Refined Lower Bounds on the 2-Class Number of the Hilbert 2-Class Field of Imaginary Quadratic Number Fields with Elementary 2-Class Group of Rank 3

Elliot Benjamin
Associate Professor of Mathematics, Unity College, Unity, Maine 04988-9502

and

Charles J. Parry
Professor of Mathematics, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061-0123

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Let \( \mathbb{C}_k \) be an imaginary quadratic number field with \( \mathbb{C}_k \), the 2-Sylow subgroup of its ideal class group, isomorphic to \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \). By the use of various versions of the Kuroda class number formula, we improve significantly upon our previous lower bound for \( |\mathbb{C}_k^{1,2}| \), the 2-class number of the Hilbert 2-class field of \( k \).

1. INTRODUCTION

Throughout this paper we let \( k \) denote an imaginary quadratic number field and \( \mathbb{C}_k \), the 2-Sylow subgroup of its ideal class group, i.e., the 2-Sylow subgroup of the ideal class group, \( \mathbb{C}_k \), (in the wide sense) of \( k \). We assume that \( \mathbb{C}_k \) is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) (elementary of rank 3) and that at most one prime \( q \equiv 3 \mod 4 \) divides \( d_k \), the discriminant of \( k \). We let \( \mathbb{k}_1 \) denote the Hilbert 2-class field of \( k \), i.e., the maximal unramified (including the infinite primes) abelian field extension of \( k \) which has degree a power of 2. In our previous work (cf. [1, 2]) we have shown that \( |\mathbb{C}_{\mathbb{k}_1}^{1,2}| \geq 8 \). By the use of various versions of the Kuroda class number formula, we shall prove that \( |\mathbb{C}_{\mathbb{k}_1}^{1,2}| \geq 64 \).
We first fix some notation; we make use of genus theory and Rédei and Reichardt conditions [10, 15, 20] to determine congruence relations and the values of Legendre symbols of the primes dividing \(d_k\).

\[ k = \mathbb{Q}(\sqrt{-p_1 p_2 p_3 q_1}), \quad p_1 \equiv p_2 \equiv 1 \mod 4, \quad p_3 \equiv 1 \mod 4 \text{ or } p_3 = 2, \]
\[ q_1 \equiv 3 \mod 4 \text{ or } [(q_1 = 1 \text{ or } q_1 = 2) \text{ and } p_3 \equiv 1 \mod 4], \]
where \(p_1, p_2, p_3, q_1\) are distinct primes.

\[ k_0 = \mathbb{Q}(\sqrt{p_1 p_2 p_3}), \]
\[ k_1 = \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{p_3}, \sqrt{-q_1}). \]

\[ E = \text{Group of units in } k_1, \]
\[ E^+ = \text{Group of units in } k_1^+. \]

\[ K_i = k_0(\sqrt{p_i}), \quad i = 1, 2, 3, \] are the three quartic subfields.

of \(k_1^+\) which contain \(k_0\).

\[ E_{K_i} = \text{Group of units in } K_i, \]
\[ h(K_i) = 2\text{-Class number of } K_i, \]
\[ k^{(i)}, \quad 0 \leq i \leq 14, \] are the 15 quadratic subfields of \(k_1\).

\[ k^{(0)} = k_0 \text{ and } k^{(1)}, ..., k^{(14)} \text{ are real.} \]
\[ e_i = \text{Group of units of } k^{(i)}, \]
\[ h_i = 2\text{-Class number of } k^{(i)}. \]
\[ \text{and } h^*(K_i) = h(Q(\sqrt{p_j p_k})), \quad \{i, j, k\} = \{1, 2, 3\}, = 2\text{-class number of } Q(\sqrt{p_j p_k}). \]

We always assume (unless specified otherwise) that \( C_{k_1^2} \cong \langle 2, 2, 2 \rangle \) \((\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})\) and that at most one prime \(q \equiv 3 \mod 4\) divides \(d_k\); we let \(\{i, j, k\} = \{1, 2, 3\}\). We make use of the following two Kuroda class number formulas (cf. [14, 18]), utilizing the fact that \(k_1\) is the genus field of \(k\):

\[
(1) \quad |C_{k_1^2}| = 2^{-16} \cdot |E : \prod_{i=0}^{14} e_i| \cdot \prod_{i=0}^{14} h_i,
\]
\[
(2) \quad h(K_i) = \frac{1}{2} q(K_i) \cdot h_{i_0}^{(i)} \cdot h_{i_2}^{(i)}, \quad \text{where } q(K_i) = |E_{K_i} : e_0 e_1^{(i)} e_2^{(i)}|; \quad e_{i_0}^{(i)} \text{ (resp., } h_{i_0}^{(i)} \text{) is the unit group (resp., 2-class number) of } Q(\sqrt{p_i}), \text{ and } e_{i_2}^{(i)} \text{ (resp., } h_{i_2}^{(i)} \text{) is the unit group (resp., 2-class number) of } Q(\sqrt{p_j p_k}), \quad \{i, j, k\} = \{1, 2, 3\}.
\]

We also make use of a generalization of the Kuroda class number formula for \(V_4\) extensions, i.e., normal extensions \(L/F\) with \(F\) a number field and \(\text{Gal}(L/F) \cong \langle 2, 2 \rangle\) (cf. [18]).
(3) \( h(L) = \) the 2-class number of \( L = 2^{d-x-2-v} \cdot q(L/F) \cdot (h(L_1) \cdot h(L_2) \cdot h(L_3)/\eta^2) \), where
\[
d = \text{the number of infinite places ramified in } L/F;
v = 1 \text{ if } L = F(\sqrt{\epsilon}, \sqrt{\eta}) \text{ with units } \epsilon, \eta \in E_F, \text{ and};
v = 0 \text{ otherwise};\EF \text{ is the group of unbits of } F;\x \text{ is the } \mathbb{Z} - \text{rank of } E_F;\h \text{ is the 2-class number of } F;\h(L_i), i = 1, 2, 3, \text{ is the 2-class number of the intermediate field } L_i;\q(L/F) = q(L) \text{ is the unit index of } L/F, \text{ i.e., } q(L) = [E_L : E_{L_1} \cdot E_{L_2} \cdot E_{L_3}] \text{ where } E_L \text{ is the group of units of } L, i = 1, 2, 3.

In our case, by letting \( F = k_0, L = k_1^+, \) and \( L_i = K_i, i = 1, 2, 3, \) formula (3)

implies the following formula:
\[
q(k_1^+) \geq 8h_0^2/h(K_1)^1 \cdot h(K_2)^1 \cdot h(K_3)^1.
\]

2. MAIN RESULTS

We begin with our previous lower bound on \(|C_{k_1,2}|\), given our above assumptions on \( k \) (cf. [1]).

**Lemma 1.** \( |C_{k_1,2}| \geq 8. \)

In order to extend this lower bound on \(|C_{k_1,2}|\), we make use of all four above versions of the Kuroda class number formula.

**Lemma 2.** Let \( \prod_{i=0}^{14} h_i = 2^m \) and \( |C_{k_1,2}| = 2^m, \ m \in \mathbb{N}, \ n \in \mathbb{N}. \) Then \( \prod E : \prod_{i=0}^{14} e_i = 2^{16 + 2^m - n}. \)

**Proof.** We apply formula (1) to \( k_1 = k_{\text{gen}} \) where \( \text{Gal}(k_1/Q) \cong (2, 2, 2, 2);\)
\[
|C_{k_1,2}| = 2^m = 2^{-16} \cdot \prod_{i=0}^{14} e_i \cdot \prod_{i=0}^{14} h_i = \prod_{i=0}^{14} e_i \cdot 2^{2^m - 16}
\]
and our result immediately follows.

**Remark.** The only assumption upon \( k \) necessary for Lemma 2 is that \( k \) is an imaginary quadratic number field with \( C_{k,2} \cong (2, 2, 2, 2). \)

**Lemma 3.** Let \( |C_{k_1,2}| = 2^m \) (note \( m \geq 3 \) by [1]). Then \( \prod E : \prod_{i=0}^{14} e_i \) \( \leq 2^m - 1. \)

**Proof.** Since at most one prime \( q \equiv 3 \mod 4 \) divides \( d_k, \) we see from genus theory that \( \prod_{i=0}^{14} h_i \geq 2^{17}. \) From formula 1 we therefore have \( 2^m \geq 2 \cdot \prod E : \prod_{i=0}^{14} e_i \) and our result immediately follows.
We now make use of the fact that \( k_1 \) is a CM field, i.e., a totally complex quadratic extension of a totally real number field, in order to reduce \( E/\prod_{i=0}^{14} e_i \) to \( E^+/\prod_{i=0}^{6} e_i \) where \( \prod_{i=0}^{6} e_i \) is the composition of the unit groups of the 7 real quadratic subfields of \( k_1^+ \).

**Lemma 4.** \( E/\prod_{i=0}^{14} e_i \cong E^+/\prod_{i=0}^{6} e_i \).  

**Proof.** Since \( k_1 \) is an abelian CM field that is the compositum of \( k_1^+ \) and \( \mathbb{Q}(\sqrt{-q_1}) \) which have pairwise different prime power conductors, we know that \( E/W_{k_1}: E^+ = I \) where \( W_{k_1} \) is the group of roots of unity in \( k_1 \) (cf. \( [11, 16] \)). If \( W_{k_1} = \{1, -1\} \) then it follows that \( E/\prod_{i=0}^{14} e_i = E/\prod_{i=0}^{6} e_i \). We now assume that \( W_{k_1} \not\subseteq \{1, -1\} \). Since \( E/W_{k_1} \), \( E^+ = I \), we have \( E/\prod_{i=0}^{12} e_i = \mathbb{Q}_k^+ \). Since at most one prime \( q \equiv 3 \) mod 4 divides \( d_k \), we know given our original assumptions on \( k \)—that \( \mathbb{W}_{k_1} \subseteq \mathbb{Q}(\sqrt{-1}) \) or \( W_{k_1} \subseteq \mathbb{Q}(\sqrt{-3}) \). We thus have

\[
E/\prod_{i=0}^{14} e_i = W_{k_1} \cdot E^+ / \prod_{i=0}^{14} e_i / W_{k_1} \cong W_{k_1} \cdot E^+ / W_{k_1} \cdot \prod_{i=0}^{6} e_i / W_{k_1} \cong E^+ / \prod_{i=0}^{6} e_i
\]

and our result follows.

Before proving our main theorems we establish the following lemma.

**Lemma 5.** \( \prod_{i=1}^{3} E_{K_i} \cdot \prod_{i=0}^{6} e_i = \prod_{i=1}^{3} q(K_i) \).  

**Proof.** We have

\[
\prod_{i=1}^{3} q(K_i) = \left| \frac{E_{K_1}}{e_0 e_1 e_2} \times \frac{E_{K_2}}{e_0 e_3 e_4} \times \frac{E_{K_3}}{e_0 e_5 e_6} \right|
\]

where without the loss of generality we specify that \( k^{(i)}, 1 \leq i \leq 6 \), have been chosen such that

\[E_{K_i} \cong e_1 e_2, \quad E_{K_i} \cong e_3 e_4, \quad \text{and} \quad E_{K_i} \cong e_5 e_6.\]

We set up the surjective homomorphism

\[f: E_{K_1} \times E_{K_2} \times E_{K_3} \to E_{K_1}E_{K_2}E_{K_3}/E_0,\]
where \( E_0 = e_0 e_1 e_2 e_3 e_4 e_5 e_6 \) is the compositum of the unit groups of the 7 real quadratic subfields of \( k^+_1 \). The map is defined as follows:

\[
f(\eta_1, \eta_2, \eta_3) = \eta_1 \eta_2 \eta_3 E_0.
\]

We proceed to show that the kernel of this map is precisely \( e_0 e_1 e_2 \times e_0 e_3 e_4 \times e_0 e_5 e_6 \), which would give us

\[
E_{K_1} \times E_{K_2} \times E_{K_3} / e_0 e_1 e_2 \times e_0 e_3 e_4 \times e_0 e_5 e_6 \cong E_{K_1} E_{K_2} E_{K_3} / E_0,
\]

and prove our lemma.

It is immediate that \( e_0 e_1 e_2 \times e_0 e_3 e_4 \times e_0 e_5 e_6 \equiv \ker(f) \). Assume that \((\eta_1, \eta_2, \eta_3) \in \ker(f)\). Then \( \eta_1 \eta_2 \eta_3 = u_{0 j} u_{1 i} u_{2} u_{3} u_{4} u_{5} u_{6} \) for some \( u_j \in e_j, \ 0 \leq j \leq 6 \). We know that \( N_{K_1/K_0}(\eta_1 \eta_2 \eta_3) = \eta_1^2 \eta_2 \eta_3, \ \eta_4 \in e_0 \), since \( K_1 \cap K_2 = K_1 \cap K_3 = K_0 \). However, \( N_{K_1/K_0}(u_{0 j} u_{1 i} u_{2} u_{3} u_{4} u_{5} u_{6}) = \pm u_{0 j}^2 u_{1 i}^2 u_{2}^2 \) since \( K_1 \cap e_i = \pm 1, 3 \leq i \leq 6 \). We thus have \( \eta_1^2 \eta_2 = \pm u_{0 j}^2 u_{1 i}^2 u_{2}^2 \) and \( \eta_2^3 = \pm w_0 u_{1 i}^2 u_{2}^2 \) for some \( w_0 \in e_0, w_0 > 0 \). Since \( \eta_1 \) is real we obtain \( \eta_1 = \pm \sqrt{w_0 u_{1 i} u_{2}} \). An analogous argument yields \( \eta_2 = \pm \sqrt{w_0 u_{1 i} u_{2}^3} \) and \( \eta_3 = \pm \sqrt{w_0 u_{1 i} u_{2}^3} \) for some \( x_0, y_0 \in e_0, x_0 > 0, y_0 > 0 \).

We now let \( e = e_0 e_1 e_2 \times e_0 e_3 e_4 \times e_0 e_5 e_6 \) and assume that \( (\eta_1, \eta_2, \eta_3) \notin e \). We claim that at least two of \((\eta_1, 1, 1), (1, \eta_2, 1), (1, 1, \eta_3)\) are not in \( e \). To see this assume that (without loss of generality) \((\eta_1, 1, 1) \in e \) and \((1, \eta_2, 1) \in e \), which implies that \((1, 1, \eta_3) \notin e \). We thus have \( \eta_3 = \sqrt{\epsilon_0 t_0 u_{1 i} u_{2}} \) for some \( t_0 \in e_0 \) where \( \epsilon_0 \) is the fundamental unit of \( e_0 \). Consequently \( u_{0 j} u_{1 i} u_{2} u_{3} u_{4} u_{5} u_{6} = \eta_1 \eta_2 \eta_3 = \sqrt{\epsilon_0 t_0 u_{1 i} u_{2}} \) for some \( \epsilon_0 \in e_0 \), \( 0 \leq i \leq 6 \). This yields \( \epsilon_0 = \epsilon_0^2 t_0^2 \), \( t_0 \in e_0 \), \( 0 = 0, 2, 5, 6 \), and therefore \( \sqrt{\epsilon_0} = \pm t_0 t_4 \). However, this contradicts \((1, 1, \eta_3) \notin e \), our claim is therefore established.

We now make use of our claim and assume (again without loss of generality) that \((\eta_1, 1, 1) \notin e \) and \((1, 1, \eta_3) \notin e \). Then as above, we have \( \eta_3 = \sqrt{\epsilon_0 t_0 u_{1 i} u_{2}} \) and \( \eta_2 = \sqrt{\epsilon_0 x_0 u_{3} u_{4}} \) for some \( \epsilon_0, x_0 \in e_0 \). Thus \( \sqrt{\epsilon_0} \in K_1 \) and therefore \( K_1 = K_0(\sqrt{\epsilon_0}) \). Similarly, \( \sqrt{\epsilon_0} \in K_2 \) and \( K_2 = K_0(\sqrt{\epsilon_0}) \), which implies that \( K_0 = K_3 \), a contradiction. We therefore conclude that \((\eta_1, \eta_2, \eta_3) \in e \) and our lemma is proved.

**Theorem 1.** \( |C_{k_1,2}| \geq 32 \).

We apply formulas (1), (2), and (3*) to the following two cases:

**Case 1.** \( \prod_{i=1}^{3} q(K_i) = 1 \). From formula (3*), Lemma 4, and Lemma 5 we see that

\[
\begin{align*}
E : \prod_{i=0}^{14} e_i & = E^* : \prod_{i=0}^{6} e_i = q(k_1^+) \geq \frac{8h_0^2}{h(K_1) h(K_2) h(K_3)}.
\end{align*}
\]
From formula (2) we know that

\[ h(K_i) = \frac{1}{2}q(K_i) \cdot h_k \cdot h(Q(\sqrt{p_i}) \cdot h(Q(\sqrt{p_j}))), \{i, j, k\} = \{1, 2, 3\}. \]

We recall that we have defined for \( K_i \), \( h(Q(\sqrt{p_j} \cdot p_k)) = h(K_i) \) (see the Introduction). From genus theory we see that \( \prod_{i=1}^{3} h^{*}(K_i) \geq 8 \); if \( \prod_{i=1}^{3} h^{*}(K_i) \geq 128 \) then \( \prod_{i=0}^{14} h_i \geq 2^{21} \) and formula (1) implies that \( |C_{h_i, 2}| \geq 32 \). We therefore assume that \( 8 \leq \prod_{i=1}^{3} h^{*}(K_i) \leq 64 \). In a similar way we see that \( h_k \geq 4 \), and that if \( h_k \geq 64 \) then \( |C_{h_k, 2}| \geq 32 \); we therefore assume that \( 4 \leq h_k \leq 32 \). We let \( \prod_{i=1}^{3} h^{*}(K_i) = 2^n \), \( 3 \leq n \leq 6 \), and \( h_k = 2^m \), \( 2 \leq m \leq 5 \). From repeated application of formulas (2) and (3*) we find that \( \prod_{i=1}^{3} h^{*}(K_{ij}) = 2^q \), \( 1 \leq q \leq 8 \) and \( |E : \prod_{i=0}^{14} E_{ij} | \geq 2^{9 - m - n} \). We now apply formula (1) to conclude that \( |C_{h_i, 2}| \geq 32 \).

Case 2. \( \prod_{i=1}^{3} q(K_i) \geq 2 \). From Lemmas 4 and 5 and formula (1) we see that if \( \prod_{i=1}^{3} q(K_i) \geq 16 \) then \( |C_{h_i, 2}| \geq 32 \); we therefore assume that \( 2 \leq \prod_{i=1}^{3} q(K_i) \leq 8 \). We make the same assumptions on \( \prod_{i=1}^{3} h^{*}(K_i) \) and \( h_k \) as in Case 1, and we again conclude that \( \prod_{i=0}^{14} h_i \geq 2^{12 + m + n} \). By Lemmas 4 and 5 we know that \( |E : \prod_{i=0}^{14} E_{ij} | = |E^{*} : \prod_{i=0}^{14} E_{ij} | = q(k_{ij}) \cdot \prod_{i=1}^{3} q(K_i) \). We thus find that through repeated application of formulas (2) and (3*) in conjunction with Lemmas 4 and 5, we are again able to conclude, as in Case 1, that \( |E : \prod_{i=0}^{14} E_{ij} | \geq 2^{9 - m - n} \) and therefore \( |C_{h_i, 2}| \geq 32 \).

In order to further extend our lower bound on \( |C_{h_i, 2}| \), we make use of the following results.

**Lemma 6.** \( |E : \prod_{i=0}^{14} E_{ij} | \geq 8 \).

**Proof.** We show that each of the fields \( Q(\sqrt{p_i}, \sqrt{p_j}) \) for \( 1 \leq i < j \leq 3 \) must contain a unit that is not in a quadratic subfield.

Let \( k^{(i)} = Q(\sqrt{p_i}, \sqrt{p_j}) \). Then \( K_i = Q(\sqrt{p_i}, \sqrt{p_j}) \) is an unramified quadratic extension of \( k^{(i)} \). From formula 2 it follows that \( h(k^{(i)}) = h(K_i)/q(K_i) \). Since \( h(k^{(i)}) \leq 2h(K_i) \), this implies that \( q(K_i) \geq 2 \). We therefore are able to conclude (cf. [4, 13]) that \( |E : \prod_{i=0}^{14} E_{ij} | \geq 8 \).

**Lemma 7.** \( \prod_{i=0}^{14} h_i \geq 2^{18} \).

**Proof.** We demonstrate this result through a case by case analysis of the three Legendre symbols \( (p_i/p_j) \) for \( 1 \leq i < j \leq 3 \).

Case (a) \( (p_1/p_2) = (p_1/p_3) = (p_2/p_3) = -1 \).

Case (b). (Without loss of generality) \( (p_1/p_2) = (p_1/p_3) = -1, (p_2/p_3) = 1 \).
Case (c). (Without loss of generality) \((p_1/p_2) = (p_1/p_3) = 1, (p_2/p_3) = -1\).

Case (d). \((p_1/p_2) = (p_1/p_3) = (p_2/p_3) = 1\).

Case (a). Since \(C_{k_{2.2}} \cong (2, 2, 2)\) there are no \(d_k\)-splittings of the second kind (cf. [15, 20]) and since we are in Case (a) it follows that at least two Legendre symbols \((p_i/q), (p_j/q)\), are equal to 1.

By genus theory we see that \(h(Q(\sqrt{-p_1q})) \geq 4\) and \(h(Q(\sqrt{-p_2q})) \geq 4\). We are thus able to conclude in this case that \(\prod_{i=0}^{14} h_i \geq 2^{19}\).

Case (b). We use the same theory as Case (a) to conclude that either \((p_2/q) = 1\) or \((p_3/q) = 1\), and consequently either \(h(Q(\sqrt{-p_2q})) \geq 4\) or \(h(Q(\sqrt{-p_3q})) \geq 4\). It follows that \(\prod_{i=0}^{14} h_i \geq 2^{18}\).

Case (c). Through genus theory and a combinatorial analysis utilizing biquadratic residue symbols and criteria for the norm of the fundamental unit of a real quadratic number field being equal to \(-1\), it follows that either \(h(Q(\sqrt{p_1p_2p_3})) \geq 8\) or \(h(Q(\sqrt{p_1p_3})) \geq 4\) or \(h(Q(\sqrt{p_1p_2})) \geq 4\) (cf. [5, 6, 12, 17] for details).

Case (d). Since there are four \(d_k\)-splittings of the second kind, where \(k_0 = Q(\sqrt{p_1p_2p_3})\), it follows that \(h(Q(\sqrt{p_1p_2})) \geq 8\) and consequently \(\prod_{i=0}^{14} h_i \geq 2^{18}\).

Remark. One must be careful when using Buell [6] to include the omitted biquadratic residue symbol requirement \((p_2p_3/p_1) = -1\) in Case (c) of Lemma 7, in order to insure that the norm of the fundamental unit of \(Q(\sqrt{p_1p_2p_3})\) is equal to \(-1\).

We can make use of Theorem 1 and Lemma 7 to strenghten Lemma 3 as follows.

**Corollary 1.** Let \(|C_{k_{1.2}}| = 2^m\) (note \(m \geq 5\) by Theorem 1). Then \(|E: \prod_{i=0}^{14} e_i| \leq 2^{m-2}\).

**Proof.** By applying Lemma 7 to formula (1) our result immediately follows.

We will now employ Lemma 6 and Lemma 7 to prove that \(|C_{k_{1.2}}| \geq 64\). We note that by applying Lemma 6 and Lemma 7 to formula (1), we have a second proof that \(|C_{k_{1.2}}| \geq 32\).

**Theorem 2.** \(|C_{k_{1.2}}| \geq 64\).

**Proof.** We prove our result through a case by case analysis of the three Legendre symbols \((p_i/p_j)\) for \(1 \leq i < j \leq 3\), as was done in the proof of Lemma 7.
Case (a) \((p_1/p_2) = (p_1/p_3) = (p_2/p_3) = -1\).

Case (b). (Without loss of generality) \((p_1/p_2) = (p_1/p_3) = -1, (p_2/p_3) = 1\).

Case (c). (without loss of generality) \((p_1/p_2) = (p_1/p_3) = 1, (p_2/p_3) = -1\).

Case (d). \((p_1/p_2) = (p_1/p_3) = (p_2/p_3) = 1\).

Case (a). We see from the proof of Lemma 7, Case (a), that \(\Pi_{i=0}^{14} h_i \geq 2^{19}\). We apply Lemma 6 to formula (1) and immediately conclude that \(|C_{k_1,2}| \geq 64\).

Case (b). If \(|E^+|\Pi_{i=0}^{14} e_i| > 16\) then by Lemma 4, Lemma 7, and formula (1) we can conclude that \(|C_{k_1,2}| \geq 64\).

Assume \(|E^+|\Pi_{i=0}^{14} e_i| < 16\); by Lemma 6 and Lemma 4 this implies that \(|E^+|\Pi_{i=0}^{14} e_i| = 8\). By genus theory applied to formula (2) we obtain \(h(K_i) = 2q(K_i)\) for \(i = 2, 3\).

From the proof of Lemma 6 and our assumption that \(|E^+|/\Pi_{i=0}^{14} e_i| = 8\), it follows that \(q(K_i) = 1\) and consequently \(h(K_i) = 2\) for \(i = 2, 3\) (cf. [4, 13]). If \(h(Q(\sqrt{p_1 p_2})) \geq 4\) then from the proof of Lemma 7, Case (b), it follows that \(\Pi_{i=0}^{14} h_i \geq 2^{19}\), and from Lemma 4 and formula (1) we conclude that \(|C_{k_1,2}| \geq 64\).

If \(h(Q(\sqrt{p_1 p_2})) = 2\) then it follows as above, that \(q(K_i) = 1\) and \(h(K_i) = 2\) for \(i = 1, 2, 3\). Since \(h(k_0) = 4\) and \(h(K_i) = 2\) for \(i = 1, 2, 3\), we know by [3] (Proposition 7), that the 2-class field tower of \(k_0\) terminates at \(k_{0,1}\), the Hilbert 2-class field of \(k_0\).

Since \(k_{0,1} = k^+_1\), by formula (3) we obtain \(1 = h(k^+_1) = 1/4 q(K_i) \cdot \frac{1}{\phi}\), which implies \(q(k^+_1) = 16\) and consequently, using Lemma 5, \(|E^+|/\Pi_{i=0}^{14} e_i| = 16\), which is a contradiction, and we therefore conclude that \(|C_{k_1,2}| \geq 64\).

Case (c). If \(q(K_i) > 1\) for \(i = 1, 2, 3\), then from Lemma 7, the proof of Lemma 6, and formula (1) we conclude that \(|C_{k_1,2}| \geq 64\) (cf. [4, 13]).

We therefore assume that \(q(K_i) = 1\) for \(i = 1, 2, 3\). If \(h(k_0) = 4\) then from formula (2) we see that \(h(K_1) = 2\), and from [3, Proposition 7], the 2-class field tower of \(k_0\) terminates at \(k_{0,1}\), the Hilbert 2-class field of \(k_0\).

From [3] (Theorem 1), this implies that \((p_1/p_2)_{4} \cdot (p_1/p_3)_{4} \cdot (p_2/p_3)_{4} = -1\), where \((p_i/p_j)_{4}, 1 \leq i < j \leq 3\), is the biquadratic residue symbol. It follows that \(h(Q(\sqrt{p_1 p_2})) = (h(Q(\sqrt{p_1 p_3})) = 2\) (cf. [5, 17]). However, from the proof of Lemma 7, Case (c), we see that \(h(Q(\sqrt{p_1 p_2})) \geq 4\) or \(h(Q(\sqrt{p_1 p_3})) \geq 4\), which is a contradiction and we therefore conclude that \(|C_{k_1,2}| \geq 64\).
If \( h(k_0) \geq 8 \), we know that if \( h(Q(\sqrt{p_1 p_2})) \geq 4 \) or \( h(Q(\sqrt{p_1 p_3})) \geq 4 \) then \( \prod_{i=1}^{14} h_i \geq 2^{19} \). By Lemma 6 and formula (1) we conclude that \( |C_{k_1,2}| \geq 64 \). We therefore assume that \( h(k_0) \geq 8 \) and \( h(Q(\sqrt{p_1 p_2})) = h(Q(\sqrt{p_1 p_3})) = 2 \). By formula (2) we see that \( h(K_0) = \frac{1}{2} h(k_0) \), \( i = 1, 2, 3 \), and once again by [3] (Proposition 7), we know that the 2-class field tower of \( k_0 \) terminates at \( k_{0,1} \). It follows by genus theory that the Hilbert 2-class field of \( k_1^* \) is precisely equal to \( k_{0,1} \), and therefore since \( h(k_1^*) = \frac{1}{2} h(k_0) \), \( i = 1, 2, 3 \), we have \( h(k_1^*) = \frac{1}{2} h(k_0) \). From formula (3) we therefore obtain

\[
h(k_1^*) = \frac{1}{2} h(k_0) = \frac{1}{2} q(k_1^*) \left( \frac{1}{2} (h(k_0))^2 \right) \frac{1}{2} q(k_1^*) \cdot h(k_0) \text{ which implies that } q(k_1^*) = 16 \text{ and consequently, using Lemma 5, } [E : \prod_{i=0}^{1} e_i] = 16. \]

By Lemma 4, Lemma 7, and formula (1) we conclude that \( |C_{k_1,2}| \geq 64 \).

Case (d). From genus theory we know that \( h(k_0) \geq 8 \). We can therefore apply the same argument as in Case (c), under the assumption that \( h(k_0) \geq 8 \), to conclude that \( |C_{k_1,2}| \geq 64 \).

We thus see that in all possible cases we have \( |C_{k_1,2}| \geq 64 \) and our theorem is proved.

3. EXAMPLES

**Example 1.** \( Q(\sqrt{-2 \cdot 5 \cdot 29}) = Q(\sqrt{-8 \cdot 120}) \).

Since \( \prod_{i=1}^{14} h_i = 2^{24} \), Lemma 6 and formula (1) yield \( |C_{k_1,2}| \geq 2^{-16} \cdot 2^{3} \cdot 2^{24} = 2^{11} = 2048 \).

**Example 2.** \( Q(\sqrt{-2 \cdot 5 \cdot 17 \cdot 7}) = Q(\sqrt{-7 \cdot 760}) \).

Since \( (\frac{2}{5}) = (\frac{5}{2}) = -1 \), \( (\frac{17}{2}) = 1 \), and \( (\frac{7}{2}) \cdot (\frac{2}{2}) \cdot (\frac{17}{2}) \cdot (\frac{7}{2}) = 1 \), \( (a/b) \cdot (c/d) \cdot (e/f) = 1 \), where \( (a/b) \) is the biquadratic residue symbol, we know from [3] that \( h(k_1^*) = 1 \), and utilizing the techniques described in [4] and [13] we obtain that \( \prod_{i=1}^{14} q(K_i) = 1 \). Since \( \prod_{i=1}^{14} h_i = 2^{19} \), \( h_{k} = 4 \), and \( \prod_{i=1}^{14} h^*(K_i) = 8 \), we see from formula (1), formula (2), formula (3), Lemma 4, and Lemma 5, that \( [E : \prod_{i=0}^{1} e_i] = 16 \) and \( |C_{k_1,2}| = 128 \).

**Example 3.** \( K = Q(\sqrt{-2 \cdot 5 \cdot 13 \cdot 29}) = Q(\sqrt{-15 \cdot 760}) \).

Utilizing techniques described in [4] and [13] we see that \( \sqrt{2} \in Q(\sqrt{29} , \sqrt{5 \cdot 13}) \) and \( \sqrt{2} \in Q(\sqrt{5} , \sqrt{13 \cdot 29}) \), where \( \varepsilon_0 = 521 + 12 \sqrt{5 \cdot 13} \) and \( \varepsilon_1 = 233 + 12 \sqrt{13 \cdot 29} \), and thus \( \prod_{i=1}^{14} q(K_i) = 4 \). Since \( \prod_{i=1}^{14} h^*(K_i) = 16 \) and \( h_{k} = 4 \), we see from formulas (2) and (3) that \( q(k_1^*) = 2 \cdot h(k_1^*) \). Since \( \prod_{i=1}^{14} h_i = 2^{18} \), we know from formula (1) that \( |C_{k_1,2}| = 4 \cdot [E : \prod_{i=0}^{1} e_i] \). From Lemmas 4 and 5 we therefore have
\( E: \prod_{i=0}^{\infty} e_i = 8 \cdot h(k_i^+) \) and \( |C_{k_i,2}| = 32 \cdot h(k_i^+) \). Since \( k_i^+ = \mathbb{Q}(\sqrt{5}, \sqrt{13}, \sqrt{29}) \) and \( C_{k_i,2} \cong (2, 2) \) (cf. [6, 12]) we know that \( \text{Gal}(k_i^+)/k \) is dihedral and \( \mathbb{Q}(\sqrt{13}, \sqrt{29}) \) is the fixed field of the maximal cyclic subgroup of \( \text{Gal}(k_i^+)/k \) (cf. [8]). Since \( q(\mathbb{Q}(\sqrt{13}, \sqrt{29}) = 1 \) we see from formula (2) that \( h(\mathbb{Q}(\sqrt{13}, \sqrt{29}) = 4 \). Since the 2-class group of \( \mathbb{Q}(\sqrt{13}, \sqrt{29}) \) is cyclic, we know that the 2-class field tower of \( \mathbb{Q}(\sqrt{13}, \sqrt{29}) \) terminates at its first Hilbert 2-class field. It follows that \( \text{Gal}(k_i^+)/k \) is the dihedral group of order 8; therefore \( h(\mathbb{Q}(\sqrt{5}, \sqrt{13}, \sqrt{29}) = 2 \) and we are able to conclude that \( |C_{k_i,2}| = 64 \).

**Note.** By utilizing the technique in Example 3 we can obtain a whole family of fields \( k \) for which \( |C_{k_i,2}| = 64 \).

**Remark.** We take this opportunity to mention that in the first author’s previous work on obtaining lower bounds on \( |C_{k_i,2}| \) for imaginary quadratic number fields with \( C_{k_i,2} \cong (2, 2, 2) \) (cf. [1, 2]), we have incorrectly stated that there are 10 non-isomorphic groups \( G \) of order 64 with \( G/G' \cong (2, 2, 2) \) and \( G' \cong (2, 2, 2) \), when in actuality there are 12 such groups. However, the two groups we omitted each have a maximal subgroup such that 8 ideal classes capitulate (become principal) in its corresponding unramified quadratic extension, and therefore these two groups cannot occur for \( k \) imaginary, leaving all our results intact.

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