Hopf Algebras of Low Dimension

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Communicated by Michel Van den Bergh

Received September 26, 1997

The main aim of this paper is to classify all types of Hopf algebras of dimension less than or equal to 11 over an algebraically closed field of characteristic 0. If $A$ is such a Hopf algebra that is not semisimple, then we shall prove that $A$ or $A^*$ is pointed. This property will result from the fact that, under some assumptions, any Hopf algebra that is generated as an algebra by a four-dimensional simple subcoalgebra is a Hopf quotient of the coordinate ring of quantum $SL_2(k)$. The first result allows us to reduce the classification to the case of pointed Hopf algebras of dimension 8. We shall describe their types in the last part of the paper.

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INTRODUCTION

The problem of classification of all Hopf algebras of low dimension was posed by Kaplansky in [5] and solved by R. Williams in the case when the dimension is less than or equal to 11. Because the arguments used in [24] are very computational, in this paper we reconsider the classification of these Hopf algebras, proposing a new method that is based on some rather recent results concerning the structure and the classification of finite-dimensional Hopf algebras. The key step of our approach will be the following.

**Theorem 2.8.** Let $A$ be a Hopf algebra of dimension $\leq 11$ that is not semisimple. Then $A$ or $A^*$ is pointed.

Therefore, if we are interested in Hopf algebras of dimension $\leq 11$, we deal with either semisimple or pointed Hopf algebras and duals of them. First, the classification of semisimple Hopf algebras of dimension $p^2$, $p^3$,
and $2p$ ($p$ is a prime number) is known by Masuoka’s work (see [9–13]). Second, we proved in [21] that a pointed Hopf algebra of dimension $pq$ is semisimple ($p$ and $q$ are different prime numbers). Finally, the types of pointed Hopf algebras of dimension $p^2$ were described by N. Andruskiewitsh and W. Chin in an unpublished paper, but they can be found in [1], [3], and [21]. Since any Hopf algebra of prime dimension is the group algebra of the cyclic group, it remains, by the above theorem, to enumerate the isomorphism classes of pointed Hopf algebras of dimension 8. This will be done in the last section of this paper (after submitting this paper, pointed Hopf algebras of dimension $p^3$ have been classified independently in [2], [4] and [20]).

In the first section of the paper we define and study certain “canonical” bases on the simple coalgebra $M_m(k)^*$, the dual of the $k$-algebra of $m \times m$ matrices. Following [8], a subset $e = \{e_{ij} | i, j = 1, m\}$ of a coalgebra $C$ will be called a multiplicative matrix if $\Delta(e_{ij}) = \sum_{p=1}^{m} e_{ip} \otimes e_{pj}$ and $e(e_{ij}) = \delta_{ij}$, for $i, j = 1, m$. Basically, if $f$ is an (anti-)automorphism of $M_m(k)^*$, we look for bases $e$ that are multiplicative matrices on $M_m(k)^*$ such that $f$ has a “nice” form with respect to $e$. More precisely, we construct multiplicative matrices by applying the Skolem–Noether theorem to $f^*$, the transposed map of $f$ (Lemma 1.2). We then specialize this result to the particular case in which $m = 2$ and $f$ is an (anti-)automorphism of finite order (Theorem 1.4). Furthermore, let $A$ be a Hopf algebra (not necessarily finite dimensional) with antipode $S$. Suppose that $A$ contains a four-dimensional simple subcoalgebra $C$ such that $S(C) = C$ and $\text{ord}(S^2 | C) = n < \infty$ and $n > 1$. Then we shall see that there are a root of unity $\omega$ and a multiplicative matrix $e = \{e_{ij} | i, j = 1, 2\}$ in $C$ such that $\text{ord}(\omega^2) = n$ and $e$ satisfies all relations from the definition of $O_{\sqrt{-\omega}}(SL_2(k))$, the coordinate ring of quantum $SL_2(k)$. In particular, if $C$ generates $A$ as an algebra, then $A$ is a Hopf quotient of $O_{\sqrt{-\omega}}(SL_2(k))$. We shall end this section by proving, in the case in which $\text{ord}(\omega) = 2n$, that the subalgebra generated by $\{e_{ij}^* | i, j = 1, 2\}$ is a commutative Hopf subalgebra $B$.

The second section is devoted to the study of Hopf algebras of dimension $\leq 11$. Let $A$ be such a Hopf algebra that is not semisimple. We first show that $A$ fulfills the requirements necessary for applying the results of the preceding section, namely, $A$ contains a simple subcoalgebra of dimension 4, say $C$, which is $S$ invariant (i.e., $S(C) = C$) and generates $A$ as an algebra. Actually, this is a simple consequence of

**Theorem 2.2.** Let $A$ be an $n$-dimensional Hopf algebra over an algebraically closed field $k$ of characteristic 0. If $A$ is not cosemisimple, then $\dim(A_0) \leq n - 2$, where $A_0$ is the coradical of $A$. 
Let us turn back to the case of Hopf algebras of dimension \(\leq 11\) that are not semisimple. If, in addition, \(A\) is not pointed, then we also obtain as a corollary that \(\dim(A) \geq 8\) and \(A_0 = k[G] \oplus M_2(k)^s\), where \(G\) is a group with \(|G| \leq 3\). We end this section by proving Theorem 2.8.

In the last section of the paper, using the same techniques as in [21], we describe the types of 8-dimensional pointed Hopf algebras \(A\). We can assume that \(A\) is not semisimple (otherwise \(A\) is a group algebra). Let \(G(A)\) be the set of group-like elements of \(A\). Obviously, \(G(A)\) is a group, and by the Nichols–Zoeller theorem, \(|G(A)| \in \{2, 4\}\). The most difficult case to handle is \(|G(A)| = 2\). For we need the technical Lemma 3.1, which asserts that \(A\) is generated as an algebra by \(A_1\), where \((A_n)_{n \geq 0}\) is the coradical filtration of \(A\). A careful analysis of the proof of Lemma 3.1 reveals the strong connection between the coradical filtration and the Hochschild cohomology of a pointed coalgebra (of arbitrary dimension), but this subject will be discussed in a subsequent paper.

If \(|G(A)| = 4\), then the classification is obtained by using [21, Theorem 2] (see Theorems 3.3 and 3.4). Finally, the types of all Hopf algebras of dimension \(\leq 11\) are described in Theorem 3.5.

**Notation and Prerequisites**

Throughout this paper \(k\) will denote an algebraically closed field of characteristic 0. All vector spaces and, in particular, all algebras will be finite-dimensional over \(k\). The tensor product \(\otimes\) will denote \(\otimes_k\). Let \(A\) be a Hopf algebra over \(k\). A nonzero element \(g \in A\) is called a group-like element if \(\Delta(g) = g \otimes g\). The set (actually the group) of all group-like elements of \(A\) will be denoted by \(G(A)\). If \(x \in C\) and \(\Delta(x) = x \otimes \sigma + \tau \otimes x\), then \(x\) will be called \((\sigma, \tau)\) skew-primitive. The set of all \((\sigma, \tau)\) skew-primitive elements will be denoted by \(\mathcal{P}_{\sigma, \tau}(C)\). For each pair \(\sigma, \tau \in G(C)\) we choose a subspace \(\mathcal{P}_{\sigma, \tau}'(C)\) such that \(k(\sigma - \tau) \oplus \mathcal{P}_{\sigma, \tau}'(C) = \mathcal{P}_{\sigma, \tau}(C)\). The coradical of \(A\) is, by definition, the sum of its simple subcoalgebras and will be denoted by \(A_0\). Recall that a Hopf algebra is called pointed if \(A_0 = k[G(A)]\), the vector space spanned by group-like elements. At the other extreme, \(A\) is called cosemisimple if \(A_0 = A\) or, equivalently, if \(A^*\) is semisimple (in the finite-dimensional case). The coradical filtration \((C_n)_{n \in \mathbb{N}}\) of a coalgebra \(C\) is defined inductively as follows. Let \(C_0\) be the coradical of \(C\) and define \(C_{n+1} = \Delta^{-1}(C \otimes C_n + C_0 \otimes C)\), for all \(n \in \mathbb{N}\).

For the reader's convenience we shall recall some well-known results about finite-dimensional Hopf algebras that will be used in this paper. If \(C\) is pointed, then we have the following description of the coradical filtration.
Theorem 0.1 [22]. Let $C$ be a pointed coalgebra and let $(C_n)_{n \in \mathbb{N}}$ be its coradical filtration.

(a) $C_1 = k[G(C)] \oplus (\bigoplus_{\sigma, \tau \in G(C)} P_{\sigma, \tau}(C))$.

(b) If $n \geq 2$ and $C^\sigma \tau \delta = \{x \in C | x \Delta = x \otimes \sigma + \tau \otimes x + C_{n-1} \otimes C_{n-1}\}$, then $C_n = \sum_{\sigma, \tau \in G(C)} C^\sigma \tau \delta$.

The fact that the antipode of a finite-dimensional Hopf algebra is of finite order will play a central role in this paper. For instance, this result will help us to prove the main results of the first section.

Theorem 0.2 [19, Theorem 1, Corollary 7]. Let $A$ be a finite-dimensional Hopf algebra. Then the antipode of $A$ has finite order and $\text{ord}(S)$ divides $4 \text{lcm}(|G(A)|, |G(A^*)|)$.

Furthermore, in characteristic zero we can check if a Hopf algebra is semisimple or not by computing the order of its antipode. Actually, we have the following theorem due to R. G. Larson and D. E. Radford.

Theorem 0.3 [6, 7]. Let $A$ be a finite-dimensional Hopf algebra over a field of characteristic 0. Then the following assertions are equivalent:

(a) $A$ is semisimple.

(b) $A$ is cosemisimple.

(c) $\text{tr}(S^2) \neq 0$.

(d) $S^2 = \text{id}_A$.

In particular, if $A$ is commutative then $A$ is semisimple, since $S^2 = \text{id}_A$ and $\text{tr}(S^2) = \dim(A)1_k$ is not 0. Moreover, if $k$ is algebraically closed, then $A = k[G]^s$.

For finite-dimensional Hopf algebras a kind of Lagrange's Theorem holds.

Theorem 0.4 [17]. Let $A$ be a finite-dimensional Hopf algebra. If $B$ is a Hopf subalgebra, then $A$ is a free left $B$-module. In particular, $\dim(B)$ and $|G(A)|$ divide $\dim(A)$.

Hopf algebras of dimension that is a prime number were classified by Zhu.

Theorem 0.5 [23, Theorem 2]. If $A$ is Hopf algebra of prime dimension $p$ over an algebraically closed field of characteristic 0, then $A$ is the group algebra of the cyclic group $\mathbb{C}_p$ of order $p$.

For bialgebras Zhu's theorem does not hold. Interesting examples of noncommutative and noncocommutative 5-dimensional bialgebras are constructed in [14, 15]. In conclusion, the existence of the antipode for a
bialgebra is a very strong condition, having profound implications for its structure.

1. AUTOMORPHISMS OF SIMPLE COALGEBRAS

The main goal of this section is to prove Theorem 1.5, which basically asserts that the Hopf algebras we are interested in are Hopf quotients of $O_{\sqrt{\omega}}(SL_2(k))$, the coordinate ring of quantum $SL_2(k)$, where $\omega$ is a certain root of unity. The main idea of the proof is to associate with any coalgebra (anti-)automorphism $f$ of $M^n(k)^*$ a certain basis $e$, such that the matrix of $f$ with respect to $e$ has a "canonical" form. Throughout this section $C$ will denote the simple coalgebra $M^n(k)^*$, the dual of the matrix ring $M^n(k)$.

Definition 1.1. A basis $e = \{e_{ij} | i, j = \overline{1, m}\}$ of $C$ will be called a multiplicative matrix if

$$\Delta(e_{ij}) = \sum_{p=1}^{m} e_{ip} \otimes e_{pj}$$

and

$$g(e_{ij}) = \delta_{ij},$$

for $i, j = \overline{1, m}$.

Remark 1. (a) We are only interested in multiplicative matrices that are bases of $M^n(k)^*$, so we have changed the usual definition slightly (see, for example, [8]) such that they are always linearly independent.

(b) Take $\{E_{i,j} | i, j = \overline{1, m}\}$ to be the canonical basis of $M^n(k)$ and denote its dual basis by $e = \{e_{ij} | i, j = \overline{1, m}\} \subset C$. Then $e$ is a multiplicative matrix in $C$.

Lemma 1.2. Let $C = M^n(k)^*$ and let $\{e_{ij} | i, j = \overline{1, m}\}$ be a multiplicative matrix in $C$.

(a) If $f$ is a coalgebra automorphism of $C$, then there is an invertible matrix $A = (a_{ij})_{i,j = \overline{1, m}}$ such that $f(e_{ij}) = \sum_{p,q=1}^{m} a_{ip} e_{pq} a'_{jq}$, where $A^{-1} = (a'_{ij})_{i,j = \overline{1, m}}$.

(b) If $f$ is a coalgebra anti-automorphism of $C$, then there is an invertible matrix $A = (a_{ij})_{i,j = \overline{1, m}}$ such that $f(e_{ij}) = \sum_{p,q=1}^{m} a_{ip} e_{pq} a'_{jq}$, where $A^{-1} = (a'_{ij})_{i,j = \overline{1, m}}$.

Proof. (a) This is the dual Skolem–Noether theorem. The map $f^*: C^* \to C^*$ is an algebra morphism, so there is $b \in C^*$, an invertible element, such that $f(x) = bxb^{-1}$, for all $x \in C^*$. If $\{e_{ij}^* | i, j = \overline{1, m}\}$ is the dual basis of $\{e_{ij} | i, j = \overline{1, m}\}$, then $e_{ij}^* e_{pq}^* = \delta_{ip} e_{jq}^*$ for all $i, j, p, q = \overline{1, m}$. We can easily check that the unit of $C^*$ is $\sum_{i=1}^{m} e_{ii}^*$ and

$$e_{pq}^* \circ f = f^*(e_{pq}^*) = \sum_{i=1}^{m} a_{ip} e_{ij}^* a'_{jq},$$

(1)
where \( b = \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij} e_{ij}^* \) and \( b^{-1} = \sum_{i=1}^{m} \sum_{j=1}^{m} a'_{ij} e_{ij}^* \). By computing the element \( (e_{pq}^* \circ f)(e_{ij}) \) one gets \( e_{pq}^* (f(e_{ij})) = a_{ip} a'_{jq} \), so
\[
 f(e_{ij}) = \sum_{p,q=1}^{m} a_{ip} e_{pq}^* a'_{jq}.
\]
Therefore we can take \( A = (a_{ij})_{i,j=1}^{m} \). Note that \( A' = (a_{ij})_{i,j=1}^{m} \) is the inverse of \( A \), since \( b^{-1} \) is the inverse of \( b \) in \( C^* \).

(b) Let \( \tilde{f}: C \to C \) be the coalgebra map that sends \( e_{ij} \) to \( f(e_{ij}) \). By the first part of the lemma there is a matrix \( B = (b_{ij})_{i,j=1}^{m} \in GL_m(k) \) such that \( \tilde{f}(e_{ij}) = \sum_{p,q=1}^{m} b_{ip} e_{pq}^* b'_{jq} \). Then we can take \( A = (B^{-1})' \).

**Definition 1.3.** If \( f \) is a coalgebra automorphism (respectively, antiautomorphism) of \( C \) and \( A \) is a matrix as in the preceding lemma, then we shall say that \( f \) is afforded by \( A \) with respect to the basis \( e \), or that \( A \) is the matrix of \( f \) corresponding to the basis \( e \), and we shall write \( f(e) = AeA^{-1} \) (respectively, \( f(e) = AeA^{-1} \)).

**Remark 2.** (a) Two matrices \( A \) and \( B \) afford the same (anti-)automorphism of \( C \) if and only if there is \( \alpha \in k^* \), such that \( B = \alpha A \). In this case we shall say that \( A \) and \( B \) are equivalent, and we shall write \( A \sim B \).

(b) If \( f \) and \( g \) are automorphisms (respectively, antiautomorphisms) of \( C \) that are afforded by \( A \) and \( B \), then the matrix of \( f \circ g \) corresponding to the basis \( e \) is \( BA \) (respectively, \( B(A^{-1})' \)).

(c) If \( e \) and \( e' \) are two multiplicative matrices, then there is an invertible matrix \( B \), such that \( e' = BeB^{-1} \). Indeed, the map \( f: C \to C \), which sends \( e_{ij} \) to \( e'_{ij} \) for all \( i,j \), is an automorphism, so the assertion follows by Lemma 1.2(b).

(d) Let \( e \) be a multiplicative matrix and \( e' = BeB^{-1} \), where \( B \) is an invertible matrix. Suppose that \( f \) is an automorphism of \( C \) such that \( f(e) = AeA^{-1} \). Then \( f(e') = (BAB^{-1})e(BAB^{-1})^{-1} \).

(e) Let \( e \) be a multiplicative matrix and \( e' = BeB^{-1} \), where \( B \) is an invertible matrix. Suppose that \( f \) is an antiautomorphism of \( C \) such that \( f(e) = AeA^{-1} \). Then \( f(e') = (BAB^{-1})e'(BAB^{-1})^{-1} \).

Throughout the rest of the paper we shall focus on the case \( m = 2 \), so \( C \) will denote the coalgebra \( M_2(k)^* \). If \( f \) is a \( k \)-linear automorphism of \( C \), then \( \text{ord}(f) \) denotes the order of \( f \) as an element in the group \( GL(C) \). Similarly, if \( \omega \in k^* \), we shall use the notation \( \text{ord}(\omega) \) for the order of \( \omega \) in the group \( k^* \).

**Theorem 1.4.** (a) Let \( f \) be an antiautomorphism of \( C = M_2(k)^* \) such that \( \text{ord}(f^2) = n < \infty \) and \( n > 1 \). Then there are a multiplicative matrix \( e \) in \( C \) and a root of unity \( \omega \) such that \( f(e_{12}) = \omega^{-1} e_{12} \), \( f(e_{21}) = \omega e_{21} \), \( f(e_{11}) = e_{22} \), \( f(e_{22}) = e_{11} \), and \( \text{ord}(\omega^2) = n \).
(b) Let $f$ be an automorphism of $C$ of finite order $n$. Then there are a multiplicative matrix $e$ on $C$ and a root of unity $\omega$ such that $f(e_{ij}) = \omega^{i-j}e_{ij}$ and $\text{ord}(\omega) = n$.

Proof. Fix a multiplicative matrix $e'$ on $C$ and suppose that $f$ is afforded by $A'$ with respect to $e'$. Obviously, $f^2$ is afforded by $B = A'(A'^{-1})'$. Since $n > 1$, it follows that $B \neq I_2$. Moreover, $B^n$ is equivalent to the unit matrix $I_2$. In particular, there are an invertible matrix $U \in GL_2(k)$ and $\omega_1, \omega_2 \in k^*$ such that

$$UBU^{-1} = \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix}.$$

Note that either $\omega_1$ or $\omega_2$ is not 1, since $B \neq I_2$. If we take $e = Ue'U^{-1}$, then $f$ is afforded by $A = UA'U'$ with respect to $e'$; see Remark 2(d). We get

$$A(A^{-1})' = UBU^{-1} = \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix},$$

so $a_{ij} = \omega_1 a_{ji}$, for all $i, j = 1, 2$, where $a_{ij}$ are the elements of the matrix $A$. If $\omega_1 = 1$ and $\omega_2 \neq 1$, then $a_{12} = a_{21} = a_{22} = 0$; hence we get a contradiction with $A \in GL_2(k)$. Similarly, we can proceed when $\omega_1 \neq 1$ and $\omega_2 = 1$, so both $\omega_1$ and $\omega_2$ are not equal to 1. Thus $a_{11} = a_{22} = 0$, which implies that

$$f(e_{12}) = (a_{12}a_{21}^{-1})^{-1}e_{12}, f(e_{21}) = (a_{12}a_{21}^{-1})e_{21}, f(e_{11}) = e_{22}, \text{ and } f(e_{22}) = e_{11}. \text{ We can conclude by taking } \omega = a_{12}a_{21}^{-1}.$$

(b) The proof of this part is similar, so it will be omitted.

We use the preceding result in the case in which $C$ is an $S$-invariant four-dimensional simple subcoalgebra of a Hopf algebra $A$ (not necessarily finite-dimensional) with antipode $S$ and $f = SI_c$. Recall that $O_{\sqrt{-\omega}}(SL_2(k))$, the coordinate ring of quantum $SL_2(k)$, is by definition the Hopf algebra generated as an algebra by the elements $x_{11}, x_{12}, x_{21}, x_{22}$ subject to the following seven relations:

$$x_{22}x_{11} + \omega^{-1}x_{12}x_{21} = 1, \quad x_{22}x_{12} + \omega^{-1}x_{12}x_{22} = 0,$$

$$\omega x_{21}x_{11} + x_{11}x_{21} = 0, \quad \omega^{-1}x_{11}x_{12} + x_{12}x_{11} = 0,$$

$$x_{21}x_{22} + \omega x_{22}x_{21} = 0, \quad x_{12}x_{21} = x_{21}x_{12},$$

$$x_{11}x_{22} - x_{22}x_{11} = (\omega^{-1} - \omega)x_{12}x_{21}.$$

The coalgebra structure is defined such that $\{x_{11}, x_{12}, x_{21}, x_{22}\}$ is a multiplicative matrix. Note that the antipode is uniquely determined such that the matrix $(S(x_{ij}))_{i,j=1,2}$ is the inverse of $(x_{ij})_{i,j=1,2}$.
THEOREM 1.5. Let $A$ be a Hopf algebra that contains an $S$-invariant 4-dimensional simple subcoalgebra $C$. If $\operatorname{ord}(S^2 | C) = n < \infty$ and $n > 1$, then there are a root of unity $\omega$ and a multiplicative matrix $e$ such that $\operatorname{ord}(\omega^2) = n$ and the elements $e_{ij} \in e$ satisfy all relations defining $O_{\sqrt[n]{\omega}}(SL_2(k))$. In particular, there is a Hopf algebra morphism $\pi: O_{\sqrt[n]{\omega}}(SL_2(k)) \to A$. Moreover, if $C$ generates $A$ as an algebra, then $A$ is a quotient Hopf algebra of $O_{\sqrt[n]{\omega}}(SL_2(k))$.

Proof. By the preceding theorem (applied for $f = S | C$) there are a multiplicative matrix $e$ in $C$ and a root of unity $\omega$ such that $\operatorname{ord}(\omega^2) = n$ and $S(e_{11}) = e_{22}$, $S(e_{22}) = e_{11}$, $S(e_{12}) = \omega^{-1}e_{12}$, $S(e_{21}) = \omega e_{21}$. From the definition of the antipode we get

\[
\begin{pmatrix}
S(e_{11}) & S(e_{12}) \\
S(e_{21}) & S(e_{22})
\end{pmatrix}
\begin{pmatrix}
e_{11} & e_{12} \\
e_{21} & e_{22}
\end{pmatrix} = I_2,
\]

which impose the following relations on $e$:

\begin{align*}
e_{22}e_{11} + \omega^{-1}e_{12}e_{21} & = 1, \\
e_{22}e_{12} + \omega^{-1}e_{12}e_{22} & = 0, \\
\omega e_{21}e_{11} + e_{11}e_{21} & = 0, \\
\omega e_{21}e_{12} + e_{11}e_{22} & = 1, \\
e_{11}e_{22} + \omega e_{12}e_{21} & = 1, \\
\omega^{-1}e_{11}e_{12} + e_{12}e_{11} & = 0, \\
e_{21}e_{22} + \omega e_{22}e_{21} & = 0, \\
\omega^{-1}e_{21}e_{12} + e_{22}e_{11} & = 1.
\end{align*}

By (2), (9) and, respectively, (2), (6) this results in

\begin{align*}
e_{12}e_{21} & = e_{21}e_{12}, \\
e_{11}e_{22} - e_{22}e_{11} & = (\omega^{-1} - \omega)e_{12}e_{21}.
\end{align*}

Therefore, by the definition of $O_{\sqrt[n]{\omega}}(SL_2(k))$ there is a unique Hopf algebra morphism $\pi: O_{\sqrt[n]{\omega}}(SL_2(k)) \to A$ mapping $x_{ij}$ to $e_{ij}$. Obviously, if $C$ generates $A$ as an algebra, then $\pi$ is surjective, because $e$ is a set of algebra generators of $A$.

THEOREM 1.6. Let $A$ be a Hopf algebra as in Theorem 1.5. Suppose that $e$ is a multiplicative matrix in $C$ that satisfies the relations (2)–(11). Then the
HOPF ALGEBRAS OF LOW DIMENSION

subspace of $A$ spanned by $e_{11}^n$, $e_{12}^n$, $e_{21}^n$, and $e_{22}^n$ is an $S$-invariant subcoalgebra $D$. Moreover, the subalgebra generated by $D$ is a Hopf subalgebra that is commutative if $\text{ord}(\omega) = 2n$.

**Proof.** From relations (7) and (4) we have $(e_{11} \otimes e_{11})(e_{12} \otimes e_{21}) = \omega^2(e_{12} \otimes e_{21})(e_{11} \otimes e_{11})$, so by [18, Proposition 1] we get

$$\Delta(e_{11}^n) = e_{11}^n \otimes e_{11}^n + e_{12}^n \otimes e_{21}^n. \quad (12)$$

Similarly, we can show that

$$\Delta(e_{12}^n) = e_{11}^n \otimes e_{12}^n + e_{12}^n \otimes e_{22}^n, \quad (13)$$

$$\Delta(e_{21}^n) = e_{21}^n \otimes e_{11}^n + e_{22}^n \otimes e_{21}^n, \quad (14)$$

$$\Delta(e_{22}^n) = e_{21}^n \otimes e_{12}^n + e_{22}^n \otimes e_{22}^n. \quad (15)$$

Hence $D$ is a subcoalgebra that is obviously $S$-invariant. The elements $e_{11}^n$ and $e_{22}^n$ commute, since we can see by induction that

$$e_{11}^i e_{22}^j = (1 - \omega^{2i-1} e_{21} e_{12}) e_{11}^{i-1} e_{22}^{j-1},$$

$$e_{22}^i e_{11}^j = (1 - \omega^{2(n-i)+1} e_{21} e_{12}) e_{11}^{i-1} e_{22}^{j-1}.$$  

Note that for $i = 1$ these relations come from (6) and (9). It follows that

$$e_{11}^n e_{22}^n = (1 - \omega^{2n-1} e_{21} e_{12})(1 - \omega^{2n-3} e_{21} e_{12}) \cdots (1 - \omega e_{21} e_{12}),$$

$$e_{22}^n e_{11}^n = (1 - \omega e_{21} e_{12})(1 - \omega^3 e_{21} e_{12}) \cdots (1 - \omega^{2n-1} e_{21} e_{12}),$$

which implies $e_{11}^n e_{22}^n = e_{22}^n e_{11}^n$, as $e_{12}$ and $e_{21}$ commute. Furthermore, if $\text{ord}(\omega) = 2n$, then

$$e_{11}^n e_{12}^n = (-\omega)^n e_{12} e_{21} e_{11}^n = (-1)^n e_{12} e_{11}^n = e_{12}^n e_{11}^n.$$  

The other commutation relations can be proved similarly.

2. HOPF ALGEBRAS OF DIMENSION $\leq 5$

Let $A$ be a Hopf algebra, $\dim A \leq 11$. The main aim of this paragraph is to prove that if $A$ is not semisimple, then $A$ or $A^*$ is pointed. This result will lead us to the classification of all Hopf algebras of dimension $\leq 11$. Indeed, the classification of semisimple or/and pointed Hopf algebras of dimension $\leq 11$ is already known, except for the case of pointed
8-dimensional ones. To complete the picture, their types will be described in the next section.

We begin with a result on the dimension of the coradical of an arbitrary finite-dimensional Hopf algebra that is not semisimple.

**Lemma 2.1.** Let C be an n-dimensional coalgebra. If I is a coideal in C such that dim(I) = 1, then there is a subcoalgebra D such that I ⊂ D and 2 ≤ dim(D) ≤ 4. Dually, if A is an n-dimensional algebra and B is a subalgebra of codimension 1, then there is an ideal I in A such that I ⊂ B and n - 4 ≤ dim(I) ≤ n - 2.

**Proof.** We shall proceed as in the proof of [21, Theorem 5]. Let x be a nonzero element in I and write \( Δ(x) = x ⊗ a + b ⊗ x \), with \( a, b ∈ C \). Since I is a coideal, it follows that \( ε(x) = 0 \). Thus \( ε(a) = 1 \), and the elements \( a, x \) are linearly independent. We pick up a basis \( \{ e_i \}_{i=1}^{n} \) such that \( e_1 = x \) and \( e_2 = a \). Writing \( Δ(a) = \sum_{i=1}^{n} a_i ⊗ e_i \) and \( Δ(b) = \sum_{i=1}^{n} e_i ⊗ b \), and replacing them in the equality \( (Δ ⊗ id)Δ(x) = (id ⊗ Δ)Δ(x) \), we obtain

\[
Δ(a) = a ⊗ a + a_1 ⊗ x \\
Δ(b) = b ⊗ b + x ⊗ a_1.
\]

By \( (Δ ⊗ id)Δ(a) = (id ⊗ Δ)Δ(a) \) it follows that \( Δ(a_1) = a ⊗ a_1 + a_1 ⊗ b \), so the subspace generated by \( x, a, b, \) and \( a_1 \) is a subcoalgebra \( D \) containing \( I \). The second assertion of the lemma follows by duality. 

**Theorem 2.2.** Let \( A \) be an n-dimensional Hopf algebra over an algebraically closed field \( k \) of characteristic 0. If \( A \) is not cosemisimple, then \( \dim(A_0) ≤ n - 2 \).

**Proof.** Let us suppose that \( \dim(A_0) = n - 1 \). Then by [16, Theorem 5.4.2] there is a coideal \( I \) in \( A \) such that \( A = A_0 ⊕ I \). Let \( C \) be a subcoalgebra of \( A \) such that \( I ⊂ C \) and \( 2 ≤ \dim(C) ≤ 4 \). Since \( I \) is contained in \( C \), it follows that \( C \) is not cosemisimple; otherwise \( C ⊂ A_0 \).

Let \( a, b, x, a_1 \) be generators of \( C \), as in the proof of the preceding lemma. The \( k \)-linear map \( φ: M_2(k)^* → C \), which sends the canonical multiplicative matrix \( e_{11}, e_{21}, e_{12}, e_{22} \) respectively to \( b, x, a_1, a \), is a surjective morphism of coalgebras. If \( \dim(C) = 4 \), then \( φ \) is an isomorphism, so \( C \) is simple, a contradiction. It results that \( \dim(C) ≤ 3 \) and \( C \) is pointed.

Suppose now that \( \dim(C) = 2 \). Since \( C \) is not cosemisimple, by (0.1) there is a group-like element \( g ∈ G(A) \) such that \( C = kg ⊕ P'_g ⊕ P''_g(C) \). But, for any \( a ∈ P'_g \), we have \( g^{-1}a ∈ P(A) \). Therefore \( a = 0 \), since \( P(A) = 0 \) (see [21, Proposition 1(b)]). It results in \( C = kg \), a contradiction.

Finally, let us study the case \( \dim(C) = 3 \). We have \( \dim(C_0) ≤ 2 \), and we can prove, as in the preceding case, that \( \dim(C_0) ≥ 2 \). Let \( g, h \) be the
Hopf Algebras of Low Dimension

353

group-like elements of C. Then, again by (0.1), \( C_1 = C_0 \oplus P_{g, h}(C) \oplus P_{h, g}(C) \). Since \( C \neq C_0 \), it follows that either \( P_{g, h}(C) \neq 0 \) or \( P_{h, g}(C) \neq 0 \), so there is \( \sigma \in G(A) \) such that \( P_{\sigma, 1}(A) \neq 0 \). Let \( y \in P_{\sigma, 1}(A) \) be a nonzero element. The subalgebra \( B \) generated by \( \sigma \) and \( y \) is a Hopf subalgebra. The coradical of \( B \) is the subalgebra generated by \( \sigma \) (see [18, Lemma 1]). On the other hand, \( B_0 = B \cap A_0 \), so either \( B \subset A_0 \) or \( \dim(B_0) = \dim(B) - 1 \). In the first case we get \( y \in A_0 \), and in the second one we have \( \dim(B_0) = 1 \), since \( \dim(B_0) \) divides \( \dim(B) \). Both conclusions lead us to a contradiction, so \( \dim(A_0) < n - 2 \).

**Corollary 2.3.** Let \( A \) be a Hopf algebra such that \( \dim(A) \leq 11 \). Suppose that \( A \) is neither semisimple nor pointed. Then \( \dim(A) \geq 8 \) and \( A_0 = k[G] \oplus M_2(k)^* \), where \( G \) is a group with \( |G| \leq 3 \). The subcoalgebra \( M_2(k)^* \) is \( S \) invariant and generates \( A \) as an algebra.

**Proof.** Of course, if \( \dim(A) \leq 5 \), then \( A \) is pointed. By [21, Theorem 5] any Hopf algebra of dimension 6 is semisimple, so \( \dim(A) \geq 8 \) (Hopf algebras of prime dimension are group algebras; hence they are pointed). Let us suppose that \( |G| \geq 4 \). Then \( \dim(A_0) \geq 8 \), so \( \dim(A) \geq 10 \). That forces \( \dim(A) = 10 \) and \( \ord(G) = 4 \), which is impossible by Theorem 0.4. This results in \( \ord(G) \leq 3 \). Since \( S(M_2(k)^*) \) is a simple subcoalgebra of dimension 4, we have \( S(M_2(k)^*) = M_2(k)^* \). Let \( B \) be the subalgebra generated by \( M_2(k)^* \). Since \( B \) is a Hopf subalgebra and \( B \) includes \( k \oplus M_2(k)^* \), it follows that \( \dim(B) > 5 \); therefore \( B = A \).

Throughout the rest of this section we focus on a Hopf algebra \( A \) of dimension \( \leq 11 \) such that \( A \) is neither pointed nor semisimple. Hence the coradical of \( A \) contains a unique four-dimensional simple subcoalgebra \( C \) that generates \( A \) as an algebra. It follows that \( \ord(S \mid C) = \ord(S) \). If the order of the antipode \( S \) is \( 2n \), then \( 2n > 2 \), since \( A \) is not semisimple (see Theorem 0.3). By Theorem 1.5, we can pick up a root of unity \( \omega \) and a multiplicative matrix \( e \) on \( C \) such that \( \ord(\omega^2) = n \) and \( e \) satisfies the relations (2)-(10). Let \( D \) be the subcoalgebra spanned as a vector space by \( e_{11}^n, e_{12}^n, e_{21}^n, \) and \( e_{22}^n \) and take \( B \) to be the subalgebra generated by \( D \). Note that \( B \) is a Hopf subalgebra. We wish to show that \( B \) is commutative. By Theorem 1.6 it is enough to show that \( \ord(\omega) = 2n \). We need the following proposition.

**Proposition 2.4.** There is no Hopf algebra \( A \) of dimension 9 such that \( \dim(A_0) = 7 \).

**Proof.** Suppose that there is a such Hopf algebra \( A \). The coradical of \( A \) is the direct sum \( k[G] \oplus M_2(k)^* \), where \( G \) is cyclic of order 3. Let \( g \) be a generator of \( G \). The multiplication by \( g \) to the right is a coalgebra automorphism of \( C \) of order 1 or 3. Then, by Lemma 1.2, there exist a
multiplicative matrix $e'$ in $C$ and a third root of unity $\zeta$ (not necessarily primitive) such that $e'_{ij}g = \zeta^{i-j}e'_{ij}$, for all $i, j$. By [12, Proposition 3.8 and Theorem 4.2] there is a right $k[G]$-linear map $\phi: G \rightarrow k[G]$ that is invertible under the convolution product. Let $\phi^{-1}: x \rightarrow k[G]$ be the $*$-inverse of $\phi$. Let $a_{ij} = \phi(e'_{ij}) \in k[G]$ and $b_{ij} = \phi^{-1}(e'_{ij}) \in k[G]$, for $i, j = 1, 2$. Since $\phi^{-1}$ is the $*$-inverse of $\phi$, it results that the matrix $X = (a_{ij})_{i,j=1,2}$ is invertible and its inverse is $Y = (b_{ij})_{i,j=1,2}$. On the other hand, $\phi$ is $k[G]$-linear; hence

$$a_{ij}g = \phi(e'_{ij})g = \phi(e'_{ij}g) = \zeta^{i-j}a_{ij},$$

for all $i, j$. Write $a_{ij} = \sum_{p=0}^{2} a_{ij0}g_{p}$, with $a_{ijp} \in k$. From the above relation we obtain $a_{ij} = \alpha_{ij0}g_{0} + \alpha_{ij1}g_{1} + \alpha_{ij2}g_{2}$. Since

$$\det(X) = \left[3\alpha_{110}\alpha_{220} - \alpha_{120}\alpha_{210}(1 + \zeta + \zeta^2)\right](1 + g + g^2),$$

we get the desired contradiction by remarking that the determinant of $X$ is a zero divisor in $k[G]$.

**Proposition 2.5.** Let $A$ be a Hopf algebra that is not semisimple. If neither $A$ nor $A^*$ is pointed, then the root of unity $\omega$ from the remarks preceding Proposition 2.4 is primitive of order $2n$. In particular, the Hopf algebra $B$ is commutative and strictly included in $A$.

**Proof.** Since ord($\omega^2$) = $n$, it results that either ord($\omega$) = 2 or ord($\omega$) = $n$ and $n$ is odd. Suppose that ord($\omega$) = $n$ and $n$ is odd. Since $A$ is not semisimple, it follows that $n > 1$. By Theorem 0.2, the order of the antipode divides $4\text{lcm}(|G(A)|, |G(A^*)|)$; hence any prime factor of $n$ divides either $|G(A)|$ or $|G(A^*)|$. In conclusion, by Corollary 2.3, $|G(A)| = 3$ or $|G(A^*)| = 3$. This forces dim($A$) = 9 = dim($A^*$) and dim($A^\circ$) = 3 or dim($A^\circ$) = 7, contradicting the preceding proposition. Finally, $B$ is commutative by Theorem 1.6, and it is strictly included in $A$, because $A$ is not commutative (in characteristic 0 commutative Hopf algebras are semisimple).

**Corollary 2.6.** The Hopf subalgebra $B$ is one or two-dimensional.

**Proof.** As dim($B$) divides dim($A$), it follows that dim($B$) = 3. Since a commutative Hopf algebra of dimension $\leq 5$ is cosemisimple (the base field is algebraically closed of characteristic 0), it follows that $B$ is a group algebra, so dim($B$) = $|G(B)| \leq |G(A)| \leq 3$. If dim($B$) = 3, then $B = k[G]$, dim($A$) = 9, and dim($A^\circ$) = 7, which leads us to a contradiction of Proposition 2.4.

**Proposition 2.7.** Let $A$ be a Hopf algebra such that dim($A$) = 11. Suppose that $A$ and $A^*$ are neither pointed nor semisimple. If $e$ is a multiplicative
matrix satisfying (2)-(9), then the left ideal generated by $e_{12}$ is a nilpotent Hopf ideal.

**Proof.** The Hopf subalgebra $B$ is commutative and $\dim(B) \leq 2$. Let $I = Ae_{12}$ be the left ideal generated by $e_{12}$. By (3), (7), and (10) it follows that $Ae_{12} = e_{12}A$ (the basis $e$ generates $A$ as an algebra), so $I$ is a two-sided ideal. The vector space $ke_{12}$ is a coideal in $A$, so the left ideal $I = Ae_{12} = A(ke_{12})$ is a coideal, too. Moreover, $I$ is $S$-invariant, because $S(e_{12}) = \omega^{-1}e_{12}$ and $Ae_{12} = e_{12}A$. In conclusion, $I$ is a Hopf ideal. Let us show that $I$ is nilpotent. For it is enough to prove that $e_{12}^n = 0$, as $Ae_{12} = e_{12}A$. We know that $\dim(B) \leq 2$, where $B$ is the Hopf subalgebra generated by $e_{11}, e_{12}, e_{21},$ and $e_{22}$. This implies $S = id_B$ on $B$, so $S(e_{12}^n) = e_{12}^n$. On the other hand, $S(e_{12}^n) = \omega^{-n}e_{12}^n = -e_{12}^n$, that is, $e_{12}^n = 0$.

**THEOREM 2.8.** Let $A$ be a Hopf algebra of dimension $\leq 11$ that is not semisimple. Then $A$ or $A^*$ is pointed.

**Proof.** Obviously $\dim(A) \leq 10$, since $A$ is not semisimple. Suppose that $A$ and $A^*$ are not pointed; hence $\dim(A) \geq 8$. Because $A$ is not semisimple, it results that $A^*$ is not semisimple, too. If $J$ is the Jacobian radical of $A$, we have

$$(A/J)^* = (A^*)_0 = k[G(A^*)] \oplus M_2(k)^*;$$

therefore, $\dim(J) \leq \dim(A) - 5$. The two-sided ideal $I = Ae_{12}$ is nilpotent, so $I \subset J$. Let us prove that we obtain a contradiction for each possible dimension of $A$.

**The case** $\dim(A) = 8$. We have $\dim(I) \leq \dim(J) \leq 3$ and $A/I$ is a Hopf algebra, so $\dim(A/I)$ divides $\dim(A)$, a contradiction, because $\dim(A/I) \geq 5$.

**The case** $\dim(A) = 9$. We can obtain a contradiction similarly: $\dim(I) \leq \dim(J) \leq 4$; consequently, $\dim(A/I) \geq 5$. Thus $\dim(A/I)$ does not divide $\dim(A)$.

**The case** $\dim(A) = 10$. If $\dim(I) \leq 4$, we can proceed as above. Suppose that $\dim(I) = 5$. It follows that $I = J$ and $J$ is a Hopf ideal. Hence $(A/J)^*$ is a Hopf algebra of dimension 5 that is isomorphic as a coalgebra to $k[G(A^*)] \oplus M_2(k)^*$, a contradiction. 

3. POINTED HOPF ALGEBRAS OF DIMENSION 8

In this section we shall complete the classification of Hopf algebras of dimension $\leq 11$ by describing the types of pointed 8-dimensional ones. We shall start by studying those Hopf algebras $A$ with $G(A) = \{1, g\}$, a
group of order 2. By [21, Theorem 2] there is a Hopf subalgebra $B$ of $A$ that is isomorphic to Sweedler's noncommutative and noncocommutative Hopf algebra of dimension 4. Let $x$ be a nonzero $(g, 1)$-primitive element in $B$. Our main aim is to prove that under these assumptions $A_1 \neq B$, where $(A_i)_{i \geq 0}$ is the coradical filtration of $A$.

By Theorem 0.1, $A_1 = A_0 \oplus (\bigoplus_{g, \tau \in G(A)} P_{g, \tau}(A))$, where $P_{g, \tau}(A)$ is a $k$-complement of $k(g - \tau)$ in $P_{g, \tau}(A)$. Furthermore, if we take $A_{n+1}^{\sigma, \tau}$ to be by definition the set

$$A_{n+1}^{\sigma, \tau} = \{ x \in A \mid \Delta(x) = x \otimes \sigma + \tau \otimes x + A_n \otimes A_n \},$$

then $A_{n+1} = \sum_{\sigma, \tau \in G(A)} A_{n+1}^{\sigma, \tau}$, for any $n \geq 1$. Let $d_{\sigma, \tau} : A \rightarrow A \otimes A$ denote the $k$-linear map given by $d_{\sigma, \tau}(y) = \Delta(y) - y \otimes \sigma - \tau \otimes y$. Note that $d_{\sigma, \tau}(y) = 0$ if and only if $y \in A_1$. It is easy to see that $d_{\sigma, \tau}$ is $k$-linear and

$$\tau \otimes d_{\sigma, \tau}(y) - (\Delta \otimes I)(d_{\sigma, \tau}(y)) + (I \otimes \Delta)(d_{\sigma, \tau}(y)) - d_{\sigma, \tau}(y) \otimes \sigma = 0. \quad (16)$$

Moreover, we have $d_{\sigma, \tau}(y) \in A_n \otimes A_n$, for any $y \in A_{n+1}^{\sigma, \tau}$. We now turn back to the particular case of pointed Hopf algebras of dimension 8.

**Lemma 3.1.** Let $A$ be a pointed Hopf algebra of dimension 8 such that $|G(A)| = 2$. Then $B \neq A_1$.

**Proof.** Suppose that $B = A_1$. We first prove that there is $y \in A_2 \setminus B$ and $\alpha \in k^*$ such that $\Delta(y) = y \otimes 1 + 1 \otimes y + \alpha x \otimes gx$. By definition we have $\tau^{-1} A_2^{\sigma, \tau} = A_2^{\tau^{-1}, \sigma}$; therefore there is $\sigma \in G(A)$ such that $A_2^{\sigma, 1} \subseteq B$. Let $y'$ be in $A_2^{\sigma, 1} \setminus B$. Since $d_{\sigma, \tau}(y') \in B \otimes B$, there are $a_{i, j} \in B$ such that

$$d_{\sigma, \tau}(y') = \sum_{i, j=0} g_i^j x^j \otimes a_{i, j},$$

By replacing $d_{\sigma, \tau}(y')$ in (16) we get

$$d_{\sigma, \tau}(y') = 1 \otimes a_{00} - \Delta(a_{00}) + a_{00} \otimes \sigma + x \otimes a_{01}$$

$$= -d_{\sigma, \tau}(a_{00}) + x \otimes a_{01}.$$ 

Thus $d_{\sigma, \tau}(y' + a_{00}) = x \otimes a_{01}$, and therefore $x \otimes a_{01}$ satisfies (16). This implies

$$\Delta(a_{01}) = a_{01} \otimes \sigma + g \otimes a_{01}.$$ 

If we suppose that $\sigma = g$, then the element $g^{-1} a_{01}$ is a primitive element, so $a_{01} = 0$, since $P(A) = 0$. Then $d_g(y' + a_{00}) = 0$, and it results that
\[y' + a_{00} \in A_1,\] which gives us a contradiction, because \(y' \notin B.\) In conclusion, \(\sigma = 1\) and \(a_{01}\) is \((1, g)\)-primitive in \(B.\) Hence \(a_{01} = \alpha gx + \beta(g - 1),\) where \(\alpha, \beta\) are in \(k.\) We end this part of the proof by taking \(y = y' + a_{00} - \beta x.\)

Let \(y \in A_2 \setminus B\) be such that \(\Delta(y) = y \otimes 1 + 1 \otimes y + \alpha x \otimes gx.\) The subalgebra generated by \(B \oplus ky\) is a Hopf subalgebra of \(A\) that strictly includes \(B.\) Hence \(A\) is generated as an algebra by \(x, g,\) and \(y.\)

We now prove that \(B\) is a normal Hopf subalgebra of \(A,\) that is, \(AB^+\) is a Hopf ideal in \(A.\) In particular, it results that the quotient \(\overline{A} = A/AB^+\) is a Hopf algebra. It is well known that \(B\) is normal in \(A\) if and only if \(B\) is invariant with respect to the adjoint action, which is defined by

\[\text{ad} a(b) = \sum_{(a)} a_{(1)}bS(a_{(2)}), \quad a \in A, \quad b \in B.\]

One can check easily that

\[\Delta(gy - yg) = (gy - yg) \otimes g + g \otimes (gy - yg),\]

so \(gy = yg.\) Thus we have

\[\Delta(xy - yx) = (xy - yx) \otimes g + 1 \otimes (xy - yx).\]

It results that \(xy - yx = \beta x,\) since we have supposed that \(A_1 = B.\) Obviously \(S(y) = -y;\) therefore one gets

\[\text{ad} y(x) = yx - xy = \beta x \in B,\]

\[\text{ad} y(g) = gy - yg = 0 \in B.\]

These relations prove that \(B\) is normal in \(A,\) because on the one hand \(x, g,\) and \(y\) generate \(A\) as an algebra, and on the other hand \(\{g, x\}\) is a generating set for the algebra \(B.\)

By [12, Theorem 3.5] \(A\) is isomorphic to \(B \otimes \overline{A}\) as a left \(B\)-module and a right \(\overline{A}\)-comodule. Here, \(B \otimes \overline{A}\) is endowed with the natural left \(B\)-module and right \(\overline{A}\)-comodule structures. In particular, we obtain \(\dim(\overline{A}) = 2.\) On the other hand, the algebra \(\overline{A}\) is generated by \(\bar{g}, \bar{x}, \bar{y},\) the classes of \(x, g,\) and \(y,\) respectively. But \(\bar{g} = \bar{1}, \bar{x} = 0,\) and

\[\overline{\Delta}(\bar{y}) = \bar{y} \otimes \bar{1} + \bar{1} \otimes \bar{y},\]

so \(\bar{y} = 0,\) as \(\bar{y}\) is a primitive element in the Hopf algebra \(\overline{A}.\) We end the proof by remarking that \(\overline{A}\) is the \(k\)-vector generated by \(\{\bar{1}\},\) a contradiction.
We can now describe the types of 8-dimensional pointed Hopf algebras that have two group-like elements. Let us denote the free algebra in the indeterminates $X, Y, G$ by $k\langle X, Y, G \rangle$. This algebra can be uniquely endowed with a bialgebra structure such that $G$ is a group-like element and $X, Y$ are $(G, 1)$-primitive. The two-sided ideal generated by the set \{ $G^2 - 1, X^2, Y^2, GX + XG, YG + GY, XY + YX$ \} is a bi-ideal; therefore the corresponding quotient algebra is a bialgebra $A_{C_2}$. Moreover, one can see easily that $A_{C_2}$ is an 8-dimensional Hopf algebra. Its antipode $S$ maps the classes of $G, X, Y$, respectively, to the classes of $G, -GX, -GY$.

**Theorem 3.2.** Let $A$ be a pointed Hopf algebra such that $\dim(A) = 8$ and $G(A) \cong C_2$. Then $A \cong A_{C_2}$.

**Proof.** By the preceding lemma it results that $\dim(P_{g,1}(A)) \geq 2$. The inner automorphism $\phi_g$ afforded by $g$ is semisimple and $P_{g,1}(A)$ is $\phi_g$-invariant, so we can choose a basis of $P_{g,1}(A)$ that contains only eigenvectors of $\phi_g$. Let $x$ and $y$ be two different elements that belong to this basis. There are $a, b \in \{-1, 1\}$ such that $gx = axg$ and $gy = byg$. If $a = 1$, then the subalgebra generated by $x$ and $g$ is a commutative Hopf subalgebra of dimension 4. Since any commutative Hopf algebra of dimension 4 is a group algebra, it results that $|G(A)| = 4$, a contradiction. In conclusion, $a = -1$ and, similarly, $b = -1$, so $gx = -xg$ and $gy = -yg$.

We get

$$
\Delta(x^2) = x^2 \otimes 1 + 1 \otimes x^2,
\Delta(y^2) = y^2 \otimes 1 + 1 \otimes y^2,
\Delta(xy + yx) = (xy + yx) \otimes 1 + 1 \otimes (xy + yx).
$$

We conclude by remarking that a finite-dimensional Hopf algebra has no primitive elements in characteristic 0 and that the subalgebra generated by $x, y$, and $g$ is $A$.

**Remark 3.** One can see easily that $A_{C_2}$ is self-dual, i.e., $(A_{C_2})^* \cong A_{C_2}$.

Let us now consider a pointed Hopf algebra of dimension 8 with $G(A) = \langle g \rangle$, a cyclic of order 4 generated by $g$. If $\sigma \in G(A)$ is of order $n$ and $P_{\sigma,1}(A) \not\subseteq A_0$, then, by the proof of [21, Theorem 2], there are a $(\sigma, 1)$-primitive element $x \not\in A_0$ and a natural number $m > 1$ such that $m \mid n$, $gx = qxg$ and $x^m \in \{0, \sigma^m - 1\}$. Here $q$ is a root of unity with $\text{ord}(q) = m$.

If $\sigma \in G(A)$ is a generator, then the subalgebra $B$ generated by $x$ and $\sigma$ is a Hopf subalgebra of $A$ that strictly includes $A_0$ and $\dim(B) \mid 8$. Hence $A = B$, $n = 4$, and $m = 2$. In conclusion, there are two types of
pointed 8-dimensional Hopf algebras with \( \sigma \) a generator of \( G(A) \), namely,

\[
A'_{C_4} = \frac{k \langle X, G \rangle}{\langle G^4 - 1, X^2, GX + XG \rangle},
\]

\[
A''_{C_4} = \frac{k \langle X, G \rangle}{\langle G^4 - 1, X^2 - G^2 + 1, GX + XG \rangle}.
\]

The coalgebra structure of \( A'_{C_4} \) and \( A''_{C_4} \) is induced by the unique coalgebra structure on \( k \langle X, G \rangle \) such that \( G \) is a group-like element and \( X \) is \( (G, 1) \)-primitive. Their antipodes map the classes of \( G \) and \( X \), respectively, to the class of \( G^3 \) and \( -XG^3 \).

If \( P_{\sigma, \tau}(A) \subset A_0 \), for all \( \sigma \in \{g, g^3\} \), then it results that \( P_{g^2, 1}(A) \notin A_0 \). Proceeding as in the proof of [21, Theorem 2], there exist \( x \in P_{g^2, 1}(A) \setminus A_0 \) and \( q \in k^* \) such that \( gx = qg \), and \( q^4 = 1 \). If \( q^2 = 1 \), then \( g^2x = xg^2 \) and the subalgebra \( B \) generated by the coalgebra \( kx \oplus kg^2 \) is a commutative Hopf algebra of dimension 4. It follows that \( B \subset A_0 \), since \( B \) is cosemisimple, too. It results that \( x \in A_0 \), which is impossible by the assumptions on \( x \). Therefore we have \( g^2x = q^2xg^2 = -xg^2 \) and

\[
\Delta(x^2) = x^2 \otimes 1 + 1 \otimes x^2.
\]

The last relation implies \( x^2 = 0 \), so

\[
A''_{C_4, q} = \frac{k \langle X, G \rangle}{\langle G^4 - 1, X^2, GX - qXG \rangle},
\]

where \( q \) is a primitive root of unity of order 4. In this case \( k \langle X, G \rangle \) is a bialgebra such that \( G \) is a group-like element and \( X \) is \( (G^2, 1) \)-primitive. As before, the antipode maps the classes of \( G \) and \( X \), respectively, to the classes of \( G^3 \) and \( -XG^3 \). In conclusion, we have proved the following.

**Theorem 3.3.** Let \( A \) be a pointed 8-dimensional Hopf algebra \( A \) such that \( G(A) \cong C_4 \). Then \( A \) is isomorphic to one of the following three Hopf algebras: \( A'_{C_4, q}, A''_{C_4, q}, A''_{C_4, q} \).

**Remark 4.** The map \( A''_{C_4, q} \rightarrow A''_{C_4, -q} \), sending \( \hat{G} \) to \( \hat{G}^3 \) and \( \hat{X} \) to \( q\hat{X} \), is an isomorphism of Hopf algebras. The Hopf dual of \( A'_{C_4} \) is \( A''_{C_4, q} \), and obviously, the set of group-like elements of \( (A''_{C_4})^* \) has two elements. Hence \( (A''_{C_4})^* \) is neither pointed nor semisimple.

We now focus on 8-dimensional Hopf algebras with \( G(A) = C_2 \times C_2 \). In the next theorem we shall prove that there are two types of such Hopf algebras. Let \( k \langle G, H, X \rangle \) be the free algebra in the indeterminates \( G, H, X \). We define a bialgebra structure on \( k \langle G, H, X \rangle \) such that \( G \) and \( H \) are group-like elements and \( X \) is \( (G, 1) \)-primitive. The two-sided ideal \( \langle G^2 - 1, H^2 - 1, X^2, GX + XG, HX + XH, GH - HG \rangle \) is a bi-ideal such that the quotient bialgebra is an 8-dimensional Hopf algebra. The antipode
sends the classes of $G, H, X$, respectively, to the classes of $G, H, -XG$. We denote this Hopf algebra by $A_{C_2 \times C_2}$.

**Theorem 3.4.** If $A$ is a pointed Hopf algebra of dimension 8 such that $G(A) = C_2 \times C_2$, then $A = A_{C_2 \times C_2}$.

**Proof.** Using [21, Theorem 2] we find $g \in G(A)$ and $x \in P_{g,1}(A) \setminus A_0$ such that $gx = -xg$. Counting the dimension of $A$, one can see that $\dim(P_{g,0}(A)) = 1$, for all $\sigma \in G(A)$. Let $h$ be a group-like element that is not in $\{1, g\}$. Then $hx = qxh$, where $q \in \{-1, 1\}$. The corresponding Hopf algebras for $q = 1$ and $q = -1$ are isomorphic, so $A = A_{C_2 \times C_2}$. 

We are now able to describe the types of all Hopf algebras of dimension $\leq 11$ over an algebraically closed field.

**Theorem 3.5.** If $A$ is a Hopf algebra of dimension $\leq 11$, then $A$ is isomorphic with one and only one Hopf algebra from Table I.

**Table I**

<table>
<thead>
<tr>
<th>$n = \dim(A)$</th>
<th>$A$</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n \in {2, 3, 5, 7, 11}$</td>
<td>$k[C_n]$</td>
<td>Pointed and semisimple$^a$</td>
</tr>
<tr>
<td>$n \in {4, 9}$</td>
<td>$k[G],</td>
<td>G</td>
</tr>
<tr>
<td></td>
<td>$T_n$</td>
<td>Pointed$^c$</td>
</tr>
<tr>
<td>$n \in {6, 10}$</td>
<td>$k[G],</td>
<td>G</td>
</tr>
<tr>
<td></td>
<td>$k[D_n]^e$</td>
<td>Semisimple$^e$</td>
</tr>
<tr>
<td>$n = 8$</td>
<td>$k[G],</td>
<td>G</td>
</tr>
<tr>
<td></td>
<td>$k[G]^+, G$ nonabelian</td>
<td>Semisimple$^g$</td>
</tr>
<tr>
<td></td>
<td>$A_8$</td>
<td></td>
</tr>
<tr>
<td>$n = 8$</td>
<td>$A_{C_2}$</td>
<td>Pointed$^h$</td>
</tr>
<tr>
<td></td>
<td>$A_{C_2 \times C_2}$</td>
<td>Pointed and self-dual</td>
</tr>
<tr>
<td></td>
<td>$A_{C_4}, A_{C_6}, A_{C_4 \times C_2}$</td>
<td>Pointed</td>
</tr>
<tr>
<td></td>
<td>$(A_{C_4}^<em>)^</em>$</td>
<td>Its dual is pointed</td>
</tr>
</tbody>
</table>

$^a$ See [23].  
$^b$ See [10].  
$^c$ $T_p^2$ is the unique noncommutative and noncocommutative Hopf algebra of dimension $p^2$ (see [21]).  
$^d$ See [9] and [21].  
$^e$ $D_n$ is the dihedral group of order $n$.  
$^f$ See [11].  
$^g$ $A_8$ is the unique noncommutative and noncocommutative semisimple Hopf algebra of dimension 8 (see [11]).  
$^h$ See Theorems 3.2, 3.3, and 3.4.
HOPF ALGEBRAS OF LOW DIMENSION

REFERENCES