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Quantum Lévy Laplacian and associated heat equation

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Abstract

As a non-commutative extension of the Lévy Laplacian for entire functions on a nuclear space, we define the quantum Lévy Laplacian acting on white noise operators. We solve a heat type equation associated with the quantum Lévy Laplacian and study its relation to the classical Lévy heat equation. The solution to the quantum Lévy heat equation is obtained also from a normal-ordered white noise differential equation involving the quadratic quantum white noise.

Keywords: Quantum Lévy Laplacian; Heat equation; Quantum white noise; Wick symbol; White noise operator

1. Introduction

As an infinite-dimensional generalization of the usual Laplacian on an Euclidean space the so-called Lévy Laplacian:

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$$\Delta_{\mathrm{L}} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \frac{\partial^2}{\partial x_n^2}$$

was introduced and studied by Lévy in his famous books [21,22], and has been investigated from various aspects by many authors, see e.g., Feller [13], Polishchuk [29] and references cited therein for the works until the mid-1980s. In recent years the Lévy Laplacian has afforded us much interest for its newly discovered relations with certain stochastic processes [1,3,30,32], Yang–Mills equations [4,20], Gross Laplacian [19], infinite-dimensional rotation group [24], quadratic quantum white noise [26,27], Poisson noise functionals [31], and further relevant questions [2,5–7,9,11,23].

In this paper, we introduce a non-commutative generalization of the Lévy Laplacian, called the *quantum Lévy Laplacian*, acting on white noise operators. Given a real Gelfand triple $E \subset H \subset E^*$, its complexification $N \subset H_{\mathbb{C}} \subset N^*$ and a certain Young function θ, we construct a Gelfand (white noise) triple

$$\mathcal{W} = F_\theta(N) \subset \Gamma(H_{\mathbb{C}}) \subset G_\theta(N^*) = \mathcal{W}^*$$

by means of infinite-dimensional holomorphic functions [14], see Section 3. The Wick symbol of a white noise operator $\Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ is defined by

$$w\Xi(\xi, \eta) = \langle\langle \Xi\phi_\xi, \phi_\eta\rangle\rangle e^{-\langle \xi,\eta\rangle}, \quad \xi, \eta \in N,$$

where ϕ_ξ, ϕ_η are exponential vectors. It is known (see Theorem 6.1) that $\Xi \mapsto w\Xi$ gives rise to a topological isomorphism from $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ onto $\mathcal{G}_{\theta^*}(N \oplus N)$, which is the space of two-variable entire functions on N with certain growth rates. The Lévy Laplacian acting on such functions is naturally defined as soon as an infinite sequence in $E \oplus E$ is specified. A natural choice is $\{e_1 \oplus 0,\ 0 \oplus e_1,\ e_2 \oplus 0,\ 0 \oplus e_2, \ldots\}$, where $\{e_n\} \subset E$ is a given infinite sequence. The associated Lévy Laplacian Δ_{L} acts on entire functions in two variables. If a white noise operator Ξ satisfies conditions (i) $w\Xi$ belongs to the domain of Δ_{L}; and (ii) $\Delta_{\mathrm{L}}(w\Xi) \in \mathcal{G}_{\theta^*}(N \oplus N)$, then there exists a unique white noise operator whose Wick symbol is $\Delta_{\mathrm{L}}(w\Xi)$. The quantum Lévy Laplacian Δ_{QL} is thus defined by

$$w(\Delta_{\mathrm{QL}}\Xi) = \Delta_{\mathrm{L}}(w\Xi).$$

This definition is reasonable from several viewpoints. For example, for a multiplication operator $M_\Phi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ by $\Phi \in \mathcal{W}^*$, we have

$$\Delta_{\mathrm{QL}} M_\Phi = M_{\Delta_{\mathrm{L}}\Phi},$$

whenever $\Delta_{\mathrm{L}}\Phi$ is defined (Theorem 7.5). Namely, we have a natural quantum-classical correspondence. It is noticed that the non-commutative extension of the Lévy Laplacian introduced by Accardi et al. [5] is different from our quantum Lévy Laplacian. For the precise definition of the quantum Lévy Laplacian and its relevant properties see Section 7.

We are interested in the Cauchy problem associated with the quantum Lévy Laplacian:

$$\frac{\partial \Xi}{\partial t} = \alpha\Delta_{\mathrm{QL}}\Xi, \qquad \Xi(0) = \Xi_0 \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*),$$

where $\alpha \in \mathbb{C}$ is a constant. In fact, by means of the Wick symbol the above equation is brought into a Cauchy problem for entire functions in two-variables. We solve the above Cauchy problem with $\alpha > 0$ and a positive initial condition Ξ_0 by means of the integral representation for positive white noise operators [28], see Section 8. Another noteworthy aspect of the above Cauchy problem with an arbitrary $\alpha \in \mathbb{C}$ is found in the relation to the normal-ordered white noise differential equation with quadratic quantum white noise:

$$\frac{dZ_t}{dt} = \left(a_t^2 + a_t^{*2}\right) \diamond Z_t, \quad Z_0 = I,$$

where $\diamond$ stands for the normal-ordered (or Wick) product. A similar relation between the quadratic quantum white noise and the Cauchy problem associated with the (classical) Lévy Laplacian has been already observed in [26,27]. In this paper we realize its quantum counterpart in terms of the newly introduced quantum Lévy Laplacian, see Section 9.

2. Entire functions on a nuclear space

Let E be a real countably Hilbert nuclear space equipped with the defining Hilbertian norms $|\cdot|_p$, where $p \geqslant 0$. Without loss of generality we assume that the Hilbertian norms are linearly ordered so that

$$E = \operatorname*{proj\,lim}_{p\to\infty} E_p,$$

where E_p is the Hilbert space obtained by completing E with respect to $|\cdot|_p$. We set $H = E_0$ as a distinguished Hilbert space, of which the inner product is denoted by $\langle\cdot,\cdot\rangle$. Let E^* be the dual space of E and assume that E^* is equipped with the strong dual topology unless otherwise stated. By the standard argument we obtain an increasing chain of Hilbert spaces $\{E_p;\ p \in \mathbb{R}\}$ with the inclusion relation:

$$E \subset \cdots \subset E_p \subset \cdots \subset H \subset \cdots \subset E_{-p} \subset \cdots \subset E^*, \tag{2.1}$$

where the real Hilbert space H is identified with its dual space by the Riesz theorem. Moreover, it holds that

$$E^* = \operatorname*{ind\,lim}_{p\to\infty} E_{-p}.$$

The canonical bilinear form on $E^* \times E$, being compatible with the inner product of H, is also denoted by $\langle\cdot,\cdot\rangle$. For complexification of (2.1) we set

$$N = E + iE, \qquad H_{\mathbb{C}} = H + iH, \qquad N^* = E^* + iE^*.$$

The canonical $\mathbb{C}$-bilinear form on $N^* \times N$ is again denoted by $\langle\cdot,\cdot\rangle$ so that $|\xi|_0^2 = \langle\bar{\xi},\xi\rangle$ for $\xi \in H_{\mathbb{C}}$.

We are interested in entire functions on N. In general, a $\mathbb{C}$-valued function F defined on a complex topological vector space X is called *Gâteaux-entire* if for each $\xi, \eta \in X$, the $\mathbb{C}$-valued function $z \mapsto F(\xi + z\eta)$ is holomorphic at every $z \in \mathbb{C}$. A Gâteaux-entire function F is called *entire* if it is continuous on X, or equivalently if it is locally bounded, i.e., every point of X is contained in a neighborhood on which F is bounded. We refer to Dineen [12, Chapter 3] for generalities.

A continuous, convex, increasing function $\theta : [0,\infty) \to [0,\infty)$ is called a *Young function* if

$$\theta(0) = 0 \quad \text{and} \quad \lim_{x\to\infty} \frac{\theta(x)}{x} = \infty.$$

For $p \in \mathbb{R}$ and $m > 0$ let $\mathrm{Exp}(N_p, \theta, m)$ be the space of Gâteaux-entire functions F on N_p satisfying

$$\|F\|_{\theta,p,m} = \sup_{\xi\in N_p} \big|F(\xi)\big| e^{-\theta(m|\xi|_p)} < \infty.$$

Then, $\mathrm{Exp}(N_p, \theta, m)$ becomes a Banach space equipped with the norm $\|\cdot\|_{\theta,p,m}$. An element of $\mathrm{Exp}(N_p, \theta, m)$ is considered as an entire function on N. We set

$$\mathcal{F}_\theta(N^*) = \operatorname*{proj\,lim}_{\substack{p\to\infty \\ m\to+0}} \mathrm{Exp}(N_{-p}, \theta, m).$$

The polar function of θ is defined by

$$\theta^*(x) = \sup_{t\geqslant 0} \big(tx - \theta(t)\big), \quad x \geqslant 0,$$

which becomes again a Young function. We set

$$\mathcal{G}_{\theta^*}(N) = \operatorname*{ind\,lim}_{\substack{p\to\infty \\ m\to\infty}} \mathrm{Exp}(N_p, \theta^*, m).$$

Thus, $\mathcal{F}_\theta(N^*)$ and $\mathcal{G}_{\theta^*}(N)$ are classes of entire functions on N^* and N, respectively, with particular growth rates.

3. Taylor expansion and weighted Fock space

In general, an entire function F on N admits the Taylor expansion

$$F(\xi) = \sum_{n=0}^{\infty} \langle F_n, \xi^{\otimes n}\rangle, \quad F_n \in \big(N^{\widehat{\otimes} n}\big)^*,$$

where $N^{\widehat{\otimes} n}$ is the n-fold symmetric tensor power and $N^{\otimes 0} = \mathbb{C}$. The map

$$T : F \mapsto (F_n)_{n=0}^{\infty}$$

is called the *Taylor series map*. Entire functions in $\mathcal{F}_\theta(N^*)$ and $\mathcal{G}_\theta(N)$ are characterized in terms of their Taylor coefficients.

Given a Young function θ and $m > 0$, we define a weight sequence by

$$\alpha(n) = \alpha_{\theta,m}(n) = \frac{\theta_n^{-2}}{m^n n!}, \quad \theta_n = \inf_{r>0} \frac{e^{\theta(r)}}{r^n}, \ n = 0, 1, 2, \ldots. \tag{3.1}$$

For $p \geqslant 0$ let $F_{\theta,m}(N_p)$ be the weighted Fock space with weight sequence $\{\alpha(n)\}$, i.e.,

$$F_{\theta,m}(N_p) = \left\{ \phi = (f_n);\ f_n \in N_p^{\widehat{\otimes} n},\ \|\phi\|_{\alpha,p}^2 = \sum_{n=0}^{\infty} n!\alpha(n)|f_n|_p^2 < \infty \right\},$$

which is a Hilbert space equipped with the norm $\|\cdot\|_{\alpha,p}$. Define

$$F_\theta(N) = \operatorname*{proj\,lim}_{\substack{p\to\infty\\ m\to+0}} F_{\theta,m}(N_p),$$

which becomes a countably Hilbert nuclear space. Similarly, we set

$$G_{\theta,m}(N_{-p}) = \left\{ \Phi = (F_n);\ F_n \in N_{-p}^{\widehat{\otimes} n},\ \|\Phi\|_{\alpha^{-1},-p}^2 = \sum_{n=0}^{\infty} \frac{n!}{\alpha(n)} |F_n|_{-p}^2 < \infty \right\},$$

$$G_\theta(N^*) = \operatorname*{ind\,lim}_{\substack{p\to\infty\\ m\to\infty}} G_{\theta,m}(N_{-p}).$$

The (boson) Fock space is defined by

$$\Gamma(H_{\mathbb{C}}) = \left\{ \phi = (f_n);\ f_n \in H_{\mathbb{C}}^{\widehat{\otimes} n},\ \|\phi\|_0^2 = \sum_{n=0}^{\infty} n!|f_n|_0^2 < \infty \right\}.$$

From now on, we assume that the Young function θ satisfies

$$\sup_{x>0} \frac{\theta(x)}{x^2} < \infty$$

so as to have the inclusion relation:

$$F_\theta(N) \subset \Gamma(H_{\mathbb{C}}) \subset G_\theta(N^*). \tag{3.2}$$

It is verified that $F_\theta(N)$ and $G_\theta(N^*)$ are mutually dual spaces with respect to the canonical $\mathbb{C}$-bilinear form of $\Gamma(H_{\mathbb{C}})$, and hence (3.2) is a Gelfand triple. We write the canonical $\mathbb{C}$-bilinear form on $G_\theta(N^*) \times F_\theta(N)$ as

$$\langle\langle \Phi, \phi \rangle\rangle = \sum_{n=0}^{\infty} n!\,\langle F_n, f_n \rangle, \quad \Phi = (F_n) \in G_\theta(N^*),\ \phi = (f_n) \in F_\theta(N).$$

The next result is due to Gannoun et al. [14].

Theorem 3.1. *The Taylor series map T is a topological isomorphism from $\mathcal{F}_\theta(N^*)$ onto $F_\theta(N)$ and from $\mathcal{G}_{\theta^*}(N)$ onto $G_\theta(N^*)$.*

Remark 3.2. In order to construct a Gelfand triple as in (3.2), we may start with a weight sequence $\{\alpha(n)\}$ not necessarily defined by a Young function. However, to ensure important properties of (3.2), for example, characterization theorems of the S-transform and of the operator symbol, we need to assume certain conditions for $\{\alpha(n)\}$, see e.g., [10,15] and references cited therein. It then turns out, as is seen from Asai et al. [8] and Gannoun et al. [14], that these two approaches are equivalent.

4. Lévy Laplacian

The Lévy Laplacian is defined as an operator acting on functions on a nuclear space [2,19]. Let $F \in C^2(N)$. Then for each $\xi \in N$ there exist $F'(\xi) \in N^*$ and $F''(\xi) \in (N \otimes N)^*$ such that

$$F(\xi+\eta) = F(\xi) + \langle F'(\xi), \eta\rangle + \frac{1}{2}\langle F''(\xi), \eta\otimes\eta\rangle + o\big(|\eta|_p^2\big), \quad \eta \in N, \tag{4.1}$$

for some $p \geqslant 0$. Moreover, both maps $\xi \mapsto F'(\xi) \in N^*$ and $\xi \mapsto F''(\xi) \in (N \otimes N)^*$ are continuous. For notational simplicity, taking into account the canonical isomorphism $(N \otimes N)^* \cong \mathcal{L}(N, N^*) \cong \mathcal{B}(N, N)$, which follows from the kernel theorem for a nuclear space, we write $\langle F''(\xi), \eta\otimes\eta\rangle = \langle F''(\xi)\eta, \eta\rangle = F''(\xi)(\eta, \eta)$.

Let $\{e_n\} \subset E$ be an arbitrary infinite sequence, where we recall that $N = E + iE$. The Cesàro mean of $\langle F''(\xi), e_n \otimes e_n\rangle$ is defined by

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N}\langle F''(\xi), e_n\otimes e_n\rangle. \tag{4.2}$$

Note that the limit does not necessarily exist. Moreover, the limit depends not only on the choice of the sequence $\{e_n\}$ but also its arrangement. Let $\mathcal{D}_{\mathrm{L}}(N, \{e_n\})$ be the subspace of $F \in C^2(N)$ for which the limit (4.2) exists for all $\xi \in N$. If there is no danger of confusion, we write simply $\mathcal{D}_{\mathrm{L}}(N)$ for $\mathcal{D}_{\mathrm{L}}(N, \{e_n\})$. For $F \in \mathcal{D}_{\mathrm{L}}(N)$ we define

$$\Delta_{\mathrm{L}} F(\xi) = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N}\langle F''(\xi), e_n\otimes e_n\rangle, \quad \xi \in N.$$

The operator Δ_{L} is called the *Lévy Laplacian* on N associated with $\{e_n\}$.

Remark 4.1. Our definition of the Lévy Laplacian is quite general in the sense that we do not require any specific properties of the sequence $\{e_n\}$. (In some statements below additional conditions are required, which will be stated explicitly therein. In Section 9 the original definition due to Lévy [21,22] will appear.) In particular, we do not assume that $\{e_n\}$ forms an orthonormal basis of a Hilbert space, which is a standard condition in many literatures. Thus, the following case falls within our framework. For each $k = 1, 2, \ldots,$ let $M_k \subset H_{\mathbb{C}}$ be the orthogonal complement of the linear span of $\{e_1, \ldots, e_{k-1}, e_{k+1}, \ldots\}$. Suppose that $M_k \neq \{0\}$ for all $k = 1, 2, \ldots.$ Then we may choose a nonzero vector $f_k \in M_k$ with $\langle f_k, e_k\rangle \neq 0$. Define

$$x_c = \sum_{k=1}^{\infty} \frac{c_k}{\langle f_k, e_k\rangle} f_k,$$

where the coefficients $c_k \in \mathbb{C}$ is taken in order that $x_c \in N^*$ and the limit

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} c_n^2$$

exists. Then $\langle x_c, e_n\rangle = c_n$ and a function of the form $F(\xi) = f(\langle x_c, \xi\rangle)$, where $f \in C^2(\mathbb{C})$, belongs to $\mathcal{D}_{\mathrm{L}}(N)$.

Following Accardi, Smolyanov [2, Definition 3] we define the *Lévy trace* with respect to $\{e_n\}$ by

$$\langle f\rangle_{\mathrm{L}} = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \langle f, e_n\otimes e_n\rangle, \qquad f \in (N\otimes N)^*,$$

whenever the limit exists. The set of $f \in (N\otimes N)^*$ which admit the Lévy trace forms a linear subspace and is denoted by $(N\otimes N)^*_{\mathrm{L}}$. The Lévy trace is a linear form on $(N\otimes N)^*_{\mathrm{L}}$. With this notation the following assertion is obvious.

Lemma 4.2. *A function $F \in C^2(N)$ belongs to $\mathcal{D}_{\mathrm{L}}(N)$ if and only if $F''(\xi) \in (N\otimes N)^*_{\mathrm{L}}$ for all $\xi \in N$. In that case,*

$$\Delta_{\mathrm{L}} F(\xi) = \big\langle F''(\xi)\big\rangle_{\mathrm{L}}.$$

Eigenfunctions of the Lévy Laplacian will play an important role later. The following formula is easily checked, see also [2,27].

Proposition 4.3. *Let $p \in \mathcal{D}_{\mathrm{L}}(N)$ and assume that $p'(\xi)\otimes p'(\xi) \in (N\otimes N)^*_{\mathrm{L}}$ for all $\xi \in N$. Then $e^p \in \mathcal{D}_{\mathrm{L}}(N)$ and*

$$\Delta_{\mathrm{L}} e^{p(\xi)} = \big(\Delta_{\mathrm{L}} p(\xi) + \big\langle p'(\xi)\otimes p'(\xi)\big\rangle_{\mathrm{L}}\big)\, e^{p(\xi)}.$$

In particular, for $a \in N^*$ such that $a\otimes a \in (N\otimes N)^*_{\mathrm{L}}$ we have

$$\Delta_{\mathrm{L}} e^{\langle a,\xi\rangle} = \langle a\otimes a\rangle_{\mathrm{L}}\, e^{\langle a,\xi\rangle}.$$

The Lévy Laplacian is naturally defined for functions in two variables. Consider two real countably Hilbert nuclear spaces E_1 and E_2, and their complexifications $N_1 = E_1 + iE_1$ and $N_2 = E_2 + iE_2$. A function in two variables $(\xi_1, \xi_2) \in N_1 \times N_2$ is identified in an obvious manner with a single-variable function on the direct sum $N = N_1 \oplus N_2$, which is again a countably Hilbert nuclear space. Then, for $F \in C^2(N)$, (4.1) is written in the following form:

$$F(\xi_1+\eta_1, \xi_2+\eta_2) = F(\xi_1,\xi_2) + \sum_{i=1}^{2}\big\langle F_i'(\xi_1,\xi_2), \eta_i\big\rangle + \frac{1}{2}\sum_{i,j=1}^{2}\big\langle F_{ij}''(\xi_1,\xi_2)\eta_i, \eta_j\big\rangle + o\big(|\eta_1|_p^2 + |\eta_2|_p^2\big),$$

where $F_i'(\xi_1, \xi_2) \in N_i^*$ and $F_{ij}''(\xi_1, \xi_2) \in \mathcal{L}(N_i, N_j^*)$ for $i, j = 1, 2$. We also use the notation:

$$F'(\xi_1, \xi_2) = \begin{pmatrix} F_1'(\xi_1, \xi_2) \\ F_2'(\xi_1, \xi_2) \end{pmatrix}, \qquad F''(\xi_1, \xi_2) = \begin{pmatrix} F_{11}''(\xi_1, \xi_2) & F_{12}''(\xi_1, \xi_2) \\ F_{21}''(\xi_1, \xi_2) & F_{22}''(\xi_1, \xi_2) \end{pmatrix}.$$

Suppose we are given two infinite sequences: $\{e_n^{(i)}\} \subset E_i$ for $i = 1, 2$. Then the associated Lévy Laplacian acts on $F(\xi_1, \xi_2)$ regarded as a function of $\xi_i \in N_i$, which we denote by $\Delta_{\mathrm{L}}^{(i)}$. In other words,

$$\Delta_{\mathrm{L}}^{(i)} F(\xi_1, \xi_2) = \big\langle F_{ii}''(\xi_1, \xi_2)\big\rangle_{\mathrm{L}}.$$

Identifying N_i with a subspace of $N_1 \oplus N_2$ in the canonical manner, we write $e_n^{(1)}$ and $e_n^{(2)}$ for $e_n^{(1)} \oplus 0$ and $0 \oplus e_n^{(2)}$, respectively. Then the following assertion is straightforward from the definitions.

Proposition 4.4. *Let Δ_{L} be the Lévy Laplacian associated with an infinite sequence $\{e_1^{(1)}, e_1^{(2)}, e_2^{(1)}, e_2^{(2)}, \ldots\} \subset E_1 \oplus E_2$. Then we have*

$$\Delta_{\mathrm{L}} F = \frac{1}{2}\, \Delta_{\mathrm{L}}^{(1)} F + \frac{1}{2}\, \Delta_{\mathrm{L}}^{(2)} F$$

for all $F \in C^2(N_1 \oplus N_2)$ for which each term of the right-hand side exists.

Similarly, the Lévy trace $\langle\cdot\rangle_{\mathrm{L}}$ is decomposed:

$$\langle f\rangle_{\mathrm{L}} = \frac{1}{2}\, \langle f\rangle_{\mathrm{L}}^{(1)} + \frac{1}{2} \langle f\rangle_{\mathrm{L}}^{(2)}, \quad f = (f_{ij}) \in \mathcal{L}\big(N_1 \oplus N_2, N_1^* \oplus N_2^*\big),$$

where

$$\langle f\rangle_{\mathrm{L}}^{(i)} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \big\langle f_{ii}, e_n^{(i)} \otimes e_n^{(i)}\big\rangle.$$

The next result is a direct consequence of Propositions 4.3 and 4.4.

Proposition 4.5. *Let $p \in \mathcal{D}_{\mathrm{L}}(N_1 \oplus N_2)$ and assume that $p_i'(\xi_1, \xi_2) \otimes p_i'(\xi_1, \xi_2) \in (N_i \otimes N_i)_{\mathrm{L}}^*$ for all $\xi_1 \in N_1$ and $\xi_2 \in N_2$. Then $e^p \in \mathcal{D}_{\mathrm{L}}(N_1 \oplus N_2)$ and*

$$\Delta_{\mathrm{L}} e^{p(\xi_1, \xi_2)} = \left\{ \Delta_{\mathrm{L}} p(\xi_1, \xi_2) + \frac{1}{2} \sum_{i=1}^{2} \big\langle p_i'(\xi_1, \xi_2) \otimes p_i'(\xi_1, \xi_2)\big\rangle_{\mathrm{L}}^{(i)} \right\} e^{p(\xi_1, \xi_2)},$$

for $\xi_1 \in N_1$ and $\xi_2 \in N_2$. In particular, for each $a = (a_1, a_2) \in (N_1 \oplus N_2)^*$ satisfying $a_i \otimes a_i \in (N_i \otimes N_i)_{\mathrm{L}}^*$ we have

$$\Delta_{\mathrm{L}} e^{\langle a_1, \xi_1\rangle + \langle a_2, \xi_2\rangle} = \frac{1}{2}\big(\langle a_1 \otimes a_1\rangle_{\mathrm{L}}^{(1)} + \langle a_2 \otimes a_2\rangle_{\mathrm{L}}^{(2)}\big) e^{\langle a_1, \xi_1\rangle + \langle a_2, \xi_2\rangle}.$$

Remark 4.6. In Proposition 4.4 the Lévy Laplacian Δ_L on $N = N_1 \oplus N_2$ is defined, being associated with the sequence obtained by merging two given sequences $\{e_n^{(1)}\}$ and $\{e_n^{(2)}\}$ in such a way that each $\{e_n^{(i)}\}$ appears as a subsequence in the merged sequence at equal asymptotic density. It is possible to merge the two sequences in such a way that $\{e_n^{(1)}\}$ appears at asymptotic density α and $\{e_n^{(2)}\}$ at $1-\alpha$. Then the associated Lévy Laplacian becomes

$$\Delta_L = \alpha \Delta_L^{(1)} + (1-\alpha)\Delta_L^{(2)}.$$

Remark 4.7. The Lévy Laplacian for functions in two variables was discussed also by Accardi et al. [5]. Their definition is different from ours (see Definition 3 and Remark 4 in [5]) and this difference will be important for the later discussion on the quantum Lévy Laplacian.

5. Lévy Laplacian for entire functions

It is natural to ask whether the Lévy Laplacian of an entire function is again entire. A partial answer to this question is given in the following proposition.

Proposition 5.1. *Let Δ_L be the Lévy Laplacian with respect to an infinite sequence $\{e_n\} \subset E$ and let $F \in \mathcal{D}_L(N, \{e_n\})$. If there exists $p \geqslant 0$ such that $F \in \mathrm{Exp}(N_p, \theta, m)$ and*

$$M \equiv \sup_{N \geqslant 1} \frac{1}{N} \sum_{n=1}^{N} |e_n|_p^2 < \infty, \tag{5.1}$$

then $\Delta_L F \in \mathrm{Exp}(N_p, \theta, m')$ for any $m' > m$.

Proof. By the Cauchy integral formula we have

$$\langle F''(\xi), e_n \otimes e_n \rangle = \frac{2!}{2\pi i} \int_{|z|=R} \frac{F(\xi + z e_n)}{z^3}\, dz, \qquad R > 0.$$

Since

$$|F(\xi)| \leqslant \|F\|_{\theta,p,m} e^{\theta(m|\xi|_p)}, \qquad \xi \in N,$$

by assumption, noting that θ is an increasing function, we obtain

$$|\langle F''(\xi), e_n \otimes e_n \rangle| \leqslant \frac{2}{R^2} \|F\|_{\theta,p,m}\, e^{\theta(m|\xi|_p + mR|e_n|_p)}. \tag{5.2}$$

Let $m' > m$. Define $0 < \delta, \delta' < 1$ by $\delta = m/m'$ and $\delta + \delta' = 1$. Since θ is convex, we obtain

$$\begin{aligned} \theta(m|\xi|_p + mR|e_n|_p) &\leqslant \delta\theta(m'|\xi|_p) + \delta'\theta\left(\frac{mR}{\delta'}|e_n|_p\right) \\ &\leqslant \theta(m'|\xi|_p) + \theta\left(\frac{mR}{\delta'}|e_n|_p\right). \end{aligned}$$

Then (5.2) becomes

$$\left|\left\langle F''(\xi), e_n \otimes e_n\right\rangle\right| \leqslant 2\|F\|_{\theta,p,m} \frac{e^{\theta(\frac{mR}{\delta'}|e_n|_p)}}{(\frac{mR}{\delta'}|e_n|_p)^2} \left(\frac{m}{\delta'}\right)^2 e^{\theta(m'|\xi|_p)}|e_n|_p^2.$$

Minimizing the right-hand side by letting R run over $(0, +\infty)$, we obtain

$$\left|\left\langle F''(\xi), e_n \otimes e_n\right\rangle\right| \leqslant C\|F\|_{\theta,p,m} e^{\theta(m'|\xi|_p)}|e_n|_p^2, \tag{5.3}$$

where $C = 2\theta_2(m/\delta')^2$ which is independent of n (see (3.1) for θ_2). Set

$$S_N(\xi) = \frac{1}{N}\sum_{n=1}^{N}\left\langle F''(\xi), e_n \otimes e_n\right\rangle.$$

We see from (5.1) and (5.3) that

$$\left|S_N(\xi)\right| \leqslant CM\|F\|_{\theta,p,m} e^{\theta(m'|\xi|_p)}. \tag{5.4}$$

Since $\lim_{N\to\infty} S_N(\xi) = \Delta_{\mathrm{L}} F(\xi)$, we also have

$$\left|\Delta_{\mathrm{L}} F(\xi)\right| \leqslant CM\|F\|_{\theta,p,m} e^{\theta(m'|\xi|_p)}. \tag{5.5}$$

We shall prove that $\Delta_{\mathrm{L}} F$ is Gâteaux-entire. Since S_N is Gâteaux-entire, for any $\xi, \eta \in N$ and any closed curve $\gamma \subset \mathbb{C}$ we have

$$\int_\gamma \Delta_{\mathrm{L}} F(\xi + z\eta)\, dz = \int_\gamma \left\{\Delta_{\mathrm{L}} F(\xi + z\eta) - S_N(\xi + z\eta)\right\} dz.$$

Taking (5.4) and (5.5) into account, we apply Lebesgue's dominated convergence theorem to get

$$\int_\gamma \Delta_{\mathrm{L}} F(\xi + z\eta)\, dz = 0,$$

which means that $z \mapsto \Delta_{\mathrm{L}} F(\xi + z\eta)$ is holomorphic at all $z \in \mathbb{C}$. Together with (5.5) we conclude that $\Delta_{\mathrm{L}} F \in \mathrm{Exp}(N_p, \theta, m')$. □

Remark 5.2. In fact, we have proved that $F \mapsto \Delta_{\mathrm{L}} F$ is a continuous linear map from $\mathrm{Exp}(N_p, \theta, m) \cap \mathcal{D}_{\mathrm{L}}(N, \{e_n\})$ into $\mathrm{Exp}(N_p, \theta, m')$.

Corollary 5.3. *Let Δ_{L} be the Lévy Laplacian with respect to an infinite sequence $\{e_n\} \subset E$ satisfying the same condition as in Proposition* 5.1. *Let $F \in \mathrm{Exp}(N_p, \theta, m)$ of which the Taylor expansion is given by*

$$F(\xi) = \sum_{n=0}^{\infty} F_n(\xi), \qquad F_n(\xi) = \left\langle F_n, \xi^{\otimes n}\right\rangle. \tag{5.6}$$

If $F_n \in \mathcal{D}_{\mathrm{L}}(N)$ *for all n, then* $F \in \mathcal{D}_{\mathrm{L}}(N)$ *and*

$$\Delta_{\mathrm{L}} F(\xi) = \sum_{n=0}^{\infty} \Delta_{\mathrm{L}} F_n(\xi), \quad \xi \in N.$$

Proof. It is easy to see that the partial sum of the Taylor expansion (5.6) converges to F in $\mathrm{Exp}(N_p, \theta, m)$. Then the assertion follows by Proposition 5.1. □

Corollary 5.4. *Let* Δ_{L} *be the Lévy Laplacian with respect to an infinite sequence* $\{e_n\} \subset E$ *satisfying the same condition as in Proposition* 5.1. *Set*

$$\mathcal{G}_{\theta,p}(N) = \operatorname*{ind\,lim}_{m\to\infty} \mathrm{Exp}(N_p, \theta, m).$$

Then Δ_{L} *is a continuous linear operator from* $\mathcal{G}_{\theta,p}(N) \cap \mathcal{D}_{\mathrm{L}}(N, \{e_n\})$ *into* $\mathcal{G}_{\theta,p}(N)$.

6. White noise operators

From now on, the Gelfand triple (3.2) is denoted simply by

$$\mathcal{W} = F_\theta(N) \subset \Gamma(H_{\mathbb{C}}) \subset G_\theta(N^*) = \mathcal{W}^*.$$

By the celebrated Wiener–Itô–Segal isomorphism, $\Gamma(H_{\mathbb{C}})$ is isomorphic to $L^2(E^*, \mu)$, where μ is the Gaussian measure on E^*. So, in the context of stochastic analysis, each $\Phi \in \mathcal{W}^*$ is considered as a generalized random variable and is called a *white noise function*. For $\xi \in N$ the *exponential vector* is defined by

$$\phi_\xi = \left(1, \xi, \frac{\xi^{\otimes 2}}{2!}, \ldots, \frac{\xi^{\otimes n}}{n!}, \ldots\right).$$

It is proved that $\phi_\xi \in \mathcal{W}$ and $\{\phi_\xi;\ \xi \in N\}$ spans a dense subspace of $\mathcal{W}$. The *S-transform* of $\Phi \in \mathcal{W}^*$ is defined by

$$S\Phi(\xi) = \langle\langle \Phi, \phi_\xi \rangle\rangle = \sum_{n=0}^{\infty} \langle F_n, \xi^{\otimes n} \rangle, \quad \Phi = (F_n) \in \mathcal{W}^*.$$

Since the S-transform and the Taylor series map T are mutually inverse, Theorem 3.1 realizes an analytic characterization of the S-transform.

A continuous linear operator Ξ from $\mathcal{W}$ into $\mathcal{W}^*$ is called a *white noise operator*. Let $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ denote the space of white noise operators and equip it with the topology of bounded convergence. In the context of quantum stochastic analysis, each white noise operator is considered as a (generalization of) quantum random variable with respect to the vacuum state $\phi_0 = (1, 0, 0, \ldots)$. The *Wick symbol* of $\Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ is defined by

$$w\Xi(\xi, \eta) = \langle\langle \Xi\phi_\xi, \phi_\eta \rangle\rangle\, e^{-\langle \xi, \eta \rangle}, \quad \xi, \eta \in N. \tag{6.1}$$

If the Fock expansion (see [15,25]) of Ξ is given by

$$\Xi = \sum_{l,m}^{\infty} \Xi_{l,m}(\kappa_{l,m}), \quad \kappa_{l,m} \in \left(N^{\otimes(l+m)}\right)^*,$$

the Wick symbol becomes

$$w\Xi(\xi, \eta) = \sum_{l,m=0}^{\infty} \left\langle \kappa_{l,m}, \eta^{\otimes l} \otimes \xi^{\otimes m} \right\rangle, \quad \xi, \eta \in N.$$

The Wick symbol is characterized in a similar manner as the S-transform. The next result is shown by Ji et al. [18].

Theorem 6.1. *The Wick symbol $\Xi \mapsto w\Xi$ gives rise to a topological isomorphism from $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ onto $\mathcal{G}_{\theta^*}(N \oplus N)$.*

Remark 6.2. As a concept similar to the Wick symbol (6.1), the *symbol* of a white noise operator $\Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ is defined by $\widehat{\Xi}(\xi, \eta) = \langle\langle \Xi\phi_\xi, \phi_\eta \rangle\rangle$. The symbol is also widely used, particularly in quantum white noise calculus [10,15,16,18,25].

7. Quantum Lévy Laplacian

Let $\Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ be a white noise operator. Then, by Theorem 6.1, the Wick symbol $w\Xi$ is an entire function on $N \oplus N$ and hence belongs to $C^2(N \oplus N)$. Let $\{e_n\} \subset E$ be an infinite sequence, where $N = E + iE$. As in the second half of Section 4, we define the Lévy Laplacian Δ_{L} acting on functions in two variables. If $w\Xi \in \mathcal{D}_{\mathrm{L}}(N \oplus N)$, we obtain $\Delta_{\mathrm{L}}(w\Xi)$. If, in addition, $\Delta_{\mathrm{L}}(w\Xi)$ is the Wick symbol of a white noise operator, or equivalently if $\Delta_{\mathrm{L}}(w\Xi) \in \mathcal{G}_{\theta^*}(N \oplus N)$, there exists a unique white noise operator, denoted by $\Delta_{\mathrm{QL}}\Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$, satisfying

$$w(\Delta_{\mathrm{QL}}\Xi) = \Delta_{\mathrm{L}}(w\Xi). \tag{7.1}$$

We call Δ_{QL} the *quantum Lévy Laplacian*. By definition the domain of Δ_{QL}, denoted by $\mathrm{Dom}(\Delta_{\mathrm{QL}}) = \mathrm{Dom}(\Delta_{\mathrm{QL}}; \mathcal{L}(\mathcal{W}, \mathcal{W}^*))$, consists of white noise operators Ξ for which $\Delta_{\mathrm{QL}}\Xi$ is defined as in (7.1).

For $\kappa_{l,m} \in (N^{\otimes(l+m)})^*$ we define $\tau_{\mathrm{L}} * \kappa_{l,m} \in (N^{\otimes(l-2+m)})^*$ by

$$\begin{aligned}&\langle \tau_{\mathrm{L}} * \kappa_{l,m}, \eta_1 \otimes \cdots \otimes \eta_{l-2} \otimes \xi_1 \otimes \cdots \otimes \xi_m \rangle \\ &\quad = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle \kappa_{l,m}, e_n \otimes e_n \otimes \eta_1 \otimes \cdots \otimes \eta_{l-2} \otimes \xi_1 \otimes \cdots \otimes \xi_m \rangle,\end{aligned}$$

if the limit exists and is a continuous linear form in $\eta_1, \ldots, \eta_{l-2}, \xi_1, \ldots, \xi_m$. Similarly, $\kappa_{l,m} * \tau_{\mathrm{L}} \in (N^{\otimes(l+m-2)})^*$ is defined. In particular, for $\kappa \in (N \otimes N)^*_{\mathrm{L}}$ we have

$$\kappa * \tau_{\mathrm{L}} = \tau_{\mathrm{L}} * \kappa = \langle \kappa \rangle_{\mathrm{L}}.$$

With these notations we claim the following.

Lemma 7.1. *Let* $\kappa_{l,m} \in (N^{\otimes(l+m)})^*$ *for which both* $\tau_{\mathrm{L}} * \kappa_{l,m}$ *and* $\kappa_{l,m} * \tau_{\mathrm{L}}$ *are defined. Then,* $\Xi_{l,m}(\kappa_{l,m}) \in \mathrm{Dom}(\Delta_{\mathrm{QL}})$ *and*

$$2\Delta_{\mathrm{QL}} \Xi_{l,m}(\kappa_{l,m}) = l(l-1)\Xi_{l-2,m}(\tau_{\mathrm{L}} * \kappa_{l,m}) + m(m-1)\Xi_{l,m-2}(\kappa_{l,m} * \tau_{\mathrm{L}}).$$

Proof. The Wick symbol of an integral kernel operator $\Xi = \Xi_{l,m}(\kappa_{l,m})$ is given by

$$w\Xi(\xi,\eta) = \langle \kappa_{l,m}, \eta^{\otimes l} \otimes \xi^{\otimes m} \rangle.$$

Applying the Lévy Laplacian Δ_{L} for two variables, we obtain

$$\begin{aligned} 2\Delta_{\mathrm{L}} w\Xi(\xi,\eta) &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \{ l(l-1)\langle \kappa_{l,m}, e_n \otimes e_n \otimes \eta^{\otimes(l-2)} \otimes \xi^{\otimes m} \rangle \\ &\quad + m(m-1)\langle \kappa_{l,m}, \eta^{\otimes l} \otimes \xi^{\otimes(m-2)} \otimes e_n \otimes e_n \rangle \} \\ &= l(l-1)\langle \tau_{\mathrm{L}} * \kappa_{l,m}, \eta^{\otimes(l-2)} \otimes \xi^{\otimes m} \rangle \\ &\quad + m(m-1)\langle \kappa_{l,m} * \tau_{\mathrm{L}}, \eta^{\otimes l} \otimes \xi^{\otimes(m-2)} \rangle, \end{aligned}$$

which proves the assertion. □

We shall obtain an algebraic expression for the quantum Lévy Laplacian Δ_{QL}. With each $\zeta \in N$ we associate the annihilation operator $A(\zeta)$ and the creation operator $A^*(\zeta)$ in the standard manner. By means of integral kernel operators we have

$$A(\zeta) = \Xi_{0,1}(\zeta), \qquad A^*(\zeta) = \Xi_{1,0}(\zeta).$$

These are white noise operators. In fact, they are more regular in the following sense: $A(\zeta)$ is a continuous linear operator from $\mathcal{W}$ into itself and extends to a continuous linear operator from $\mathcal{W}^*$ into itself (denoted by the same symbol). Hence $A^*(\zeta)$ is a continuous linear operator from $\mathcal{W}$ into itself as well as from $\mathcal{W}^*$ into itself, see e.g., [10,15,25]. Thus, for an arbitrary white noise operator $\Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$, the commutators

$$[A(\zeta), \Xi] = A(\zeta)\Xi - \Xi A(\zeta), \qquad [\Xi, A^*(\zeta)] = \Xi A^*(\zeta) - A^*(\zeta)\Xi,$$

are defined and become again white noise operators. We define

$$D_\zeta^+ \Xi = [A(\zeta), \Xi], \qquad D_\zeta^- \Xi = [\Xi, A^*(\zeta)],$$

which are respectively called the *creation derivative* and *annihilation derivative* of Ξ, and both together the *quantum white noise derivatives* (*qwn-derivatives* for brevity) of Ξ, see [17]. Furthermore, $D_\zeta^\pm$ becomes a continuous linear map from $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ into itself.

We shall prove the following formula:

$$\Delta_{\mathrm{QL}} = \lim_{N\to\infty} \frac{1}{2N} \sum_{n=1}^{N} (D_{e_n}^+ D_{e_n}^+ + D_{e_n}^- D_{e_n}^-) \quad \text{on } \mathrm{Dom}(\Delta_{\mathrm{QL}}). \tag{7.2}$$

However, the right-hand side does not make sense. The precise assertion is given in the following

Theorem 7.2. *Let $\{e_n\} \subset E$ be an infinite sequence and Δ_{QL} the associated quantum Lévy Laplacian. Then for $\Xi \in \mathrm{Dom}(\Delta_{\mathrm{QL}})$ it holds that*

$$\langle\langle(\Delta_{\mathrm{QL}}\Xi)\phi_\xi, \phi_\eta\rangle\rangle = \lim_{N\to\infty} \frac{1}{2N} \sum_{n=1}^{N} \langle\langle\big(D_{e_n}^+ D_{e_n}^+ \Xi + D_{e_n}^- D_{e_n}^- \Xi\big)\phi_\xi, \phi_\eta\rangle\rangle, \quad \xi, \eta \in N. \tag{7.3}$$

Before going into the proof, we prepare a useful formula.

Lemma 7.3. *For $\eta, \zeta \in N$ and $z \in \mathbb{C}$ we have*

$$\phi_{\eta+z\zeta} = \sum_{n=0}^{\infty} \frac{z^n}{n!} A^*(\zeta)^n \phi_\eta, \tag{7.4}$$

where the right-hand side converges in $\mathcal{W}$ uniformly in z running over a compact set in $\mathbb{C}$. Therefore,

$$\frac{d^n}{dz^n}\bigg|_{z=0} \phi_{\eta+z\zeta} = A^*(\zeta)^n \phi_\eta$$

holds in $\mathcal{W}$.

Proof. Let $\omega_n = \omega_n(\eta, \zeta, z)$ be the partial sum of (7.4). By a direct verification we see that for any $p \geqslant 0$ and $m > 0$,

$$\lim_{n\to\infty} \|S\phi_{\eta+z\zeta} - S\omega_n\|_{\theta,p,m} = 0$$

uniformly in z running over a compact set in $\mathbb{C}$. Since the S-transform is a topological isomorphism between $\mathcal{W}$ and $\mathcal{F}_\theta(N^*)$, the assertion follows. □

Lemma 7.4. *Let $\zeta \in N$ and $\Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$. Then*

$$w\big(D_\zeta^+ D_\zeta^+ \Xi\big)(\xi, \eta) = \frac{d^2}{dz^2}\bigg|_{z=0} w\Xi(\xi, \eta + z\zeta), \tag{7.5}$$

$$w\big(D_\zeta^- D_\zeta^- \Xi\big)(\xi, \eta) = \frac{d^2}{dz^2}\bigg|_{z=0} w\Xi(\xi + z\zeta, \eta). \tag{7.6}$$

Proof. We have by definition

$$D_\zeta^+ D_\zeta^+ \Xi = \big[A(\zeta), \big[A(\zeta), \Xi\big]\big] = A(\zeta)^2 \Xi - 2A(\zeta)\Xi A(\zeta) + \Xi A(\zeta)^2.$$

Then, using $A(\zeta)\phi_\xi = \langle\xi, \zeta\rangle\, \phi_\xi$, we obtain

$$\begin{aligned}
w\big(D_\zeta^+ D_\zeta^+ \varXi\big)(\xi,\eta) &= \big\langle\!\big\langle \big(A(\zeta)^2 \varXi - 2A(\zeta)\varXi A(\zeta) + \varXi A(\zeta)^2\big)\phi_\xi, \phi_\eta \big\rangle\!\big\rangle e^{-\langle \xi,\eta\rangle} \\
&= \big\langle\!\big\langle \varXi\phi_\xi, A^*(\zeta)^2\phi_\eta \big\rangle\!\big\rangle e^{-\langle \xi,\eta\rangle} - 2\langle \xi,\zeta\rangle \big\langle\!\big\langle \varXi\phi_\xi, A^*(\zeta)\phi_\eta \big\rangle\!\big\rangle e^{-\langle \xi,\eta\rangle} \\
&\quad + \langle \xi,\zeta\rangle^2 \langle\!\langle \varXi\phi_\xi, \phi_\eta\rangle\!\rangle e^{-\langle \xi,\eta\rangle}.
\end{aligned}$$

In view of Lemma 7.3 the last expression becomes

$$\begin{aligned}
&= \frac{d^2}{dz^2}\bigg|_{z=0} \langle\!\langle \varXi\phi_\xi, \phi_{\eta+z\zeta}\rangle\!\rangle e^{-\langle \xi,\eta\rangle} - 2\langle \xi,\zeta\rangle \frac{d}{dz}\bigg|_{z=0} \langle\!\langle \varXi\phi_\xi, \phi_{\eta+z\zeta}\rangle\!\rangle e^{-\langle \xi,\eta\rangle} \\
&\quad + \langle \xi,\zeta\rangle^2 \langle\!\langle \varXi\phi_\xi, \phi_\eta\rangle\!\rangle e^{-\langle \xi,\eta\rangle}. \qquad (7.7)
\end{aligned}$$

On the other hand, by definition we have

$$\frac{d^2}{dz^2}\bigg|_{z=0} w\varXi(\xi,\eta+z\zeta) = \frac{d^2}{dz^2}\bigg|_{z=0} \langle\!\langle \varXi\phi_\xi, \phi_{\eta+z\zeta}\rangle\!\rangle e^{-\langle \xi,\eta+z\zeta\rangle}. \qquad (7.8)$$

It is then straightforward to check that (7.7) and (7.8) coincide, which proves (7.5). The proof of (7.6) is similar. □

Proof of Theorem 7.2. By Lemma 7.4 we see that

$$\begin{aligned}
\big\langle\!\big\langle \big(D_{e_n}^+ D_{e_n}^+ \varXi\big)\phi_\xi, \phi_\eta \big\rangle\!\big\rangle e^{-\langle \xi,\eta\rangle} &= w\big(D_{e_n}^+ D_{e_n}^+ \varXi\big)(\xi,\eta) = \frac{d^2}{dz^2}\bigg|_{z=0} w\varXi(\xi,\eta+ze_n), \\
\big\langle\!\big\langle \big(D_{e_n}^- D_{e_n}^- \varXi\big)\phi_\xi, \phi_\eta \big\rangle\!\big\rangle e^{-\langle \xi,\eta\rangle} &= w\big(D_{e_n}^- D_{e_n}^- \varXi\big)(\xi,\eta) = \frac{d^2}{dz^2}\bigg|_{z=0} w\varXi(\xi+ze_n,\eta).
\end{aligned}$$

If $\varXi \in \mathrm{Dom}(\Delta_{\mathrm{QL}})$, the limit

$$\lim_{N\to\infty} \frac{1}{2N} \sum_{n=1}^{N} \left\{ \frac{d^2}{dz^2}\bigg|_{z=0} w\varXi(\xi,\eta+ze_n) + \frac{d^2}{dz^2}\bigg|_{z=0} w\varXi(\xi+ze_n,\eta) \right\}$$

exists and coincides with $w(\Delta_{\mathrm{QL}}\varXi)(\xi,\eta)$. Hence

$$\lim_{N\to\infty} \frac{1}{2N} \sum_{n=1}^{N} \big\langle\!\big\langle \big(D_{e_n}^+ D_{e_n}^+ \varXi + D_{e_n}^- D_{e_n}^- \varXi\big)\phi_\xi, \phi_\eta \big\rangle\!\big\rangle e^{-\langle \xi,\eta\rangle} = w(\Delta_{\mathrm{QL}}\varXi)(\xi,\eta),$$

from which (7.3) follows. □

The above argument suggests to introduce two operators: for a white noise operator $\varXi$ we define $\Delta_{\mathrm{AL}}\varXi$ and $\Delta_{\mathrm{CL}}\varXi$ by

$$w(\Delta_{\mathrm{AL}}\varXi) = \Delta_{\mathrm{L}}^{(1)}(w\varXi) \quad \text{and} \quad w(\Delta_{\mathrm{CL}}\varXi) = \Delta_{\mathrm{L}}^{(2)}(w\varXi),$$

respectively, whenever the right-hand sides are the Wick symbols of white noise operators. We see easily that

$$\Delta_{\mathrm{AL}} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} D_{e_n}^- D_{e_n}^-, \qquad \Delta_{\mathrm{CL}} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} D_{e_n}^+ D_{e_n}^+,$$

where the limit is understood in a similar manner as in Theorem 7.2. We call Δ_{AL} and Δ_{CL} the quantum Lévy Laplacians associated with the annihilation derivatives and the creation derivatives, respectively. Moreover,

$$\Delta_{\mathrm{QL}} = \frac{1}{2}\Delta_{\mathrm{AL}} + \frac{1}{2}\Delta_{\mathrm{CL}} \tag{7.9}$$

holds on $\mathrm{Dom}(\Delta_{\mathrm{AL}}) \cap \mathrm{Dom}(\Delta_{\mathrm{CL}})$. (7.9) corresponds to the decomposition of the Lévy Laplacian acting on a function in two variables, see Proposition 4.4.

To end this section we mention the classical–quantum correspondence. For $\Phi \in \mathcal{W}^*$ we define a multiplication operator $M_\Phi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ by

$$\langle\langle M_\Phi \phi, \psi \rangle\rangle = \langle\langle \Phi, \phi\psi \rangle\rangle, \quad \phi, \psi \in \mathcal{W},$$

where $\phi\psi$ is the pointwise multiplication as white noise functions, see e.g., [15,25]. Moreover, $\Phi \mapsto M_\Phi$ yields a continuous injection from $\mathcal{W}^*$ into $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ and obviously, $M_\Phi \phi_0 = \Phi$.

Let $\mathrm{Dom}(\Delta_{\mathrm{L}}) = \mathrm{Dom}(\Delta_{\mathrm{L}}; \mathcal{W}^*)$ be the subspace of white noise functions $\Phi \in \mathcal{W}^*$ such that $S\Phi \in \mathcal{D}(\Delta_{\mathrm{L}})$ and $\Delta_{\mathrm{L}}(S\Phi)$ is the S-transform of a white noise function in $\mathcal{W}^*$. In this case we define $\Delta_{\mathrm{L}}\Phi$ by

$$S(\Delta_{\mathrm{L}}\Phi) = \Delta_{\mathrm{L}} S\Phi.$$

Theorem 7.5. *Let* $\Phi \in \mathrm{Dom}(\Delta_{\mathrm{L}}; \mathcal{W}^*)$. *Then* $M_\Phi \in \mathrm{Dom}(\Delta_{\mathrm{AL}}) \cap \mathrm{Dom}(\Delta_{\mathrm{CL}})$ *and*

$$\Delta_{\mathrm{AL}} M_\Phi = \Delta_{\mathrm{CL}} M_\Phi = M_{\Delta_{\mathrm{L}}\Phi}. \tag{7.10}$$

In particular, $M_\Phi \in \mathrm{Dom}(\Delta_{\mathrm{QL}})$ *and*

$$\Delta_{\mathrm{QL}} M_\Phi = M_{\Delta_{\mathrm{L}}\Phi}. \tag{7.11}$$

Proof. For any $\xi, \eta \in N$ we have

$$w M_\Phi(\xi, \eta) = \langle\langle \Phi, \phi_\xi \phi_\eta \rangle\rangle\, e^{-\langle \xi, \eta \rangle} = \langle\langle \Phi, \phi_{\xi+\eta} \rangle\rangle = S\Phi(\xi+\eta), \tag{7.12}$$

where the obvious identity $\phi_\xi \phi_\eta = \phi_{\xi+\eta}\, e^{\langle \xi, \eta \rangle}$ is used. Since $\Phi \in \mathrm{Dom}(\Delta_{\mathrm{L}}; \mathcal{W}^*)$ by assumption, applying the Lévy Laplacians with respect to ξ and η, we obtain

$$\Delta_{\mathrm{L}}^{(1)}(w M_\Phi)(\xi, \eta) = \Delta_{\mathrm{L}}^{(2)}(w M_\Phi)(\xi, \eta) = \Delta_{\mathrm{L}} S\Phi(\xi+\eta) = S(\Delta_{\mathrm{L}}\Phi)(\xi+\eta).$$

Moreover, by definition

$$S(\Delta_{\mathrm{L}}\Phi)(\xi+\eta) = \langle\langle \Delta_{\mathrm{L}}\Phi, \phi_{\xi+\eta} \rangle\rangle = \langle\langle \Delta_{\mathrm{L}}\Phi, \phi_\xi \phi_\eta \rangle\rangle\, e^{-\langle \xi, \eta \rangle} = w M_{\Delta_{\mathrm{L}}\Phi}(\xi, \eta).$$

Thus,

$$\Delta_{\mathrm{L}}^{(1)}(wM_{\Phi})(\xi,\eta) = \Delta_{\mathrm{L}}^{(2)}(wM_{\Phi})(\xi,\eta) = wM_{\Delta_{\mathrm{L}}\Phi}(\xi,\eta),$$

which means that $M_{\Phi} \in \mathrm{Dom}(\Delta_{\mathrm{AL}}) \cap \mathrm{Dom}(\Delta_{\mathrm{CL}})$ and (7.10) holds. Then (7.11) is obvious. □

Remark 7.6. It is also noted that

$$D_{\zeta}^{+}M_{\Phi} = D_{\zeta}^{-}M_{\Phi} = M_{A(\zeta)\Phi}, \quad \zeta \in N,\ \Phi \in \mathcal{W}^*,$$

which is verified by the operator symbols.

Remark 7.7. As a classical counterpart of the formula (7.2) we have

$$\Delta_{\mathrm{L}} = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} A(e_n)^2 \quad \text{on } \mathrm{Dom}(\Delta_{\mathrm{L}}; \mathcal{W}^*),$$

which is well known in various contexts. In fact, the right-hand side is given a meaning in a similar manner as in Theorem 7.2. Namely, for $\Phi \in \mathrm{Dom}(\Delta_{\mathrm{L}}; \mathcal{W}^*)$ we have

$$\langle\langle \Delta_{\mathrm{L}}\Phi, \phi_{\xi}\rangle\rangle = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \langle\langle A(e_n)^2\Phi, \phi_{\xi}\rangle\rangle, \quad \xi \in N.$$

8. Quantum Lévy heat equation

In this and the next sections we study the Cauchy problem associated with the quantum Lévy Laplacian:

$$\frac{\partial \Xi}{\partial t} = \alpha\Delta_{\mathrm{QL}}\Xi, \qquad \Xi(0) = \Xi_0 \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*), \tag{8.1}$$

where $\alpha \in \mathbb{C}$ is a constant.

An element of $\mathcal{F}_{\theta}(N^*)$, being an entire function on $N = E + iE$, is called *positive* if it takes non-negative values on E. Let $\mathcal{F}_{\theta}(N^*)_+$ be the set of such positive elements. Taking the isomorphism $\mathcal{W} = F_{\theta}(N) \cong \mathcal{F}_{\theta}(N^*)$ into account (Theorem 3.1), we say that $\phi \in \mathcal{W}$ is *positive* if the corresponding function (i.e., $S\phi = T^{-1}\phi$) in $\mathcal{F}_{\theta}(N^*)$ is positive. Let $\mathcal{W}_+$ denote the set of positive elements. A white noise operator $\Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ is called *positive* if

$$\langle\langle \Xi\phi, \psi\rangle\rangle \geqslant 0, \quad \phi, \psi \in \mathcal{W}_+.$$

The set of positive white noise operators is denoted by $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)_+$. A positive white noise operator admits a useful integral representation.

Theorem 8.1. *For a positive white noise operator $\Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)_+$ there exists a unique positive finite Radon measure $\mu = \mu_{\Xi}$ on $(E \oplus E)^*$ such that*

$$\langle\langle \Xi\phi, \psi\rangle\rangle = \int_{(E\oplus E)^*} \phi(x)\psi(y)\mu(dx\,dy), \quad \phi, \psi \in \mathcal{W}. \tag{8.2}$$

Moreover, there exist $p > 0$, $q > 0$ *and* $m > 0$ *such that* μ *is supported by the subspace* $E_{-p} \oplus E_{-q}$ *and*

$$\int_{E_{-p}\oplus E_{-q}} e^{\theta(m|x|_{-p})+\theta(m|y|_{-q})}\mu(dx\,dy) < \infty. \tag{8.3}$$

Conversely, such a positive finite measure μ *on* $(E \oplus E)^*$ *defines a positive white noise operator* Ξ *by* (8.2).

The proof is given in [5, Theorem 8]. In fact, we need only to employ the representation theorem for positive distributions due to Ouerdiane, Rezgui [28] and a one-to-one correspondence between $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)_+$ and $\mathcal{F}^*_\theta(N^* \oplus N^*)_+$ derived from the canonical isomorphism:

$$\mathcal{L}(\mathcal{W}, \mathcal{W}^*) \cong \mathcal{G}_{\theta^*}(N \oplus N) \cong \mathcal{F}^*_\theta(N^* \oplus N^*),$$

where the first isomorphism is given by the Wick symbol (Theorem 6.1) and the second by the Laplace transform [18].

Now we go back to the Cauchy problem (8.1). When $\alpha > 0$ and $\Xi_0 \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)_+$, the Cauchy problem (8.1) looks like a heat equation and the method of Fourier transform for the Lévy heat equation [3,27] can be applied. Recall that the Lévy Laplacian depends on the fixed sequence $\{e_n\} \subset E$. If there exists a continuous operator $S \in \mathcal{L}(E, E)$ such that $Se_n = e_{n+1}$ for all n, we say that S is the shift operator associated with $\{e_n\}$. In this case, $S \oplus S \in \mathcal{L}(E \oplus E, E \oplus E)$ so that $(S \oplus S)^*$ becomes a continuous transformation on $(E \oplus E)^*$.

Theorem 8.2. *Let* Δ_{QL} *be the quantum Lévy Laplacian associated with an infinite sequence* $\{e_n\} \subset E$ *for which the shift operator* $S \in \mathcal{L}(E, E)$ *exists. Let* $\alpha > 0$ *and* $\Xi_0 \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)_+$. *Assume that the corresponding Radon measure* $\mu = \mu_{\Xi_0}$ *is invariant under* $(S \oplus S)^*$. *Then for* μ*-a.e.* $x \oplus y \in (E \oplus E)^*$,

$$\|x \oplus y\|_{\mathrm{L}}^2 \equiv \lim_{N\to\infty} \frac{1}{2N} \sum_{n=1}^{N} \bigl\{\langle x, e_n\rangle^2 + \langle y, e_n\rangle^2\bigr\} \tag{8.4}$$

exists and the Wick symbol of a solution to (8.1) *is given by*

$$A_t(\xi, \eta) = \int_{(E\oplus E)^*} e^{-t\alpha\|x\oplus y\|_{\mathrm{L}}^2} e^{i\langle x\oplus y,\, \xi\oplus\eta\rangle}\mu(dx\,dy), \quad \xi, \eta \in N,\ t \geqslant 0. \tag{8.5}$$

Proof. We set

$$G(x \oplus y) = \frac{1}{2}\bigl\{\langle x, e_1\rangle^2 + \langle y, e_1\rangle^2\bigr\}.$$

By (8.3) we can verify $G \in L^1((E \oplus E)^*, \mu)$. Since μ is invariant under $(S \oplus S)^*$, we see by the ergodic theorem that

$$\bar{G}(x \oplus y) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} G\bigl((S \oplus S)^{*(n-1)}(x \oplus y)\bigr) \tag{8.6}$$

converges μ-a.e. $x \oplus y \in (E \oplus E)^*$. Moreover, the convergence holds in the L^1-sense and $\bar{G} \in L^1((E \oplus E)^*, \mu)$. On the other hand, by definition we have

$$\begin{aligned} G\big((S \oplus S)^{*(n-1)}(x \oplus y)\big) &= G\big(S^{*(n-1)}x \oplus S^{*(n-1)}y\big) \\ &= \frac{1}{2}\big\{\big\langle S^{*(n-1)}x, e_1\big\rangle^2 + \big\langle S^{*(n-1)}y, e_1\big\rangle^2\big\} \\ &= \frac{1}{2}\big\{\langle x, e_n\rangle^2 + \langle y, e_n\rangle^2\big\}. \end{aligned}$$

Then (8.6) becomes $\bar{G}(x \oplus y) = \|x \oplus y\|_{\mathrm{L}}^2$ and (8.4) exists for μ-a.e. $x \oplus y \in (E \oplus E)^*$. It is straightforward to verify that A_t defined in (8.5) is a solution to

$$\frac{\partial}{\partial t} A_t(\xi, \eta) = \Delta_{\mathrm{L}} A_t(\xi, \eta), \qquad A_0(\xi, \eta) = w\Xi_0(\xi, \eta), \quad \xi, \eta \in N. \tag{8.7}$$

Moreover, we see easily that $A_t \in \mathcal{G}_{\theta^*}(N \oplus N)$ for all $t \geqslant 0$. Hence for each $t \geqslant 0$ there exists a white noise operator $\Xi_t \in \mathrm{Dom}(\Delta_{\mathrm{QL}}; \mathcal{L}(\mathcal{W}, \mathcal{W}^*))$ whose Wick symbol is A_t. Finally, we show that $t \mapsto \Xi_t$ is differentiable. For $t > 0$ and $t + h > 0$ we have

$$\begin{aligned} \left|\frac{A_{t+h}(\xi, \eta) - A_t(\xi, \eta)}{h}\right| &\leqslant \int_{(E\oplus E)^*} \alpha\|x \oplus y\|_{\mathrm{L}}^2 e^{-(t+\delta h)\alpha\|x\oplus y\|_{\mathrm{L}}^2}\big|e^{i\langle x\oplus y,\, \xi\oplus\eta\rangle}\big|\,\mu(dx\,dy) \\ &\leqslant M(t)\int_{(E\oplus E)^*} \big|e^{i\langle x\oplus y,\, \xi\oplus\eta\rangle}\big|\,\mu(dx\,dy), \end{aligned} \tag{8.8}$$

where $0 < \delta < 1$ is chosen by the mean value theorem and $M(t) = \sup\{re^{-tr}; r > 0\}$. Take $p > 0$, $q > 0$ and $m > 0$ as in Theorem 8.1. Then,

$$\begin{aligned} \int_{(E\oplus E)^*} \big|e^{i\langle x\oplus y,\, \xi\oplus\eta\rangle}\big|\mu(dx\,dy) &\leqslant \int_{E_{-p}\oplus E_{-q}} e^{|x|_{-p}|\xi|_p + |y|_{-q}|\eta|_q}\mu(dx\,dy) \\ &\leqslant Ce^{\theta^*(|\xi|_p/m)+\theta^*(|\eta|_p/m)}, \end{aligned} \tag{8.9}$$

where C is the constant defined by the integral in (8.3). Combining (8.8) and (8.9), we come to

$$\left|\frac{A_{t+h}(\xi, \eta) - A_t(\xi, \eta)}{h}\right| \leqslant CM(t)e^{\theta^*(|\xi|_p/m)+\theta^*(|\eta|_p/m)}.$$

Therefore, by applying Lebesgue's dominated convergence theorem we prove that $t \mapsto A_t \in \mathcal{G}_{\theta^*}(N \oplus N)$ is differentiable and so is $t \mapsto \Xi_t \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ by Theorem 6.1. Consequently, Ξ_t is the solution to (8.1) and the proof is complete. □

As for the quantum Lévy Laplacian associated with the creation derivatives Δ_{CL} we note the following.

Proposition 8.3. *Let $\Xi_t \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ be a solution to the following Cauchy problem*:

$$\frac{\partial \Xi}{\partial t} = \alpha \Delta_{\mathrm{CL}} \Xi, \qquad \Xi_0 \in \mathrm{Dom}(\Delta_{\mathrm{CL}}). \tag{8.10}$$

Then $\Phi_t = \Xi_t \phi_0$ is a solution to the Cauchy problem associated with the (*classical*) *Lévy Laplacian:*

$$\frac{\partial \Phi}{\partial t} = \alpha \Delta_{\mathrm{L}} \Phi, \qquad \Phi_0 = \Xi_0 \phi_0 \in \mathrm{Dom}(\Delta_{\mathrm{L}}; \mathcal{W}^*). \tag{8.11}$$

Proof. We first note that

$$(\Delta_{\mathrm{CL}} \Xi)\phi_0 = \Delta_{\mathrm{L}}(\Xi \phi_0), \quad \Xi \in \mathrm{Dom}(\Delta_{\mathrm{CL}}).$$

In fact, for $\xi \in N$ we have

$$\begin{aligned} \big\langle\!\big\langle (\Delta_{\mathrm{CL}} \Xi)\phi_0, \phi_\xi \big\rangle\!\big\rangle &= w(\Delta_{\mathrm{CL}} \Xi)(0, \xi) = \Delta_{\mathrm{L}}^{(2)} w\Xi(0, \xi) = \Delta_{\mathrm{L}} \langle\!\langle \Xi \phi_0, \phi_\xi \rangle\!\rangle \\ &= \Delta_{\mathrm{L}} S(\Xi \phi_0)(\xi) = S\big(\Delta_{\mathrm{L}}(\Xi \phi_0)\big)(\xi) = \big\langle\!\big\langle \Delta_{\mathrm{L}}(\Xi \phi_0), \phi_\xi \big\rangle\!\big\rangle. \end{aligned}$$

Now let Ξ_t be a solution to (8.10). Then we have

$$\frac{\partial(\Xi_t \phi_0)}{\partial t} = \left(\frac{\partial \Xi_t}{\partial t}\right)\phi_0 = (\alpha \Delta_{\mathrm{CL}} \Xi_t)\phi_0 = \alpha \Delta_{\mathrm{L}}(\Xi \phi_0),$$

which means that $\Phi_t = \Xi_t \phi_0$ is a solution to (8.11). □

Remark 8.4. Uniqueness of a solution to (8.1) is unclear.

9. Quadratic quantum white noise

In this section we consider a particular Gelfand triple:

$$E = \mathcal{S}(\mathbb{R}) \subset H = L^2(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R}) = E^*,$$

where $\mathcal{S}(\mathbb{R})$ is the space of rapidly decreasing functions on $\mathbb{R}$ and $\mathcal{S}'(\mathbb{R})$ the dual space, i.e., the space of tempered distributions. The underlying space $\mathbb{R}$ stands often for the time axis. Let $\{a_t, a_t^*; t \in \mathbb{R}\}$ be the quantum white noise, that is, a_t is the annihilation operator at $t \in \mathbb{R}$ and a_t^* the creation operator. Both are white noise operators. In fact, they are more regular in the sense that $a_t \in \mathcal{L}(\mathcal{W}, \mathcal{W})$ and $a_t^* \in \mathcal{L}(\mathcal{W}^*, \mathcal{W}^*)$.

It is noteworthy that the quadratic quantum white noise $\{a_t^2, a_t^{*2}\}$ is well defined. Let us consider the following normal-ordered white noise equation:

$$\frac{dZ_t}{dt} = \big(a_t^2 + a_t^{*2}\big) \diamond Z_t, \qquad Z_0 = I. \tag{9.1}$$

It is known that (9.1) has a unique solution in $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$. In fact, the solution to (9.1) is given by the Wick exponential:

$$Z_t = \sum_{n=0}^{\infty} \frac{1}{n!} \left\{ \int_0^t \left(a_s^2 + a_s^{*2}\right) ds \right\}^{\diamond n}. \tag{9.2}$$

In this connection see [10,26] for details.

Now we consider the Lévy Laplacian associated with an infinite sequence $\{e_n\} \subset E$ having specific properties. We assume that $\{e_n\} \subset E$ is a complete orthonormal basis of $L^2([0,1])$ and that for each $t \in [0,1]$ the following limit exists:

$$c(t) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int_0^t e_n(s)^2 \, ds. \tag{9.3}$$

See also Remark 9.5.

Lemma 9.1. *It holds that*

$$\Delta_{\mathrm{L}} \exp\left\{ \int_0^t \xi(s)^2 \, ds \right\} = 2c(t) \exp\left\{ \int_0^t \xi(s)^2 \, ds \right\}, \quad \xi \in N, \ t \in [0,1].$$

Proof. For simplicity we set

$$p_t(\xi) = \int_0^t \xi(s)^2 \, ds, \quad \xi \in N, \ t \in [0,1].$$

We see easily by direct calculation that $p_t'(\xi) = 2\,1_{[0,t]}\xi \in N^*$ and $p_t''(\xi) = 2\,\tau_{[0,t]} \in (N \otimes N)^*$. Then by (9.3) we obtain

$$\Delta_{\mathrm{L}} p_t(\xi) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle p_t''(\xi), e_n \otimes e_n \rangle = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} 2 \int_0^t e_n(s)^2 \, ds = 2c(t). \tag{9.4}$$

On the other hand,

$$\begin{aligned} \langle p_t'(\xi) \otimes p_t'(\xi) \rangle_{\mathrm{L}} &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle p_t'(\xi) \otimes p_t'(\xi), e_n \otimes e_n \rangle \\ &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle 2\,1_{[0,t]}\xi, e_n \rangle^2 = 0, \end{aligned} \tag{9.5}$$

because $\langle 2\,1_{[0,t]}\xi, e_n \rangle$ is the Fourier coefficient of $2\,1_{[0,t]}\xi \in L^2([0,1])$ and vanishes as $n \to \infty$. The assertion then follows from Proposition 4.3 with (9.4) and (9.5). □

Theorem 9.2. *Let Z_t be the solution to the normal-ordered white noise differential equation* (9.1). *Then*

$$\Delta_{\mathrm{AL}} Z_t = \Delta_{\mathrm{CL}} Z_t = \Delta_{\mathrm{QL}} Z_t = 2c(t) Z_t, \quad t \in [0,1]. \tag{9.6}$$

Proof. The Wick symbol of the solution Z_t in (9.2) is easily obtained:

$$wZ_t(\xi,\eta) = \exp\left\{ \int_0^t \bigl(\xi(s)^2 + \eta(s)^2\bigr)\, ds \right\}, \quad \xi,\eta \in N.$$

It follows from Lemma 9.1 that

$$\Delta_{\mathrm{L}}^{(1)}(wZ_t)(\xi,\eta) = \Delta_{\mathrm{L}}^{(2)}(wZ_t)(\xi,\eta) = 2c(t)(wZ_t)(\xi,\eta).$$

Hence $Z_t \in \mathrm{Dom}(\Delta_{\mathrm{AL}}) \cap \mathrm{Dom}(\Delta_{\mathrm{CL}})$ and

$$\Delta_{\mathrm{AL}} Z_t = \Delta_{\mathrm{CL}} Z_t = 2c(t) Z_t.$$

The rest of (9.6) is clear from (7.9). □

Thus, the solution Z_t to the normal-ordered white noise differential equation (9.1) is an eigenvector of the quantum Lévy Laplacian with eigenvalue $2c(t)$. The following result is now clear.

Theorem 9.3. *Let Z_t be the solution to the normal-ordered white noise differential equation* (9.1). *Let ν be a finite measure on* $[0,1]$ *and $\alpha \in \mathbb{C}$. Define*

$$\Xi_t = \int_0^1 e^{2\alpha t c(s)} Z_s\, \nu(ds), \quad t \in \mathbb{R}.$$

Then $\Xi_t \in \mathcal{L}(\mathcal{W},\mathcal{W}^*)$ is a solution to the following Cauchy problem:

$$\frac{\partial \Xi}{\partial t} = \alpha \Delta_{\mathrm{QL}} \Xi, \qquad \Xi_0 = \int_0^1 Z_s \nu(ds) \in \mathcal{L}(\mathcal{W},\mathcal{W}^*).$$

As a classical reduction as in Proposition 8.3, we readily reproduce the following result due to Obata, Ouerdiane [27], see also [26].

Corollary 9.4. *Let Ξ_t be as in Theorem* 9.3 *and set $\Phi_t = \Xi_t \phi_0$. Then $\Phi_t \in \mathcal{W}^*$ and is a solution to the following Cauchy problem*:

$$\frac{\partial \Phi}{\partial t} = \alpha \Delta_{\mathrm{L}} \Phi, \qquad \Phi_0 = \int_0^1 Z_s \phi_0 \nu(ds) \in \mathcal{W}^*. \tag{9.7}$$

Remark 9.5. Note that $c(t)$ defined in (9.3) becomes a non-decreasing function such that $c(0) = 0$ and $c(1) = 1$. Hence there exists a probability measure μ on $[0, 1]$ whose distribution function is (the right-continuous modification of) $c(t)$. It is shown that

$$\int_0^1 f(t)\,\mu(dt) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int_0^1 f(t)e_n(t)^2\,dt, \qquad f \in C[0, 1]. \tag{9.8}$$

Moreover, with usual approximation argument by step functions we see that the limit in (9.8) exists for all $f \in L^\infty([0, 1], dt)$ and becomes a continuous linear functional on $L^\infty([0, 1], dt)$. In the case of $c(t) = t$, i.e., when μ is the Lebesgue measure, $\{e_n\}$ is called *equally dense* by Lévy [21,22].

Remark 9.6. Another approach to the Cauchy problem (9.7) is studied by Chung et al. [9], where the initial value $\Phi_0 \in \mathcal{W}^*$ is assumed to be a normal function of Lévy (or its generalization).

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