(X\) of \(G\) which is not \(P_\kappa\)-small.

**Proof.** We consider two cases: \(\kappa = \omega\) and \(\kappa > \omega\).

Case \(\kappa = \omega\). We enumerate \(G = \{G_n : n < \omega\}\), put \(K_n = \{g_0, \ldots, g_n\}\) and fix a sequence \((y_n)_{n<\omega}\) in \(G\) such that every element \(g \in G\) appears in \((y_n)_{n<\omega}\) infinitely many often. We put \(F_n = [e, y_n]\), where \(e\) is the identity of \(G\), and choose inductively a sequence \((x_n)_{n<\omega}\) in \(G\) and a sequence \((H_n)_{n<\omega}\) of \((K_n)_{n<\omega}\) such that, for each \(n < \omega\),

(1) \(\bigcup_{\gamma < n} H_n F_\gamma X_\gamma \cap H_n F_n x_\gamma = \emptyset\);
(2) \(H_{n+1} x \setminus \bigcup_{\gamma < n} H_n F_\gamma x_\gamma \neq \emptyset\) for each \(x \in \bigcup_{\gamma < n} F_\gamma x_\gamma\);
(3) \(|H_{n+1}| > 2|H_n|\).

We put \(X = \bigcup_{\gamma < n} F_\gamma x_\gamma\). By the choice of \((y_n)_{n<\omega}\), \(gX \cap X\) is infinite for each \(g \in G\). It follows that \(X\) is not \(P_\omega\)-small.

To show that \(X\) is \(\omega\)-small, we assume the contrary and choose a finite subset \(Y\) of \(G\) such that \(YX\) is \(\omega\)-thick. Then we pick \(n\) such that \(Y \subseteq H_n\), so \(H_n Y\) is \(\omega\)-thick, and choose \(x \in X\) such that \(H_{n+1} x \subseteq H_n X\). If \(x \in \bigcup_{\gamma < n} F_\gamma x_\gamma\), by (1), \(H_{n+1} x \subseteq H_n \bigcup_{\gamma < n} F_\gamma x_\gamma\) contradicting (2). Thus, \(x \notin F_\gamma x_\gamma\) for some \(m > n\), by (1), \(H_{n+1} x \subseteq K_m x_\gamma\), contradicting (3).

Case \(\kappa > \omega\). We fix a \(\kappa\)-sequence \((y_\alpha)_{\alpha<\kappa}\) in \(G\) such that, for every \(g \in G\), the subset \(\{\alpha < \kappa : y_\alpha = g\}\) is cofinal in \(G\), and \(F_\alpha = [e, y_\alpha]\). Then we choose inductively an increasing family \(\{G_\alpha : \alpha < \kappa\}\) of subgroups of \(G\) an a \(\kappa\)-sequence \((x_\alpha)_{\alpha<\kappa}\) in \(G\), satisfying (1)-(4) from proof of Theorem 3.1.

We put \(X = \bigcup_{\alpha < \kappa} F_\alpha x_\alpha\). Since \(\kappa\) is regular, by the choice of \((y_\alpha)_{\alpha<\kappa}\), \(|gX \cap X| = \kappa\) for every \(g \in G\). Hence, \(X\) is not \(P_\kappa\)-small. To see that \(X\) is \(\kappa\)-small, it suffices to repeat the arguments proving Theorem 3.1. \(\square\)

**Theorem 4.4.** Let \(G\) be an infinite group, \(\kappa\) be an infinite cardinal such that \(\kappa \leq |G|\), \(A\) be a \(\kappa\)-thin subset of \(G\). Then the following statements hold:
(i) if $\kappa$ is regular then $A$ is $\kappa$-small;
(ii) if $\kappa'$ is a cardinal such that $\kappa < \kappa' \leq |G|$, then $A$ is $\kappa'$-small;
(iii) if $\kappa = \omega$ then $A$ is $\kappa'$-small for each cardinal $\kappa'$ such that $\omega \leq \kappa' \leq |G|$.

**Proof.** (i) We take an arbitrary $F \in [G]^{|\kappa|}$. If $|A| < |G|$, we can choose $g \in G$ such that $gFA \cap FA = \emptyset$. Then $|e, g^{-1}|(G \setminus FA) = G$ so $A$ is $\kappa$-small.

Now let $|A| = |G|$. We pick an arbitrary $g \in G \setminus F$. Since $A$ is $\kappa$-thin, $|gA \cap xA| < \kappa$ for each $x \in F$. Since $\kappa$ is regular, $|gA \cap FA| < \kappa$. It follows that $|FA \setminus Fg^{-1}(G \setminus FA)| < \kappa$. We put $K = FA \setminus Fg^{-1}(G \setminus FA)$, choose any $x \in G \setminus FA$ and note that $G = (|e| \cup Fg^{-1} \cup Kx^{-1})(G \setminus FA)$, so $G \setminus FA$ is $\kappa$-large and $A$ is $\kappa$-small.

(ii) We may suppose (see proof of (i)) that $|A| = |G|$. Given any $F \in [G]^{<\kappa}$, we pick an arbitrary $g \in G \setminus F$. Since $A$ is $\kappa$-thin, $|gA \cap xA| < \kappa$ for each $x \in F$. Since $|F| < \kappa'$ and $\kappa < \kappa'$, $|gA \cap FA| < \kappa'$ and above arguments proving (i) conclude the proof of (ii).

(iii) Apply (i) and (ii). □

**Theorem 4.5.** Every group $G$ can be generated by some 2-thin subset $X$ of $G$.

**Proof.** We put $G_0 = \{e\}$, fix an arbitrary $x_0 \neq e$ and construct inductively a system of subgroups $\{G_\alpha: \alpha < \kappa\}$ and a $\kappa$-sequence $(x_\alpha)_{\alpha < \kappa}$ in $G$ such that $G = \bigcup_{\alpha < \kappa} G_\alpha$, $G_\alpha = \bigcup_{\beta < \alpha} x_\beta$ for a limit ordinal $\alpha$, $x_\alpha \notin G_\alpha$ and $G_{\alpha+1}$ is a subgroup generated by $G_\alpha \cup \{x_\alpha\}$. We put $X = \{x_\alpha: \alpha < \kappa\}$ and note that $X$ generates $G$.

To show that $X$ is 2-thin, we suppose the contrary: $|gX \cap X| > 2$ for some $g \in G$, $g \neq e$. Then there exist distinct $\alpha, \beta < \kappa$ such that, for $x_\alpha' = gx_\alpha$, $x_\beta' = gx_\beta$, either $\alpha < \alpha'$ and $\beta < \beta'$ or $\alpha > \alpha'$ and $\beta > \beta'$. In the first case, $x_\alpha'x_\beta'^{-1} \notin G_{\alpha'} \setminus G_{\alpha'+1}$ and $x_\beta'x_\alpha'^{-1} \notin G_{\beta'} \setminus G_{\beta'+1}$, which implies $\alpha' = \beta'$ and so $\alpha = \beta$, contradicting $\alpha \neq \beta$. The second case is analogous. □

**Corollary 4.1.** Every infinite group $G$ can be generated by a subset which is $\kappa$-small for each $\kappa$ such that $\omega \leq \kappa \leq |G|$.

**Proof.** We use Theorem 4.5 to take a 2-thin subset $X$ generating $G$. Then $X$ is $\omega$-thin and we can apply Theorem 4.4(iii). □

**5. Partition in $\kappa$-small subsets**

Let $G$ be an infinite group with the identity $e$. A filtration of $G$ is a family $\{G_\alpha: \alpha < |G|\}$ of subgroups of $G$ such that

1. $G_0 = \{e\}$ and $G = \bigcup\{G_\alpha: \alpha < |G|\}$;
2. $G_\alpha \subset G_\beta$ for all $\alpha < \beta < |G|$;
3. $\bigcup\{G_\alpha: \alpha < \beta\} = G_\beta$ for every limit ordinal $\beta$;
4. $|G_\alpha| < |G|$ for every $\alpha < |G|$.

Using a minimal well-ordering of $G$ it is easy to construct a filtration of $G$ provided that $G$ is not finitely generated. In particular, every uncountable group $G$ admits a filtration.

For each $\alpha < |G|$, we decompose $G_{\alpha+1} \setminus G_\alpha$ in right cosets by $G_\alpha$ and fix some system $X_\alpha$ of representatives so $G_{\alpha+1} \setminus G_\alpha = G_\alpha X_\alpha$. Take an arbitrary element $g \in G \setminus \{e\}$ and choose the smallest subgroup $G_\alpha$ with $g \in G_\alpha$. By (3), $\alpha = \alpha_1 + 1$ for some ordinal $\alpha_1 < |G|$. Hence, $g \in G_{\alpha_1+1} \setminus G_{\alpha_1}$, and there exist $g_1 \in G_{\alpha_1}$, $x_\alpha_1 \in X_{\alpha_1}$ such that $g = g_1x_\alpha_1$. If $g_1 \neq e$, we choose the ordinal $\alpha_2$ and the elements $g_2 \in G_{\alpha_2+1} \setminus G_{\alpha_2}$ and $x_\alpha_2 \in X_{\alpha_2}$ such that $g_1 = g_2x_\alpha_2$. Since the set of ordinals $< |G|$ is well-ordered, after finite member $s(g)$ of steps we get the representation

$$g = x_{\alpha_1}x_{\alpha_1+1} \ldots x_{\alpha_2}x_{\alpha_2}, \quad x_{\alpha_i} \in X_{\alpha_i},$$

We note that this representation is unique and put

$$\gamma_1(g) = \alpha_1, \quad \gamma_2(g) = \alpha_2, \quad \ldots, \quad \gamma_{s(g)}(g) = \alpha_{s(g)}, \quad \Gamma(g) = \{\gamma_1(g), \ldots, \gamma_{s(g)}(g)\}$$

and, for every natural number $n$, put

$$D_n = \{g \in G: \ s(g) = n\}.$$

**Theorem 5.1.** Every infinite group $G$ can be partitioned in $\omega$ subsets which are $\kappa$-small for every cardinal $\kappa$ such that $\omega \leq \kappa \leq \text{cf}|G|$, where $\text{cf}|G|$ is cofinality of $|G|$. 
Proof. If \( G \) is countable, the statement is trivial because every singleton is \( \kappa \)-small. We suppose that \( G \) is uncountable, use above filtration \( \{G\alpha : \alpha < |G|\} \) of \( G \) and note \( G \setminus \{\varepsilon\} = \bigcup_{n=1}^{\infty} D_n. \) We fix a natural number \( n \) and show that \( D_n \) is \( \kappa \)-small. Take an arbitrary \( F \in [G]\)}\(^\kappa\)\(^*\). Since \( \kappa \leq \text{cf} |G| \), there exists \( \beta < |G| \) such that \( F \subseteq G_\beta \) so \( FD_n \subseteq G_\beta D_n. \) Now it suffices to prove that \( G \setminus G_\beta D_n \) is \( \omega \)-large. We choose the elements \( a_1, a_2, \ldots, a_{n+1} \) in \( G \) such that
\[
a_1 \in G_{\beta+1} \setminus G_\beta, \quad a_2 \in G_{\beta+2} \setminus G_{\beta+1}, \ldots, \quad a_{n+1} \in G_{\beta+n+1} \setminus G_{\beta+n}.
\]
We take an arbitrary element \( g \in G_\beta D_n \) and put \( g = g_0 \). If \( \beta + n \in \Gamma(g) \), we put \( \epsilon_0 = 0 \), otherwise \( \epsilon_0 = 1 \). Note that \( \beta + n \in \Gamma(a_{n+1}^0 g_0) \) and put \( g_1 = a_{n+1}^0 g_0 \). If \( \beta + n - 1 \in \Gamma(g_1) \), we put \( \epsilon_1 = 0 \), otherwise \( \epsilon_1 = 1 \). Note that \( \{\beta + n - 1, \beta + n\} \in \Gamma(a_{n+1}^1 g_1) \) and put \( g_2 = a_0^1 g_1 \). After \( n + 1 \) steps we get
\[
\{\beta, \beta + 1, \ldots, \beta + n\} \subseteq \Gamma(a_1^{n+1} a_2^{n+1} \ldots a_{n+1}^{n+1} g).
\]
If follows that \( a_1^{n+1} a_2^{n+1} \ldots a_{n+1}^{n+1} g \notin G_\beta D_n \) and put \( A = \{a_1, a_2, \ldots, a_{n+1}\}, \: K = A^n. \) We have shown that \( G_\beta D_n \subseteq K^{-1}(G \setminus G_\beta D_n). \) Hence, \( G \subseteq K^{-1}(G \setminus G_\beta D_n). \) This shows that \( G \setminus G_\beta D_n \) is \( \omega \)-large. \( \square \)

6. Duality

Theorem 6.1. Let \( G \) be an infinite group of regular cardinality \( \kappa \). Then \( S_\kappa^* = [G]\)}\(^\kappa\)\(^*\).

Proof. If \( X \in [G]\)}\(^\kappa\)\(^*\) then, clearly, \( X \subseteq S_\kappa^* \). Let \( |X| = \kappa \). To show that \( X \notin S_\kappa^* \), we enumerate \( G = \{g_\alpha : \alpha < \kappa\} \) and, for each \( \alpha < \kappa \), put \( K_\alpha = \{g_\gamma : \gamma < \alpha\} \). Then we choose inductively a \( \kappa \)-sequence \( (x_\alpha)_{\alpha < \kappa} \) in \( X \) and a \( \kappa \)-sequence \( (y_\alpha)_{\alpha < \kappa} \) in \( G \) such that the following conditions are satisfied for each \( \alpha < \kappa \):
\[
\begin{align*}
(1) & \quad K_\alpha \{x_\gamma g_\gamma : \gamma < \alpha\} \cap K_\alpha x_\alpha g_\alpha = \emptyset; \\
(2) & \quad y_\alpha \{x_\gamma g_\gamma : \gamma < \alpha\} \cap K_\alpha \{x_\gamma g_\gamma : \gamma < \kappa\} = \emptyset.
\end{align*}
\]
At the \( \alpha \)th step we choose \( x_\alpha \) to satisfy (1) and, for each \( \lambda < \alpha \),
\[
y_\lambda \{x_\gamma g_\gamma : \gamma < \lambda\} \cap K_\lambda x_\lambda g_\lambda = \emptyset.
\]
Then we choose \( y_\alpha \) such that
\[
y_\alpha \{x_\gamma g_\gamma : \gamma < \alpha\} \cap K_\alpha \{x_\gamma g_\gamma : \gamma < \kappa\} = \emptyset.
\]
We put \( A = \{x_\alpha g_\alpha : \alpha < \kappa\}. \) Since \( \{x_\alpha : \alpha < \kappa\} \subseteq X \), we have \( G = X^{-1}A. \) Now we show that \( A \) is \( \kappa \)-small. Let \( K \in [G]\)}\(^\kappa\)\(^*\). Since \( \kappa \) is regular, there is \( \alpha < \kappa \) such that \( K \subseteq K_\alpha \). So \( g_\alpha \notin K. \) By (1), \( g_\alpha x_\alpha g_\alpha \notin KA \) for each \( \gamma > \alpha \), so
\[
\{x_\gamma g_\gamma : \gamma > \alpha\} \subseteq g_\alpha^{-1}(G \setminus KA).
\]
By (2), \( \{x_\gamma g_\gamma : \gamma < \alpha\} \subseteq y_{\alpha+1}^{-1}(G \setminus KS) \) and \( A \) is \( \kappa \)-small. \( \square \)

Lemma 6.1. Let \( G \) be an infinite group, \( \kappa \) be an infinite cardinal such that \( \kappa \leq |G| \) and \( X \) be a subset of \( G \) such that \( |X| = \kappa \). Then there exists a \( P_\kappa \)-small subset \( A \) of \( G \) such that \( G = X^{-1}A. \)

Proof. First we suppose that \( \kappa = |G| \) and fix some enumeration \( \{g_\alpha : \alpha < \kappa\} \) of \( G. \) We construct inductively a \( \kappa \)-sequence \( (x_\alpha)_{\alpha < \kappa} \) in \( X \) and a \( \kappa \)-sequence \( (y_\alpha)_{\alpha < \kappa} \) in \( G \) such that, for each \( \alpha < \kappa \) and each \( A_\alpha = \{x_\gamma g_\gamma : \gamma < \alpha\} \), the subsets \( \{y_\gamma A_\gamma : \gamma < \alpha\} \) are pairwise disjoint. To this end, at \( \alpha \)th step, we put \( B_\alpha = \bigcup_{\gamma < \alpha} A_\gamma \) and choose \( x_\alpha \) so that
\[
\{y_\gamma : \gamma < \alpha\} x_\alpha g_\alpha \cap \{y_\gamma : \gamma < \alpha\} B_\alpha = \emptyset.
\]
Then we put \( A_\alpha = B_\alpha \cup \{x_\alpha g_\alpha\} \) and pick \( y_\alpha \) so that
\[
y_\alpha A_\alpha \cap y_\alpha A_\alpha = \emptyset
\]
for every \( \gamma < \alpha \). After \( \kappa \)-steps, we put \( A = \{x_\alpha g_\alpha : \alpha < \kappa\}. \) Since \( x_\alpha \in X \) for each \( \alpha < \kappa \), we have \( X^{-1}A. \) By construction of \( A \), the subset \( \{y_\alpha A_\alpha : \alpha < \kappa\} \) are pairwise disjoint, so \( A \) is \( P_\kappa \)-small.

In general case, we denote by \( Y \) the subgroup of \( G \) generated by \( X. \) Since \( |X| = |Y| = \kappa \), we can choose a subset \( B \) of \( Y \) and a \( \kappa \)-sequence \( (y_\alpha)_{\alpha < \kappa} \) in \( Y \) such that \( Y = X^{-1}B \) and the subsets \( \{y_\alpha B : \alpha < \kappa\} \) are pairwise disjoint. We decompose \( G \) in right cosets by \( Y \) and fix some system \( Z \) of representatives, so \( G = YZ. \) Then \( G = X^{-1}(BZ) \) and the subsets \( \{y_\alpha BZ : \alpha < \kappa\} \) are pairwise disjoint. Put \( A = BZ. \) Then \( G = X^{-1}A \) and \( A \) is \( P_\kappa \)-small. \( \square \)

Theorem 6.2. Let \( G \) be an infinite Abelian group and \( \kappa \) be a limit cardinal such that \( \kappa \leq |G| \). Then \( [G]\)}\(^\kappa\)\(^*\) = \( S_\kappa^* \).
Proof. Apply Lemma 6.1 and Theorem 4.1. □

Remark 6.1. In view of Corollary 3.1 from [17] and Theorem 6.2, for every infinite Abelian group \( G \) and every infinite cardinal \( \kappa \) such that \( \kappa \leq |G| \), there exists a translation invariant ideal \( I \) of subsets of \( G \) such that \( |G|^{-\kappa} = I^* \). I do not know whether Theorem 6.2 is true for every non-limit cardinal. The ideal \( I \) constructed by Solecki contains a piecewise \( \kappa \)-large subset.

Theorem 6.3. Let \( G \) be a divisible Abelian group and \( \kappa \) be an infinite cardinal such that \( \kappa \leq |G| \). Then \( S_\kappa = |G|^{-\kappa} \).

Proof. In view of Theorem 6.2, we may suppose that \( X \) is \( \kappa \)-regular. Let \( X \) be a subset of \( G \) such that \( |X| = \kappa \). We assume the contrary: \( \kappa < \kappa \). But in contrast to Remark 6.2, \( \kappa \) is regular. Then \( \kappa \) is not \( \kappa \)-large. Then \( \kappa \neq |G| \).

Let \( G \) be a divisible Abelian group and \( \kappa \) be an infinite cardinal such that \( \kappa \leq |G| \). Then \( S_\kappa = |G|^{-\kappa} \).

Proof. In view of Theorem 6.2, we may suppose that \( X \) is \( \kappa \)-small subset of \( G \) such that \( |X| < \kappa \), and \( Y \) is \( \kappa \)-large. Then \( \kappa \neq |G| \).

Remark 6.2. Let \( G \) be an infinite Abelian group and \( \kappa \) be an infinite cardinal such that \( \kappa \leq |G| \). Then \( S_\kappa = |G|^{-\kappa} \).

Proof. In view of Theorem 6.2, we may suppose that \( X \) is \( \kappa \)-small subset of \( G \) such that \( |X| < \kappa \), and \( Y \) is \( \kappa \)-large. Then \( \kappa \neq |G| \).

References