Note

Digraphs with unique minimal king sets

Stephen Bowser, Charles Cable
Department of Mathematics, Allegheny College, 520 N. Main Street, Meadville 16335, USA

Received 14 October 2003; received in revised form 22 July 2005; accepted 10 October 2005
Available online 17 November 2005

Abstract

If $D$ is a digraph, then $K \subseteq V(D)$ is a king set of $D$ if $D[K]$ is discrete and for each $y \in V(D) - K$ there is $x \in K$ such that the directed distance from $x$ to $y$ is less than three. A king set $K$ will be called minimal if no proper subset of $K$ is a king set. We characterize both the digraphs which have a unique king set and those which have a unique minimal king set.

© 2005 Elsevier B.V. All rights reserved.

Keywords: Digraphs; King sets

In 1980 Maurer [3] introduced the notion of a king. The investigation in [3] was carried out in the setting of tournaments, but the idea makes sense in an arbitrary digraph: a king is a vertex whose directed distance to every other vertex is either one or two. Maurer includes citations to numerous related investigations. See also [4]. In particular, Laudau [2] proved a result that implies that every tournament has a king and Maurer [3] proved that every tournament without a source has at least three kings. Neither of these results extend to arbitrary digraphs. (The digraph with order four and arc set $\{x_1 \to x_2, x_2 \to x_3, x_3 \to x_4\}$ has no king whereas if the arc $x_3 \to x_1$ is added, one obtains a digraph with no sources and yet exactly two kings.) The guarantee of existence can be recovered for arbitrary digraphs by generalizing the notion of king to that of king set (defined below). Moreover, it is possible to give a simple characterization of those digraphs with unique king sets as well as those which have a unique minimal king set.
Notation. For a (simple) digraph $D$, $V(D)$ and $A(D)$ denote its vertex set and arc set, respectively. If $U \subseteq V(D)$, then $D[U]$ denotes the subdigraph of $D$ induced by $U$.

**Definition 1.** A king set $K$ of a digraph $D$ is a subset of $V(D)$ satisfying two properties:

1. the ruling property: $\forall y \in V(D) - K, \exists x \in K$ such that the directed distance in $D$ from $x$ to $y$ is either one or two.
2. the royal courtesy property: $D[K]$ is discrete, i.e. $K$ is an independent set in the underlying graph of $D$.

If $K$ is a king set for digraph $D$, $x \in K$ and $y \in V(D) - K$ and the directed distance from $x$ to $y$ in $D$ is either one or two, then we will say that $y$ is ruled by $x$ (in one or two steps, respectively).

**Definition 2.** A king set $K$ of a digraph $D$ is called minimal if no proper subset of $K$ is a king set of $D$.

Due to the condition of royal courtesy, it is evident that the king sets in a tournament are exactly those singletons containing a king. In this sense, the definition of king set specializes to that of king when restricted to tournaments. On the other hand, although, as already mentioned, there are digraphs with no king, Chvátal and Lovász [1] proved a result that implies that every digraph has at least one king set. Our goals are to describe precisely which digraphs have exactly one king set and those which have precisely one minimal king set. The following lemma will be useful for both results.

**Lemma.** If $x$ is a non-source vertex of digraph $D$, then $D$ has a king set which does not contain $x$.

**Proof.** Let $D$ be a digraph with vertex set $V$ and king set $K$. (By Chvátal and Lovász [1], such a king set exists.) Let $x \in V$ such that $\text{in}(x) \neq \emptyset$. If $x \notin K$, then we are done, so suppose that $x \in K$, and let $y \in \text{in}(x)$. If $V - y - \text{out}(y) = \emptyset$, then $\{y\}$ is a king set of $D$ which does not contain $x$. Otherwise, $D[V - y - \text{out}(y)]$ has a (nonempty) king set $K'$. If a vertex of $K'$ beats $y$, then $K'$ is a king set of $D$ and does not contain $x$. If no vertex of $K'$ beats $y$, then $K' \cup \{y\}$ is a king set of $D$ and, again, does not contain $x$. □

**Theorem 1.** A digraph $D$ has a unique king set $\iff V(D) = S \cup \text{out}(S)$, where $S$ is the set of sources of $D$. In this case, $S$ is the unique king set.

**Proof.** Let $D$ be a digraph with vertex set $V$ and let $S$ be the set of sources of $D$. If $V = S \cup \text{out}(S)$, then it is clear that $S$ is the only king set of $D$, so suppose that $R = V - (S \cup \text{out}(S)) \neq \emptyset$, let $K$ be a king set of $D[R]$, and fix $x \in K$. Then $S \cup K$ is a king set of $D$. Since $x$ is not a source in $D$, the lemma provides a king set of $D$ distinct from $S \cup K$ and the proof is complete. □

**Theorem 2.** A digraph $D$ has a unique minimal king set if and only if $V(D) = S \cup \text{out}(S) \cup \text{out}(\text{out}(S))$, where $S$ is the set of sources in $D$. In this case, $S$ is the unique minimal king set.
Proof. Let $D$ be a digraph with vertex set $V$ and let $S$ be the set of sources of $D$. Since every king set of $D$ must contain $S$, if $V = S \cup \text{out}(S) \cup \text{out}(\text{out}(S))$, then $S$ is the unique minimal king set.

Now suppose that $R = V - (S \cup \text{out}(S) \cup \text{out}(\text{out}(S))) \neq \emptyset$, let $K$ be a minimal king set of $D[R]$, and fix $x \in K$. Clearly, $S \cup K$ is a king set of $D$ which is minimal and, since $x$ is not a source in $D$, the lemma again provides a king set of $D$ which does not include $x$ and so does not contain $S \cup K$. Thus, $D$ has a second minimal king set. □

Corollary. If $D$ has no sources, then $D$ has at least two minimal king sets.

As we mentioned in the introduction, Maurer proved that a tournament without sources has at least three kings. Note that the directed 4-cycle (1 $\to$ 2 $\to$ 3 $\to$ 4 $\to$ 1) has no sources and exactly two king sets, each of which is minimal. Thus, the most obvious adaptation to generalized digraphs of Maurer’s result does not hold.

References