Fuzzy Topology. II. Product and Quotient Spaces

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In [6], after introducing the fundamental concepts of fuzzy points which take crisp singletons or ordinary points as special cases, q-relations and Q-neighborhoods of fuzzy points, we built up a satisfactory theory of neighborhood structures and generalized many fundamental notions and theorems, especially those theorems in the theory of Moore-Smith's convergence such as those contained in [5, Chaps. I, II]. The purpose of the present paper is to generalize all the theorems concerning product spaces and quotient spaces contained in [5, Chap. III]. Our treatment will sharpen the related results contained in the literature [1-4] so that these problems in fuzzy topology will be solved to the same degree as the corresponding ones in general topology. The present work is the continuation of [6] and hence all the conventions in [6] still hold good in the present paper, especially that \((X, \mathcal{T})\) and \((Y, \mathcal{U})\) denote fuzzy topological spaces.

1. FUZZY CONTINUITY

**Definition 1.1.** Let \(f: X \rightarrow Y\) be a function. For a fuzzy set \(B\) in \(Y\), we define \(f^{-1}(B)\) by the formula
\[
 f^{-1}(B)(x) = B(f(x)) \quad \text{for} \quad x \in X
\]
Obviously, \(f^{-1}(B)\) is a fuzzy set in \(X\). For a fuzzy set \(A\) in \(X\), \(f(A)\) is defined as follows:
\[
 f(A)(y) = \sup\{A(x) \mid x \in X \text{ and } f(x) = y\} \quad \text{when } f^{-1}(y) \neq \Phi
\]
\[
 = 0 \quad \text{when } f^{-1}(y) = \Phi.
\]

It is clear that \(f(A)\) is a fuzzy set in \(Y\).

**Lemma 1.1.** Let \(f: X \rightarrow Y\) be a function and let \(A\) and \(B\) be fuzzy sets in \(X\) and \(Y\), respectively; then the following properties hold:

1. \(f^{-1}(A) \supseteq A\) and \(f^{-1}(f(A)) = A\) iff, for every \(y \in Y\), when \(f^{-1}(y) \neq \Phi\), \(A\) is a constant function on \(f^{-1}(y)\). In particular, if \(f\) is injective, then \(f^{-1}(A) = A\).

2. \(f^{-1}(B) \subseteq B\) and \(f^{-1}(f^{-1}(B)) = B\) iff \(f(X) \supseteq \text{supp } B\). In particular, when \(f\) is surjective, \(f^{-1}(B) = B\).
(3) For a fuzzy point \( x_h \) in \( X \), \( f(x_h) \) is a fuzzy point in \( Y \) and \( f(x_h) = (f(x))_h \).

(4) When \( f(A) \subseteq B \), \( A \subseteq f^{-1}(B) \).

(5) Let \( \{A_\alpha : \alpha \in I\} \) be a family of fuzzy sets in \( X \), then \( f(U_\alpha A_\alpha) = U_\alpha f(A_\alpha) \).

The proof of this lemma is straightforward and is left for the readers.

**DEFINITION 1.2.** [1] The function \( f: (X, \mathcal{J}) \rightarrow (Y, \mathcal{W}) \) is called fuzzy continuous (or \( F \)-continuous) iff, for every \( B \in \mathcal{W} \), \( f^{-1}(B) \in \mathcal{J} \). The function \( f: (X, \mathcal{J}) \rightarrow (Y, \mathcal{W}) \) is called fuzzy homeomorphic (or \( F \)-homeomorphic) iff \( f: X \rightarrow Y \) is a bijection (i.e., \( f \) is both injective and surjective) and both \( f \) and \( f^{-1} \) are \( F \)-continuous.

**THEOREM 1.1.** Let \( f: (X, \mathcal{J}) \rightarrow (Y, \mathcal{W}) \) be a function; then the following are equivalent:

1. \( f \) is \( F \)-continuous.
2. For every \( \mathcal{W} \)-closed \( A \), \( f^{-1}(A) \) is \( \mathcal{J} \)-closed.
3. For each member \( V \) of a subbase \( \mathcal{S} \) for \( \mathcal{W} \), \( f^{-1}(V) \) is \( \mathcal{J} \)-open.
4. For each fuzzy point \( e \) in \( X \) and each neighborhood \( V \) of \( f(e) \), there exists a neighborhood \( U \) of \( e \) such that \( f(U) \subseteq V \).
5. For each fuzzy point \( e \) in \( X \) and each \( \mathcal{Q} \)-neighborhood \( V \) of \( f(e) \), there exists a \( \mathcal{Q} \)-neighborhood of \( e \) such that \( f(U) \subseteq V \).
6. For each fuzzy net \( S = \{S_n : n \in D\} \), if \( S \) converges to \( e \), then \( f \circ S = \{f(s_n) : n \in D\} \) is a fuzzy net in \( Y \) and converges to \( f(e) \).
7. For any fuzzy set \( A \) in \( X \), \( f(A) \subseteq f^{-1}(A) \).
8. For any fuzzy set \( B \) in \( Y \), \( f^{-1}(B) \subseteq f^{-1}(B) \).

**Proof.** Conditions (1) and (2) are obviously equivalent (cf. [1]). The other equivalences and implications such as: \((1) \iff (3), (1) \iff (4) \) and \((6) \Rightarrow (7) \Rightarrow (8) \Rightarrow (1)\) can be proved in a similar way as in general topology. Hence it suffices to show the new characterization of \( F \)-continuity in terms of the \( \mathcal{Q} \)-neighborhood, that is, we have to prove the implications \((1) \Rightarrow (5) \Rightarrow (6)\). As for \((1) \Rightarrow (5)\), let \( e = x_h \), \( f(x) = y \), then \( f(e) = y_h \). From the definition of \( \mathcal{Q} \)-neighborhood, there is a \( \mathcal{Q} \)-neighborhood \( V \) of \( f(e) \) such that \( \bar{V} \subseteq f^{-1}(V) \). Let \( f^{-1}(\bar{V}) = U \); we have \( U(x) + \lambda = \bar{V}(y) + \lambda > 1 \), i.e., \( U \) is a \( \mathcal{Q} \)-neighborhood of \( e \) and \( f(U) \subseteq f^{-1}(V) \) (Lemma 1.1). As for \((5) \Rightarrow (6)\), when the fuzzy net \( S \) is eventually quasi-coincident with \( A \), it is easily seen that the fuzzy net \( f \circ S \) is eventually quasi-coincident with \( f(A) \). From \((5)\) and the definition of convergence of fuzzy nets ([6 definition 11.4]), we obtain \((6)\).

**THEOREM 1.2.** Let \( A \) be a connected fuzzy set in \( (X, \mathcal{J}) \), and let \( f: (X, \mathcal{J}) \rightarrow (Y, \mathcal{W}) \) be \( F \)-continuous; then \( f(A) \) is a connected fuzzy set in \( (Y, \mathcal{W}) \).
Proof. Suppose that $f(A)$ is not connected. In the light of Lemma 10.1 in [6] there are non-empty relative closed sets $C$ and $D$ in the subspace $(f(A))_0(=\text{supp} f(A))$ such that $C \cup D \supset f(A)$, and $C \cap D = \emptyset$. Denote the restriction of $f$ on $A_0$ as $g: A_0 \to Y$. $g$ is, of course, also $F$-continuous. Since $C$ and $D$ are relative closed sets in $(f(A))_0$, $g^{-1}(C)$ and $g^{-1}(D)$ are relative closed sets in the subspace $A_0$. Evidently they are non-empty and disjoint (non-intersecting). Moreover, $g^{-1}(C) \cup g^{-1}(D) = g^{-1}(C \cup D) \supset g^{-1}(f(A)) \supset A$, that is to say, $A$ is disconnected, which contradicts the hypothesis.

Theorem 1.3. Let $A$ and $B$ be fuzzy sets in $(X, \mathcal{F})$ such that $X = A \cup B$ and $A \sim B$ and $B \sim A$ are $Q$-separated or separated. If $f$ is a function on $X$ such that the restrictions of $f$ on $A_0$ and $B_0$ are $F$-continuous, respectively, then $f$ is $F$-continuous on $X$.

Proof. This theorem follows directly from Proposition 9.4 and Theorem 9.3 in [6].

Definition 1.5. A family $Q = \{e_\alpha\}$ of fuzzy points $e_\alpha$ in $X$ is said to be dense in $(X, \mathcal{F})$ iff every non-empty $\mathcal{F}$-open set contains some member of $Q$. $Q$ is said to be $Q$-dense iff every non-empty $\mathcal{F}$-open set is quasi-coincident with some member of $Q$.

It is easy to construct examples to show that the concept of being dense and that of being $Q$-dense do not imply each other.

Lemma 1.2. In a fuzzy topological space $(X, \mathcal{F})$, a family $Q$ of fuzzy points in $X$ is $Q$-dense iff $\bigcup Q = X$.

Proof. The necessity of obvious, because at this time each fuzzy point on $X$ is obviously an adherence point of $\bigcup Q$ and hence belongs to $\bigcup Q$. We shall now show the sufficiency. Let $A$ be any non-empty $\mathcal{F}$-open set. For $x \in \text{supp} A$, putting $\lambda = A(x) > 0$, $\mu = 1 - \lambda/2 > 0$. Since $A(x) = \lambda > 1 - \mu$, $A$ is an open $Q$-neighborhood of the fuzzy point $x_\alpha$. Since $x_\alpha \in X = \bigcup Q$, by Theorem 4.1' in [6], there exists $y \in X$ such that $A(y) + (\bigcup Q)(y) > 1$. And hence there exists some $e \in Q$ such that $A(y) + e(y) > 1$, that is to say, $e$ is quasi-coincident with $A$.

Theorem 1.4. Let $f, g: (X, \mathcal{F}) \to (Y, \mathcal{U})$ be continuous and let $(Y, \mathcal{U})$ be a fuzzy $T_2$-space. Let $\Omega = \{e \mid e$ is a fuzzy point in $X$ such that $f(e) = g(e)\}$, then $\bigcup \Omega$ is $\mathcal{F}$-closed and if $\Omega$ is $Q$-dense, we have $f = g$.

Proof. For any $x \in X$, $f(x) = g(x)$ iff $f(x) = g(x)$. Hence, if a fuzzy point $d = y_\mu \not\in \Omega$, we have $f(y) \neq g(y)$. Since $(Y, \mathcal{U})$ is fuzzy $T_2$, there are $\mathcal{U}$-open $Q$-neighboors $U$ and $V$ of $f(y)_n$ and $g(y)_n$, respectively, such that $U \cap V = \emptyset$. From (5) of Theorem 1.1, thus there exists a $\mathcal{F}$-open $Q$-neighborhood $W$
of \(d\) such that \(f(W) \subseteq U, g(W) \subseteq V\). In the light of \(U \cap V = \emptyset\), it is easily seen that \(W \cap \bigcup \Omega = \emptyset\), therefore \(W\) and \(\bigcup \Omega\) are not quasi-coincident. That is to say, \(d\) is not an accumulation point of \(\bigcup \Omega\). By the corollary of Theorem 5.1 in [6], \(\bigcup \Omega\) is \(\mathcal{T}\)-closed.

The proof of the second statement: From Lemma 1.2, \(X = \bigcup \Omega = \bigcup \Omega\), hence for each \(x \in X\), there exists some \(e = x \in \Omega\). Then \(f(e) = g(e)\) and consequently \(f(x) = g(x)\) for each \(x \in X\). We have thus proved that \(f = g\).

2. Product Spaces

**Definition 2.1** [2-3]. Let \((X_\alpha, \mathcal{T}_\alpha)\) be a fuzzy topological space for each \(\alpha \in I\). Let \(X\) be the Cartesian product of \(\{X_\alpha \mid \alpha \in I\}\) and let \(P_\alpha\) be the projection of the product \(X\) into the \(\alpha\)th coordinate set \(X_\alpha\). Let \(\mathcal{P}_I\) denote the family of all finite subsets of \(I\). Putting \(\mathcal{B} = \{\bigcap_{\alpha \in F} P_\alpha^{-1}(U_\alpha) \mid U_\alpha \in \mathcal{T}_\alpha, F \in \mathcal{P}_I\}\), we call the topology \(\mathcal{T}\) which takes \(\mathcal{B}\) as a base the product topology for \(X\), \(\mathcal{B}\) the defining base for the product topology. The pair \((X, \mathcal{T})\) is called the product space of the fuzzy topological spaces \((X_\alpha, \mathcal{T}_\alpha), \alpha \in I\).

**Lemma 2.1.** Let \(U = \bigcap_{\alpha \in F} \{P_\alpha^{-1}(U_\alpha)\}\) be a member of the defining base \(\mathcal{B}\) for the product topology, then, when \(\beta \notin F, P_\beta(U)\) is the fuzzy set in \(X_\beta\) which takes the constant value \(\lambda\) on \(X_\mu\), where

\[
\lambda = \min_{\alpha \in F}\{\sup_{\alpha}(U_\alpha(x_\alpha) \mid x_\alpha \in X_\alpha)\},
\]

when \(\beta \in F\), let \(F_1 = F - \{\beta\}\), then \(P_\beta(U) = U_\beta \cap A\), where \(A\) is the fuzzy set in \(X\), taking the constant value \(\mu\):

\[
\mu = \min_{\alpha \in F_1}\{\sup_{\alpha}(U_\alpha(x_\alpha) \mid x_\alpha \in X_\alpha)\}.
\]

**Proof.** This lemma can be directly verified.

**Definition 2.2.** A fuzzy topological space \((X, \mathcal{T})\) is called a fully stratified space iff every fuzzy set taking a constant value on \(X\) belongs to \(\mathcal{T}\).

**Definition 2.3.** Let \(X = \times \{Z_\alpha \mid \alpha \in I\}, x = \{x_\alpha \mid \alpha \in I\} \in X\), and \(\beta \in I\), the subset \(X_\beta\) of \(X\) defined by

\[
X_\beta = \{y = \{y_\alpha\} \mid \alpha \neq \beta, y_\alpha = x_\alpha\},
\]

is called the section through \(x = \{x_\alpha\}\) and parallel to \(X_\beta\).
In ordinary general topology, the subspaces $X_a$ and $X_\beta$ are naturally considered to be homeomorphic. But this is not true generally in fuzzy topology. On account of this fact, we must take care in deducing some property $P$ for the coordinate space $(X_a, \mathcal{F}_a)$ from the product space $X$ which enjoys the same property $P$.

**Theorem 2.1.** Let $\mathcal{F}_\beta$ be the relative topology of the subspace $X_\beta$. Then there is an $F$-continuous bijection $\varphi: (X_\beta, \mathcal{F}_\beta) \rightarrow (X_\beta, \mathcal{F}_\beta)$. Moreover, when $(X_\beta, \mathcal{F}_\beta)$ is fully stratified $\varphi$ is a fuzzy homeomorphism.

**Proof.** Let $y = \{ y_a \}$ be an arbitrary point in $X_\beta$. Let $\varphi(y) = y_\beta$. It is easily seen that $\varphi$ is a bijection. From the definition of product topology, $\varphi$ is a bijection. From the definition of product topology, $\varphi$ is $F$-continuous. Suppose that $(X_\beta, \mathcal{F}_\beta)$ is fully stratified, we show that $\varphi^{-1}$ is also $F$-continuous. In fact, $\mathcal{F} = \{ A \mid A = P_a^{-1}(U_a) \cap X_\beta, U_a \in \mathcal{F}_a, \alpha \in I \}$ is a subbase of $\mathcal{F}_\beta$. From (3) of Theorem 1.1, in order to prove that $\varphi^{-1}$ is $F$-continuous, it suffices to show that $\varphi(A) \in \mathcal{F}_\beta$. Noting that $(\varphi^{-1})^{-1} = \varphi$. Obviously $\varphi(A) = P_\beta P_a^{-1}(U_a)$. When $\alpha \neq \beta$, from Lemma 2.1, $\varphi(A)$ is the fuzzy set taking the constant value on $X_\beta$; by the definition of fully stratified spaces, $\varphi(A) \in \mathcal{F}_\beta$, when $\alpha = \beta$, evidently $\varphi(A) = U_\beta \in \mathcal{F}_\beta$.

**Note.** In the light of the correspondence $\varphi$ constructed above, we may directly consider the subspace $(X_\beta, \mathcal{F}_\beta)$ of the product space $(X, \mathcal{F})$ to be a fuzzy topological space $(X, \mathcal{F}, \mathcal{F}_\beta)$, where $\mathcal{F}_\beta$ is generated by the family of fuzzy sets which consists of all the members of $\mathcal{F}_\beta$ and in addition, some fuzzy sets taking constant value on $X_\beta$ (Of course $\mathcal{F}_\beta$ is finer than $\mathcal{F}_\beta$.)

**Theorem 2.2.** Let $(X, \mathcal{F}) = \times \{(X_a, \mathcal{F}_a) \mid \alpha \in I\}$. Thus

1. The product topology is the coarsest topology for $X$ such that each projection $P_\alpha: (X, \mathcal{F}) \rightarrow (X_a, \mathcal{F}_a)$ is $F$-continuous.

2. If $(X_\beta, \mathcal{F}_\beta)$ is fully stratifiable, then the projection $P_\beta: (X, \mathcal{F}) \rightarrow (X_\beta, \mathcal{F}_\beta)$ is an open mapping.

**Proof.** (1) is obvious or see Theorem 3.1 in [3]. Part (2) can be proved as follows: In view of (5) of Lemma 1.1, it suffices to show that for a member $U = \bigcap_{\alpha \in F} P_\alpha^{-1}(U_a)$ of the defining base $\mathcal{B}, P_\beta(U)$ is open. But this can be obtained from Lemma 2.1.

The following theorem was given in [3]. For completeness, we restate it as follows:

**Theorem 2.3 [3].** Let $(Y, \mathcal{U})$ and $(X_\alpha, \mathcal{F}_\alpha), \alpha \in I$ be fuzzy topological spaces, then $f: (Y, \mathcal{U}) \rightarrow \times \{(X_\alpha, \mathcal{F}_\alpha) \mid \alpha \in I\}$ is $F$-continuous iff each $P_\alpha \circ f$ is $F$-continuous.
THEOREM 2.4. A fuzzy net \( \{S_n, n \in D\} \) in the product space \((X, \mathcal{F}) \equiv \times \{X_\alpha, \mathcal{F}_\alpha\} | \alpha \in I\) converges to a fuzzy point \( e \) iff, for each projection \( P_\alpha : X \rightarrow X_\alpha \), the fuzzy net \( P_\alpha \circ S = \{P_\alpha(S_n), n \in D\} \) in \( X_\alpha \) converges to the fuzzy point \( P_\alpha(e) \).

Applying the foregoing results, we can prove this theorem by a method similar to that of Theorem 3.4 in [5].

THEOREM 2.5. (1) Let \( \{(X_\alpha, \mathcal{F}_\alpha) | \alpha \in I\} \) be a collection of fuzzy topological spaces, among which there is at least one quasi-\( T_0 \) space, then the product space \((X, \mathcal{F})\) is quasi-\( T_0 \).

(2) If each \( (X_\alpha, \mathcal{F}_\alpha) \) is \( T_0 \) (resp. \( T_1 \)) space, the product space \((X, \mathcal{F})\) is a \( T_0 \) (resp. \( T_1 \)) space.

(3) The product space \((X, \mathcal{F})\) is a \( T_2 \) space iff each coordinate space \((X_\alpha, \mathcal{F}_\alpha)\) is a \( T_2 \) space.

Proof. (1) follows from Theorem 6.1 in [6]. (2) When all the coordinate spaces are \( T_0 \) spaces, from Theorem 6.2 in [6] and part (1) of the present theorem, \((X, \mathcal{F})\) is a \( T_0 \) space. Consider the case in which all the coordinate spaces are \( T_1 \) spaces. Note that when \( P_\alpha \) is a \( \mathcal{F}_\alpha \)-closed set, \( P_\alpha^{-1}(P_\alpha) \) is a \( \mathcal{F} \)-closed set. Taking intersection. it is easily seen that each fuzzy point in \((X, \mathcal{F})\) is closed. That is to say, the product space is a \( T_1 \) space. (3) The sufficiency is obvious. We shall now show its necessity. Consider an arbitrary coordinate space \((X_\alpha, \mathcal{F}_\beta)\). Let \( e \) and \( d \) be two fuzzy points on \( X_\beta \), whose supports are \( x_\beta \) and \( y_\beta \), respectively, such that \( x_\beta \neq y_\beta \). Let \( e(x_\alpha) = \lambda \) and \( d(y_\beta) = \mu \). For each \( \alpha \in I \) which is different from \( \beta \), take \( x_\alpha = y_\alpha \in X_\alpha \). Thus, we obtain two points \( x = \{x_\alpha\} \) and \( y = \{y_\alpha\} \) of \( X = \times \{X_\alpha | \alpha \in I\} \). Since \( x \neq y \), for the fuzzy points \( x_\alpha \) and \( y_\alpha \) in \( X \), there exist \( Q \)-neighborhoods \( U \) and \( V \) of \( x \) and \( y \), respectively, such that \( U \cap V = \emptyset \). From the definition of the product topology and a property of the least upper bound, in the defining base we have \( \overline{U} = \bigcap_{\alpha \in F} P_\alpha^{-1}(U_\alpha) \subset U \) such that \( \overline{U}(x) + \lambda > 1 \) and \( \overline{V} = \bigcap_{\alpha \in E} P_\alpha^{-1}(V_\alpha) \subset V \) such that \( \overline{V}(y) + \mu > 1 \). We claim that \( \beta \in F \), if \( \beta \notin F \) since the coordinates of \( x \) and \( y \) are different only on the coordinate set \( X_\beta \), it is obvious that \( \overline{U}(x) = \overline{U}(y) > 0 \). On the other hand, since \( \overline{V}(y) > 0 \), \( \overline{U} \cap \overline{V} \neq \emptyset \). This contradicts the fact that \( U \cap V = \emptyset \). We have thus proved that \( \beta \in F \). Similarly we can show that \( \beta \in E \). Now we show that \( U_\beta \cap V_\beta = \emptyset \). If there is \( z_\alpha \in X_\alpha \) such that \( (U_\beta \cap V_\beta)(z_\alpha) > 0 \), then for each \( \alpha \neq \beta \), letting \( z_\alpha = x_\alpha \), we get a point \( z = \{z_\alpha\} \in X = \times \{X_\alpha | \alpha \in I\} \). Obviously, we have

\[
\bigcap_{\alpha \in F} P_\alpha^{-1}(U_\alpha) (z) = \bigcap_{\alpha \in F} P_\alpha^{-1}(U_\alpha) (x) \supseteq \overline{U}(x) > 0,
\]

\[
\bigcap_{\alpha \in E} P_\alpha^{-1}(V_\alpha) (z) = \bigcap_{\alpha \in E} P_\alpha^{-1}(V_\alpha) (y) \supseteq \overline{V}(y) > 0.
\]
Therefore $U \cap \bar{V}(x) > 0$. This is in contradiction with $U \cap \bar{V} = \emptyset$. Hence $U_\beta \cap V_\beta = \emptyset$. In view of the fact that $U_\beta(x_\beta) = P_{\beta}^{-1}(U_\beta(x)) \geq \bar{U}(x) > 1 - \lambda$, $U_\beta$ is an open $Q$-neighborhood. Similarly $V_\beta$ is also an open $Q$-neighborhood. We have thus proved that $(X_\beta, \mathcal{T}_\beta)$ is a $T_2$ space.

When the product space enjoys anyone of the properties of being quasi-$T_0$, $T_0$ and $T_1$, each coordinate space does not necessarily enjoy the corresponding separation property. For example, let $X_1 = X_2 = \{x\}$, $\mathcal{F}_1 = \{\emptyset\} \cup \{x_\lambda \mid 0 < \lambda \leq \frac{1}{2} \text{ and } \lambda = 1\}$, $\mathcal{F}_2 = \{\emptyset\} \cup \{x_\lambda \mid \frac{1}{2} \leq \lambda \leq 1\}$. From Theorem 6.1 in [6], $(X_1, \mathcal{F}_1)$ and $(X_2, \mathcal{F}_2)$ are not quasi-$T_0$ space (and hence are not $T_0$ and $T_1$ spaces), but their product space is evidently a $T_2$-space. However, these separation properties are all hereditary and hence in the light of Theorem 2.1, we know that if a coordinate space is fully stratified, it enjoys the same separation property as the product space does.

### 3. PRODUCT OF $C_\Pi$-SPACES

**Definition 3.1.** A fuzzy topological space $(X, \mathcal{F})$ is called a purely stratified space iff, for each $U \in \mathcal{F}$, there is $\alpha \in [0, 1]$ such that $U(x) = \alpha$ for each $x \in X$. In particular, $(X, \mathcal{F})$ is called simply stratified iff $\mathcal{F} = \{\emptyset, X\}$.

In fuzzy topology, a purely stratified space plays the role analogous to that played by an indiscrete space in ordinary general topology.

**Lemma 3.1.** Let $K$ be an uncountable index set. If, for each $\alpha \in K$ there corresponds a positive number $\delta_\alpha$, then there exists $\delta > 0$ and an interval $\bar{\Delta} = ((5/6) \delta, \delta)$ such that $\{\alpha \in K \mid \delta_\alpha \in \bar{\Delta}\}$ is uncountable.

**Proof.** Since all $\delta_\alpha > 0$, it is obvious that there exists a positive integer $n$ such that $\bar{\Delta} = \{\alpha \in K \mid \delta_\alpha \in (1/(n + 2), 1/n)\}$ is uncountable. Since different $\alpha$'s may correspond to the same $\delta_\alpha$, there are two possible cases: (1) the set $\{\alpha \in K \mid \delta_\alpha \in \Delta\}$ is countable and (2) $\{\alpha \in K \mid \delta_\alpha \in \bar{\Delta}\}$ is uncountable. In either of these cases, it is not difficult to find a real number $\lambda > 0$ such that for any open interval $\Omega$ containing $\lambda$, $\{\alpha \in K \mid \delta_\alpha \in \Omega\}$ is uncountable. Now putting $\delta = (7/6)\lambda$, we have $(5/6) \delta < \lambda < \delta$. Thus, the interval $((5/6)\delta, \delta)$ is the required one.

**Lemma 3.2.** Let $K$ be an uncountable set. If for each $\alpha \in K$, $(X_\alpha, \mathcal{F}_\alpha)$ is not a purely stratified space, then there exist an uncountable set $J \subset K$ and an open interval $\Delta = (\rho, \rho - \delta/6)$, where $\delta > 0$, $0 < \rho < 1$, such that for each $\alpha \in J$, there are points $\bar{x}_\alpha$, $\bar{y}_\alpha \in X_\alpha$ and $U_\alpha \in \mathcal{F}_\alpha$ satisfying $U_\alpha(\bar{y}_\alpha) < U_\alpha(\bar{x}_\alpha) - \delta/2$ and $U_\alpha(\bar{x}_\alpha) \in \Delta$.

**Proof.** Since each $(X_\alpha, \mathcal{F}_\alpha)$ is not purely stratified, there exists $U_\alpha \in \mathcal{F}_\alpha$ which does not take constant value on $X_\alpha$. Therefore

$$\delta_\alpha = \sup\{|U_\alpha(x^1) - U_\alpha(x^2)| : x^1, x^2 \in X_\alpha\} > 0.$$
From Lemma 3.1, there exist $\delta > 0$ and $\overline{\delta} = ((5/6)\delta, \delta)$ such that $J_1 = \{x \in K \mid \delta_x \in \overline{\delta}\}$ is uncountable. For $x \in J_1$ since the amplitude $\delta_x$ of $U_x$ on $X_x$ belongs to $\overline{\delta}$, then $\delta_x > (5/6)\delta$. Consequently there are points $x_\alpha, y_\alpha \in X_\alpha$ such that $U_\alpha(y_\alpha) < U_\alpha(x_\alpha) - \delta/2$. On the other hand, it is not difficult to find an open interval, $\Delta = (\rho, \rho - \delta/6)$ of length $\delta/6$ such that $J = \{x \in J_1 \mid U_\alpha(x_\alpha) \in \Delta\}$ is uncountable. Moreover, because all $U_\alpha(x_\alpha) > \delta/2$, we may assume $\rho \in (0, 1)$. The lemma is thus proved.

The following theorem concerning the necessary and sufficient condition that the second countability be productive is a strengthening of Theorems 3.2 and 3.3 in [3].

**Theorem 3.1.** Let $(X_\alpha, \mathcal{T}_\alpha) (\alpha \in I)$ be second countable ($\mathcal{C}_{II}$) and let $(X, \mathcal{G})$ be their product space. Then $(X, \mathcal{G})$ is second countable iff all but a countable number of coordinate spaces are purely stratified.

**Proof. Necessity.** Let $\mathcal{B} = \{B_\alpha\}$ be a countable base for $\mathcal{G}$. If the condition is not necessary, then according to Lemma 3.2, there exists an uncountable set $J \subset I$, $\rho \in (0, 1)$, $\delta > 0$ such that for each $\alpha \in J$, there are $x_\alpha, y_\alpha \in X_\alpha$ and $U_\alpha \in \mathcal{T}_\alpha$ satisfying $U_\alpha(y_\alpha) < U_\alpha(x_\alpha) - \delta/2$ and $U_\alpha(x_\alpha) \in \Delta = (\rho, \rho - \delta/6)$.

We shall now define a mapping $J \rightarrow \mathcal{B}$ as follows: For each $\alpha \in J$, let $y = \{y_\alpha\} \in X$, where $y_\alpha = x_\alpha$, $y_\beta = y_\beta$ (when $\beta \in J$ and $\beta \neq \alpha$), when $\gamma \notin J$, $y_\gamma$ may be arbitrarily assigned. Since $\mathcal{B}$ is the countable base for $\mathcal{G}$ there corresponds $B_\alpha \in \mathcal{B}$ such that $B_\alpha \subset P_{a}^{-1}(U_\alpha)$ and $B_\alpha(y) \supseteq P_{a}^{-1}(U_\alpha)(y) - \delta/6 = U_\alpha(x_\alpha) - \delta/6$. We say that the correspondence is one to one. In fact, for $\beta \in J$, $\beta \neq \alpha$, noting that $U_\alpha(x_\alpha)$ and $U_\beta(x_\beta)$ both belong to the open interval $\Delta$ (cf. Lemma 3.2) and the absolute value of their difference is less than $\delta/6$, we have

$$P_\beta(B_\alpha)(\tilde{y}_\beta) \geq B_\alpha(y) \geq U_\alpha(x_\alpha) - \frac{\delta}{6} \geq U_\beta(x_\beta) - \frac{\delta}{3}.$$ 

On the other hand, from Lemma 3.2, $U_\beta(\tilde{y}_\beta) < U_\beta(\tilde{x}_\beta) - \delta/2$, and hence

$$P_\beta(B_\alpha)(\tilde{y}_\beta) \leq P_\beta(P_{\beta}^{-1}(U_\beta)(\tilde{y}_\beta) = U_\beta(\tilde{y}_\beta) < U_\beta(\tilde{x}_\beta) - \frac{\delta}{2}.$$ 

Consequently, $P_\beta(B_\beta)(\tilde{y}_\beta) < P_\beta(B_\alpha)(\tilde{y}_\beta)$, that is to say, $B_\beta \neq B_\alpha$. Since $J$ is uncountable and $\mathcal{B}$ is countable, this is a contradiction.

**Sufficiency.** The proof of this part goes along with the idea of that in ordinary general topology. From Theorem 3.2 in [3], it suffices to prove the following two lemmas.

**Lemma 3.3.** If, for each $\alpha \in J$, $(X_\alpha, \mathcal{T}_\alpha)$ is purely stratified, so is the product space $(X, \mathcal{G}) = \times \{X_\alpha, \mathcal{T}_\alpha\} | \alpha \in J$.

**Lemma 3.4.** A purely stratified space is $\mathcal{C}_{II}$. 

The former holds because \( \mathcal{S} \) has a subbase whose members are fuzzy sets taking constant value on \( X \). The proof of the latter can be obtained from the following simple property of the unit interval \([0, 1]\) of the real line: For any set \( \Omega = \{ \lambda_n \} \subset [0, 1] \), there exists a countable subset \( \Omega_1 \subset \Omega \) such that for each \( \lambda_n \in \Omega \), there is a sequence \( \lambda_n \in \Omega_1 \) \( (n = 1, 2, \ldots) \) satisfying \( \lambda_n \leq \lambda_n \) and \( \{ \lambda_n \} \) converges to \( \lambda \) under the usual topology of the real line.

4. Product of \( Q - C_1 \) spaces and product of \( C_1 \)-spaces

Theorems 6.1 and 6.2 in [4] gave partial results concerning the problem on products of so called \( C_1 \)-spaces, but the definition of \( C_1 \)-space given there does not take the \( C_1 \)-space in ordinary general topology as a special case and the result obtained there are not necessary and sufficient conditions for productiveness. In the present paper, both the concepts of \( C_1 \)-spaces and \( Q - C_1 \) spaces take the \( C_1 \)-space in ordinary general topology as a special case (cf. [6, Sect. 3]). For \( Q - C_1 \) spaces, we give a necessary and sufficient condition for productiveness; for \( C_1 \)-spaces, we give a necessary condition for productiveness and construct a counterexample to show that the product space of two \( C_1 \)-spaces may fail to be \( C_1 \).

**Theorem 4.1.** Let \((X, \mathcal{S})\) be the product space of the \( Q - C_1 \) (or \( C_1 \)) spaces \((X_\alpha, \mathcal{S}_\alpha)\) \((\alpha \in I)\). If \((X, \mathcal{S})\) is \( Q - C_1 \) (or \( C_1 \))-space; then all but a countable number of coordinate spaces are purely stratified spaces.

**Proof.** According to Proposition 3.3 in [6], each \( C_1 \) space is \( Q - C_1 \) space. Hence it suffices to prove the theorem for the case in which all \((X_\alpha, \mathcal{S}_\alpha)\) and \((X, \mathcal{S})\) are \( Q - C_1 \) spaces. If the conclusion is not true, then there is an uncountable index set \( K \subset I \), such that for each \( \alpha \in K \), \((X_\alpha, \mathcal{S}_\alpha)\) is not purely stratified.

In the light of Lemma 3.2, there exists an uncountable set \( J \subset K \), \( \rho \in (0, 1) \), \( \delta > 0 \) such that for each \( \alpha \in J \), there are \( \tilde{x}_\alpha, y_\alpha \in X \) and \( U_\alpha \in \mathcal{S}_\alpha \) satisfying

\[
U_\alpha(y_\alpha) < U_\alpha(\tilde{x}_\alpha) - \frac{\delta}{2} \quad \text{and} \quad U_\alpha(\tilde{x}_\alpha) \in \Delta = (\rho, \rho - \frac{\delta}{6}).
\]

For \( \alpha \in J \), let \( x_\alpha = \tilde{x}_\alpha \); for \( \alpha \in I - J \), let \( x_\alpha \) be some point of \( X_\alpha \), then we obtain \( x = (x_\alpha) \in X \). Take \( \lambda = 1 - \rho \in (0, 1) \). Let the \( \mathcal{S} \)-open \( Q \)-neighborhood base of the fuzzy point \( e = x \) be \( \{ B_\alpha \} \}_{\alpha \in N} \), i.e., \( B_\alpha(x) > 1 - \lambda = \rho \). Since the product topology is determined by its defining base, from the definition of the least upper bound, we may assume that the \( \mathcal{S} \)-open \( Q \)-neighborhood \( B_\alpha \) of \( x_\alpha \) is a member of the defining base, i.e., \( B_\alpha = \bigcap_{n \in F(t_\alpha)} P^{-1}(U_{t_\alpha}, n) \), where \( F_\alpha \) is a finite subset of \( I \) and \( U_{\alpha t_\alpha} \in \mathcal{S}_\alpha \). Since \( J \) is uncountable, take \( \beta \in J \setminus \bigcup_{n=1}^{\infty} F_n \), let \( U = P^{-1}(U_\beta) \). From Lemma 3.2, \( U(x) = U_\beta(\tilde{x}_\beta) \in \Delta \), i.e., \( U(x) \geq \rho = 1 - \lambda \), hence \( U \) is an open \( Q \)-neighborhood of \( x_\beta \) and there is some \( B_\beta \subset U \). Since \( \beta \not\in F_\alpha \), from
Lemma 2.1, $P_\theta(B_n)$ takes the constant value $\mu$ and $\mu = P_\theta(B_n)(\tilde{x}_\theta) \geq B_n(x) > 1 - \lambda = \rho$. Consequently,

$$P_\theta(B_n)(\tilde{y}_\theta) > \rho > \rho + \frac{\delta}{\theta} \geq U_\theta(\tilde{x}_\theta) \geq U_\theta(\tilde{y}_\theta) = P_\theta(U)(\tilde{y}_\theta).$$

Hence, $P_\theta(B_n)$ is not contained in $P_\theta(U)$ and $B_n$ is not contained in $U$. There is a contradiction.

**Theorem 4.2.** Let $(X_\alpha, \mathcal{F}_\alpha)(\alpha \in I)$ be $Q - C_1$ spaces such that all but a countable number of them are purely stratified; then their product space $(X, \mathcal{F})$ is a $Q - C_1$ space.

**Proof.** Owing to Proposition 3.2 in [6] and Lemmas 3.3 and 3.1 in the present paper, it is easily seen that this theorem can be reduced to the case where the index set $I$ is countable. Take an arbitrary fuzzy point $x_\alpha$, where $x = (x_\alpha) \in X$. Let $U_{\alpha,n}$ ($n = 1, 2,...$) be a countable open $Q$-neighborhood base of the fuzzy point $(x_\alpha)_0$ in $X_\alpha$; then the collection $\mathcal{B}$ of all the finite intersections of members of $\mathcal{B} = \{P_\alpha^{-1}(U_{\alpha,n}) : \alpha \in I, n = 1, 2,...\}$ is a countable $Q$-neighborhood base of the fuzzy point $x_\lambda$. In fact, since $U_{\alpha,n}$ is an $\mathcal{F}_\alpha$-open $Q$-neighborhood of the fuzzy point $x_\alpha$, it is obvious that each member of $\mathcal{B}$ together with each member of $\mathcal{B}$ is an open $Q$-neighborhood of the fuzzy point $x_\alpha$. Let $V$ be an arbitrary open $Q$-neighborhood of $x_\alpha$; from the definition of product topology, we may assume that $V$ is a member of its defining base: $V = \bigcap_{\alpha \in F} P_\alpha^{-1}(V_\alpha)$, where $F$ is a finite subset of $I$ and $V_\alpha \in \mathcal{F}_\alpha$. For $\alpha \in F$, evidently we have $V_\alpha(x_\alpha) = P_\alpha^{-1}(V_\alpha)(x) \geq V(x) > 1 - \lambda$. Hence $V_\alpha$ is an open $Q$-neighborhood of $(x_\alpha)_\alpha$. Then there is an open $Q$-neighborhood $U_{\alpha,n}$ of $(x_\alpha)_\alpha$ such that $U_{\alpha,n} \subset V$. Putting $W = \bigcap_{\alpha \in F} P_\alpha^{-1}(U_\alpha, n_\alpha)$, we have $W \subset V$ and $W \in \mathcal{B}$. Therefore $\mathcal{B}$ is a countable $Q$-neighborhood base of $x_\lambda$.

By constructing a counterexample, we establish the following:

**Theorem 4.3.** There exist $C_1$-spaces $(X_1, \mathcal{F}_1)$ and $(X_2, \mathcal{F}_2)$ whose product space is not $C_1$.

First we state the following Lemma 4.1, which may be verified directly from the definition of product topology.

**Lemma 4.1.** Let $\mathcal{B}_i$ be a base for the fuzzy topology $\mathcal{F}_i$ for $X_i$ ($i = 1, 2$). Let $(X, \mathcal{F}) = (X_1, \mathcal{F}_1) \times (X_2, \mathcal{F}_2)$ and let $B$ be an open neighborhood of a fuzzy point $e$ in $X$. Then there is $\bar{B} \in \mathcal{F}$, which is of the form

$$\bar{B} = \bigcup_{\alpha} (P_i^{-1}(g_\alpha) \cap P_\alpha^{-1}(f_\alpha)),$$

where $P_i$ are projections ($i = 1, 2$), $g_\alpha \in \mathcal{B}_1$, $f_\alpha \in \mathcal{B}_2$ such that $\bar{B} \subset B$ and $\bar{B}$ is an open neighborhood of $e$. 
Example 1. (1) Construction of \((X_1, \mathcal{T}_1)\). Let \(X_1\) be the unit interval \([0, 1]\) of the real line. Denote the zero point by \(x\). For the positive integers \(m, n\) such that \(n \geq m\), we define the fuzzy set \(g_{m,n}\) on \(X_1\) as follows:

\[
\begin{align*}
g_{m,n}(x) &= 1 - \frac{1}{1 + m} & \text{for} & \quad x \in \left[0, \frac{1}{n + 1}\right], \\
&= 0 & \text{for} & \quad x \in X_1 - \left[0, \frac{1}{n + 1}\right].
\end{align*}
\]

Denote the fuzzy point \(x_{1-1/(n+1)}\) by \(h_n\). It is obvious that when \(n_1 \geq n_2\), \(m = \min(m_1, m_2)\), we have \(g_{n_1,m_1} \cap g_{n_2,m_2} = g_{n_1,m}, \ h_{n_1} \cap h_{n_2} = h_{n_2}\) and \(g_{n_1, m_1} \cap h_{n_2} = h_i\), where \(l = \min(m_1, n_2)\). Hence, the collection \(\mathcal{B}_1\) consisting of all the \(g_{m,n}, h_n\), \(\Phi\), and \(X_1\) is a base for some fuzzy topology \(\mathcal{T}_1\) for \(X_1\). It is easily seen that \((X_1, \mathcal{T}_1)\) is both \(C_1\) and \(C_{II}\). For convenience, let us write \(\mathcal{B}_1 = \{\Phi, g_1, g_2, \ldots\}\), where each \(g_i \neq \Phi\).

(2) Construction of \((X_2, \mathcal{T}_2)\). Let \(X_2 = X_1\) and denote the zero point of \(X_2\) by \(y\). For each natural number \(n\), define the fuzzy set \(f_n\) on \(X_2\) as follows:

\[
f_n(x) = 1 \quad \text{for} \quad x \in [0, 1/n] \\
= 0 \quad \text{for} \quad x \in X_2 - [0, 1/n].
\]

It is easily verified that the family \(\mathcal{T}_2\) consisting of all the \(f_n\)'s and \(\Phi\) is a fuzzy topology \(\mathcal{T}_2\) for \(X_2\). Obviously, \((X_2, \mathcal{T}_2)\) is both \(C_1\) and \(C_{II}\).

(3) \((X, \mathcal{T}) = (X_1, \mathcal{T}_1) \times (X_2, \mathcal{T}_2)\) is not \(C_1\).

Consider the fuzzy point \(e\) in \(X = X_1 \times X_2\) whose support is \((x, y)\) and whose value is 1. We shall now show that the neighborhood system of \(e\) has no countable base. If this is not the case, there exists an open countable neighborhood base \(\{B_n \mid n = 1, 2, 3, \ldots\}\) of \(e\). From Lemma 4.1, we may assume

\[
B_n = \bigcup_{i=1}^{\infty} (P_1^{-1}(g_{\lambda(n,i)})) \cap P_2^{-1}(f_{\mu(n,i)}), \quad (*)
\]

where \(\lambda(n, i)\) and \(\mu(n, i)\) are natural numbers, \(\Phi \neq g_{\lambda(n,i)} \in \mathcal{B}_1\) and \(\Phi \neq f_{\mu(n,i)} \in \mathcal{T}_2\). For any \(B_n\), since \(B_n(x, y) = 1\), there is a term \(P_1^{-1}(g_{\lambda(n,i)}) \cap P_2^{-1}(f_{\mu(n,i)})\) in the expression (*) for \(B_n\), which takes its value at \((x, y)\) greater than \(1 - 1/(n + 1)\). Consequently, \(g_{\lambda(n,i)}(x) > 1 - 1/(n + 1)\). Now we take a strictly monotonic increasing sequence of natural numbers \(\{S_n\}\) such that \(S_n > \mu(n, i)\). Let

\[
A = \bigcup_{m=1}^{\infty} P_1^{-1}(h_m) \cap P_2^{-1}(f_{S_m}).
\]

Since \(h_m(x) = 1 - 1/(n + 1)\), \(A\) is evidently a neighborhood of \(e\) (note that
A(x, y) = 1). Take an arbitrary \( B_n \); there corresponds a point \( g_n \in (1/S_n, 1/(S_n^{-1})) \subset X \). Then we have

\[
B_n(x, y_n) \supseteq (P_1^{-1}(g_{x(n, i_n)}) \cap P_2^{-1}(f_{x(n, i_n)})) (x, y_n)
\]

\[
\supseteq \min(g_{x(n, i_n)}(x), f_{x(n, i_n)}(y_n))
\]

\[
= g_{x(n, i_n)}(x) > 1 - \frac{1}{n + 1}.
\]

On the other hand, since \( m \geq n \), evidently \( f_{x(n, i_n)}(y_n) = 0 \), we have

\[
A(x, y_n) = \sup \{\min(h_m(x), f_{x(n, i_n)}(y_n))\} \leq \sup \{h_m(x)\}
\]

\[
= 1 - \frac{1}{n}.
\]

Consequently, \( B_n(x, y_n) > A(x, y_n) \), that is to say, \( B_n \) is not contained in \( A \). This contradicts the fact that \( \{B_n \mid n = 1, 2, \ldots\} \) is an open neighborhood base of \( e \).

The space \( (X, \mathcal{T}) \) in the above example is a \( C_\Pi \) space according to Theorem 3.1. Hence this example provides another counterexample showing that a \( C_\Pi \) space (hence a \( Q - C_\Pi \) space) need not be a \( C_1 \)-space. Note that the counterexample given in [6, Theorem 3.1] with these same properties was constructed by means of an induced fuzzy topological space.

5. Separability and Connectedness of Product Spaces

**Definition 5.1.** A fuzzy topological space \((X, \mathcal{T})\) is said to be separable (resp. \( Q \)-separable) iff there exists a countable family of fuzzy points in \( X \) which is dense (resp. \( Q \)-dense) in \((X, \mathcal{T})\).

Although the concept of being dense and that of being \( Q \)-dense do not imply each other, as we already pointed out in Section 1, but we have

**Proposition 5.1.** A fuzzy topological space \((X, \mathcal{T})\) is separable iff it is \( Q \)-separable.

**Proof.** Let the countable family \( \Omega = \{e_n\} \) of fuzzy points in \( X \) be dense. Let \( \text{Supp } e_n = x_n \). Let \( \tilde{e}_n = (x_n)_{1/1} \), i.e., a crisp singleton with support \( x_n \). It is evident that the family \( \tilde{\Omega} = \{\tilde{e}_n\} \) is countable \( \tilde{Q} \)-dense. Conversely, suppose that \( \Omega = \{e_n\} \) is \( Q \)-dense. Let \( \text{Supp } e_n = x_n \) and let \( e_{n,m} = (x_n)_{1/m} \), i.e., a fuzzy point whose value at its support \( x_n \) is \( 1/m \). It is obvious that \( \Omega = \{e_{n,m} \mid n, m = 1, 2, \ldots\} \) is countable dense.

In the light of this proposition, we shall, from now on, make no difference
between separable spaces and Q-separable spaces. For convenience, they are both called separable spaces.

**Theorem 5.1.** Let each coordinate space \((X_\alpha, \mathcal{T}_\alpha)\) \((\alpha \in I)\) be separable and \(|I| \leq 2^{|\alpha|}\), then the product space \((X, \mathcal{T})\) is separable.

**Proof.** Since \(|I| \leq 2^{|\alpha|}\) we may assume that \(I\) is a subset of the real line and hence \(I\) has a natural order between its elements. Let \(\{e_\alpha^k \mid k = 1, 2, \ldots\}\) denote a countable dense family of fuzzy points \(e_\alpha^k\) in \((X_\alpha, \mathcal{T}_\alpha)\), where the support of \(e_k\) is denoted by \(x_k \in (X_\alpha^\alpha)\) and the value of \(e_\alpha^k\) at \(x_k^\alpha\) by \(\lambda_k^\alpha\). Consider the index set

\[ J = \{(\tau = (r_1, \ldots, r_{n-1}; k_1, \ldots, k_n, l)\),

where all \(k_i\)'s are natural numbers, all \(r_i\)'s are rational numbers such that \(r_1 < r_2 < \cdots < r_{n-1}\) \((n \geq 2)\), and \(l\) is a rational number in \((0, 1]\). \(J\) is evidently countable. For \(\tau \in J\), we can obtain a point \(x_\tau = (x_\alpha^\alpha) \in X\) as follows: For \(\alpha \in J\), let

\[ x_\alpha^\tau = x_k^\alpha \quad \text{when} \quad \alpha \leq r_1, \]

\[ = x_k^{\alpha_m} \quad \text{when} \quad r_{m-1} \leq \alpha < r_m \quad (2 \leq m < m - 1), \]

\[ = x_k^{\alpha_n} \quad \text{when} \quad r_{n-1} < \alpha, \]

where \(x_k^\alpha = \text{supp} e_k^\alpha\), defined above. Let \(e_\tau\) be the fuzzy point in \(X\) whose value at \(x_\tau\) is \(l\). We claim that the family \(\{e_\tau \mid \tau \in J\}\) is dense in \((X, \mathcal{T})\). In fact, take an arbitrary non-empty \(\mathcal{T}\)-open set \(U = \bigcap \{\mathcal{F}_i^{-1}(U_i)\}\), in the defining base for \(\mathcal{F}\), where \(F = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}\) is a finite subset of \(I\) (we may assume \(\alpha_1 < \alpha_2 < \cdots < \alpha_n\) without loss of generality) and \(U_i \in \mathcal{F}_i\) for each \(i \in F\). We may take \(n - 1\) rational numbers \(r_1, \ldots, r_{n-1}\) such that

\[ \alpha_1 < \gamma_1 < \alpha_2 < \gamma_2 < \cdots < \alpha_{n-1} < \gamma_{n-1} < \alpha_n. \]

For each non-empty \(U_m \in \mathcal{T}_m\) \((1 \leq m \leq n)\), there is a natural number \(k_m\) such that \(e_m^{2m} \in U_m\). Then choose a natural number \(l\) such that \(l = \min(\lambda_m^{2m} \mid m = 1, 2, \ldots, n)\) and consequently we obtain a member \(\tau = (r_1, r_2, \ldots, r_{n-1}; k_1, \ldots, k_n, l)\). It is easily seen that for each \(m \quad (1 \leq m \leq n)\), \(e_\tau \in P_{1-m}(e_m^{2m}) \subseteq P_{1-m}(U_m)\). Hence \(e_\tau \in U\). The proof is completed.

**Theorem 5.2.** Let \(I\) be an indexed set. Let each \((X_\alpha, \mathcal{T}_\alpha)\) \((\alpha \in I)\) be a fuzzy topological space such that \(\mathcal{T}_\alpha\) has non-intersecting non-empty members \(U_\alpha, V_\alpha\). If the product space \((X, \mathcal{T}) = \times \{(X_\alpha, \mathcal{T}_\alpha) \mid \alpha \in I\}\) is separable, then each \((X_\alpha, \mathcal{T}_\alpha)\) is separable, and \(|I| \leq 2^{|\alpha|}\).

**Proof.** Since an \(F\)-continuous mapping preserves separability and each
projection $P_{a}$ is $F$-continuous, each $(X_{a}, \mathcal{T}_{a})$, being the projection of the product space $(X, \mathcal{F})$ under $P_{a}$, is separable. Let $\Omega$ be a countable dense family of fuzzy points in $X$. For each $\alpha \in I$, we may define a mapping $f_{\alpha}: \Omega \rightarrow \{0, 1\}$ as follows: For $e \in \Omega$, with $\text{supp } e = \{x_{\alpha}\}$, define

$$f_{\alpha}(e) = 1, \quad \text{if } x_{\alpha} \in \text{supp } U_{\alpha},$$

$$= 0, \quad \text{otherwise}.$$ Moreover, when $\beta \in I$ such that $\beta \neq \alpha$, $f_{\beta} \neq f_{\alpha}$. In fact, it is obvious that $p_{\alpha}^{-1}(U_{\alpha}) \cap p_{\beta}^{-1}(U_{\beta}) \neq \emptyset$, and hence there exists $e \in \Omega$ such that $e \in p_{\alpha}^{-1}(U_{\alpha}) \cap p_{\beta}^{-1}(V_{\beta})$. Then $f_{\alpha}(e) = 1, f_{\beta}(e) = 0$. Since the power of $\{0, 1\}^{\Omega} = \{f | f: \Omega \rightarrow \{0, 1\}\}$ is at most $2^{\aleph_{0}}$, the power of $I$ is also at most $2^{\aleph_{0}}$.

**Proposition 5.2.** Let $\mathcal{F}_{1}$ and $\mathcal{F}_{2}$ be two fuzzy topologies on $X$ such that $\mathcal{F}_{2}$ is finer than $\mathcal{F}_{1}$ and $\mathcal{F}_{2}$ is generated by the family $\mathcal{P}$ of fuzzy sets on $X$ which consists of all the members of $\mathcal{F}_{1}$ and a number of fuzzy sets each of which takes constant value on $X$. Then $(X, \mathcal{F}_{1})$ is connected iff $(X, \mathcal{F}_{2})$ is connected.

**Proof.** It is obvious that $\mathcal{F}_{2}$ is finer than $\mathcal{F}_{1}$. When $(X, \mathcal{F}_{2})$ is connected, according to the definition of connectedness given in [6, Sect. 10], $(X, \mathcal{F}_{1})$ is also connected. Conversely, suppose $(X, \mathcal{F}_{1})$ is connected, then $(X, \mathcal{F}_{2})$ is connected too. In fact, if $(X, \mathcal{F}_{2})$ is not connected, there are non-empty closed sets $A$ and $B$ such that $A \cap B = \emptyset$ and $A \cup B = X$. It is obvious that $A$ and $B$ are crisp sets in $X$ and hence their corresponding complements $A'$ and $B'$ are crisp $\mathcal{F}_{2}$-open sets. According to the construction of $\mathcal{F}_{2}$, a $T_{S}$-open set $A' = \bigcup_{\alpha \in J} (U_{\alpha} \cap C_{\alpha})$, where $f$ is an index set, $U_{\alpha}$ is $\mathcal{F}_{1}$-open and $C_{\alpha}$ is a fuzzy set taking non-zero constant value on $X$, for each $\alpha \in I$. Since $A'(x) = 0$ for $x \in \text{supp } A$ and $C_{\alpha}$ takes non-zero constant value, it is obvious that $U_{\alpha}$ and hence $U = \bigcup_{\alpha \in J} U_{\alpha}$ take the constant value 0 on $\text{supp } A$. That is to say, $U \subseteq A'$. On the other hand, from the expression of $A'$, it follows that $A' \subseteq U$. Consequently $A' = U \subseteq \mathcal{F}_{1}$. Similarly, we can prove that $B' \subseteq \mathcal{F}_{1}$. Therefore $A$ and $B$ are $\mathcal{F}_{1}$-closed. This contradicts the hypothesis that $(X, \mathcal{F}_{1})$ is connected.

**Theorem 5.3.** A product space $(X, \mathcal{F}) = \times \{(X_{\alpha}, \mathcal{T}_{\alpha}) | \alpha \in I\}$ is connected iff each coordinate space $(X_{\alpha}, \mathcal{T}_{\alpha})$ is connected.

**Proof.** When $(X, \mathcal{F})$ is connected, from Theorem 1.2 and $P_{\alpha}(X) = X_{\alpha}$, it follows that $(X_{\alpha}, \mathcal{T}_{\alpha})$ is connected. We shall now prove that the condition is sufficient. Suppose that each $(X_{\alpha}, \mathcal{T}_{\alpha})$ is connected. We note first that from Proposition 5.2 and the Note after Theorem 2.1, it is easily seen that the section $X_{\beta}$ through a point $x \in X$ and parallel to $X_{\beta}$, considered as a subspace of $(X, \mathcal{F})$, is connected. Now we take a point $x = \{x_{\alpha}\} \in X$. Consider the component $D$ which contains $x$, where $x$ is considered as a crisp singleton in $X$. From the connectedness of the section just mentioned above, it is obvious that if $a$
point \( y = \{ y_{a} \} \in X \) is different from \( x = (x_{a}) \) in only one coordinate, \( y \) is also contained in \( D \). Furthermore, if \( y \) is different from \( x \) in only a finite number of coordinates, it can be inductively proved that \( y \) is also contained in \( D \). Finally, we shall show that \( \overline{D} = X \). Let \( z = \{ z_{a} \} \) be any point of \( X \). Take an arbitrary \( Q \)-neighborhood \( V \) of \( z \), \( z \) being considered as a crisp singleton in \( X \) here. From the definition of product topology, there is a member \( U = \bigcap_{a \in F} p_{a}^{-1}(U_{a}) \) of the defining base for \( \mathcal{T} \), where \( U_{a} \in \mathcal{T}_{a} \), \( a \in F \), \( F \) being a finite subset of the subset of the index set \( I \), such that \( U \subset V \) and \( U \) is also a \( Q \)-neighborhood of \( z \). Then \( U(z) = \min_{a \in F} [U_{a}(z_{a})] > 0 \). For \( \alpha \notin F \), let \( y_{a} = x_{a} \); for \( \alpha \in F \), let \( y_{a} = z_{a} \), then we get a point \( y = \{ y_{a} \} \in X \) such that \( y \) is different from \( x \) in only a finite number of coordinates. According to what we have just proved, \( y \in D \), i.e., \( D(y) = 1 \) (there \( y \) is considered as crisp singleton in \( X \)). Meanwhile, since \( U(y) = \min_{a \in F} [U_{a}(y_{a})] = \min_{a \in F} [U_{a}(z_{a})] > 0 \), hence \( D(y) + U(y) > 1 \), i.e., \( U \) and \( D \) are quasi-coincident at \( y \). Therefore \( V \) and \( D \) are quasi-coincident. Since \( V \) is an arbitrary \( Q \)-neighborhood of \( z \), \( z \in D \). Consequently \( \overline{D} = X \). From Theorem 10.1 in [6], \( U \) is connected and hence \( X \) is connected.

6. Quotient Spaces

Theorems 8, 9 and 10 concerning quotient spaces in [5, Chap. III] were generalized to fuzzy topological spaces in [3, Sect. 4]. In this section, we shall generalize the remaining two theorems (Theorems 11 and 12) to fuzzy topology.

**Definition 6.1.** Let \((X, \mathcal{T})\) be a fuzzy topological space. Let \( R \) be an equivalence relation on \( X \). Let \( X/R \) be the quotient set, and let \( \pi: X \to X/R \) be the projection (quotient map). Let \( \mathcal{U} \) be the family of fuzzy sets in \( X \) defined by

\[
\mathcal{U} = \{ B | \pi^{-1}(B) \in \mathcal{T} \}.
\]

Then \( \mathcal{U} \) is obviously a fuzzy topology, called the quotient topology for \( X/R \) and \( (X/R, \mathcal{U}) \) is called the quotient space of \( (X, \mathcal{T}) \) (relative to the quotient map \( \pi \)).

When no confusion may arise, the quotient space is briefly denoted by \( X/R \).

**Theorem 6.1.** If the quotient space \((X/R, \mathcal{U})\) is a \( T_{2} \)-space, then the corresponding equivalence relation \( R \) is closed in the product space \((X, \mathcal{T}) \times (X, \mathcal{T})\).

If an equivalence relation \( R \) on \( X \) is closed in the product space \((X, \mathcal{T}) \times (X, \mathcal{T})\), and the projection \( \pi: X \to X/R \) is open, then \( X/R \) is a \( T_{2} \)-space.

**Proof.** Let \((Z, \mathcal{T}) = (X, \mathcal{T}) \times (X, \mathcal{T})\). The two projections of \( Z \) onto the two coordinate spaces are denoted in order: \( \pi_{i} Z \to X \) \((i = 1, 2)\). Suppose that \( X/R \) is \( T_{2} \). Take an arbitrary fuzzy point \( e = (x, y)_{\lambda} \) in \( Z \), where \( x, y \in X \), \( \lambda \in (0, 1] \). If \( e \notin R \), we shall prove that there is a \( Q \)-neighborhood of \( e \) which is not
quasi-coincident with $R$ and hence $e \notin \bar{R}$. This shows that $R$ is closed in $(Z, \mathcal{T})$.

Since $e \notin R$, obviously $p(x) \neq p(y)$, where $p$ is the projection $X \to X/R$. Then, for the fuzzy points $(p(x))_a$ and $(p(y))_a$ in $X/R$, owing to the hypothesis that $X/R$ is $T_2$, there exist open $Q$-neighborhoods $\hat{U}$ and $\hat{V}$ of $(p(x))_a$ and $(p(y))_a$, respectively, such that $\hat{U} \cap \hat{V} = \emptyset$. It is easily seen that $W = p^{-1}(\hat{U}) \cap p^{-1}(\hat{V})$ is an open $Q$-neighborhood of $(x, y)_a$. Since $\hat{U} \cap \hat{V} = \emptyset$, we can show, by the indirect method, that $W \cap R = \emptyset$ and hence $R$ and $W$ are not quasi-coincident.

The proof of the first part of the theorem is thus completed. Let $e_x$ and $d_y$ be two fuzzy points in $X/R$ with $e = p(x)$, $d = p(y)$ in $X/R$ such that $p(x) \neq p(y)$. Hence $(x, y) \notin R$. Let $\nu = \min(\lambda, \mu) > 0$. Since $R$ is closed in $(Z, \mathcal{T})$, $(x, y)_a \notin R = R$, and hence there is an open $Q$-neighborhood $W$ which is not quasi-coincident with $R$. But $R$ is a crisp set in $(X, \mathcal{T})$; it follows that $W$ and $R$ do not intersect. It is obvious that there are two $\mathcal{T}$-open sets $U$, $V$ such that $G = p^{-1}(U) \cup p^{-1}(V)$ is an open $Q$-neighborhood of $(x, y)_a$, and $G \subseteq W$. From $G(x, y) + \nu > 1$, it is easily seen that $U(x) + \nu > 1$, $V(y) + \nu > 1$, and hence $p(U)(p(x)) + \lambda > 1$, $p(V)(p(y)) + \mu > 1$. By hypothesis, $p$ is an open mapping; therefore $p(U)$ and $p(V)$ are open $Q$-neighborhoods of $e_x$ and $d_y$, respectively. Meanwhile, from $W \cap R = \emptyset$, we have $G \cap R = \emptyset$. It is easy to show, by an indirect argument, that $P(U) \cap P(V) = \emptyset$. Consequently, $(X/R, \mathcal{U})$ is a $T_{\infty}$-space.

**Definition 6.2.** Let $X$ be a set. A decomposition of $X$ is disjoint family $\mathcal{D}$ of subsets of $X$ whose union is $X$. Given a decomposition $\mathcal{D} = \{X_i \mid i \in I\}$ of $X$, we can define a relation $R$ on $X$ as follows:

$$(x, y) \in R \iff \text{there exists } X_i \in \mathcal{D} \text{ such that } (x, y) \in X_i$$

$R$ is obviously an equivalence relation on $X$, called the relation of the decomposition $\mathcal{D}$. For simplicity and convenience, this relation $R$ is still denoted by $\mathcal{D}$ from now on. The corresponding quotient set is denoted by $X/R$.

**Definition 6.3.** Let $(X, \mathcal{T})$ be a fuzzy topological space. Let $\mathcal{D} = \{X_i\}$ be a decomposition of $X$. $\mathcal{D}$ is called upper semi-continuous iff, for each fuzzy set $D$ in $X$ such that $D$ takes a constant value on some member $X_i$ of $\mathcal{D}$ and the value 0 on the complement of $X_i$ and any $\mathcal{T}$-open set $U$ containing $D$, there exists an $\mathcal{T}$-open set $V$ such that $D \subseteq V \subseteq U$ and $V$ takes a constant value $c_i$ on each $X_i \in \mathcal{D}$ ($c_i$ may be different from $c_j$ for $i \neq j$).

**Lemma 6.1.** Let $\mathcal{D}$ be an upper semi-continuous decomposition of $(X, \mathcal{T})$. Let the fuzzy set $\hat{D}$ in $X$ take some constant value on each member of $\mathcal{D}$. Let $U$ be an $\mathcal{T}$-open set such that $\hat{D} \subseteq U$. Then there exists an $\mathcal{T}$-open set $V$ such that $\hat{D} \subseteq V \subseteq U$ and $V$ takes some constant value on each member of $\mathcal{D}$.
It is obvious that there exists a family $\mathcal{D} = \{D_i \mid i \in I\}$ of fuzzy sets on $X$ such that $\tilde{D} = \bigcup \{D_i \mid i \in I\}$ and each $D_i$ takes a constant value on one and only one member of $\mathcal{D}$ and the value 0 on its complement. According to Definition 6.3, it is easy to find the required $V$.

The following lemma is obvious.

**Lemma 6.2.** Let $X$ be a non-empty set. Let $p: X \to X/R$ be the projection. Then $p^{-1}p(A)$ takes the constant value $\sup\{A(x) \mid x \in D\}$ on each $D \in X/R$, and hence, $p^{-1}p(A) = A$ iff $A$ takes a constant value on each $D \in X/R$.

**Theorem 6.2.** Let $(X, \mathcal{F})$ be a fuzzy topological space. Then a decomposition $\mathcal{D}$ of $X$ is upper semi-continuous iff the projection $p: X/\mathcal{D}$ is a closed mapping.

**Proof.** Sufficiency. Let $\mathcal{D}$ be a decomposition of $X$. Let $D$ be a fuzzy set in $X$ such that $D$ takes a constant value on some member $X_i \in D$ and the value 0 on $Z_i$. Let $U \in \mathcal{F}$ such that $D \subseteq U$. Obviously, $D' \supseteq U'$ and $p^{-1}p(D') \supseteq p^{-1}p(U')$. But since $D'$ takes a constant value on each member of $\mathcal{D}$, from Lemma 6.2, we have $p^{-1}p(D') = D'$. Let $V = (p^{-1}p(U'))'$, then $D = (p^{-1}p(D'))' \subseteq V$, from $p^{-1}p(U') \supseteq U'$, we have $V \subseteq U$. Since $p$ is a closed mapping, $V$ is $\mathcal{F}$-open and $V$ takes a constant value on each member of $\mathcal{D}$ according to Lemma 6.2. Thus, $\mathcal{D}$ is upper semi-continuous.

Necessity. Let $\mathcal{D}$ be a decomposition of $X$. Let $D$ be a fuzzy set in $X$ such that $D$ takes a constant value on some member $X_i \in D$ and the value 0 on $Z_i$. Let $U \in \mathcal{F}$ such that $D \subseteq U$. Obviously, $D' \supseteq U'$ and $p^{-1}p(D') \supseteq p^{-1}p(U')$. But since $D'$ takes a constant value on each member of $\mathcal{D}$, from Lemma 6.2, we have $p^{-1}p(D') = D'$. Let $V = (p^{-1}p(U'))'$, then $D = (p^{-1}p(D'))' \subseteq V$, from $p^{-1}p(U') \supseteq U'$, we have $V \subseteq U$. Since $p$ is a closed mapping, $V$ is $\mathcal{F}$-open and $V$ takes a constant value on each member of $\mathcal{D}$ according to Lemma 6.2. Thus, $\mathcal{D}$ is upper semi-continuous.

**Remark.** A summary containing the main results of the present paper has been recently published in Kexue Tongbao (see [7]).

**References**