Spaces of holomorphic functions in regular domains

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Abstract

Let \( \Omega \) be a regular domain in the complex plane \( \mathbb{C} \), \( \Omega \neq \mathbb{C} \). Let \( G_b(\Omega) \) be the linear space over \( \mathbb{C} \) of the holomorphic functions \( f \) in \( \Omega \) such that \( f^{(n)} \) is bounded in \( \Omega \) and is continuously extendible to the closure \( \overline{\Omega} \) of \( \Omega \), \( n = 0, 1, 2, \ldots \). We endow \( G_b(\Omega) \), in a natural manner, with a structure of Fréchet space and we obtain dense subspaces \( F \) of \( G_b(\Omega) \), with good topological linear properties, also satisfying that each function \( f \) of \( F \), distinct from zero, does not extend holomorphically outside \( \Omega \).

\( \Omega \) is a regular domain in \( \mathbb{C} \), that is, a domain which coincides with the interior of its closure. We assume that \( \Omega \) is distinct from \( \mathbb{C} \). We write \( G_b(\Omega) \) to denote the linear space over \( \mathbb{C} \) of the holomorphic functions in \( \Omega \) such that \( f^{(n)} \) is bounded in \( \Omega \) and extends by continuity to its closure, \( n = 0, 1, 2, \ldots \). Given \( m \) in \( \mathbb{N} \) and \( f \) in \( G_b(\Omega) \), we put

\[
q_m(f) := \sup \left\{ \sum_{j=0}^{m} |f^{(j)}(z)| : z \in \Omega \right\}.
\]

1. Introduction and notation

We write \( \mathbb{N} \) for the set of positive integers. By \( \omega \) we denote the linear space \( \mathbb{C}^\mathbb{N} \) of the complex sequences provided with the topology of pointwise convergence. We have that \( \omega \) is a Fréchet space whose topological dual \( \varphi \) is identified with the space given by the elements \( (b_j) \) of \( \mathbb{C}^\mathbb{N} \) whose terms are all zero from a certain subindex on, and the duality \( \langle \omega, \varphi \rangle \) being given by

\[
\langle (a_j), (b_j) \rangle := \sum_{j=1}^{\infty} a_j b_j, \quad (a_j) \in \omega, \quad (b_j) \in \varphi.
\]

Let \( \Omega \) be a regular domain in \( \mathbb{C} \), that is, a domain which coincides with the interior of its closure. We assume that \( \Omega \) is distinct from \( \mathbb{C} \). We write \( G_b(\Omega) \) to denote the linear space over \( \mathbb{C} \) of the holomorphic functions in \( \Omega \) such that \( f^{(n)} \) is bounded in \( \Omega \) and extends by continuity to its closure, \( n = 0, 1, 2, \ldots \). Given \( m \) in \( \mathbb{N} \) and \( f \) in \( G_b(\Omega) \), we put

\[
q_m(f) := \sup \left\{ \sum_{j=0}^{m} |f^{(j)}(z)| : z \in \Omega \right\}.
\]

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Then

\[ q_1, q_2, \ldots, q_m, \ldots \]

is a fundamental system of norms in \( G_b(\Omega) \) which provides this space with a structure of Fréchet space. In what follows, we assume \( G_b(\Omega) \) endowed with this structure. If \( z_0 \) is any point in the boundary \( \partial \Omega \) of \( \Omega \), we set

\[ f^{(n)}(z_0) := \lim_{z \to z_0, z \in \Omega} f^{(n)}(z), \quad n = 0, 1, 2, \ldots \]

We say that an element \( f \) of \( G_b(\Omega) \) extends holomorphically outside \( \Omega \) whenever there is a domain \( \Omega_1 \) in \( \mathbb{C} \) such that \( \Omega_1 \supset \Omega, \Omega_1 \neq \Omega \), and a holomorphic function \( g \) in \( \Omega_1 \) whose restriction to \( \Omega \) coincides with \( f \).

We say that a point \( z_0 \) of \( \partial \Omega \) is \( C^\infty \)-regular for \( G_b(\Omega) \), or simply \( C^\infty \)-regular when no confusion occurs, if, given an arbitrary sequence \((a_j)_{j=0}^\infty\) of complex numbers, there is an element \( f \) of \( G_b(\Omega) \) such that

\[ f^{(j)}(z_0) = a_j, \quad j = 0, 1, 2, \ldots \]

We shall need later the following result found in [2]:

(a) There exists a dense subset \( \{z_j: j \in \mathbb{N}\} \) in \( \partial \Omega \) such that, for any of its arbitrary subsets \( \{u_j: j \in \mathbb{N}\} \) and any infinite-dimensional triangular matrix of complex numbers

\[
\begin{array}{cccccccc}
a_{1,0} & a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} & a_{1,n+1} & \cdots \\
a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} & a_{2,n+1} & \cdots \\
a_{3,2} & a_{3,3} & \cdots & a_{3,n} & a_{3,n+1} & \cdots \\
a_{4,3} & \cdots & a_{4,n} & a_{4,n+1} & \cdots \\
& \cdots & \cdots & \cdots & \cdots & \cdots \\
a_{n+1,n} & a_{n+1,n+1} & \cdots \\
& \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{array}
\]

there is an element \( f \) of \( G_b(\Omega) \) such that

\[ f^{(j)}(u_1) = a_{1,j}, \quad j = 0, 1, 2, 3, \ldots, n, n+1, \ldots \]

\[ f^{(j)}(u_2) = a_{2,j}, \quad j = 1, 2, 3, \ldots, n, n+1, \ldots \]

\[ f^{(j)}(u_3) = a_{3,j}, \quad j = 1, 2, 3, \ldots, n, n+1, \ldots \]

\[ f^{(j)}(u_4) = a_{4,j}, \quad j = 2, 3, \ldots, n, n+1, \ldots \]

\[ \vdots \]

\[ f^{(j)}(u_{n+1}) = a_{n+1,j}, \quad j = n, n+1, \ldots \]

\[ \vdots \]

It is an immediate consequence of the previously cited result that all points in the set \( \{z_j: j \in \mathbb{N}\} \) are \( C^\infty \)-regular. We may thus assert the following:

(b) There exists in \( \partial \Omega \) a dense subset of \( C^\infty \)-regular points.

The fundamental problem that we shall study in this article consists in determining dense subspaces inside \( G_b(\Omega) \) with good linear topological properties so that their elements, except the origin, cannot be extended holomorphically outside \( \Omega \).
2. Functions which do not extend holomorphically outside $\Omega$

Let us consider the space $G_b(\Omega)$ as described in the previous section. We take a dense subset $\{v_j: j \in \mathbb{N}\}$ in $\partial \Omega$. For every pair of positive integers $m, n$, we put $B_{m,n}$ to denote the set of elements $f$ of $G_b(\Omega)$ such that

$$|f^{(j)}(z)| \leq m^{j+1} \cdot j!, \quad j = 0, 1, 2, \ldots, z \in \Omega, \quad |z - v_n| < \frac{1}{m}.$$

It is easy to see that $B_{m,n}$ is a closed absolutely convex subset of $G_b(\Omega)$. We show that $B_{m,n}$ is not absorbent. After result (b), we may take a $C^\infty$-regular point $v$ in $\partial \Omega$ such that $|v - v_n| < 1/m$. We find an element $g$ of $G_b(\Omega)$ such that

$$g^{(j)}(v) = (j!)^2, \quad j = 0, 1, 2, \ldots.$$

Assuming that $B_{m,n}$ absorbs $g$, we find $\lambda > 0$ such that $\lambda g \in B_{m,n}$. Then

$$|\lambda g^{(j)}(z)| \leq m^{j+1} \cdot j!, \quad j = 0, 1, 2, \ldots, z \in \Omega, \quad |z - v_n| < 1/m,$$

and so

$$\lambda(j!)^2 = \lambda g^{(j)}(v) \leq m^{j+1} \cdot j!, \quad j = 0, 1, 2, \ldots,$$

and, consequently,

$$\lambda \leq \lim_{j \to \infty} \frac{m^{j+1}}{j!} = 0,$$

which is a contradiction. Hence, $B_{m,n}$ is not a zero-neighborhood in $G_b(\Omega)$ and, since this set is absolutely convex, it has no interior points. Setting

$$B := \bigcup\{B_{m,n}: m, n \in \mathbb{N}\}$$

we have that $B$ is a first category set in $G_b(\Omega)$.

We consider now a function $f$ of $G_b(\Omega)$ which extends holomorphically outside $\Omega$. We find $n \in \mathbb{N}$, $\rho > 0$ and a holomorphic function $g$ in the disk $D(v_n, \rho) := \{z \in \mathbb{C}: |z - v_n| < \rho\}$ which coincides with $f$ in $\Omega \cap D(v_n, \rho)$. Let

$$h := \sup\{|g(z)|: z \in D(v_n, \rho/2)\}.$$

We find $r \in \mathbb{N}$ such that $h < r$ and take $m \in \mathbb{N}$ so that

$$\frac{r + 1}{m} < \frac{\rho}{2}.$$

If we take $z$ in $\Omega$ such that $|z - v_n| < 1/m$, then $g$ is holomorphic in the disk $D(z, \rho - 1/m)$ and, since $\frac{2r+1}{m} < \rho - 1/m$, Cauchy’s formula applies to obtain that, for $j = 0, 1, 2, \ldots$,

$$|f^{(j)}(z)| := |g^{(j)}(z)| \leq \frac{j!}{(2r+1)^{j+1}}h \leq m^{j+1} \cdot j!$$

and so $f \in B_{m,n} \subset B$, thus the functions of $G_b(\Omega)$ that extend holomorphically outside $\Omega$ form a set of the first category. This result appears, in a more general form, in [2].

If $M$ is the subset of $G_b(\Omega)$ formed by the functions that do not extend holomorphically outside $\Omega$, then $M$ is a residual set of $G_b(\Omega)$. We shall obtain later dense subspaces of $G_b(\Omega)$ which, except the origin, are contained in $M$, with a number of good properties both linear and topological. These properties are described in the next section.
3. Nearly-Baire spaces

The linear spaces that we use in this section are assumed to be defined over the field \( \mathbb{F} \) of the real or complex numbers. We say that a subset \( A \) of a locally convex space \( E \) is sum-absorbing whenever there is \( \lambda > 0 \) such that \( \lambda(A + A) \) is contained in \( A \). We say that \( E \) is nearly-Baire if, given a sequence \( (A_j) \) of sum-absorbing balanced closed subsets which covers \( E \), there is \( j_0 \) such that \( A_{j_0} \) is a zero-neighborhood. In [4], we have proved the following result:

(c) Let \( F \) be a closed subspace of a locally convex space \( E \). If \( F \) and \( E/F \) are nearly-Baire, then \( E \) is also nearly-Baire.

A locally convex space \( E \) is said to be unordered Baire-like whenever, for any sequence \( (A_j) \) of closed absolutely convex subsets covering \( E \), there is \( j \) such that \( A_j \) is a zero-neighborhood [1].

Clearly, there are unordered Baire-like locally convex spaces which are not nearly-Baire [5].

Let \( I \) be a non-empty set. For each \( i \) in \( I \), let \( E_i \) be a locally convex space. Let \( E := \prod_{i \in I} E_i \). We consider, in the usual way, \( E_i \) as a subspace of \( E \). If \( M \) is a non-empty subset of \( I \), we identify \( \prod_{i \in M} E_i \), in the usual fashion, with a subspace of \( E \).

**Proposition 1.** Let \( I \) be a non-empty finite set. For each \( i \in I \), let \( E_i \) be a locally convex space. Let \( A \) be a closed sum-absorbing subset of \( E := \prod_{i \in I} E_i \). If \( A \cap E_i \) is a zero-neighborhood in \( E_i \), \( i \in I \), then \( A \) is a zero-neighborhood in \( E \).

**Proof.** If \( I \) has only one element there is nothing to be shown. Proceeding recurrently, let us assume that the property holds when the cardinal of \( I \) is \( n \). We take now \( I \) having \( n + 1 \) elements, \( i_1, i_2, \ldots, i_{n+1} \). It follows that

\[
B = A \cap \prod_{i \in I \setminus \{i_{n+1}\}} E_i
\]

is a closed sum-absorbing subset of \( \prod_{i \in I \setminus \{i_{n+1}\}} E_i \) which meets \( E_i \) in a zero-neighborhood when \( i \) is in \( I \setminus \{i_{n+1}\} \). Then, after the induction hypothesis, \( B \) is a zero-neighborhood in

\[
G := \prod_{i \in I \setminus \{i_{n+1}\}} E_i.
\]

We then have that \( A \) is a closed sum-absorbing subset of \( E = G \times E_{i_{n+1}} \) such that \( A \cap G \) is a zero-neighborhood in \( G \) and \( A \cap E_{i_{n+1}} \) is a zero-neighborhood in \( E_{i_{n+1}} \). We now find \( \lambda > 0 \) such that \( \lambda(A + A) \subset A \). Then, in \( G \times E_{i_{n+1}} \), we have that

\[
\lambda(A \cap G) \times \lambda(A \cap E_{i_{n+1}}) = \lambda(A \cap G) + \lambda(A \cap E_{i_{n+1}}) \subset \lambda(A + A) \subset A.
\]

Thus, \( A \) is a zero-neighborhood in \( E \). \( \square \)

**Proposition 2.** Let \( E_i \) be a nearly-Baire space, \( i \in I \), \( I \neq \emptyset \). Let \( (A_j) \) be a sequence of closed balanced sum-absorbing subsets of \( E := \prod_{i \in I} E_i \) such that it covers \( E \) and so that, for each \( j \), all the sets homothetics to \( A_j \) with ratio a positive integer are also contained in the sequence \( (A_j) \). If \( A \) denotes the subfamily of \( \{A_j: j \in \mathbb{N}\} \) formed by those \( A_j \) which intersect every \( E_i \) in a zero-neighborhood, then \( A \) covers \( E \).

**Proof.** Let us assume that the property is not true. If \( B := \{A_j: j \in \mathbb{N}\} \setminus A \), we have that \( B \) is not empty. We show that \( B \) does not cover \( E \). Let \( r_1 \) be the first positive integer such that \( A_{r_1} \in B \), we take \( i_1 \) in \( I \) such that \( A_{r_1} \cap E_{i_1} \) is not a zero-neighborhood in \( E_{i_1} \). Let \( B_1 \) be the subfamily of \( B \) formed by those elements whose intersection with \( E_{i_1} \) is not a zero-neighborhood in \( E_{i_1} \). Proceeding by recurrence, let us suppose that, for a positive integer \( q \), we have found the subfamily \( B_s \) of \( B \), \( s = 1, 2, \ldots, q \). If

\[
B \setminus \bigcup \{B_s: s = 1, 2, \ldots, q \}
\]
is not empty, then we find the smallest positive integer \( r_q+1 \) for which \( A_{r_q+1} \) is in (1). We choose now \( i_q+1 \) in \( I \) such that \( A_{i_q+1} \cap E_{i_q+1} \) is not a zero-neighborhood in \( E_{i_q+1} \). We put \( \mathcal{B}_{i_q+1} \) to denote the subfamily of (1) formed by all those elements whose intersection with \( E_{i_q+1} \) are not zero-neighborhoods in \( E_{i_q+1} \). This concludes the complete induction process. If after \( p \) steps the family \( \mathcal{B} \setminus \bigcup \{ \mathcal{B}_s: s = 1, 2, \ldots, p \} \) is empty, then we set \( M := \{1, 2, \ldots, p\} \). If, for no \( p \in \mathbb{N} \), the family \( \mathcal{B} \setminus \bigcup \{ \mathcal{B}_s: s = 1, 2, \ldots, p \} \) is empty, then we write \( M := \mathbb{N} \). It is then clear that

\[
\mathcal{B} := \bigcup \{ \mathcal{B}_s: s \in M \}.
\]

We fix \( s \) in \( M \). We put

\[
S := \{ j \in \mathbb{N}: A_j \in \mathcal{B}_s \}.
\]

Since \( A_j \cap E_{i_j} \) is not a zero-neighborhood in \( E_{i_j} \), \( j \in S \), and since \( E_{i_j} \) is nearly-Baire, it follows that \( \{ A_j: j \in S \} \) does not cover \( E_{i_j} \). Besides, if \( A_j \) is in \( \mathcal{B}_s \), then all homothetics of \( A_j \), with ratio a positive integer, are in \( \mathcal{B}_s \) and so there is a linear subspace \( L_s \) of dimension one in \( E_{i_j} \) such that, if \( O \) denotes the origin of \( E \), we have that

\[
A_j \cap L_s = \{ O \}, \quad j \in S.
\]

It is immediate that, if \( s_1, s_2 \) are in \( M \), \( s_1 \neq s_2 \), then \( i_{s_1} \) is distinct from \( i_{s_2} \). Hence,

\[
L := \prod \{ L_s: s \in M \}
\]

is a Fréchet subspace of \( E \). Let us assume now that \( \mathcal{B} \) covers \( E \). Then \( E \) covers \( L \) and there is \( j_0 \in \mathbb{N} \) such that \( A_{j_0} \in \mathcal{B} \) and \( A_{j_0} \cap L \) has interior points in \( L \) and, since \( A_{j_0} \cap L \) is sum-absorbing, it easily follows that \( A_{j_0} \cap L \) is a zero-neighborhood in \( L \). So, there is \( s \) in \( M \) such that \( A_{j_0} \in \mathcal{B}_s \), and thus \( A_{j_0} \cap L_s = \{ O \} \), which is a contradiction.

If \( A_j \) is in \( \mathcal{A} \), then \( m A_j \) is also in \( \mathcal{A} \). Thus, there is a countable set \( P \) and a family \( \{ F_p: p \in P \} \) of subspaces of \( E \) such that

\[
F := \bigcup \{ F_p: p \in P \} = \bigcup \{ A_j: A_j \in \mathcal{A} \}.
\]

Similarly, there is a countable set \( Q \) and a family \( \{ G_q: q \in Q \} \) of subspaces of \( E \) such that

\[
G := \bigcup \{ G_q: q \in Q \} = \bigcup \{ A_j: A_j \in \mathcal{B} \}.
\]

We take \( x, y \in E \) such that

\[
x \notin F, \quad y \notin G.
\]

The vectors \( x, y \) define a one-dimensional real linear manifold \( H \). Clearly, \( H \) is not contained in \( F_p \), nor in \( G_q \), \( p \in P \), \( q \in Q \). Consequently, \( H \cap F_p \) is either the empty set or has only one element. Analogously, \( H \cap G_q \) is either empty or is a singleton. On the other hand, \( H \) coincides with

\[
\left( \bigcup \{ H \cap F_p: p \in P \} \right) \cup \left( \{ H \cap G_q: q \in Q \} \right)
\]

which is a countable set, thus achieving a contradiction. \( \square \)

**Theorem 1.** If \( E_i \) is a nearly-Baire space, \( i \in I \), \( I \neq \emptyset \), then \( E := \prod_{i \in I} E_i \) is nearly-Baire.

**Proof.** Let \( \{ A_j \} \) be a sequence of balanced sum-absorbing closed subsets of \( E \) such that they cover \( E \). We want to prove that some element of the former sequence is a zero-neighborhood in \( E \) and so we may assume that the homothetics of every \( A_j \), with ratio a positive integer, also belong to \( \{ A_j \} \). After the previous proposition, we may also assume that \( A_j \cap E_i \) is a zero-neighborhood in \( E_i \), \( i \in I \), \( j \in \mathbb{N} \).

If \( I \) is finite, we choose any \( j_0 \) in \( \mathbb{N} \). Then \( A_{j_0} \cap E_i \) is a zero-neighborhood in \( E_i \), \( i \in I \), and, after Proposition 1, \( A_{j_0} \) is a zero-neighborhood in \( E \).

Let us assume now that \( I \) is an infinite set. For each \( j \in \mathbb{N} \), we assume that the set

\[
I_j := \{ i \in I: A_j \nsubseteq E_i \}
\]
is infinite. We take an element \( i_{1,1} \) of \( I_1 \). Proceeding recurrently, let us assume that, for an integer \( r \geq 2 \), we have already found \( i_{s,t} \in I_s \), for \( s, t \in \mathbb{N} \), \( 2 \leq s + t \leq r \). We then find elements

\[
i_{1,r} \in I_1, \quad i_{2,r-1} \in I_2, \quad \ldots, \quad i_{r,1} \in I_r,
\]

which are pairwise distinct and distinct from \( i_{s,t} \), for \( 2 \leq s + t \leq r \). This concludes the complete induction process.

Given an arbitrary element \((j, m)\) in \( \mathbb{N} \times \mathbb{N} \), we may find, since \( A_j \) is sum-absorbing, a one-dimensional subspace \( L_{j,m} \) of \( E_{j,m} \) such that \( A_j \not\supset L_{j,m} \). We write

\[
L := \prod_{(j,m) \in \mathbb{N} \times \mathbb{N}} L_{j,m}.
\]

We have that \( L \) is a Fréchet subspace of \( E \). Since \((A_j)\) covers \( L \), there is \( j_0 \) in \( \mathbb{N} \) such that \( A_{j_0} \cap L \) has interior points in \( L \) and, having in mind that \( A_{j_0} \) is sum-absorbing, \( A_{j_0} \cap L \) is a zero-neighborhood in \( L \). Thus, \( A_{j_0} \) contains all the subspaces \( L_{j,m} \), \((j, m) \in \mathbb{N} \times \mathbb{N} \), except for a finite number, which contradicts that

\[
A_{j_0} \not\supset L_{j_0,m}, \quad m \in \mathbb{N}.
\]

Therefore, we may assert that there is \( j_1 \in \mathbb{N} \) such that \( I_{j_1} \) is finite. After Proposition 1,

\[
A_{j_1} \cap \prod_{i \in I_{j_1}} E_i
\]

is a zero-neighborhood in \( \prod_{i \in I_{j_1}} E_i \). On the other hand, \( A_{j_1} \supset E_i, \quad i \in I \setminus I_{j_1} \) and, since \( A_{j_1} \) is sum-absorbing, it follows that

\[
A_{j_1} \supset \prod_{i \in I \setminus I_{j_1}} E_i.
\]

It now can be easily seen that \( A_{j_1} \) is a zero-neighborhood in \( E \). \( \square \)

For the proof of the coming proposition we shall make use of the following result which can be found in [3, p. 24].

(d) Let \( E \) be a locally convex space such that, if \((E_j)\) is an arbitrary sequence of linear subspaces of \( E \) covering \( E \), there is \( p \) in \( \mathbb{N} \) for which \( E_p \) is dense in \( E \). Then, if \( F \) is a hyperplane in \( E \) and \((F_j)\) is any sequence of linear subspaces of \( F \) which cover \( F \), there is \( q \) in \( \mathbb{N} \) such that \( F_q \) is dense in \( F \).

**Theorem 2.** Let \( E \) be a nearly-Baire space. If \( F \) is a subspace of \( E \) of countable codimension, then \( F \) is nearly-Baire.

**Proof.** Let us first assume that the codimension of \( F \) in \( E \) is countably infinite. Let \( \{x_j: j \in \mathbb{N}\} \) be a cobasis of \( F \) in \( E \). We denote by \( F_j \) the linear span of \( F \cup \{x_1, x_2, \ldots, x_j\} \), \( j \in \mathbb{N} \). We now show that there is \( r \) such that \( F_r \) is nearly-Baire. Assuming this not so, for each \( m \in \mathbb{N} \) we find in \( F_m \) a sequence \((A_{m,j})\) of closed subsets of \( F_m \) which are balanced sum-absorbing covering \( F_m \) and such that they are not zero-neighborhoods in this space. We put \( B_{m,j} \) for the closure of \( A_{m,j} \) in \( E \). Thus, \( B_{m,j} \) is a closed subset of \( E \) which is balanced and sum-absorbing. On the other hand,

\[
E = \bigcup \{B_{m,j}: m, j \in \mathbb{N}\}
\]

and so there exist \( p, q \in \mathbb{N} \) such that \( B_{p,q} \) is a zero-neighborhood in \( E \), and therefore \( A_{p,q} = E_p \cap B_{p,q} \) is a zero-neighborhood in \( E_p \), obtaining a contradiction.

To finish the proof, it suffices then to see it for the case of \( F \) being a hyperplane of \( E \). We consider a sequence \((A_j)\) of closed subsets of \( F \) which are balanced sum-absorbing and covering \( F \) and such that the homothetics of each \( A_j \) are also contained in the sequence \((A_j)\). We write \( L_j \) for the linear span of \( A_j, j \in \mathbb{N} \). We define

\[
P := \{j \in \mathbb{N}: L_j \text{ is dense in } F\}, \quad Q := \{j \in \mathbb{N}: L_j \text{ is not dense in } F\}.
\]

On the other hand, let \((M_j)\) be a sequence of subspaces of \( E \) covering \( E \). If \( \overline{M_j} \) stands for the closure of \( M_j \), then \( \overline{M_j} \) is closed balanced sum-absorbing and, besides, \((\overline{M_j})\) covers \( E \). Consequently, there is \( r \in \mathbb{N} \) such that \( \overline{M_j} \) is a
zero-neighborhood in $E$, that is, $M_r$ is dense in $E$. Applying result (d), we obtain that $\{L_j: j \in Q\}$ does not cover $F$. Using a similar argument to the one used in the proof of the previous proposition we have that $\{L_j: j \in P\}$ covers $F$. Hence, we may take the sequence $(A_j)$ in such a way that $L_j$ be dense in $F$, $j \in \mathbb{N}$. We put $B_j$ for the closure of $A_j$ in $E, j \in \mathbb{N}$. We choose $z$ in $E \setminus F$ and we write $G$ for the absolutely convex hull of $\{z\}$. Then the family

$$\{B_j + mG: j, m \in \mathbb{N}\}$$

covers $E$ and is formed by closed subsets of $E$ which are also balanced and sum-absorbing. Thus, there are $r, s \in \mathbb{N}$ so that $B_r + sG$ is a zero-neighborhood in $E$. Let $T$ denote the linear span of $B_r$ in $E$. Let us first assume that $z$ is not in $T$. Then

$$(B_r + sG) \cap T = B_r \cap T$$

and so this set is a zero-neighborhood in $T$, now, since $B_r$ is closed in $E$, $T$ is a closed hyperplane of $E$. If $T$ coincides with $F$, then $B_r \cap F = A_r$ is a zero-neighborhood in $F$. If $T \neq F$, then $T \cap F$ is a closed hyperplane of $F$ that contains $A_r$, which contradicts that $L_r$ is dense in $F$. Let us assume now that $z$ is in $T$. We find a positive integer $b$ such that $G \subset bB_r$. Since $B_r$ is balanced and sum-absorbing in $E$, there is $\lambda > 0$ such that $\lambda(B_r + B_r) \subset bB_r$ and

$$B_r + sG \subset B_r + bsB_r \subset bs(B_r + B_r) \subset \frac{1}{\lambda}bsB_r.$$ 

Hence, $B_r$ is a zero-neighborhood in $E$ and thus $A_r = F \cap B_r$ is a zero-neighborhood in $F$. \square

4. Subspaces of $\omega$

By $e_r, r \in \mathbb{N}$, we represent the element of $\omega$ such that all of its terms are zero except for the one occupying the $r$th place, whose value is one. The following is a slight modification of [5, Lemma 1]:

(e) There exists an uncountable subset $M$ of $\omega$ with the following properties:
1. If $(a_n)$ belongs to $M$, then each $a_n$ is a real number greater than zero, $n \in \mathbb{N}$, and $(a_n)$ increases and diverges to $\infty$.
2. If $(a_n)$ and $(b_n)$ are distinct elements of $M$, then, either $\lim_n a_n = \infty$, or $\lim_n a_n = 0$.
3. For a given positive integer $m$, if $(a_n^{(h)})$, $h = 1, 2, \ldots, m$, are elements of $M$ pairwise different, then, for any integers $0 < n_1 < n_2 < \cdots < n_m$, the determinant

$$\begin{vmatrix}
a_{n_1}^{(1)} & a_{n_2}^{(1)} & \cdots & a_{n_m}^{(1)} \\
a_{n_1}^{(2)} & a_{n_2}^{(2)} & \cdots & a_{n_m}^{(2)} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n_1}^{(m)} & a_{n_2}^{(m)} & \cdots & a_{n_m}^{(m)} \\
\end{vmatrix}$$

is not zero.

The set $M$ appearing before has the property that, if $L$ is any closed hyperplane of $\omega$, then $L \cap M$ is a finite set. In fact, there are $m$ non-zero complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_m$ and integers $0 < r_1 < r_2 < \cdots < r_m$ such that all the homothetics of

$$\alpha_1 e_{r_1} + \alpha_2 e_{r_2} + \cdots + \alpha_m e_{r_m}$$

form the subspace of $\varphi$ orthogonal to $L$. Suppose $L$ contains $m$ distinct element of $M$, $(a_n^{(s)})$, $s = 1, 2, \ldots, m$. Then

$$\langle (a_n^{(s)}), \alpha_1 e_{r_1} + \alpha_2 e_{r_2} + \cdots + \alpha_m e_{r_m} \rangle = \alpha_1 a_{r_1}^{(s)} + \alpha_2 a_{r_2}^{(s)} + \cdots + \alpha_m a_{r_m}^{(s)} = 0$$

and we achieve a contradiction noticing that the determinant
\[\begin{bmatrix}
    a^{(1)}_1 & a^{(1)}_2 & \cdots & a^{(1)}_m \\
    a^{(2)}_1 & a^{(2)}_2 & \cdots & a^{(2)}_m \\
    \vdots & \vdots & \ddots & \vdots \\
    a^{(m)}_1 & a^{(m)}_2 & \cdots & a^{(m)}_m
\end{bmatrix}\]

is distinct from zero.

Let us now consider the family \(\{B_i : i \in I\}\) of all the subsets of \(\omega\) satisfying that, for each \(i \in I\):

1. \(B_i\) contains \(M\).
2. If \((a_n) \in B_i\), then \(a_n\) is real and greater than zero, \(n \in \mathbb{N}\), and \((a_n)\) increases and diverges to \(\infty\).
3. If \((a_n)\) and \((b_n)\) are distinct elements of \(B_i\), we have

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \infty \quad \text{or} \quad \lim_{n \to \infty} \frac{a_n}{b_n} = 0.
\]

If \(i_1, i_2 \in I\), we put \(i_1 \leq i_2\) whenever \(B_{i_1}\) is contained in \(B_{i_2}\). Thus, \(\{B_i : i \in I, \leq\}\) is an inductive ordered family. Let \(B\) be a maximal element of this family. Let \(X\) be the subset of \(\omega\) formed by those sequences that take only a finite number of different values. For an arbitrary element \((b_n)\) of \(B\), we set

\[E_{(b_n)} := \{(a_n b_n) : (a_n) \in X\}.
\]

Clearly, \(E_{(b_n)}\) is a linear subspace of \(\omega\). We shall write \(E\) to denote the linear span in \(\omega\) of

\[\{E_{(b_n)} : (b_n) \in B\}\.
\]

If \((c_n)\) is a non-zero element of \(E\), there exist \((b_n^{(j)})\) in \(B\), \(j = 1, 2, \ldots, r\), pairwise distinct, and \((c_n^{(j)})\) in \(E_{(b_n^{(j)})}\), \(j = 1, 2, \ldots, r\), such that

\[c_n = \sum_{j=1}^{r} (c_n^{(j)}) = \left(\sum_{j=1}^{r} c_n^{(j)}\right).
\]

Then, if any of the elements \((c_n^{(j)}), j = 1, 2, \ldots, r\), has an infinite amount of non-zero terms, it follows that the set

\[P := \{n \in \mathbb{N} : c_n \neq 0\}
\]

is infinite, and it also happens that

\[\lim_{n \in P, n \to \infty} |c_n| = \infty.
\]

For \((m, n), (p, q) \in \mathbb{N} \times \mathbb{N}\), we define \((m, n) < (p, q)\) if, either \(m < p\), or \(m = p\) and \(n < q\).

The proof of the next theorem requires the following result [5, Proposition 8]:

(f) Let \(A\) be a closed absolutely convex subset of \(\omega\) such that it is not a neighborhood of the origin and whose linear span \(F\) is dense in \(\omega\). Given \(r \in \mathbb{N}\), there is \(q \in \mathbb{N}\) and a non-zero element \(a_{r+1} \epsilon_{r+1} + a_{r+2} \epsilon_{r+2} + \cdots + a_{r+q} \epsilon_{r+q}\) in the polar \(A^o\) of \(A\) in \(\varphi\).

The following result will also be needed [5, Corollary 2]:

(g) Let \(S\) be a barrelled subspace of \(\omega\). If \(A\) is a closed subset of \(S\) which is balanced sum-absorbing and absorbing, then \(A\) is a zero-neighborhood.

Proposition 3. Let \((E_j)\) be a sequence of subspaces of \(E\) which covers \(E\). Then, there is \(j_0 \in \mathbb{N}\) such that \(E_{j_0}\) is dense in \(E\).
Proof. Assuming the property is not true, then, for each \( j \in \mathbb{N} \), there is a closed hyperplane \( H_j \) of \( \omega \) containing \( E_j \), but not containing \( E \). It follows that \( H_j \cap M \) is finite, \( M \) is not countable and is contained in \( E \), therefore \( \bigcup \{ H_j : j \in \mathbb{N} \} \) does not contain \( E \), which is a contradiction. \( \square \)

Proposition 4. Let \( E \) be a countable family of subspaces of \( E \) which covers \( E \). Let \( F \) be the subfamily of \( E \) formed by those elements which are dense in \( E \). Then \( F \) covers \( E \).

Proof. Let us assume that the property does not hold. After the former proposition, \( E \setminus F \) does not cover \( E \). Using a similar argument to the one in the proof of Proposition 2 a contradiction is achieved. \( \square \)

Theorem 3. \( E \) is a nearly-Baire space.

Proof. Let \( (A_j) \) be a sequence of closed subsets of \( E \) which are balanced sum-absorbing and cover \( E \). Let \( L_j \) be the linear span of \( A_j \), \( j \in \mathbb{N} \). After the former proposition, we may assume that \( (A_j) \) is taken in such a way that \( L_j \) is dense in \( E \), \( j \in \mathbb{N} \). Now, let us assume that \( L_j \) is not barrelled for each \( j \in \mathbb{N} \). We find in \( L_j \) a closed absolutely convex absorbing subset \( V_j \) which is not a zero-neighborhood. We put \( W_j \) for the closure of \( V_j \) in \( E \). We write the family

\[ \{ rW_j : r, j \in \mathbb{N} \} \]

as the sequence \( (U_j) \). In \( \mathbb{N} \times \mathbb{N} \) we consider the order relation \( \leq \) before introduced. After result (f), we obtain positive integers \( 1 < p_1^{(1)} \leq q_1^{(1)} \) and complex numbers \( a_{p_1^{(1)}}, a_{p_1^{(1)}+1}, \ldots, a_{q_1^{(1)}} \) such that

\[ a_{p_1^{(1)}} e_{p_1^{(1)}} + a_{p_1^{(1)}+1} e_{p_1^{(1)}+1} + \cdots + a_{q_1^{(1)}} e_{q_1^{(1)}} \in U_1, \quad a_{q_1^{(1)}} \neq 0. \]

Clearly, we may take \( a_{q_1^{(1)}} \) to be real and positive. Proceeding by recurrence, we assume that, for \( (l, k) \in \mathbb{N} \times \mathbb{N} \), we have found the positive integers \( p_k^{(l)} \leq q_k^{(l)} \). Let \( (m, j) \) be the element of \( (\mathbb{N} \times \mathbb{N}, \leq) \) which goes right after \( (l, k) \). We apply result (f) and so obtain integers \( p_j^{(m)} \leq q_j^{(m)} \) and complex numbers \( a_{p_j^{(m)}}, a_{p_j^{(m)}+1}, \ldots, a_{q_j^{(m)}} \) so that \( d_k^{(l)} < p_j^{(m)} \) and

\[ a_{p_j^{(m)}} e_{p_j^{(m)}} + a_{p_j^{(m)}+1} e_{p_j^{(m)}+1} + \cdots + a_{q_j^{(m)}} e_{q_j^{(m)}} \in U_m, \quad a_{q_j^{(m)}} > 0. \]

We take now a positive integer \( d_{q_j^{(m)}} \) and define \( d_n := d_{q_j^{(m)}} \), \( n = 1, 2, \ldots, q_1^{(1)} \). Proceeding by recurrence, we assume that, for \( (l, k) \in \mathbb{N} \times \mathbb{N} \), we already have the integer \( d_{q_j^{(m)}} \). Let \( (m, j) \) be the element of \( (\mathbb{N} \times \mathbb{N}, \leq) \) posterior to \( (l, k) \). We find an integer \( d_{q_j^{(m)}} \) such that

\[ d_{q_j^{(m)}} > d_{q_j^{(m)}}, \quad d_{q_j^{(m)}} + a_{q_j^{(m)}} > q_j^{(m)}. \]

We write \( d_n := d_{q_j^{(m)}}, n = q_k^{(l)} + 1, q_j^{(l)} + 2, \ldots, q_j^{(m)} \). This concludes the induction process. Clearly, the sequence \( (d_n) \) increases monotonously to infinity.

Let us now take an arbitrary element \( (b_n) \) of \( B \). Let \( (\alpha_n) \) be the element of \( X \) such that

\[ \alpha_{q_j^{(m)}} = 1, \quad (m, j) \in \mathbb{N} \times \mathbb{N}, \quad \alpha_n = 0, \quad n \notin \{ q_j^{(m)} : (m, j) \in \mathbb{N} \times \mathbb{N} \} \]

It follows that \( (\alpha_n b_n) \) is in \( E \) and so there is \( s \in \mathbb{N} \) such that \( (\alpha_n b_n) \) is in \( U_s \). Consequently, for each \( l \in \mathbb{N} \),

\[ 1 \geq \| (\alpha_n b_n), a_{p_j^{(s)}} e_{p_j^{(s)}} + a_{p_j^{(s)}+1} e_{p_j^{(s)}+1} + \cdots + a_{q_j^{(s)}} e_{q_j^{(s)}} \| = b_{q_j^{(s)}} a_{q_j^{(s)}}. \]

Given any positive integer \( j > q_1^{(s)} \), we find a positive integer \( r \) such that

\[ q_r^{(s)} < j \leq q_{r+1}^{(s)}. \]

Then
\[
\frac{d_j}{b_j} \geq \frac{d_{q_j}^{(a)}}{b_{q_j+1}^{(a)}} = \frac{d_{q_j}^{(a)}a_{q_j}^{(a)}}{b_{q_j+1}^{(a)}a_{q_j+1}^{(a)}} \geq q_r^{(s)}.
\]

Consequently,
\[
\lim_j \frac{d_j}{b_j} = \infty
\]
which is a contradiction, since \(B\) is maximal.

Thus, there is \(j_0 \in \mathbb{N}\) such that \(L_{j_0}\) is barrelled. By applying result (g) we get that \(A_{j_0}\) is a zero-neighborhood in \(L_{j_0}\) and, since \(A_{j_0}\) is closed in \(E\), \(E_{j_0}\) coincides with \(E\) and so \(A_{j_0}\) is a zero-neighborhood in \(E\). \(\square\)

5. Nearly-Baire subspaces of \(\mathcal{G}_b(\Omega)\)

In this section, \(E\) is the subspace of \(\omega\) that we constructed in the previous section. Recall that \(E\) is a nearly-Baire subspace which is dense in \(\omega\). Let \(A\) be the mapping from \(\omega\) onto \(\omega\) given by
\[
A((a_n)) := (n!^2 a_n), \quad (a_n) \in \omega.
\]
We have that \(A\) is a topological isomorphism and so \(K := A(E)\) is a dense subspace of \(\omega\) which is also nearly-Baire.

Let us now consider the set
\[
H := \{(m, n) \in \mathbb{N} \times \mathbb{N}; m \leq n\}.
\]
Let \(\leq\) be the order relation induced in \(H\) by the ordering \(\leq\) in \(\mathbb{N} \times \mathbb{N}\) before defined. Let \((z_n)\) be a dense subset of \(\partial \Omega\) for which result (a) holds. If \(f\) belongs to \(\mathcal{G}_b(\Omega)\), then we write
\[
Tf := \left( f^{(n)}(z_m); (m, n) \in (H, \leq) \right).
\]
Thus
\[
T := \mathcal{G}_b(\Omega) \rightarrow \omega
\]
is linear and continuous so that, after result (a), \(T\) is an onto map. Consequently, \(T\) is a topological isomorphism from \(\mathcal{G}_b(\Omega)\) onto \(\omega\).

**Theorem 4.** There exists a dense subspace \(\mathcal{F}\) of \(\mathcal{G}_b(\Omega)\) which is nearly-Baire, so that, if \(f\) is in \(\mathcal{F}\) and it is non-zero, then \(f\) does not extend holomorphically outside \(\Omega\).

**Proof.** We put \(K := T^{-1}(K)\). Then \(K\) is dense in \(\mathcal{G}_b(\Omega)\), \(T^{-1}(0)\) is a Fréchet subspace of \(\mathcal{G}_b(\Omega)\) and \(K\) is isomorphic to \(K/T^{-1}(0)\). Applying result (c) we obtain that \(K\) is nearly-Baire. Let \(f\) be an element of \(K\) which extends holomorphically outside \(\Omega\). We find \(r \in \mathbb{N}\) and \(\rho > 0\) such that in the disk \(D(z_r, \rho)\) there exists a holomorphic function \(g\) which coincides with \(f\) in \(\Omega \cap D(z_r, \rho)\). If there is an integer \(n_0\), \(n_0 \geq r\), such that
\[
f^{(n)}(z_r) = g^{(n)}(z_r) = 0, \quad n \geq n_0,
\]
then \(f\) is the restriction to \(\Omega\) of a polynomial. If, on the contrary, there is an infinite amount of non-zero terms in the sequence \((f^{(n)}(z_r))_{n=n_0}^{\infty}\), then, since that sequence is a subsequence of
\[
\left( f^{(n)}(z_m); (m, n) \in (H, \leq) \right)
\]
it follows that
\[
\limsup_n \left| \frac{f^{(n)}(z_r)}{(n!)^2} \right| = \infty.
\]
Let
\[
h := \sup \{ |g(z)|; z \in D(z_r, \rho/2) \}.
\]
Applying Cauchy’s formula we obtain
\[ |f^{(n)}(z_r)| = |g^{(n)}(z_r)| \leq \frac{n!}{(\rho/2)^{n+1}} h. \]

Consequently,
\[ \infty = \limsup_n \frac{|f^{(n)}(z_r)|}{(n!)^2} \leq \limsup_n \frac{2^{n+1} h}{n! \rho^{n+1}} = 0, \]

which is a contradiction.

From this we deduce that the elements of \( K \) that extend holomorphically outside \( \omega \) form a linear subspace \( L \) of \( K \) of countable dimension, whether it be finite or infinite. We now find a dense subspace \( F \) of \( K \) which is the algebraic complement of \( L \). Since \( F \) is a subspace of \( K \) with countable codimension, Theorem 2 applies so that \( F \) is nearly-Baire. On the other hand, the non-zero elements of \( F \) do not extend holomorphically outside \( \Omega \).

6. The space \( G(\Omega) \)

Let \( \Omega \) be a regular domain in the complex plane, \( \Omega \neq \mathbb{C} \). By \( G(\Omega) \) we denote the linear space over \( \mathbb{C} \) formed by the functions \( f \) which are holomorphic in \( \Omega \) such that \( f^{(n)} \) extends continuously to \( \overline{\Omega} \), \( n = 0, 1, 2, \ldots \). For \( m \in \mathbb{N} \) and \( f \in G(\Omega) \), we define
\[ p_m(f) := \sup \left\{ \sum_{j=1}^{m} |f^{(j)}(z)| : z \in \Omega, |z| \leq m \right\}. \]

Then
\[ p_1, p_2, \ldots, p_m, \ldots \]
is a fundamental system of seminorms in \( G(\Omega) \) providing it with a Fréchet space structure. In what follows we shall always assume \( G(\Omega) \) endowed with this structure. If \( z_0 \) is an arbitrary point of \( \partial \Omega \) and \( f \) is in \( G(\Omega) \), we write
\[ f^{(n)}(z_0) := \lim_{z \to z_0, z \in \Omega} f^{(n)}(z), \quad n = 0, 1, 2, \ldots. \]

We consider a dense subset \( \{ v_j : j \in \mathbb{N} \} \) of \( \partial \Omega \). For every pair of positive integers \( m \) and \( n \), we put \( C_{m,n} \) to denote the subset formed by the elements of \( G(\Omega) \) such that
\[ |f^{(j)}(z)| \leq m^{j+1} j!, \quad j = 0, 1, 2, \ldots, z \in \Omega, |z - v_n| < 1/m. \]

It means no difficulty to see that \( C_{m,n} \) is a closed absolutely convex subset of \( G(\Omega) \). We see next that \( C_{m,n} \) is not absorbing in \( G(\Omega) \). We apply result (b) and obtain an element \( v \in \partial \Omega \), \( C^\infty \)-regular for \( G_0(\Omega) \), such that \( |v - v_n| < 1/m \). We find \( g \in G_0(\Omega) \) such that
\[ g^{(j)}(v) = (j!)^2, \quad j = 0, 1, 2, \ldots. \]

It follows that \( g \) is in \( G(\Omega) \). Proceeding in a similar manner to what was done in Section 2 of this article, we obtain that \( C_{m,n} \) does not absorb \( g \) and thus \( C_{m,n} \) has no interior points in \( G(\Omega) \). If we set
\[ C := \bigcup \{ C_{m,n} : m, n \in \mathbb{N} \} \]
we have that \( C \) is a set of the first category in \( G(\Omega) \).

If \( f \) is an element of \( G(\Omega) \) which extends holomorphically outside \( \Omega \), proceeding as in Section 2, we obtain that \( f \) belongs to \( C \). Hence, the subset \( M \) of \( G(\Omega) \), formed by all those functions which do not extend holomorphically outside \( \Omega \), is a residual subset. This result appears, in a more general form, in [2].

We shall obtain in the following a dense subspace \( E \) of \( G(\Omega) \) which is nearly-Baire and such that, except for the origin, is contained in \( M \).

Let \( (H, \leq) \) be the ordered set as in the previous section. Let \( \{ z_n : n \in \mathbb{N} \} \) be the subset dense in \( \partial \Omega \) for which result (a) holds. If \( f \) belongs to \( G(\Omega) \), we write
\[ Sf := \left( f^{(n)}(z_m) : (m, n) \in (H, \lesssim) \right). \]

Then

\[ S : \mathcal{G}(\Omega) \to \omega \]

is linear continuous and, since the restriction of \( S \) to \( \mathcal{G}_b(\Omega) \) is onto, we have that \( S \) is a topological homomorphism. Proceeding now as in the proof of Theorem 4, the following result obtains.

**Theorem 5.** There exists a dense subspace \( \mathcal{E} \) of \( \mathcal{G}(\Omega) \) which is nearly-Baire and whose non-zero elements do not extend holomorphically outside \( \Omega \).

We finish by posing the following open question.

**Problem.** Does there exist in \( \mathcal{G}_b(\Omega) \) (\( \mathcal{G}(\Omega) \)) a dense subspace which is Baire and such that its non-zero elements do not extend holomorphically outside \( \Omega \)?

**Note.** The referee has kindly pointed out to the author about the existence of the paper [L. Bernal-González, M.C. Calderón, W. Luh, Large linear manifolds of non-continuable boundary-regular homomorphic functions, J. Math. Anal. Appl. 341 (2008) 337–345], where the space \( \mathcal{G}(\Omega) \) is studied, which the authors denote as \( A^\infty(\Omega) \). In this paper, a dense linear subspace of \( A^\infty(\Omega) \) is constructed such that each of its non-zero elements does not extend holomorphically outside \( \Omega \). This subspace does not have additional properties of Functional Analysis, besides, it needs \( \mathbb{C} \setminus \Omega \) to be connected.

**References**