Congruence and Conjunctivity of Matrices

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ABSTRACT

Congruence of arbitrary square matrices over an arbitrary field is treated here by elementary classical methods, and likewise for conjunctivity of arbitrary square matrices over an arbitrary field with involution. Uniqueness results are emphasized, since they are largely neglected in the literature. In particular, it is shown that a matrix S is congruent [conjunctive] to $S_0 \oplus S_1$ with S_1 nonsingular, and that if S_1 here is of maximal size among all nonsingular matrices R_1 for which $R_0 \oplus R_1$ is congruent [conjunctive] to S, then the congruence [conjunctivity] class of S determines that of S_1. Partially canonical forms (most of them already known) are derived, to the extent that they do not depend on the field. Nearly canonical forms are derived for "neutral" matrices (those congruent or conjunctive with block matrices $\begin{bmatrix} 0 & N \\ M & 0 \end{bmatrix}$ with the two zero blocks being square). For a neutral matrix S over a field F, the F-congruence [F-conjunctivity] class of S is determined by the F-equivalence class of the pencil $S+tS'$ $[S+tS^*]$ and, if the pencil is nonsingular, by the $F[t]$-equivalence class of $S+tS'$ $[S+tS^*]$.

1. INTRODUCTION

Matrices S and T over a field F are called *congruent over* F provided there is a nonsingular matrix C over F such that $C'SC=T$. (Here, and throughout, prime denotes transpose.) The problem of determining, for given F,S,T, when S and T are congruent over F has not been solved in any

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satisfactory sense, except for certain very special cases. (See [7] in this connection; there the language of "equivalence of bilinear forms" is used instead of the language of "congruence of matrices.") In this paper we shall limit our treatment to the results that can be derived by elementary means for the general case. Some of these results were derived independently in [7, pp. 45–49], [5] by very different methods. Our methods are elementary classical methods of linear algebra, though our treatment for pencils is a somewhat unusual variant of classical treatments, which was used in [9] and [3], and independently in [4]. (The treatment in [5] depends on applying the Krull-Remak-Schmidt theorem to "Kronecker modules"; our approach is completely different.) We give special stress to uniqueness results, since these are largely ignored in the literature.

The settings are those of [2]: a pair (F,E) of fields F and E is *admissible* provided it is complic or simplic, it is *complic* provided F is a proper separable quadratic extension of E, and it is *simplic* provided $F=E$. Associated with each admissible (F,E) is an involutory F-automorphism (called "(F,E)-conjugation") $\alpha\mapsto\bar{\alpha}$, whose fixed field is E and which has order 2 if (F,E) is complic, order 1 if (F,E) is simplic. We let $\bar{A}$ denote the entrywise conjugate of the matrix A, and as usual we denote $\bar{A}'$ by A^*.

We shall say matrices S and T over F are *(F,E)-congruent* (or, more briefly, *$*$-congruent* when F is understood from context) provided $C^*SC=T$ for some nonsingular matrix C over F. (Thus, in the usual complex case, where F is the complex field and E is the real field, "$*$-congruence" is the same as "conjunctivity" or "hermitian congruence", though it is sometimes called merely "congruence.") Clearly "$*$-congruence" is the same as "congruence" in the simplic cases.

In this paper (F,E) will always be an admissible pair, unrestricted except as occasionally specified. The F-automorphism, (F,E)-conjugation, will often be denoted by $*$ (since $\bar{\alpha}=\alpha^*$ as a 1×1 matrix), and the pair $(F,*)$ determines the pair (F,E) as well as vice versa. Usually F, E, and $*$ will be considered as understood (and fixed except as otherwise specified), and all matrices will be over F except as otherwise specified, but we shall often mention $*$ explicitly (as in "$*$-congruence") as a reminder that the complic cases are allowed.

To each matrix S (over F) corresponds the *(F,E)-bilinear form* x^*Sy [or, more precisely, the function $(x,y)\mapsto x^*Sy$, where x and y are column vectors over F], which we shall usually call a *$*$-bilinear form*. This correspondence is well known to be one-one, and in fact the (i,j) entry of S is just $e_i^*Sf_j$ when $e_1,e_2,\ldots$ and $f_1,f_2,\ldots$ are the standard ordered bases involved. Under this correspondence, $*$-congruence of a square (say $n\times n$) matrix S corresponds to a change of basis (or also to a nonsingular linear map) in the space, say $\mathcal{V}$, of all $n \times 1$ matrices:

$$S \mapsto C^* SC$$

corresponds to

$$(x,y) \mapsto (Cx, Cy)$$

in the *-bilinear form $x^* Sy$ (which $\mapsto x^* C^* SCy$ in either process). In this context we shall often regard S as a linear map ($y \mapsto Sy$) of $\mathcal{V}$ into its *-dual space $\mathcal{V}^*$. [Here $\mathcal{V}^*$ is the space of all mappings f of $\mathcal{V}$ into F such that $f(\alpha x + \beta y) = \bar{\alpha} f(x) + \bar{\beta} f(y)$ for all $\alpha, \beta \in F$ and all $x, y \in \mathcal{V}$.] The action of $\mathcal{V}^*$ on $\mathcal{V}$ is such that Sy ($\in \mathcal{V}^*$) acts on x ($\in \mathcal{V}$) to produce $x^* Sy$ ($\in F$).

We shall also adapt the notion of "reciprocal polynomials" (the properties of which are well summarized in [8, Sec. 1]) to apply to the complic cases as well as the simplic cases. Namely, let (F,E) be admissible and let $p(t) \in F[t]$ with $p(0) \neq 0$. Denote as usual $\overline{p(t)}$ by $\bar{p}(t)$. Then we denote by $\hat{p}(t)$ the **-reciprocal* polynomial of $p(t)$: $\hat{p}(t) = \overline{p(0)}^{-1} t^d \bar{p}(t^{-1})$, where d is the degree of $p(t)$ [hence, also of $\bar{p}(t)$ and $\hat{p}(t)$]. A monic polynomial $p(t)$ is **-self-reciprocal* provided $p(t) = \hat{p}(t)$; in this case $p(0)\overline{p(0)} = 1$. Clearly, if $A = S^{*-1}S$, then $A^{*-1} = SAS^{-1}$ and hence every invariant factor of A is *-self-reciprocal. A nonconstant monic *-self-reciprocal polynomial $p(t)$ in $F[t]$ will be called ($F[t]$-) *irreducibly* *-self-reciprocal provided $p(t)$ cannot be factored in $F[t]$ as a product of two nonconstant *-self-reciprocal polynomials; if $p(t) \in E[t]$ here and cannot be factored in $E[t]$ as such a product, then $p(t)$ will be called *$E[t]$-irreducibly* (*-) self-reciprocal. Clearly, if $K \in \{F, E\}$, each monic $K[t]$-irreducibly *-self-reciprocal $p(t)$ in $K[t]$ is either itself $K[t]$-irreducible or else $= q(t)\hat{q}(t)$ for some monic $K[t]$-irreducible $q(t)$; in the latter case the factorization is unique (up to order) and the factors are distinct and hence are coprime. Finally (if $K \in \{F, E\}$) each monic *-self-reciprocal polynomial $p(t)$ in $K[t]$ has a factorization (unique up to order)

$$p(t) = p_1(t)^{m(1)} p_2(t)^{m(2)} \cdots p_k(t)^{m(k)},$$

where $p_1(t), p_2(t), \ldots, p_k(t)$ are distinct monic $K[t]$-irreducibly *-self-reciprocal in $K[t]$.

In Sec. 2 we treat the *-congruence theory for nonsingular matrices over F to the extent that it does not depend heavily on the peculiarities of (F,E)-arithmetic. In Sec. 3 we treat the matrices S for which the pencil $S + tS^*$ (with t an indeterminate over E) is nonsingular [as a matrix over the field $F(t)$] and "reduce" these cases to those treated in Sec. 2. Lastly, in Sec.

4 we treat the matrices S for which the pencil $S+tS^*$ is singular and reduce these cases to the previous cases.

2. THE *-CONGRUENCE OF NONSINGULAR MATRICES

Throughout this section and, where applicable, in later sections we shall use the following notation.

NOTATION 2.1. We denote by I_m the $m\times m$ identity matrix, by J_m the lower triangular nilpotent $m\times m$ Jordan block, by G_m the $m\times m$ unit antidiagonal matrix (whose rows are those of I_m in reverse order):

$$I_m=\begin{bmatrix} 1 & & O \\ & \ddots & \\ O & & 1 \end{bmatrix},\qquad J_m=\begin{bmatrix} 0 & & & O \\ 1 & 0 & & \\ & \ddots & \ddots & \\ O & & 1 & 0 \end{bmatrix},$$

$$G_m=\begin{bmatrix} O & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & O \end{bmatrix}.$$

We often write I, J, or G where the size is clear from context. The pair (F,E) is a fixed admissible pair of fields, arbitrary except as otherwise specified, and (in this section) S is always nonsingular over F. *Nullspace* will be abbreviated as *nsp*, elementary divisor as *e.d.*, and *nontrivial invariant factor* as *nif*. We often write $A=S^{*-1}S$; thus $(A^*)^iS=SA^{-i}$ and $(A^*)^iS^*=S^*A^{-i}$ for every integer i, and hence $f(A^*)S=Sf(A^{-1})$, etc., for every $f(t)$ in $F[t]$, in fact, for every $f(t)$ in $F(t)$ such that $f(A^*)$ exists. When $F=E,\varepsilon\in\{1,-1\}$, and $A-\varepsilon I$ is nilpotent, we shall often write $N=A-\varepsilon I$; thus $f(\varepsilon I+N')S=Sf((\varepsilon I+N)^{-1})$, etc., for every $f(t)\in F(t)$ with $f(\varepsilon)\neq\infty$ here, and hence

$$g(N')S=Sg\left(-N(I+\varepsilon N)^{-1}\right),\qquad \text{etc.,}$$

for every $g(t)$ in $F(t)$ with $g(0)\neq\infty$. Also $N^m=0$ implies that $g(N)N^{m-1}=g(0)N^{m-1}$ for every such $g(t)$ in $F(t)$. The column space of the identity matrix (usually $n\times n$ in this context) will be denoted by $\mathcal{V}$, and its *-dual space by $\mathcal{V}^*$, and often S will be regarded as a linear map of $\mathcal{V}$ into $\mathcal{V}^*$, so that $A(=S^{*-1}S)$ and $N(=A-\varepsilon I)$ will then be maps of $\mathcal{V}$ into $\mathcal{V}$. If $\mathcal{U}$ is a subspace of $\mathcal{V}$ here, its annihilator in $\mathcal{V}^*$ will be denoted by $\mathcal{U}^0$, and $S\mathcal{U}$ will be a subspace of $\mathcal{V}^*$, whose annihilator in $\mathcal{V}$ will be denoted by $(S\mathcal{U})^0$.

Observe in Notation 2.1 that $C^{-1}AC=(C^*SC)^{*-1}(C^*SC)$, so the *-congruence class of S determines the similarity class of A. Some cases where the reverse determination holds are given in Theorem 2.11 (d) below (some of the simplic cases of which are given in [7, Theorem 5, p. 48]).

Our first two decomposition results (Lemma 2.2 and Theorem 2.3) are standard.

LEMMA 2.2. *Let S be nonsingular $n\times n$ over F, and suppose the minimum polynomial of $A=S^{*-1}S$ is $p(t)q(t)$, where $p(t)$ and $q(t)$ are coprime in $F[t]$ and each is monic *-self-reciprocal.* (1) *Then S is *-congruent to a direct sum $P\oplus Q$, where $P^{*-1}P$ and $Q^{*-1}Q$ have respective minimum polynomials $p(t)$ and $q(t)$.* (2) *Also the *-congruence classes of P and Q are determined from $p(t), q(t)$, and the *-congruence class of S as follows: the $n\times n$ matrix $P\oplus 0$ is *-congruent to $q(A)^*Sq(A)$, and $0\oplus Q$ is *-congruent to $p(A)^*Sp(A)$.*

Proof. The proof of (1) is routine: the two subspaces $\operatorname{nsp} p(A)$ [which $=q(A)\mathcal{V}$] and $\operatorname{nsp} q(A)$ [which $=p(A)\mathcal{V}$] are complementary in $\mathcal{V}$ (see [6, Vol. 1, p. 179]); using bases for them (in that order) as the columns of an $n\times n$ matrix C makes $C^*SC=P\oplus Q$ with the required properties. To prove (2), let $C^*SC=T$ and $C^{-1}AC=B$ in (1). Then $T=P\oplus Q$ and $B=T^{*-1}T=P^{*-1}P\oplus Q^{*-1}Q$ and

$$\begin{aligned}C^*q(A)^*Sq(A)C&=\left[C^{-1}q(A)C\right]^*(C^*SC)\left[C^{-1}q(A)C\right]\\&=q(B)^*Tq(B)\\&=q(P^{*-1}P)^*Pq(P^{*-1}P)\oplus 0,\end{aligned}$$

which is indeed *-congruent to $P\oplus 0$. [Here we have used the fact that $q(Q^{*-1}Q)=0$ and that $q(P^{*-1}P)$ is nonsingular.] In much the same manner one sees that $p(A)^*Sp(A)$ is *-congruent to $0\oplus Q$. Also, if R is any matrix *-congruent to S, say $D^*RD=S$, then

$$q(R^{*-1}R)^*Rq(R^{*-1}R)=D^{*-1}q(A)^*Sq(A)D^{-1},$$

which is *-congruent to $q(A)^*Sq(A)$ and hence to $P\oplus 0$. [A similar result holds for $p(t), 0\oplus Q$, and R.] ■

A routine induction applied to Lemma 2.2 yields the following.

THEOREM 2.3. *Let* S *be* $n\times n$ *nonsingular over* F, *and let* $p(t)$ *be the minimum polynomial of* $A = S^{*-1}S$ *with factorization* $p(t) = p_1(t)^{m(1)}\cdots p_k(t)^{m(k)}$, *where the polynomials* $p_i(t)$ *are monic *-self-reciprocal and pairwise coprime in* $F[t]$. *Then* S *is *-congruent to* $S_1\oplus\cdots\oplus S_k$, *with* $S_i^{*-1}S_i$ *having minimum polynomial* $p_i(t)^{m(i)}$ *for each* i. *Also the *-congruence classes of* $S_1,\dots,S_k$ *are determined from [the polynomials* $p_i(t)^{m(i)}$ *and] that of* S *by the fact that for each* i *the* $n\times n$ *matrix* $S_i\oplus 0$ *is *-congruent to* $q_i(A)^*Sq_i(A)$, *where* $q_i(t)\in F[t]$ *is defined by* $p_i(t)^{m(i)}q_i(t)=p(t)$.

Theorem 2.3 reduces the study of *-congruence of nonsingular matrices S to the case where the minimum polynomial of $S^{*-1}S$ is a power of an $F[t]$-irreducibly *-self-reciprocal polynomial. We treat the latter case in the following theorem. (Much of the simplic case of this result appeared in [7, Lemma 6, p. 49].)

THEOREM 2.4. *Let* S *be nonsingular over* F, $A=S^{*-1}S$ *have minimum polynomial* $q(t)^m$, *and* $q(t)$ *be monic* $F[t]$*-irreducibly *-self-reciprocal in* $F[t]$. *Then* S *is *-congruent to* $S_1\oplus\cdots\oplus S_k$ *with the nifs (see Notation 2.1) of the matrices* $A_i=S_i^{*-1}S_i$ *being powers of* $q(t)$ *as follows.*

(1) *Each* A_i *has just one nif, but occurring (in that* A_i*) with multiplicity 1 or 2.*

(2) *Each* A_i *is nonderogatory (i.e., the multiplicity is 1 for each* A_i*) if either* (F,E) *is complic or* $q(t)$ *is nonlinear.*

(3) *When a matrix* A_i *is similar to the direct sum of two copies of a companion matrix (i.e., when the multiplicity is 2 for that* A_i*), we can take the corresponding*

$$S_i=\begin{bmatrix} P & L \\ M & Q \end{bmatrix}$$

conformably, with L *and* M *nonsingular and* P *and* Q *singular.*

(4) *When* (F,E) *is complic and* $q(t)$ *is linear, each* S_i *is *-congruent to an upper antitriangular matrix with entries on the antidiagonal equal to each other, and all entries on the first super-antidiagonal nonzero and equal to each other, and all other entries zero (the specific form is given in Lemma 2.8).*

The proofs of parts (1), (2), and (3) of Theorem 2.4 follow by routine induction from Lemma 2.7 below, and the proof of part (4) follows from the first two parts plus Lemma 2.8 below, but we need two preparatory lemmas first.

LEMMA 2.5. *If (F,E) is complic or $q(t)$ is nonlinear in Theorem 2.4, then there are an integer $h \geqslant 0$ and a vector $e \in \mathcal{V}$ such that*

$$e^* SA^{-h} q(A)^{m-1} e \neq 0.$$

Proof. First suppose (F,E) is complic. Then there is a $\theta \neq \bar{\theta} \in F$. For any such θ we have the elementary identity

$$x^* Ty = (\theta - \bar{\theta})^{-1}\big[(\bar{\theta} - 1)x^* Tx + \bar{\theta}(1-\theta) y^* Ty$$

$$-\bar{\theta}(x+y)^* T(x+y) + (x+\theta y)^* T(x+\theta y)\big]$$

for every x and y in $\mathcal{V}$ and every linear map $T: \mathcal{V} \to \mathcal{V}^*$. Thus for every nonzero such T there is an $e \in \mathcal{V}$ such that $e^* Te \neq 0$. Since $SA^{-h}q(A)^{m-1}$ is nonzero for every integer h, we can, say, take $h=0$ and put $T = Sq(A)^{m-1}$ here.

To prove the conclusion for the case where $q(t)$ is nonlinear, we may now suppose also that (F,E) is simplic, i.e., that $F=E$. Then $q(t)$ is not $t-1$ nor $t+1$ but is monic irreducibly self-reciprocal, so $q(1)q(-1) \neq 0$ and $q(t)$ must have even degree, say $2l$, and $q(0)=1$. (See [8, p. 327].) Let $r(t) = [t^{-l}q(t)]^{m-1}$. Then $r(t^{-1}) = r(t) \in F(t)$, and $r(A) \neq 0$ because $q(A)^{m-1} \neq 0$ [and $r(A)$ exists because A is nonsingular] and $A+I$ is nonsingular [because $q(-1) \neq 0$ and $q(A)^m = 0$], so (by Notation 2.1)

$$Sr(A) + r(A')S' = S'Ar(A) + S'r(A^{-1})$$

$$= S'(A+I)r(A)$$

is nonzero, and hence $Sr(A)$ is nonalternating. (An $n \times n$ matrix T is alternating (or alternate) provided $x'Tx = 0$ for every $n \times 1$ matrix x. See [1, p. 387–391] for more details.) Thus there is an $e \in \mathcal{V}$ such that $e'Sr(A)e \neq 0$, so we can take $h = l(m-1)$ here. ■

The next lemma is also technical in nature and treats a case [case (2) of the lemma] not needed for the proof of Theorem 2.4 but needed elsewhere.

LEMMA 2.6. *Let $p(t)$ and $q(t)$ be in $F[t]$ such that (at least) one of the following two cases holds: (1) $q(t)$ is monic $F[t]$-irreducibly $*$-self-reciprocal, or (2) $p(t) \in E[t]$ and $q(t)$ is monic $E[t]$-irreducibly $*$-self-reciprocal in $E[t]$. Furthermore suppose that neither t nor $q(t)$ divides $p(t)$ and that (in*

Notation 2.1) *$e\in\mathcal{V}$ and h,s,j,l are nonnegative integers such that $q(A)^{s+1}=0$,*

$$e^*SA^{-h}q(A)^se\neq 0, \qquad [p(A)A^jq(A)^le]^*S\mathcal{U}=0,$$

where $\mathcal{U}$ is the A-cyclic subspace of $\mathcal{V}$ generated by e. Then $l>s$.

Proof. From hypothesis we have

$$e^*\bar{q}(A^*)^l\bar{p}(A^*)A^{*j}SA^ie=0$$

for all integers i. We shall show that consequently the following two equations must hold:

$$0=e^*Sp(A)q(A)^l\mathcal{U}, \qquad 0=e^*S\hat{p}(A)q(A)^l\mathcal{U}.$$

To show the first equation holds, we perform * (conjugation followed by transposition) on the preceding equation:

$$0=e^*A^{*i}S^*A^jp(A)q(A)^le$$

$$=e^*SA^{j-1-i}p(A)q(A)^le$$

(because $A^{*i}S^*=A^{*i}SA^{-1}=SA^{-i-1}$; see Notation 2.1); this holds for all integers i, and hence $0=e^*Sp(A)q(A)^l\mathcal{U}$.

To show the second equation holds, we let $d=\text{degree}\,p(t)$ and $c=\text{degree}\,q(t)$ and note that

$$0=e^*\bar{q}(A^*)^l\bar{p}(A^*)A^{*j}SA^ie$$

$$=e^*S\bar{q}(A^{-1})^l\bar{p}(A^{-1})A^{i-j}e$$

$$=e^*S\,\overline{p(0)}\,\hat{p}(A)\,\overline{q(0)}^{\,l}q(A)^lA^{i-j-d-cl}e$$

[because $q(t)=\hat{q}(t)$, etc.]; this holds for all integers i, and hence $0=e^*S\hat{p}(A)q(A)^l\mathcal{U}$.

Note that this second equation differs from the first one only in the replacement of $p(A)$ by $\hat{p}(A)$. Thus (in these two equations) we can now replace $p(A)$ and $\hat{p}(A)$ by $g(A)$, where $g(t)=\gcd(p(t),\hat{p}(t))$. Namely, $g(t)=u(t)p(t)+v(t)\hat{p}(t)$ for suitable $u(t)$ and $v(t)$ in $F[t]$, and $u(A)\mathcal{U}\subseteq\mathcal{U}$ and

$v(A)\mathcal{U} \subseteq \mathcal{U}$, so

$$0 = e^* Sp(A)q(A)^l \mathcal{U} \supseteq e^* Sp(A)q(A)^l u(A)\mathcal{U},$$

$$0 = e^* S\hat{p}(A)q(A)^l \mathcal{U} \supseteq e^* S\hat{p}(A)q(A)^l v(A)\mathcal{U},$$

and hence (by addition)

$$0 = e^* S[u(A)p(A) + v(A)\hat{p}(A)]q(A)^l \mathcal{U}$$

$$= e^* Sg(A)q(A)^l \mathcal{U}.$$

However, $g(t) = \hat{g}(t)$ is *-self-reciprocal and is not divisible by $q(t)$, so it is coprime to $q(t)$: in case (1) because $q(t)$ is $F[t]$-irreducibly *-self-reciprocal, in case (2) because $g(t) \in E[t]$ and $q(t)$ is $E[t]$-irreducibly *-self-reciprocal. Thus $g(A)$ is nonsingular [because $q(A)^{s+1} = 0$] and hence $g(A)\mathcal{U} = \mathcal{U}$, so

$$0 = e^* Sq(A)^l \mathcal{U} \supseteq e^* SA^{-h} q(A)^{l+i} e$$

for every integer $i \geqslant 0$. Thus $l > s$ (otherwise a contradiction would occur for $i = s - l$). ■

LEMMA 2.7. *Under the Theorem 2.4 hypotheses, S is *-congruent to* $T \oplus R$, *where* $T^{*-1}T$ *is nonderogatory with* nif $= q(t)^m$ *in each of the following three cases*: (i) *where* (F,E) *is complic*, (ii) *where* (F,E) *is simplic and* $q(t)$ *is nonlinear, and* (iii) *where* (F,E) *is simplic and* $q(t)$ *is linear* (*hence* $q(t) = t - \varepsilon$ *with* $\varepsilon \in \{1, -1\}$) *and* $Sq(A)^{m-1}$ *is nonalternating. On the other hand, in the case* (iv) *where* (F,E) *is simplic and* $q(t)$ *is linear* ($= t - \varepsilon$) *and* $Sq(A)^{m-1}$ *is alternating, S is congruent to* $T \oplus R$, *where* $T'^{-1}T = \varepsilon I_{2m} + (J_m \oplus J_m)$ *and*

$$T = \begin{bmatrix} P & L \\ M & Q \end{bmatrix}$$

with P and Q singular $m \times m$ *and L and M nonsingular.*

Proof. We treat cases (i), (ii), and (iii) together. In each of these cases there are an integer $h \geqslant 0$ and a vector e such that $e^* SA^{-h} q(A)^{m-1} e \neq 0$ [in cases (i) and (ii) this is by Lemma 2.5; in case (iii) this is by definition of nonalternating]. Let $\mathcal{U}$ be the A-cyclic subspace generated by e. Then $\mathcal{U} \cap (S\mathcal{U})^0 = 0$ by Lemma 2.6 (with $s = m - 1$), so $\mathcal{V} = \mathcal{U} \oplus (S\mathcal{U})^0$ is an

S-orthogonal direct-sum decomposition of $\mathcal{V}$, and is also S*-orthogonal because $S^*\mathcal{U} = SA^{-1}\mathcal{U} = S\mathcal{U}$.

Finally we treat case (iv). Here $q(t) = t - \varepsilon$ with $\varepsilon \in \{1, -1\}$, and $F = E$ and $Sq(A)^{m-1} = S(A - \varepsilon I)^{m-1}$ is alternating. Let $N = A - \varepsilon I$. Since $SN^{m-1} \neq 0$, there are vectors e and f in $\mathcal{V}$ such that $e'SN^{m-1}f = 1$. Thus $f'SN^{m-1}e = -1$, since SN^{m-1} is alternating. Let $\mathcal{U}$ and $\mathcal{W}$ be the respective N-cyclic subspaces of $\mathcal{V}$ generated by e and f. Then $\mathcal{U}$ and $\mathcal{W}$ are m-dimensional because $N^m = 0$ and $N^{m-1}e \neq 0 \neq N^{m-1}f$. It suffices to show that the following properties hold:

(1) $\mathcal{U} \cap \mathcal{W} = 0 = (\mathcal{U} + \mathcal{W}) \cap [S(\mathcal{U} + \mathcal{W})]^0$,
(2) $S(\mathcal{U} + \mathcal{W}) = S'(\mathcal{U} + \mathcal{W}), S\mathcal{W} = S'\mathcal{W}$,
(3) $\mathcal{U} \cap (S\mathcal{W})^0 = 0 = \mathcal{U} \cap (S'\mathcal{W})^0$,
(4) $\mathcal{U} \cap (S\mathcal{U})^0 \neq 0, \mathcal{W} \cap (S\mathcal{W})^0 \neq 0$.

Now, (2) holds because $S = S'A$ and $S' = SA^{-1}$ and $\mathcal{U}$ and $\mathcal{W}$ are A-invariant. Also, (4) holds because $0 \neq N^{m-1}e \in \mathcal{U} \cap (S\mathcal{U})^0$ and $0 \neq N^{m-1}f \in \mathcal{W} \cap (S\mathcal{W})^0$; for example,

$$\begin{aligned}(N^{m-1}e)'S\mathcal{U} &= e'N'^{m-1}S\mathcal{U} \\ &= e'S(-I - \varepsilon N)^{1-m}N^{m-1}\mathcal{U} \\ &= e'S(-1)^{1-m}N^{m-1}\mathcal{U} = e'SN^{m-1}\mathcal{U},\end{aligned}$$

which $= 0$ because $e'SN^{m-1}N^ie = 0$ for all $i \geqslant 0$ (for $i > 0$ because $N^m = 0$, for $i = 0$ because SN^{m-1} is alternating).

To prove (3), it suffices, since $S\mathcal{W} = S'\mathcal{W}$, to show $\mathcal{U} \cap (S\mathcal{W})^0 = 0$. To show $\mathcal{U} \cap (S\mathcal{W})^0 = 0$, it suffices to assume $[N^jp(N)e]'S\mathcal{W} = 0$ for some integer $j \geqslant 0$ and some $p(t) \in F[t]$ with $p(0) \neq 0$, and to show that then j must be $\geqslant m$. To show $j \geqslant m$ here, we note that $p(N(-I - \varepsilon N)^{-1})$ is nonsingular [because $N^m = 0$ and $p(0) \neq 0$], so

$$\begin{aligned}0 = e'p(N')N'^jS\mathcal{W} &= e'Sp\big(N(-I - \varepsilon N)^{-1}\big)\big[N(-I - \varepsilon N)^{-1}\big]^j\mathcal{W} \\ &= e'SN^j\mathcal{W} \supseteq e'SN^{j+i}f\end{aligned}$$

for every integer $i \geqslant 0$, so indeed j must be $\geqslant m$.

Finally, to prove (1), it suffices to assume $x \in \mathcal{U}$ and $y \in \mathcal{W}$ and $x + y \in [S(\mathcal{U} + \mathcal{W})]^0$, and to show that then x and y must be 0. Thus $x = N^ku(N)e$ and $y = N^lv(N)f$ for some nonnegative integers k and l and some polynomials $u(t)$ and $v(t)$ in $F[t]$ with $u(0) \neq 0 \neq v(0)$. It suffices to show that k and l must both be $\geqslant m$ here. Say $k \geqslant l$; then it suffices to show $l \geqslant m$. Suppose $l \leqslant m-1$; then $N^{m-1-l}e \in \mathcal{U} + \mathcal{W}$ and we get the following contradiction:

$$0 = \left[N^k u(N)e + N^l v(N)f\right]' S(\mathcal{U} + \mathcal{W})$$

$$\supseteq \left[e' u(N') N'^k + f' v(N') N'^l\right] S N^{m-1-l} e$$

$$= e' S u\left(N(-I-\varepsilon N)^{-1}\right)\left[N(-I-\varepsilon N)^{-1}\right]^k N^{m-1-l} e.$$

$$+ f' S v\left(N(-I-\varepsilon N)^{-1}\right)\left[N(-I-\varepsilon N)^{-1}\right]^l N^{m-1-l} e.$$

The first term (in the last member) is 0 when $k \geqslant l+1$ (because $N^m = 0$) and also is 0 when $k = l$, because in the latter case it is $(-1)^{-k} u(0) e' S N^{m-1} e$, which is 0, since SN^{m-1} is alternating. On the other hand, the second term $= (-1)^{-l} v(0) f' S N^{m-1} e = (-1)^{-l} v(0)(-1)$ is nonzero. This contradiction shows that $l \geqslant m$ when $k \geqslant l$. Similarly $k \geqslant m$ when $l \geqslant k$. This completes the proof of Lemma 2.7. ■

Remarks on Lemma 2.7(iii) and (iv).

(1) Another way to see that the blocks P and Q in (iv) are singular and the blocks L and M are nonsingular is to use Jordan bases for $\mathcal{U}$ and $\mathcal{W}$: $(e, Ne, \ldots, N^{m-1}e)$ for $\mathcal{U}$ and $(f, \ldots, N^{m-1}f)$ for $\mathcal{W}$. Then all four blocks are antitriangular (because $N^m = 0$) and the antidiagonals of P and Q are 0 (because SN^{m-1} is alternating), while all the antidiagonal entries of L and M are nonzero (alternating 1 and -1) because $e' S N^{m-1} f = 1$.

(2) When $\operatorname{char} F \neq 2$, it is known that $\mathcal{U}$ and $\mathcal{W}$ in the proof of case (iv) can be chosen so that the diagonal blocks P and Q are 0.

(3) When $\operatorname{char} F \neq 2$ and $\varepsilon = (-1)^{m-1}$ in case (iii) or (iv), then it can be routinely shown that SN^{m-1} is symmetric (and nonzero) and hence nonalternating [thus excluding case (iv)]. On the other hand, when $\varepsilon = (-1)^m$ here, then SN^{m-1} must be alternating [thus excluding case (iii)]; likewise if $\operatorname{char} F = 2$ and m is even. When $\operatorname{char} F = 2$ and m is odd, then SN^{m-1} can be either alternating or nonalternating. (Most of the computations relevant to this remark are given in the proof of [2, Theorem 3.6].)

Part (4) of Theorem 2.4 follows from parts (1) and (2) and the following lemma.

Lemma 2.8. *Let (F, E) be complic, S be $n \times n$ nonsingular over F, $A = S^{*-1}S$ be nonderogatory, $\alpha \in F$, and $A - \alpha I$ be nilpotent.* (i) *Then $\alpha\bar{\alpha} = 1$,* *and* (ii) *there is at least one* $\theta \in F$ *such that* $\bar{\theta} \neq \theta \neq \alpha\bar{\theta}$. *Furthermore*, (iii) *if* (θ *is any element of* F *such that*) $\bar{\theta} \neq \theta \neq \alpha\bar{\theta}$ *and* $\beta = (\theta - \alpha\bar{\theta})^{-1}(\alpha - 1)$, *then* $\beta = \bar{\beta}$ *and* S *is* *-*congruent to* $\varepsilon G[I + \theta(\beta I + J)]$ *for some* $\varepsilon \in E$. *Likewise*, (iv) *if* $\alpha \neq 1$ *and* $\theta \neq \bar{\theta}$ *and* $\gamma = (\alpha - 1)^{-1}(\theta - \alpha\bar{\theta})$, *then* $\gamma = \bar{\gamma}$ *and* S *is* *-*congruent to* $\varepsilon G[\gamma I + J + \theta I]$ *for some* $\varepsilon \in E$.

Proof. (i) is clear because α is the only root of A (which is similar to A^{*-1}). To see (ii), note first that there is a $\delta \neq \bar{\delta} \in F$, since (F, E) is complic; if $\delta \neq \alpha\bar{\delta}$, we can take $\theta = \delta$, whereas if $\delta = \alpha\bar{\delta}$, we can take $\theta = \delta + 1$, because $\alpha \neq 1$ in the latter case. Finally, we prove (iii) [the proof of (iv) is much the same, so we omit it]. Thus, let $\bar{\theta} \neq \theta \neq \alpha\bar{\theta}$ and $\beta = (\theta - \alpha\bar{\theta})^{-1}(\alpha - 1)$, and let

$$H = (\bar{\theta} - \theta)^{-1}(\bar{\theta}S - \theta S^*), \qquad K = (\theta - \bar{\theta})^{-1}(S - S^*).$$

Then $\beta = \bar{\beta}$ (because $\alpha\bar{\alpha} = 1$), $S = H + \theta K$, $H = H^*$, and $K = K^*$. Also H is nonsingular because $\theta \neq \alpha\bar{\theta}$. Let $B = H^{-1}K$ and $N = B - \beta I$. Then

$$B = (\theta S^* - \bar{\theta}S)^{-1}(S - S^*) = (\theta I - \bar{\theta}A)^{-1}(A - I),$$

$$N = (\theta - \alpha\bar{\theta})^{-1}(\theta I - \bar{\theta}A)^{-1}\big[(\theta - \alpha\bar{\theta})(A - I) - (\alpha - 1)(\theta I - \bar{\theta}A)\big]$$

$$= (\theta - \alpha\bar{\theta})^{-1}(\theta I - \bar{\theta}A)^{-1}(\theta - \bar{\theta})(A - \alpha I),$$

and $(A - \alpha I)^n = 0 \neq (A - \alpha I)^{n-1}$ (because $A - \alpha I$ is $n \times n$ nilpotent and nonderogatory), so $N^n = 0 \neq N^{n-1}$. Also $N^* H = K - \beta H = HN$, so $N^{*i}H = HN^i$ for all $i \geq 0$. Since $HN^{n-1} \neq 0$ and (F, E) is complic, there is a vector e such that $e^* HN^{n-1}e \neq 0$ (as in the proof of Lemma 2.5). Let $\varepsilon = e^* HN^{n-1}e$; then $\bar{\varepsilon} = e^*(N^*)^{n-1}He = \varepsilon$, so $0 \neq \varepsilon \in E$. Let

$$r(t) = \varepsilon^{-1} \sum_{j=0}^{n-1} (e^* HN^{n-1-j}e)t^j = \sum_{j=0}^{n-1} r_j t^j.$$

Then $\bar{r}(t) = r(t) \in E[t]$, and $r(0) = 1$, so there is a (unique) $s(t)$ of degree $< n$ in $E[t]$ for which $r(t)s(t) - 1$ is divisible by t^n. [Just divide $r(t)$ into 1, using ascending powers of t, until the remainder is divisible by t^n; the quotient is then $s(t)$.] Then $s(0) = 1$, and there is a $p(t) \in F[t]$ such that $p(0) = 1$ and $p(t)\bar{p}(t) - s(t)$ is divisible by t^n (since every number in E can be written as $c + \bar{c}$ for some c in F). Let $d = p(N)e$; then $N^{n-1}d = p(0)N^{n-1}e = N^{n-1}e$, so $d^* HN^{n-1}d = d^* HN^{n-1}e = (N^{n-1}d)^* He = (N^{n-1}e)^* He = \varepsilon$. Let $s(t) = \sum s_i t^i$, summed over $i \in \{0, \cdots, n-1\}$. Then for $0<j<n$ we have

$$\begin{aligned} d^*HN^{n-1-j}d &= e^*\bar{p}(N^*)HN^{n-1-j}p(N)e \\ &= e^*HN^{n-1-j}p(N)\bar{p}(N)e \\ &= e^*HN^{n-1-j}s(N)e = \sum_{i=0}^{j} e^*Hs_iN^{n-1-j+i}e \\ &= \varepsilon \sum_{i=0}^{j} r_{j-i}s_i = 0. \end{aligned}$$

Thus $d^*NB4^iHN^jd = d^*HN^{i+j}d = 0$ if $i+j \neq n-1$ (and it $=\varepsilon$ if $i+j=n-1$), so the matrix of H in the N-cyclic basis $(d, Nd, \cdots, N^{n-1}d)$ is εG. Also the matrix (in this basis) of $B-\beta I(=N)$ is J, so the matrix of $S=H+\theta K=H(I+\theta B)$ is $\varepsilon G[I+\theta(\beta I+J)]$. ■

There is in general no simple *-congruence canonical form for S in Theorem 2.4, but when the *-bilinear form x^*Sy is "neutral" in the sense of [5, p. 67], we can get a fairly canonical form (modulo interchanging companion matrices belonging to elementary divisors of A that are *-reciprocal to each other). This is done in Theorem 2.11 below (and, for special cases obtained by imposing restrictions on the minimum polynomial of A, in Theorem 2.12 and Corollary 2.13). We need first to define "neutral" as we shall use the term.

DEFINITION 2.9. A partitioned matrix $\begin{bmatrix} P & N \\ M & Q \end{bmatrix}$ will be called *neutrally partitioned* provided $P=0$ and $Q=0$ and both P and Q are *square* matrices, and a square matrix over F will be called (F,E)-*neutral* [or just *neutral*, when (F,E) is understood] provided it is (F,E)-congruent to a neutrally partitioned matrix.

REMARK 2.10. (1) A matrix S is (F,E)-neutral iff the *-bilinear form x^*Sy is neutral in the sense of [5, p. 67]. (2) Each direct sum of neutral matrices is neutral. (3) If M and N are nonsingular, then the following neutrally partitioned matrices are *-congruent to each other:

$$\begin{bmatrix} 0 & N \\ M & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & N \\ M & 0 \end{bmatrix}^{*-1} = \begin{bmatrix} 0 & N^{*-1} \\ M^{*-1} & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & I \\ N^{*-1}M & 0 \end{bmatrix}.$$

[The *-congruency of the first two is from the fact that every nonsingular matrix W is *-congruent to $W^{*-1}=(W^{-1})^*WW^{-1}$.] (4) If $N=C^{-1}MC$, then the neutrally partitioned matrix $\begin{bmatrix} 0 & I \\ M & 0 \end{bmatrix}$ is *-congruent to

$$\begin{bmatrix} 0 & I \\ N & 0 \end{bmatrix}=(C^*\oplus C^{-1})\begin{bmatrix} 0 & I \\ M & 0 \end{bmatrix}(C\oplus C^{*-1}).$$

THEOREM 2.11. *Let S be nonsingular and (F,E)-neutral and $A=S^{*-1}S$, and let $f(t)$ and $g(t)$ be monic $F[t]$-irreducible.* (a) *Then S is *-congruent to a neutrally partitioned matrix $\begin{bmatrix} 0 & I \\ M & 0 \end{bmatrix}$ with $M=P\oplus Q$, the characteristic polynomials of P and P^{*-1} being coprime and every elementary divisor of Q being *-self-reciprocal.* (b) *Let k be the multiplicity of $f(t)^m$ as an e.d. (elementary divisor) of M here; then each of $f(t)^m$ and $\hat{f}(t)^m$ has multiplicity k as an e.d. of A if $f(t)\neq\hat{f}(t)$, and $f(t)^m$ has multiplicity 2k as an e.d. of A if $f(t)=\hat{f}(t)$.* (c) *Let l be the multiplicity of $g(t)^m$ as an e.d. of A here (so $\hat{g}(t)^m$ also has multiplicity l as an e.d. of A); then one of $g(t)^m$ and $\hat{g}(t)^m$ (the same one for every m) has multiplicity l and the other has multiplicity 0, as an e.d. of M if $g(t)\neq\hat{g}(t)$, and (l is even and) $g(t)^m$ has multiplicity $\frac{1}{2}l$ as an e.d. of M if $g(t)=\hat{g}(t)$.* (d) *Thus the similarity class of A determines the *-congruence class of S here.* (e) *The matrices S_i in Theorem 2.3 are (F,E)-neutral here.*

Proof. (a): Here S is *-congruent to a neutrally partitioned matrix $\begin{bmatrix} 0 & I \\ M & 0 \end{bmatrix}$ by Remark 2.10 (3), and by Remark 2.10 (4) we may assume M is the direct sum of the companion matrices of its e.d.'s (elementary divisors). Thus S is *-congruent to the direct sum of the matrices $\begin{bmatrix} 0 & I \\ H & 0 \end{bmatrix}$ for which H is such a companion matrix (and multiplicities correspond). Further, if H is the companion matrix of an e.d. $s(t)^h$ with the divisor base $s(t)$ monic $F[t]$-irreducible, then, by Remark 2.10 (3), we may replace the direct summand H in M by H^{*-1}, and hence by the companion matrix K of $\hat{s}(t)^h$ by Remark 2.10 (4) (since K is similar to H^{*-1}). Thus we may choose the direct summands in M so that, for each pair $\{s(t),\hat{s}(t)\}$ of monic $F[t]$-irreducible polynomials with $s(t)\neq\hat{s}(t)$, at most one of the two occurs as a divisor base for (the e.d.'s of) M. With this choice for M, let P be the direct sum of the companion matrices of the e.d.'s of M which are not *-self-reciprocal, and let Q be the direct sum of the companion matrices of the *-self-reciprocal e.d.'s of M. Then M is similar to $P\oplus Q$, so we may assume M itself $=P\oplus Q$ by Remark 2.10 (4). Clearly P and Q satisfy the requirements of (a).

To prove (b), let

$$T=\begin{bmatrix} 0 & I \\ M & 0 \end{bmatrix}.$$

Then A is similar to $T^{*-1}T=M\oplus M^{*-1}=P\oplus Q\oplus P^{*-1}\oplus Q^{*-1}$, so it suffices to note that the characteristic polynomials of P, P^{*-1}, Q are pairwise coprime (and that every e.d. of Q is *-self-reciprocal and hence Q is similar to Q^{*-1}). Part (c) follows routinely from (a) and (b). Part (d) follows from (c) plus the fact that the *-congruence class of S is not changed here if a companion matrix H is replaced (as a direct summand in M) by the corresponding matrix K as specified in the proof of (a).

To prove (e), let $r(t)$ be the characteristic polynomial of A and $r(t)=r_1(t)\cdots r_k(t)$ with the polynomials $r_i(t)$ monic *-self-reciprocal and pairwise coprime, and let $S_1\oplus\cdots\oplus S_k$ be the corresponding direct sum (*-congruent to S) given by Theorem 2.3; then $r_i(t)$ is the characteristic polynomial of $A_i=S_i^{*-1}S_i$. Let P and Q be as in part (a), with respective characteristic polynomials $p(t)$ and $q(t)$. Then A is similar to $P\oplus Q\oplus P^{*-1}\oplus Q^{*-1}$, and the polynomials $p(t),\hat{p}(t),q(t)$ are pairwise coprime, and $q(t)=\hat{q}(t)$ and hence $r(t)=p(t)\hat{p}(t)q(t)^2$. Let $p_i(t)=\gcd(p(t),r_i(t))$ and $q_i(t)=\gcd(q(t),r_i(t))$ for each i. Then $\hat{p}_i(t)=\gcd(\hat{p}(t),r_i(t))$ and $\hat{q}_i(t)=q_i(t)$, and hence $r_i(t)=p_i(t)\hat{p}_i(t)q_i(t)^2$, for each i. For each i let P_i be the direct sum of the companion matrices of the e.d.'s of P which divide $p_i(t)$, and define Q_i in like fashion from Q and $q_i(t)$. Then $P\oplus Q$ is similar to $\oplus_i(P_i\oplus Q_i)$, so S is *-congruent [by Remark 2.10 (4)] to

$$\bigoplus_{i=1}^{k}\begin{bmatrix} 0 & I \\ P_i\oplus Q_i & 0 \end{bmatrix}.$$

Thus S_i is *-congruent by Theorem 2.3 to

$$\begin{bmatrix} 0 & I \\ P_i\oplus Q_i & 0 \end{bmatrix},$$

and hence is neutral, for each i. ■

One gets an essentially canonical form under *-congruence for nonsingular neutral matrices from Theorem 2.11; the same applies to the next two results, which turn out to be special cases of Theorem 2.11 but are approached via the minimum polynomial of A. (In Theorem 2.11 we found it more convenient to employ the characteristic polynomial of A, but we revert to the minimum polynomial now.)

THEOREM 2.12. *Let $p(t)$ be monic $F[t]$-irreducible and coprime to $\hat{p}(t)$, and let $p(t)^m\hat{p}(t)^m$ be the minimum polynomial of $A=S^{*-1}S$. Then S is $*$-congruent to a neutrally partitioned matrix $\begin{bmatrix} 0 & I \\ M & 0 \end{bmatrix}$ where $p(t)^m$ is the minimum polynomial of M.*

Proof. Let (in Notation 2.1) $\mathcal{U}=\operatorname{nsp} p(A)^m$ and $\mathcal{W}=\operatorname{nsp}\hat{p}(A)^m$. Then (see [6, Vol. I, p. 179]) $\mathcal{V}=\mathcal{U}\oplus\mathcal{W}$, $\dim\mathcal{U}=\dim\mathcal{W}$, $p(t)^m$ is the minimum polynomial of the $\mathcal{U}$-restriction of A, $\hat{p}(t)^m$ is that of the $\mathcal{W}$-restriction of A, $\mathcal{U}=\hat{p}(A)^m\mathcal{V}$, and $\mathcal{W}=p(A)^m\mathcal{V}$. Also $\mathcal{V}^*=\mathcal{W}^0\oplus\mathcal{U}^0$, and $\mathcal{W}^0$ and $\mathcal{U}^0$ act in the usual way as the respective $*$-duals of $\mathcal{U}$ and $\mathcal{W}$. To see that $S\mathcal{W}\subseteq\mathcal{W}^0$, let $x\in\mathcal{W}$. Then $x=p(A)^m y$ for some $y\in\mathcal{V}$, and hence

$$(Sx)^*\mathcal{W}=y^*\bar{p}(A^*)^m S^* p(A)^m\mathcal{V}$$

$$=y^*S^*\bar{p}(A^{-1})^m p(A)^m\mathcal{V},$$

which $=0$ because $\bar{p}(A^{-1})=\overline{p(0)}A^{-d}\hat{p}(A)$ [where $d=\operatorname{degree} p(t)$]. In like fashion $S\mathcal{U}\subseteq\mathcal{U}^0$. Let $\mathcal{B}$ be a basis for $\mathcal{W}$, and let $\mathcal{B}^*$ be its $*$-dual basis for $\mathcal{U}^0$; then $S\mathcal{B}$ is a basis for $\mathcal{W}^0$, and its $*$-dual basis for $\mathcal{U}$ is $S^{*-1}\mathcal{B}^*$. The matrix of $S:\mathcal{V}\to\mathcal{V}^*$ in the basis $(S^{*-1}\mathcal{B}^*,\mathcal{B})$ for $\mathcal{V}$ [and its $*$-dual basis $(S\mathcal{B},\mathcal{B}^*)$ for $\mathcal{V}^*$] is thus a neutrally partitioned matrix $\begin{bmatrix} 0 & I \\ M & 0 \end{bmatrix}$ over F (see Definition 2.9). One easily calculates that in this same basis $A:\mathcal{V}\to\mathcal{V}$ has matrix $M\oplus M^{*-1}$. Thus M is the matrix of the $\mathcal{U}$-restriction of A, so its minimum polynomial is $p(t)^m$. ■

COROLLARY 2.13. Let $p(t)$ be monic and coprime to $\hat{p}(t)$ in $F[t]$, and let $p(t)\hat{p}(t)$ be the minimum polynomial of $A=S^{*-1}S$. Then S is $*$-congruent to a neutrally partitioned matrix $\begin{bmatrix} 0 & I \\ M & 0 \end{bmatrix}$, where $p(t)$ is the minimum polynomial of M.

Proof. Let $p(t)=p_1(t)^{m(1)}\cdots p_k(t)^{m(k)}$ with the k polynomials $p_i(t)$ being distinct monic $F[t]$-irreducible. Then $\hat{p}(t)=\hat{p}_1(t)^{m(1)}\cdots\hat{p}_k(t)^{m(k)}$, and the k polynomials $q_i(t)=p_i(t)\hat{p}_i(t)$ are monic $F[t]$-irreducibly $*$-self-reciprocal and are pairwise coprime, so S is $*$-congruent to a direct sum $S_1\oplus\cdots\oplus S_k$ with $q_i(t)^{m(i)}$ the minimum polynomial of $A_i=S_i^{*-1}S_i$ for each i (by Theorem 2.3). Thus S is $*$-congruent by Theorem 2.12 to a direct sum

$$\bigoplus_{i=1}^{k}\begin{bmatrix} 0 & I \\ M_i & 0 \end{bmatrix}$$

and hence to $\begin{bmatrix} 0 & I \\ M & 0 \end{bmatrix}$, where $M = M_1 \oplus \cdots \oplus M_k$ and $p_i(t)^{m(i)}$ is the minimum polynomial of M_i (for each i), and hence $p(t)$ is the minimum polynomial of M. ■

3. NONSINGULAR PENCILS $S + tS^*$

In this and the next section we shall consider matrix pencils; in nonhomogeneous notation they appear as $A - tB$, with t an indeterminate, and A and B conformable matrices, over a field F. Such a pencil is called *nonsingular* provided it is square and its determinant is nonzero in $F[t]$. Although we are concerned with square matrices in this paper, we shall introduce the following notation (Notation 3.1) and initial result (Lemma 3.2) in a way that will also apply to nonsquare matrices.

NOTATION 3.1. Let A and B be $m \times n$ matrices over F, t be an indeterminate over F, $\mathcal{V}$ be the space of all $n \times 1$ matrices over F, and $\mathcal{V}'$ be the space of all $m \times 1$ matrices over F. For a subset $\mathcal{S}$ of $\mathcal{V}$ we define $A\mathcal{S}$ and $B\mathcal{S}$ by regarding A and B as maps of $\mathcal{V}$ into $\mathcal{V}'$, and likewise for a subset $\mathcal{S}'$ of $\mathcal{V}'$ and the inverse images $A^{-1}\mathcal{S}'$ and $B^{-1}\mathcal{S}'$:

$$A\mathcal{S} = \{Ax : x \in \mathcal{S}\}, \qquad A^{-1}\mathcal{S}' = \{x \in \mathcal{V} : Ax \in \mathcal{S}'\},$$

with analogous definitions for $B\mathcal{S}$ and $B^{-1}\mathcal{S}'$. Next, for integers $i \geq 0$ we define $(AB^{-1})^i\mathcal{S}'$ and $(A^{-1}B)^i\mathcal{S}$ inductively by $(AB^{-1})^0\mathcal{S}' = \mathcal{S}'$, $(A^{-1}B)^0\mathcal{S} = \mathcal{S}$ and

$$(AB^{-1})^{i+1}\mathcal{S}' = AB^{-1}(AB^{-1})^i\mathcal{S}',$$

$$(A^{-1}B)^{i+1}\mathcal{S} = A^{-1}B(A^{-1}B)^i\mathcal{S}$$

for each $i \geq 0$. We denote the two special sequences of sets (subspaces here) starting with $0 \in \mathcal{V}$ (more properly, with $\{0\} \subseteq \mathcal{V}$, but we shall persist in this "abuse of notation" for the sake of brevity) as follows:

$$\mathcal{U}_i = (A^{-1}B)^i 0, \qquad \mathcal{W}_i = (B^{-1}A)^i 0$$

for each $i \geq 0$. (In particular, $\mathcal{U}_1 = \operatorname{nsp} A$ and $\mathcal{W}_1 = \operatorname{nsp} B$.) Thus $0 = \mathcal{U}_0 \subseteq \mathcal{U}_1 \subseteq \cdots \subseteq \mathcal{U}_n$, and if $\mathcal{U}_i = \mathcal{U}_{i+1}$, then $\mathcal{U}_i = \mathcal{U}_n$. (Hence $\mathcal{U}_n = \mathcal{U}_{n+1} = \cdots$, and $\mathcal{W}_n = \mathcal{W}_{n+1} = \cdots$.) We shall often denote $\mathcal{U}_n$ by $\mathcal{U}$ and $\mathcal{W}_n$ by $\mathcal{W}$.

Next we list some well-known elementary facts about images and inverse images for convenient reference.

FACT 3.2. (a) *In Notation* 3.1 *we have, for all subsets* $\mathcal{S}$ *and* $\mathcal{T}$ *of* $\mathcal{V}$ *and all subsets* $\mathcal{S}'$ *and* $\mathcal{T}'$ *of* $\mathcal{V}'$, *the following*:

$$A(A^{-1}\mathcal{S}')=\mathcal{S}'\cap A\mathcal{V},$$

$$A^{-1}(\mathcal{S}'\cap\mathcal{T}')=(A^{-1}\mathcal{S}')\cap(A^{-1}\mathcal{T}'),$$

$$A(\mathcal{S}\cap\mathcal{T})\subseteq(A\mathcal{S})\cap(A\mathcal{T}),$$

$$A^{-1}(A\mathcal{S})=\mathcal{S}+A^{-1}0,$$

$$A^{-1}(\mathcal{S}'+\mathcal{T}')\supseteq(A^{-1}\mathcal{S}')+(A^{-1}\mathcal{T}'),$$

$$A^{-1}[(A\mathcal{S})+\mathcal{T}']=\mathcal{S}+A^{-1}\mathcal{T}'.$$

(b) *In Notation* 3.1, *if* $\mathcal{V}'=\mathcal{V}^*$ *is the *-dual of* $\mathcal{V}$, $\mathcal{S}$ *and* $\mathcal{S}^*$ *are subsets of* $\mathcal{V}$ *and* $\mathcal{V}^*$ *respectively, and* $\mathcal{T}^*$ *is a* subspace *of* $\mathcal{V}^*$, *then*

$$(A^*\mathcal{S})^0=A^{-1}(\mathcal{S}^0),$$

$$(A^{-1}\mathcal{S}^*)^0\supseteq A^*(\mathcal{S}^{*0}),$$

$$(A^{-1}\mathcal{T}^*)^0=A^*(\mathcal{T}^{*0}).$$

LEMMA 3.3. *In Notation* 3.1 *we have* $\mathcal{U}\cap\mathcal{W}=0$ *iff* $\operatorname{rank}(A-tB)=n$ (*over the field* $F(t)$), *i.e., iff the pencil* $A-tB$ *has no column-minimal indices* [6, *Vol.* II, *p.* 38]. *Thus, when* $n=m$ *here,* $\mathcal{U}\cap\mathcal{W}=0$ *iff the pencil* $A-tB$ *is nonsingular.*

Proof. "If": Assume $\operatorname{rank}(A-tB)=n$ over $F(t)$, and let $x\in\mathcal{U}\cap\mathcal{W}$. Then there are vectors $x_0,x_1,\dots,x_{2n}$ in $\mathcal{V}$ with $x=x_n$, $x_i\in\mathcal{U}_i$, and $x_{2n-i}\in\mathcal{W}_i$ for all $i\leqslant n$, and $Ax_i=Bx_{i-1}$ for all $i\in\{1,2,\dots,2n\}$. Let

$$\tilde{x}=x_0+tx_1+\cdots+t^nx_n+\cdots+t^{2n-1}x_{2n-1}+t^{2n}x_{2n}.$$

Then $(A-tB)\tilde{x}=0$ over $F(t)$. Thus $\tilde{x}=0$ over $F(t)$ because $\operatorname{rank}(A-tB)=n$ over $F(t)$. Therefore $\tilde{x}=0$ over $F[t]$, and hence $x=x_n=0$.

"Only if": (This is essentially a reversal of steps in the "if" proof, but with complications.) Assume $\mathcal{U} \cap \mathcal{W} = 0$, and suppose $(A - tB)\tilde{x} = 0$ for some $n \times 1$ vector $\tilde{x}$ over $F(t)$. First, multiply the latter equation through by a "common denominator" in $F[t]$. Thus we may assume $\tilde{x}$ is over $F[t]$, and it suffices to prove that $\tilde{x}$ must be 0 over $F[t]$. To do this, pick out an arbitrary coefficient ($n \times 1$ over F) in $\tilde{x}$; call it x; then it suffices to show that x must be 0 (over F). Say $\tilde{x} = \cdots + t^l x + \cdots$ has degree d ($\geqslant l$). (If $\tilde{x} = 0$ here, there is nothing more to prove.) Then

$$t^{d-l}\tilde{x} = x_0 + tx_1 + \cdots + t^d x_d + \cdots + t^{2d-1}x_{2d-1} + t^{2d}x_{2d}$$

for some $n \times 1$ vectors $x_0, x_1, \ldots, x_{2d}$ over F with $x_d = x$. Thus $(A - tB)t^{d-l}\tilde{x} = 0$ over $F(t)$, and hence over $F[t]$, so $Ax_i = Bx_{i-1}$ for all $i \in \{1, 2, \ldots, 2d\}$, $Ax_0 = 0$, and $Bx_{2d} = 0$. Therefore $x_i \in \mathcal{U}_{i+1}$ and $x_{2d-i} \in \mathcal{W}_{i+1}$ for all $i \leqslant d$, so $x = x_d \in \mathcal{U}_{d+1} \cap \mathcal{W}_{d+1} \subseteq \mathcal{U} \cap \mathcal{W}$, and hence x must indeed be 0. ∎

REMARK. By the same method used in proving the "if" part of Lemma 3.3, one can show that $\mathcal{U}_1 \cap \mathcal{W} = 0$ implies $\mathcal{U} \cap \mathcal{W} = 0$ (as does $\mathcal{U} \cap \mathcal{W}_1 = 0$). Thus we have $\mathcal{U} \cap \mathcal{W} = 0$ iff $\mathcal{U}_1 \cap \mathcal{W} = 0$ iff $\mathcal{U} \cap \mathcal{W}_1 = 0$; this yields two more criteria, which are less symmetric than $\mathcal{U} \cap \mathcal{W} = 0$ but computationally simpler, for the pencil $A - tB$ to have rank n.

NOTATION 3.4. From now on, in Notation 3.1 we shall be taking $\mathcal{V}' = \mathcal{V}^* =$ the *-dual of $\mathcal{V}$ (and hence $m = n$), $A = S$, $-B = S^*$ (= the conjugate transpose of S). We shall be dealing simultaneously with all admissible pairs (F, E), both simplic and complic, and shall take t as in indeterminate over E as well as over F. (See Sec. 1 for terminology.) Thus $\mathcal{U}_i = (S^{-1}S^*)^i 0$, $\mathcal{W}_i = (S^{*-1}S)^i 0$, $\mathcal{U} = \mathcal{U}_n$, and $\mathcal{W} = \mathcal{W}_n$.

LEMMA 3.5. *In Notation* 3.4 *we have* (*for all nonnegative integers* i *and* j)

(a) $S\mathcal{U}_{i+1} = (S^*\mathcal{U}_i) \cap (S\mathcal{V})$, $S^*\mathcal{W}_{i+1} = (S\mathcal{W}_i) \cap (S^*\mathcal{V})$;
(b) $S^*\mathcal{U}_i = (S^*S^{-1})^i 0$, $S\mathcal{W}_i = (SS^{*-1})^i 0$;
(c) $\mathcal{U}_i^0 = (S^*S^{-1})^i \mathcal{V}^*$, $\mathcal{W}_i^0 = (SS^{*-1})^i \mathcal{V}^*$;
(d) $(S^*\mathcal{U}_i)^0 = (S^{-1}S^*)^i \mathcal{V}$, $(S\mathcal{W}_i)^0 = (S^{*-1}S)^i \mathcal{V}$;
(e) $S^*\mathcal{U}_i \subseteq \mathcal{U}_i^0$, $S\mathcal{W}_i \subseteq \mathcal{W}_i^0$;
(f) $S(\mathcal{U}_{i+1} \cap \mathcal{W}_j) = (S^*\mathcal{U}_i) \cap (S\mathcal{W}_j) = S^*(\mathcal{U}_i \cap \mathcal{W}_{j+1})$;
(g) $\mathcal{U} \cap (S\mathcal{W})^0 = \mathcal{U} \cap \mathcal{W} = (S^*\mathcal{U})^0 \cap \mathcal{W}$;
(h) $\mathcal{U} + (S\mathcal{W})^0 = (S^*\mathcal{U})^0 + (S\mathcal{W})^0 = (S^*\mathcal{U})^0 + \mathcal{W}$;
(i) $(\mathcal{U} + \mathcal{W}) \cap [S^*\mathcal{U} + S\mathcal{W}]^0 = \mathcal{U} \cap \mathcal{W}$; *and*
(j) $(\mathcal{U} + \mathcal{W}) + [S^*\mathcal{U} + S\mathcal{W}]^0 = [(S^*\mathcal{U}) \cap (S\mathcal{W})]^0$.

Proof. (a): We have $S\mathcal{U}_{i+1}=SS^{-1}S^*\mathcal{U}_i=(S^*\mathcal{U}_i)\cap S\mathcal{V}$ by Fact 3.2(a). [The rest of (a) is proved similarly.]

(b) follows from definition, rearrangement of parentheses, and the fact that $S^*0=0=S0$.

(c) follows from definition and recursive use of Fact 3.2(b).

(d) follows from (b) and recursive use of Fact 3.2(b).

(e) follows trivially from (b) and (c).

(f): It suffices to show $(S^*\mathcal{U}_i)\cap(S\mathcal{W}_j)\subseteq S(\mathcal{U}_{i+1}\cap\mathcal{W}_j)$, since the reverse inclusion follows routinely from (a) [and since the rest of (f) is proved in like fashion]. Thus let $x\in\mathcal{W}_j$ with $Sx\in S^*\mathcal{U}_i$. Then $x\in(S^{-1}S^*\mathcal{U}_i)\cap\mathcal{W}_j=\mathcal{U}_{i+1}\cap\mathcal{W}_j$, so $Sx\in S(\mathcal{U}_{i+1}\cap\mathcal{W}_j)$, as required.

(g): In view of (e), it suffices to show that $(S\mathcal{W})^0\cap\mathcal{U}\subseteq\mathcal{W}$ [the rest of (g) is proved similarly]. Thus let $x\in(S\mathcal{W})^0\cap\mathcal{U}$. Since $x\in(S\mathcal{W})^0=(S^{*-1}S)^n\mathcal{V}$, there are vectors $x_1,x_2,\dots,x_n$ in $\mathcal{V}$ such that

$$S^*x=Sx_n,\qquad S^*x_n=Sx_{n-1},\dots,\qquad S^*x_2=Sx_1.$$

Thus $x_1\in(S^{-1}S^*)x_2\subseteq\cdots\subseteq(S^{-1}S^*)^n x\subseteq(S^{-1}S^*)^n\mathcal{U}=\mathcal{U}$, so we have recursively (imagine $x=x_{n+1}$)

$$x_2\in S^{*-1}Sx_1\subseteq S^{*-1}S\mathcal{U}=S^{*-1}S\mathcal{U}_n\subseteq S^{*-1}(S^*\mathcal{U}_{n-1})$$

$$=\mathcal{U}_{n-1}+S^{*-1}0=\mathcal{U}_{n-1}+\mathcal{W}_1,$$

$$x_3\in S^{*-1}Sx_2\subseteq S^{*-1}[S(\mathcal{U}_{n-1}+\mathcal{W}_1)]\subseteq S^{*-1}[S^*\mathcal{U}_{n-2}+S\mathcal{W}_1]$$

$$=\mathcal{U}_{n-2}+S^{*-1}S\mathcal{W}_1=\mathcal{U}_{n-2}+\mathcal{W}_2,$$

$$\vdots$$

$$x\in S^{*-1}Sx_n\subseteq\mathcal{U}_0+\mathcal{W}_n=\mathcal{W},$$

as required.

(h) can be proved by use of (g) as follows. In view of (e), it suffices to show that $(S\mathcal{W})^0+\mathcal{U}\supseteq(S^*\mathcal{U})^0$ (etc.), i.e., that $(S\mathcal{W})\cap\mathcal{U}^0\subseteq S^*\mathcal{U}$. Thus let $x\in\mathcal{W}$ with $Sx\in\mathcal{U}^0$, i.e., $x\in[S^{-1}(\mathcal{U}^0)]\cap\mathcal{W}=(S^*\mathcal{U})^0\cap\mathcal{W}$; then $x\in\mathcal{U}$ by (g). Therefore $Sx\in S\mathcal{U}$, and hence $Sx\in S^*\mathcal{U}$ by (a).

(i): We apply the modular law (for the lattice of subspaces of $\mathcal{V}$) twice [using (e) each time]:

$$(\mathcal{U}+\mathcal{W})\cap(S^*\mathcal{U})^0\cap(S\mathcal{W})^0=\{\mathcal{U}+[\mathcal{W}\cap(S^*\mathcal{U})^0]\}\cap(S\mathcal{W})^0$$

$$=[\mathcal{U}\cap(S\mathcal{W})^0]+[\mathcal{W}\cap(S^*\mathcal{U})^0].$$

Now the required result follows from (g).

(j): Take annihilators on both sides. Then the result follows from (e) and (h) the same way (i) follows from (e) and (g). ■

LEMMA 3.6. *In Notation 3.4 the nullspace of* $S-\alpha S^*$ *is a subspace of* $[S^*\mathcal{U}+S\mathcal{W}]^0=(S^*\mathcal{U})^0\cap(S\mathcal{W})^0$ *for each nonzero* $\alpha\in F$.

Proof. Let $(S-\alpha S^*)x=0$ with $\alpha\neq 0$. We prove $x\in(S^*\mathcal{U})^0$ [the proof that $x\in(S\mathcal{W})^0$ is similar, so we omit it]. Here $Sx=\alpha S^*x$ and $S^*x=\alpha^{-1}Sx$, so for each $\mathcal{S}\subseteq\mathcal{V}$, $x\in(S\mathcal{S})^0$ iff $x\in(S^*\mathcal{S})^0$. Thus for every integer $i\geqslant 0$ we have

$$x\in(S^*\mathcal{U}_i)^0 \quad\Rightarrow\quad x\in(S\mathcal{U}_{i+1})^0 \quad\Rightarrow\quad x\in(S^*\mathcal{U}_{i+1})^0,$$

the first implication coming from Lemma 3.5(a). Applying the composite implication successively for $i=0,1,2,\cdots,n-1$, we get $x\in(S^*\mathcal{U}_n)^0=(S^*\mathcal{U})^0$, since certainly $x\in\mathcal{V}=(S^*\mathcal{U}_0)^0$.

THEOREM 3.7. *If* S *is an* $n\times n$ *matrix over* F *and the pencil* $S+tS^*$ *is nonsingular (over* $F(t)$*), then* S *is *-congruent to* $S_0\oplus S_1$*, where* $S_0-S_0^*$ *and* S_1 *are nonsingular and all roots of the matrix* $(S_0-S_0^*)^{-1}S_0$ *are in the set* $\{0,1\}$*; in fact,* S_0 *can be chosen here in the form*

$$S_0=\bigoplus_{j=1}^{n}\bigoplus_{i=1}^{k(j)}\begin{bmatrix}0 & J_j\\ I_j & 0\end{bmatrix}.$$

An alternate form for S_0 *here is* $\bigoplus_{j=1}^{n}\bigoplus_{i=1}^{k(j)}J_{2j}$.

Proof. In Notation 3.4 we have by Lemma 3.3 that $\mathcal{U}\cap\mathcal{W}=0$. Thus by Lemma 3.5(f), (i), (j) we have

$$\mathcal{V}=(\mathcal{U}\oplus\mathcal{W})\oplus[S^*\mathcal{U}+S\mathcal{W}]^0.$$

It is trivial [by Lemma 3.5(a)] to verify that this direct sum (at the second $\oplus$) is S-orthogonal and S^*-orthogonal; i.e., that $S(\mathcal{U}\oplus\mathcal{W})$ and $S^*(\mathcal{U}\oplus\mathcal{W})$ are both included in $S^*\mathcal{U}+S\mathcal{W}=[S^*\mathcal{U}+S\mathcal{W}]^{00}$. Since $[S^*\mathcal{U}+S\mathcal{W}]^0$ includes $\operatorname{nsp}(S-S^*)$ by Lemma 3.6 and $\mathcal{U}\oplus\mathcal{W}$ includes $\mathcal{U}_1=\operatorname{nsp}S$ and $\mathcal{W}_1=\operatorname{nsp}S^*$, the map $S-S^*$ (of $\mathcal{V}$ into $\mathcal{V}^*$) is nonsingular on $\mathcal{U}\oplus\mathcal{W}$ and the maps S and S^* are nonsingular on $[S^*\mathcal{U}+S\mathcal{W}]^0$. Thus we take S_0 as a matrix of the $(\mathcal{U}\oplus\mathcal{W})$-restriction of S and take S_1 as a matrix of the

$[S^*\mathcal{U}+S\mathcal{W}]^0$-restriction of S. To complete the proof, it suffices to prove the following four lemmas (Lemmas 3.8–3.11).

LEMMA 3.8. *In Notation* 3.4 *let* $\mathcal{V}=\mathcal{U}\oplus\mathcal{W}$ *and let* $K=S-S^*$. *Then* K *is nonsingular, and the map* $A=K^{-1}S$ *(of* $\mathcal{V}$ *into* $\mathcal{V}$*) has the following properties*:

(a) $S(I-A)=A^*S$, $S^*(I-A)=A^*S^*$, $K(I-A)=A^*K$;
(b) $A\mathcal{U}_{i+1}\subseteq\mathcal{U}_i$ *and* $(A-I)\mathcal{W}_{i+1}\subseteq\mathcal{W}_i$ *for all* $i\geqslant 0$;
(c) A *is nilpotent on* $\mathcal{U}$, *and* $A-I$ *is nilpotent on* $\mathcal{W}$; *and*
(d) A *is similar to* $I-A$ *(on* $\mathcal{V}$*)*.

Proof. By Lemma 3.6 the nullspace of K is included in $[S^*\mathcal{U}+S\mathcal{W}]^0$, which by Lemma 3.5(i) is 0 here; thus K is nonsingular.

(a): Note that $SA+A^*S=SK^{-1}S-S^*K^{-1}S=KK^{-1}S=S$ (because $K^*=-K$ and $A^*=-S^*K^{-1}$), so $S^*=(SA+A^*S)^*=S^*A+A^*S^*$ and $K=KA+A^*K$.

(b): By Lemma 3.5(a) we have

$$A\mathcal{U}_{i+1}=K^{-1}S\mathcal{U}_{i+1}\subseteq K^{-1}S^*\mathcal{U}_i=(A-I)\mathcal{U}_i$$

for all $i\geqslant 0$, so $A\mathcal{U}_1\subseteq(A-I)\mathcal{U}_0=0$ and we have $A\mathcal{U}_{i+1}\subseteq(A-I)\mathcal{U}_i\subseteq A\mathcal{U}_i+\mathcal{U}_i\subseteq\mathcal{U}_{i-1}+\mathcal{U}_i=\mathcal{U}_i$ successively for $i=1,2,\dots$. The other part of (b) is proved in like fashion,

(c) follows routinely from (b), since $\mathcal{U}_0=0=\mathcal{W}_0$.

(d): By (c) all roots of A are in $\{0,1\}$, and hence A is similar to A^* (e.g., because A and A^* must have the same Jordan form). But $A^*=K(I-A)K^{-1}$ is similar to $I-A$ by (a); thus A is similar to $I-A$. ■

We shall need a more algebraic condition equivalent to the "geometric" condition $\mathcal{V}=\mathcal{U}\oplus\mathcal{W}$ occurring in the hypothesis of Lemma 3.8. This is supplied by the following lemma.

LEMMA 3.9. *In Notation* 3.4 *let* $\mathcal{U}\cap\mathcal{W}=0$ *and* $K=S-S^*$ *and, when* K *is nonsingular,* $A=K^{-1}S$. (a) *If* (K *is nonsingular and*) $(A^2-A)^m=0$ *here, then* $(S^*\mathcal{U})^0\subseteq\operatorname{nsp}A^m$ *and* $(S\mathcal{W})^0\subseteq\operatorname{nsp}(A-I)^m$. (b) *The following three statements are equivalent to each other here*:

(i) (K *is nonsingular and*) A^2-A *is nilpotent*;
(ii) $([S^*\mathcal{U}+S\mathcal{W}]^0=)\ (S^*\mathcal{U})^0\cap(S\mathcal{W})^0=0$;
(iii) $\mathcal{V}=\mathcal{U}+\mathcal{W}$ $(=\mathcal{U}\oplus\mathcal{W}$ *here*).

Proof. (a): Since $\dim \mathcal{V} = n$ in Notation 3.4, it suffices to assume $m = n$ here. Thus assume $(A^2 - A)^n = 0$; we prove $(S^*\mathcal{U})^0 \subseteq \operatorname{nsp} A^n$. [We omit the proof of $(S\mathcal{W})^0 \subseteq \operatorname{nsp}(A-I)^n$, which is very similar.] Since $S = KA$ and $S^* = K(A-I)$, Lemma 3.5(d) tells us that $(S^*\mathcal{U})^0 = (S^{-1}S^*)^n\mathcal{V} = [A^{-1}K^{-1}K(A-I)]^n\mathcal{V} = [A^{-1}(A-I)]^n\mathcal{V}$. In what follows we shall write $A^{-j}0$ for $\operatorname{nsp} A^j$ [and $(A-I)^{-j}0$ for $\operatorname{nsp}(A-I)^j$] for $j \geqslant 0$, a notation consistent with Notation 3.1. The proof of (a) will be completed by showing that $[A^{-1}(A-I)]^i\mathcal{V} \subseteq A^{-n}0 + (A-I)^{i-n}0$ for $0 \leqslant i \leqslant n$, which we do by induction on i. Clearly the assertion is true for $i = 0$. In the induction step below, we assume $0 \leqslant i < n$ and use the following facts (besides our induction assertion): $(A-I)A^{-n}0 \subseteq A^{-n}0$ [because $A^{-n}0$ is A-invariant]; $(A-I)(A-I)^{i-n}0 \subseteq (A-I)^{i+1-n}0$ [obviously]; $(A-I)^{-j}0 = A(A-I)^{-j}0$ if $j = n-i-1$ [because $(A-I)^{-j}0$ is A-invariant and A is nonsingular on it]; $A^{-1}(A^{-n}0) = A^{-n}0$ [because $\dim\mathcal{V} = n$]; and the last formula in Fact 3.2(a):

$$
\begin{aligned}
\left[A^{-1}(A-I)\right]^{i+1}\mathcal{V} &\subseteq \left[A^{-1}(A-I)\right]\left[A^{-n}0 + (A-I)^{i-n}0\right] \\
&\subseteq A^{-1}\left[A^{-n}0 + (A-I)^{i+1-n}0\right] \\
&= A^{-1}\left[A^{-n}0 + A(A-I)^{i+1-n}0\right] \\
&= A^{-n}0 + (A-I)^{i+1-n}0.
\end{aligned}
$$

To prove (b), we note that (i)⇒(ii) follows immediately from (a), that (ii)⇒(iii) follows immediately from Lemma 3.5(f), (j) (plus our hypothesis that $\mathcal{U} \cap \mathcal{W} = 0$), and that (iii)⇒(i) follows from Lemma 3.8. ■

Remark on Lemma 3.9(a). It is easy to show that the inclusions here are actually equalities, in fact, that $\operatorname{nsp} A^i = \mathcal{U}_i$ and $\operatorname{nsp}(A-I)^i = \mathcal{W}_i$ for all $i \geqslant 0$. We shall not need this, however.

Lemma 3.10. *In Lemma 3.8 the minimum polynomial of A is $(t^2 - t)^m$ for some m, and there are two m-dimensional A-cyclic subspaces $\mathcal{V}_0$ and $\mathcal{V}_1$ of $\mathcal{V}$ such that $\mathcal{V}_0 \subseteq \mathcal{U}$, $\mathcal{V}_1 \subseteq \mathcal{W}$ (and hence $\mathcal{V}_0 \cap \mathcal{V}_1 = 0$), and*

(a) $(\mathcal{V}_0 + \mathcal{V}_1) \cap [K(\mathcal{V}_0 + \mathcal{V}_1)]^0 = 0$,
(b) $S(\mathcal{V}_0 + \mathcal{V}_1) \subseteq K(\mathcal{V}_0 + \mathcal{V}_1)$,
(c) $S^*(\mathcal{V}_0 + \mathcal{V}_1) \subseteq K(\mathcal{V}_0 + \mathcal{V}_1)$.
(d) *Also there is a basis for $\mathcal{V}_0 + \mathcal{V}_1$ in which (the restriction of) S has matrix $\begin{bmatrix} 0 & J \\ I & 0 \end{bmatrix}$ (with $I = I_m$ and $J = J_m$).*
(e) *There is also a basis for $\mathcal{V}_0 + \mathcal{V}_1$ in which S has matrix J_{2m}.*

Proof. Every invariant factor of A is a power of t^2-t by Lemma 3.8(d), so the minimum polynomial is $(t^2-t)^m$ for some m. Also $\mathcal{U}=(A-I)^m\mathcal{U}=(A-I)^m(\mathcal{U}+\mathcal{W})$ by Lemma 3.8(c), so $A^{m-1}\mathcal{U}=A^{m-1}(A-I)^m\mathcal{V}\neq 0$. Thus there is an $e\in\mathcal{U}$ such that $A^{m-1}e\neq 0$. Then $(A^2-A)^{m-1}e\neq 0$ because $A-I$ is nonsingular on $\mathcal{U}$ and $A^{m-1}e\in\mathcal{U}$. Thus there is a vector $f_1\in\mathcal{V}$ such that $f_1^*K(A^2-A)^{m-1}e=-1$. Write $f_1=f+f_0$ with $f\in\mathcal{W}$ and $f_0\in\mathcal{U}$. Then $f_0^*K(A^2-A)^{m-1}e\in f_0^*K\mathcal{U}=0$ because $K\mathcal{U}\subseteq\mathcal{U}^0$ by Lemma 3.5(e), (a), so $f^*K(A^2-A)^{m-1}e=-1$. Let $\mathcal{V}_0$ and $\mathcal{V}_1$ be the respective A-cyclic subspaces generated by e and f. Then $\mathcal{V}_0\subseteq\mathcal{U}$ and $\mathcal{V}_1\subseteq\mathcal{W}$, and (b) and (c) hold because $S=KA$ and $S^*=K(A-I)$.

To prove (a), let $x\in\mathcal{V}_0$ and $y\in\mathcal{V}_1$ and $(x+y)^*K(\mathcal{V}_0+\mathcal{V}_1)=0$. Then $0=x^*K\mathcal{V}_1+y^*K\mathcal{V}_0$ because $K\mathcal{V}_0\subseteq K\mathcal{U}\subseteq\mathcal{U}^0$ and $K\mathcal{V}_1\subseteq\mathcal{W}^0$ by Lemma 3.5(e), (a). Thus $x^*K\mathcal{V}_1=0=y^*K\mathcal{V}_0$ (e.g., because $0\in y^*K\mathcal{V}_0$). It suffices to show that x and y must be 0. Say $x=A^lp(A)e$ with $l\geqslant 0$ and $p(t)\in F[t]$ and $p(0)\neq 0$. Then $\bar{p}(I-A)$ is nonsingular on $\mathcal{W}$, and hence $\bar{p}(I-A)\mathcal{V}_1=\mathcal{V}_1$. Thus

$$\begin{aligned}0=x^*K\mathcal{V}_1&=e^*\bar{p}(A^*)A^{*l}K\mathcal{V}_1=e^*A^{*l}K\bar{p}(I-A)\mathcal{V}_1\\&=e^*A^{*l}K\mathcal{V}_1\supseteq e^*A^{*l}K(I-A)^iA^{m-1}f\\&=e^*(A^*)^{l+i}(I-A^*)^{m-1}Kf=\left[f^*K^*(I-A)^{m-1}A^{l+i}e\right]^*\end{aligned}$$

for all $i\geqslant 0$, so $l\geqslant m$ because $K^*=-K$ (and hence, if $l\leqslant m-1$, we would have a contradiction for $i=m-1-l$). Thus $x=0$, and $y=0$ in like fashion.

To see that (d) and (e) are equivalent, let P be the $2m\times 2m$ permutation matrix whose columns are, in order, $e_1,e_3,\dots,e_{2m-1},e_2,e_4,\dots,e_{2m}$ (where $e_1,e_2,\dots,e_{2m}$ are, in order, the columns of I_{2m}). Then

$$P^*=P^{-1}\quad\text{and}\quad P^*J_{2m}P=\begin{bmatrix}0 & J\\ I & 0\end{bmatrix}.$$

To prove (d), first note that $S\mathcal{V}_0\subseteq S\mathcal{U}\subseteq\mathcal{U}^0\subseteq\mathcal{V}_0^0$ and $S\mathcal{V}_1\subseteq\mathcal{V}_1^0$, so the two $m\times m$ blocks on the diagonal of the matrix of S will be 0 if we choose a basis for $\mathcal{V}_1\oplus\mathcal{V}_0$ by adjoining a basis for $\mathcal{V}_0$ to a basis for $\mathcal{V}_1$. In $\mathcal{V}_1$ we choose the $[A^{-1}(A-I)]$-anticyclic basis

$$\left\{\left[A^{-1}(A-I)\right]^if\right\}$$

where

$$i=m-1,\dots,2,1,0\quad\text{(in order)}$$

and in $\mathcal{V}_0$ we choose the $[(A-I)^{-1}A]$-cyclic basis

$$\{[(A-I)^{-1}A]^i e\}$$

where

$$i=0,1,2,\ldots,m-1 \quad \text{(in order)}.$$

Then in the resulting basis for $\mathcal{V}_1 \oplus \mathcal{V}_0$ (in that order) the matrix of A is

$$(I-J^*)^{-1} \oplus [J(J-I)^{-1}]$$

because the matrix of the $\mathcal{V}_0$-restriction of $A(A-I)^{-1}$ is J and the matrix of the $\mathcal{V}_1$-restriction of $A^{-1}(A-I)$ is J^*. Let the matrix of the $(\mathcal{V}_1 \oplus \mathcal{V}_0)$-restriction of S in this basis be $\begin{bmatrix} 0 & T \\ R & 0 \end{bmatrix}$. Then, since $S=KA=(S-S^*)A$, we have

$$\begin{bmatrix} 0 & T \\ R & 0 \end{bmatrix} = \begin{bmatrix} 0 & (T-R^*)J(J-I)^{-1} \\ (R-T^*)(I-J^*)^{-1} & 0 \end{bmatrix},$$

so $T=R^*J$. Also $R(I-J^*)=R-T^*$ is nonsingular (because $K=S-S^*$ is) and hence so is R. Thus the matrix of the $(\mathcal{V}_1 \oplus \mathcal{V}_0)$-restriction of S is

$$\begin{bmatrix} 0 & R^*J \\ R & 0 \end{bmatrix} = \begin{bmatrix} R^* & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & J \\ I & 0 \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & I \end{bmatrix},$$

which is *-congruent to $\begin{bmatrix} 0 & J \\ I & 0 \end{bmatrix}$, as required for (d). ■

LEMMA 3.11. *In Lemmas* 3.8 *and* 3.10 *there is a basis in which the matrix of S is in the form* $\oplus_{j=1}^{n} \oplus_{i=1}^{k(j)} J_{2j}$.

Proof. By Lemma 3.10(a), $\mathcal{V}=(\mathcal{V}_0+\mathcal{V}_1)\oplus[K(\mathcal{V}_0+\mathcal{V}_1)]^0$, and by Lemma 3.10(b), (c), the direct sum is S-orthogonal and S^*-orthogonal. Thus S has matrix $J_{2m} \oplus W$ [by Lemma 3.10(e)], where W is the matrix of the $[K(\mathcal{V}_0+\mathcal{V}_1)]^0$-restriction of S. The proof is completed by an obvious induction, once we show that W satisfies the same hypotheses that S satisfies in Lemma 3.10 (except with order $<n=\text{order}\, S$). In Lemma 3.10 the hypothesis on S is just that $\mathcal{V}=\mathcal{U}\oplus\mathcal{W}$, and by Lemma 3.9(b) this is equivalent to

requiring that $S - S^*$ be nonsingular and

$$A^2 - A = A(A - I) = (S - S^*)^{-1} S (S - S^*)^{-1} S^*$$

be nilpotent. Clearly this implies that $W - W^*$ is nonsingular and

$$(W - W^*)^{-1} W (W - W^*)^{-1} W^*$$

is nilpotent, so the proof (of Lemma 3.11 and Theorem 3.7) is complete. ■

In Theorem 3.7 the *-congruence class of S (in fact, the $F[t]$-equivalence class of the pencil $S + tS^*$) determines that of S_0 by the fact that $k(j)$ is (for each j) just the number of degree-j e.d.'s of $S + tS^*$ at $(t=)$ 0. The next result gives a somewhat more direct way in which the *-congruence classes of S_0 and S_1 are both determined from that of S.

Corollary 3.12. *In Theorem* 3.7 *the *-congruence classes of* S_0 *and* S_1 *are determined from that of* S. *In fact, if* $C^*(S_0 \oplus S_1)C = T_0 \oplus T_1$ *with* $C, S_1, T_1, S_0 - S_0^*, T_0 - T_0^*$ *nonsingular and all roots of* $(S_0 - S_0^*)^{-1} S_0$ *and* $(T_0 - T_0^*)^{-1} T_0$ *in* $\{0, 1\}$, *then* $C = C_0 \oplus C_1$ *conformably with* $S_0 \oplus S_1$ (*and hence with* $T_0 \oplus T_1$).

Proof. We sketch two proofs. In both proofs we note that S_0 is *-congruent by Theorem 3.7 to a direct sum of neutrally partitioned matrices of the form $\begin{bmatrix} 0 & J \\ I & 0 \end{bmatrix}$ and hence to a neutrally partitioned matrix $\begin{bmatrix} 0 & N \\ I & 0 \end{bmatrix}$ with N nilpotent; thus we may assume

$$S_0 = \begin{bmatrix} 0 & N \\ I & 0 \end{bmatrix}$$

here is such a matrix, and likewise for T_0.

First proof. Let $P_i = S_i + tS_i^*$ and $Q_i = T_i + tT_i^*$ for $i \in \{0, 1\}$. Then [with respect to the pair $(F(t), E(t))$, which is admissible since $t = \bar{t}$ is an indeterminate over E]

$$P_0^{*-1} P_0 = \left[(N + tI)^{-1} (I + tN) \right]^* \oplus \left[(I + tN)^{-1} (N + tI) \right],$$

and hence all its roots are in $\{t, t^{-1}\}$, and likewise for $Q_0^{*-1} Q_0$. However, no roots of $P_1^{*-1} P_1$ are in $\{t, t^{-1}\}$ since the two matrices $P_1 - tP_1^* = (1 - t^2) S_1$ and $P_1 - t^{-1} P_1^* = (t - t^{-1}) S_1^*$ are nonsingular over $F(t)$; likewise for $Q_1^{*-1} Q_1$. Let $P = P_0 \oplus P_1$ and $Q = Q_0 \oplus Q_1$. Then $C^* P C = Q$ and $C^* P^* C = Q^*$, so

$C^{-1}P^{*-1}PC=Q^{*-1}Q$, and hence the direct sums $P_0^{*-1}P_0 \oplus P_1^{*-1}P_1$ and $Q_0^{*-1}Q_0 \oplus Q_1^{*-1}Q_1$ conform and

$$\left(P_0^{*-1}P_0 \oplus P_1^{*-1}P_1\right)C = C\left(Q_0^{*-1}Q_0 \oplus Q_1^{*-1}Q_1\right).$$

Thus $C=C_0 \oplus C_1$ conforms to $P_0 \oplus P_1$ (by [6, Vol. I, top of p. 220]).

Second proof. Let $S=S_0 \oplus S_1$, and take $\mathcal{V}, \mathcal{U}, \mathcal{W}$ as in Notation 3.4. Let $\mathcal{V}_0, \mathcal{V}_1$ be the coordinate subspaces of $\mathcal{V}$ corresponding respectively to S_0, S_1 in the direct sum $S_0 \oplus S_1$, and further let $\mathcal{V}_{01}, \mathcal{V}_{02}$ be the coordinate subspaces of $\mathcal{V}_0$ for which $\mathcal{V}_0 = \mathcal{V}_{01} \oplus \mathcal{V}_{02}$ conforms to

$$S_0 = \begin{bmatrix} 0 & N \\ I & 0 \end{bmatrix}.$$

Let x, y, z be column vectors for which the $n \times 1$ column vector $v=[x^* \;\; y^* \;\; z^*]^*$ conforms to $\mathcal{V}=\mathcal{V}_{01} \oplus \mathcal{V}_{02} \oplus \mathcal{V}_1$. Then $v \in \mathcal{U} = (S^{-1}S^*)^n 0$ iff there are vectors $v_i \in \mathcal{U}_i$ for $0 \leqslant i \leqslant n$ with $v=v_n$ and $Sv_i = S^* v_{i-1}$ for $1 \leqslant i \leqslant n$. Putting $v_i = [x_i^* \;\; y_i^* \;\; z_i^*]^*$ gives

$$\begin{bmatrix} 0 & N & 0 \\ I & 0 & 0 \\ 0 & 0 & S_1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} = \begin{bmatrix} Ny_i \\ x_i \\ S_1 z_i \end{bmatrix} = \begin{bmatrix} y_{i-1} \\ N^* x_{i-1} \\ S_1^* z_{i-1} \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ N^* & 0 & 0 \\ 0 & 0 & S_1^* \end{bmatrix} \begin{bmatrix} x_{i-1} \\ y_{i-1} \\ z_{i-1} \end{bmatrix}$$

for all $i \in \{1, \dots, n\}$. Since $v_0 \in \mathcal{U}_0 = 0$ here, this shows that $v \in \mathcal{U}$ iff $x=(N^*)^n 0 = 0$ and $z=(S_1^{-1}S_1^*)^n 0 = 0$. (Note that y is arbitrary here because $N^n=0$.) Thus $\mathcal{U} = \mathcal{V}_{02}$, and a like calculation shows that $\mathcal{W} = \mathcal{V}_{01}$. The same equations, with different side conditions [namely, $v_i \in (S^*\mathcal{U}_i)^0$ for all i] can be used to calculate that $v \in (S^*\mathcal{U})^0 = (S^{-1}S^*)^n \mathcal{V}$ iff $x=0$, and hence that $(S^*\mathcal{U})^0 = \mathcal{V}_{02} \oplus \mathcal{V}_1$; a like calculation shows that $(S\mathcal{W})^0 = \mathcal{V}_{01} \oplus \mathcal{V}_1$. Thus $\mathcal{V}_0 = \mathcal{U} \oplus \mathcal{W}$ and $\mathcal{V}_1 = [S^*\mathcal{U} + S\mathcal{W}]^0$. Next, let $T=T_0 \oplus T_1$, $\mathcal{X} = (T^{-1}T^*)^n 0$, $\mathcal{Y} = (T^{*-1}T)^n 0$, and let $\mathcal{V}_0', \mathcal{V}_1'$ be the coordinate subspaces corresponding respectively to T_0, T_1 in the direct sum $T_0 \oplus T_1$. Then $\mathcal{X} \oplus \mathcal{Y} = \mathcal{V}_0'$, $[T^*\mathcal{X} + T\mathcal{Y}]^0 = \mathcal{V}_1'$, by the same proof as above (that $\mathcal{U} \oplus \mathcal{W} = \mathcal{V}_0$, etc.). Furthermore, $C^{-1}(\mathcal{U} \oplus \mathcal{W}) = \mathcal{X} \oplus \mathcal{Y}$ and $C^{-1}[S^*\mathcal{U} + S\mathcal{W}]^0 = [T^*\mathcal{X} + T\mathcal{Y}]^0$; in fact, $C^{-1}\mathcal{U} = \mathcal{X}$, $C^{-1}\mathcal{W} = \mathcal{Y}$, $C^{-1}(S^*\mathcal{U})^0 = (T^*\mathcal{X})^0$, and $C^{-1}(S\mathcal{W})^0 = (T\mathcal{Y})^0$. For example,

$$C^{-1}(S^*\mathcal{U})^0 = C^{-1}(S^{-1}S^*)^n \mathcal{V} = \left[(C^{-1}S^{-1}C^{*-1})(C^*S^*C)\right]^n C^{-1}\mathcal{V}$$

$$= (T^{-1}T^*)^n \mathcal{V} = (T^*\mathcal{X})^0.$$

Thus $\mathcal{V}_0' = C^{-1}\mathcal{V}_0$ and hence $\dim \mathcal{V}_0' = \dim \mathcal{V}_0$, so $\mathcal{V}_0 = \mathcal{V}_0' = C^{-1}\mathcal{V}_0$ and likewise $\mathcal{V}_1 = \mathcal{V}_1' = C^{-1}\mathcal{V}_1$. Therefore $C = C_0 \oplus C_1$ conformably with $\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_1$. ■

REMARK 3.13. A pencil $A + tB$ is said to have a degree-j e.d. of multiplicity k at ($t=$) ∞ provided the pencil $B + tA$ has the e.d. t^j with multiplicity k [6, Vol. II, pp. 26–27]. Unsurprisingly, for a nonsingular pencil of the form $S + tS^*$ the degree-j e.d.'s at 0 and ∞ have the same multiplicity for each j, as can be seen from Theorem 3.7. The uniqueness assured by Corollary 3.12 enables us to add the following two statements to the three equivalent statements of Lemma 3.9(b):

(iv) *every e.d. of* $S + tS^*$ *is at* 0 *or* ∞;

(v) *the direct summand* S_1 *is missing in Corollary* 3.12 (*and Theorem* 3.7).

We conclude this section with an expected result for a neutral matrix S in Theorem 3.7.

COROLLARY 3.14. *If* S *is* (F,E)*-neutral in Theorem* 3.7, *then so is* S_1.

Proof. Here S is $*$-congruent to a neutrally partitioned matrix $\begin{bmatrix} 0 & N \\ M & 0 \end{bmatrix}$ (by Definition 2.9), so $S + tS^*$ is correspondingly $*$-congruent to

$$\begin{bmatrix} 0 & N + tM^* \\ M + tN^* & 0 \end{bmatrix}.$$

The pencil $M + tN^*$ is nonsingular (because $S + tS^*$ is), and so is F-equivalent to $(R + tT^*) \oplus (V + tW^*)$ with V and W nonsingular and every e.d. of $R + tT^*$ at 0 or ∞ (by [6, Vol. II, Theorem 3, p. 28]). Thus there are nonsingular matrices P and Q over F such that

$$Q^*(M + tN^*)P = (R + tT^*) \oplus (V + tW^*),$$

so $S + tS^*$ is $*$-congruent to

$$\begin{bmatrix} P^* & 0 \\ 0 & Q^* \end{bmatrix} \begin{bmatrix} 0 & N + tM^* \\ M + tN^* & 0 \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}$$

$$= \begin{bmatrix} 0 & (T + tR^*) \oplus (W + tV^*) \\ (R + tT^*) \oplus (V + tW^*) & 0 \end{bmatrix},$$

which in turn is *-congruent to

$$\begin{bmatrix} 0 & T+tR^* \\ R+tT^* & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & W+tV^* \\ V+tW^* & 0 \end{bmatrix}.$$

Therefore S is *-congruent to

$$\begin{bmatrix} 0 & T \\ R & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & W \\ V & 0 \end{bmatrix}$$

with every e.d. of

$$\begin{bmatrix} 0 & T \\ R & 0 \end{bmatrix} + t \begin{bmatrix} 0 & T \\ R & 0 \end{bmatrix}^*$$

at 0 or ∞ and $\begin{bmatrix} 0 & W \\ V & 0 \end{bmatrix}$ nonsingular, so S_1 (in Theorem 3.7) is *-congruent to $\begin{bmatrix} 0 & W \\ V & 0 \end{bmatrix}$ by Corollary 3.12 (and Remark 3.13), and hence S_1 is (F,E)-neutral. ∎

4. SINGULAR PENCILS $S+tS^*$

In this section we continue to use Notation 3.4 and the results, especially Lemma 3.5, of Secs. 2 and 3. In addition we shall use the following notation:

NOTATION 4.1. For each positive integer m we denote by $\dot{I}_m$ and $\dot{J}_m$ the following matrices [respectively $m\times(m+1)$ and $(m+1)\times m$]:

$$\dot{I}_m = \begin{bmatrix} I_m & 0_{m\times 1} \end{bmatrix}, \qquad \dot{J}_m = \begin{bmatrix} 0_{1\times m} \\ I_m \end{bmatrix}.$$

We shall often write $\dot{I}_m$ as $\dot{I}$ and $\dot{J}_m$ as $\dot{J}$ when m is understood from context. We also adopt the convention that the $(2m+1)\times(2m+1)$ matrix $\begin{bmatrix} 0 & \dot{J} \\ \dot{I} & 0 \end{bmatrix}$ is understood as 0 (the 1×1 zero matrix) when $m=0$.

We begin with two lemmas.

LEMMA 4.2. *In Notation* 3.4 *let* $\mathcal{U}_{i+1}\cap\mathcal{W}_{q-i}=0$ *for all* $i\in\{0,1,\dots,q-1\}$. *Then* (a) *the* $q+1$ *subspaces* $\mathcal{U}_{i+1}\cap\mathcal{W}_{q-i+1}$ *with* $i\in\{0,1,\dots,q\}$ *are* *independent and each has the same dimension, say* k. *Also*, (b) *the* q *subspaces* $S(\mathcal{U}_{i+1}\cap\mathcal{W}_{q-i+1})=S^*(\mathcal{U}_i\cap\mathcal{W}_{q-i+2})$ *with* $i\in\{1,2,\dots,q\}$ *are independent and each has dimension* k.

Proof. For the independence in (b), it suffices to show that if $x_i\in\mathcal{U}_{i+1}\cap\mathcal{W}_{q-i+1}$ for each $i\in\{1,2,\dots,q\}$ and $S(x_1+x_2+\cdots+x_q)=0$, then $x_1=x_2=\cdots=x_q=0$. So suppose $S(x_1+\cdots+x_q)=0$ with $x_i\in\mathcal{U}_{i+1}\cap\mathcal{W}_{q-i+1}$ for each i. Then $x_1+\cdots+x_q\in\mathcal{U}_1=S^{-1}0$. But $x_i\in\mathcal{W}_{q-i+1}\subseteq\mathcal{W}_q$ for each i, so $x_1+\cdots+x_q\in\mathcal{U}_1\cap\mathcal{W}_q=0$. Therefore $x_1=-(x_2+\cdots+x_q)$, which is in $\mathcal{U}_2$ (because x_1 is) and also in $\mathcal{W}_{q-1}$ (because $x_i\in\mathcal{W}_{q-i+1}\subseteq\mathcal{W}_{q-1}$ for each $i\geqslant 2$), and hence is $\in\mathcal{U}_2\cap\mathcal{W}_{q-1}=0$. Thus $x_1=0$ and $x_2=-(x_3+\cdots+x_q)$. Proceeding in this way, we clearly get $x_1=\cdots=x_q=0$ eventually.

Next, to prove the independence in (a), suppose $x_0+x_1+\cdots+x_q=0$ with $x_i\in\mathcal{U}_{i+1}\cap\mathcal{W}_{q-i+1}$ for all $i\geqslant 0$. Then $Sx_0=0$ (because $x_0\in\mathcal{U}_1$), so $S(x_1+\cdots+x_q)=0$. By the preceding paragraph, we must consequently have $x_1=\cdots=x_q=0$. Therefore also $x_0=0$.

Finally, to prove the equal dimensionality in (a) and (b), let $1\leqslant i\leqslant q$ and, for this fixed i, suppose $(y_1,\dots,y_k)$ is a basis for $\mathcal{U}_i\cap\mathcal{W}_{q-i+2}$. Since $S^*(\mathcal{U}_i\cap\mathcal{W}_{q-i+2})=S(\mathcal{U}_{i+1}\cap\mathcal{W}_{q-i+1})$ by Lemma 3.5(f), there are vectors $z_1,\dots,z_k$ in $\mathcal{U}_{i+1}\cap\mathcal{W}_{q-i+1}$ such that $S^*y_j=Sz_j$ for each j. To show that $Sz_1,\dots,Sz_k$ must be linearly independent here, suppose $\alpha_1Sz_1+\cdots+\alpha_kSz_k=0$ for some scalars α_j in F. Then $S^*(\alpha_1y_1+\cdots+\alpha_ky_k)=\alpha_1S^*y_1+\cdots+\alpha_kS^*y_k=0$, so $\alpha_1y_1+\cdots+\alpha_ky_k\in\mathcal{U}_q\cap\mathcal{W}_1$ (because $\mathcal{W}_1=S^{*-1}0$ and $y_j\in\mathcal{U}_i\subseteq\mathcal{U}_q$ for each j). But $\mathcal{U}_q\cap\mathcal{W}_1=0$ by hypothesis, so $\alpha_1=\cdots=\alpha_k=0$ by the linear independence of $(y_1,\dots,y_k)$. Then $(Sz_1,\dots,Sz_k)$ is linearly independent. This shows

$$\dim(\mathcal{U}_i\cap\mathcal{W}_{q-i+2})\leqslant\dim S(\mathcal{U}_{i+1}\cap\mathcal{W}_{q-i+1}),$$

which in turn is obviously $\leqslant\dim(\mathcal{U}_{i+1}\cap\mathcal{W}_{q-i+1})$. A like argument shows

$$\begin{aligned}\dim(\mathcal{U}_{i+1}\cap\mathcal{W}_{q-i+1})&\leqslant\dim S^*(\mathcal{U}_i\cap\mathcal{W}_{q-i+2})\\&\leqslant\dim(\mathcal{U}_i\cap\mathcal{W}_{q-i+2}).\end{aligned}$$

These two chains of inequalities, and hence the corresponding equalities, hold for every $i\in\{1,\dots,q\}$, which proves the equal dimensionality in (a) and (b). ■

The major work of this section is concentrated in the following lemma.

LEMMA 4.3. *In Notation* 3.4 *suppose* $\mathcal{U}\cap\mathcal{W}\neq 0$, *and let* $q=0$ *if* $\mathcal{U}_1\cap\mathcal{W}_1\neq 0$, *and otherwise let* q *be the largest integer satisfying*

$$\mathcal{U}_1\cap\mathcal{W}_q=\mathcal{U}_2\cap\mathcal{W}_{q-1}=\cdots=\mathcal{U}_q\cap\mathcal{W}_1=0.$$

Then S *is* $*$*-congruent to* $S_0\oplus S_1$ *with*

$$S_0=\begin{bmatrix}0 & \dot{J}\\ \dot{I} & 0\end{bmatrix},$$

where $\dot{I}=\dot{I}_q$ *and* $\dot{J}=\dot{J}_q$ (*see Notation* 4.1).

Proof. Since $\mathcal{U}_n\cap\mathcal{W}_n=\mathcal{U}\cap\mathcal{W}\neq 0$, the integer q is well defined. By Lemma 4.2 there is an integer $k=\dim(\mathcal{U}_1\cap\mathcal{W}_{q+1})=\cdots=\dim(\mathcal{U}_{q+1}\cap\mathcal{W}_1)$, and $k>0$ by the definition of q. Thus there is a nonzero vector $x_0\in\mathcal{U}_1\cap\mathcal{W}_{q+1}$. Therefore there are also vectors $x_1,\dots,x_q$ in $\mathcal{V}$ such that

$$0=Sx_0,\qquad S^*x_0=Sx_1,\qquad S^*x_1=Sx_2,\dots,$$

$$S^*x_{q-1}=Sx_q,\qquad S^*x_q=0.$$

Consequently $x_i\in\mathcal{U}_{i+1}\cap\mathcal{W}_{q-i+1}$ for all $i\in\{0,1,\dots,q\}$. (We also take $x_{q+1}=0$ when it occurs later). Also each $x_i\neq 0$ for $0\leqslant i\leqslant q$, as otherwise, if $x_i=0$ for some index $i\leqslant q$, then x_0 would be $\in\mathcal{U}_1\cap\mathcal{W}_i\subseteq\mathcal{U}_1\cap\mathcal{W}_q=0$. Thus $(x_0,x_1,\dots,x_q)$ is linearly independent, because by Lemma 4.2 the $q+1$ subspaces $\mathcal{U}_1\cap\mathcal{W}_{q+1},\dots,\mathcal{U}_{q+1}\cap\mathcal{W}_1$ are independent. Also, by Lemma 3.5(f) and (e) we have

$$Sx_j\in S(\mathcal{U}_{j+1}\cap\mathcal{W}_{q-j+1})=S^*\mathcal{U}_j\cap S\mathcal{W}_{q-j+1}\subseteq\mathcal{U}_j^0\cap\mathcal{W}_{q-j+1}^0$$

for all $j\leqslant q$, whereas $x_i\in\mathcal{U}_{i+1}\cap\mathcal{W}_{q-i+1}\subseteq\mathcal{U}_j+\mathcal{W}_{q-j+1}$ for all i and j in $\{0,1,\dots,q\}$ (to see the latter inclusion, consider separately the two cases $i<j$ and $i\geqslant j$). Thus $x_i^*Sx_j=0$ for all i and j.

Next, $x_0\notin\mathcal{W}_q=S^{*-1}S\mathcal{W}_{q-1}$ (because $0\neq x_0\in\mathcal{U}_1$ and $\mathcal{U}_1\cap\mathcal{W}_q=0$), so $Sx_1=S^*x_0\notin S\mathcal{W}_{q-1}$, and hence there is a vector z_1 in $\mathcal{V}$ such that $z_1^*Sx_1=1$ and $z_1^*S\mathcal{W}_{q-1}=0$. (If $q=0$, this paragraph and the next are vacuous.) Choose such a vector z_1; then $z_1\in S(\mathcal{W}_{q-1})^0$, which $=(S^{*-1}S)^{q-1}\mathcal{V}$ by Lemma 3.5(d), so there are vectors $z_2,\dots,z_q$ with $z_i\in(S^{*-1}S)^{q-i}\mathcal{V}$ for all i

and $S^*z_i = Sz_{i+1}$ for all $i<q$. Thus by Lemma 3.5(c) we have

$$S^*z_i \in S^*(S^{-1}S^*)^{i-1}z_1 \subseteq (S^*S^{-1})^i\mathcal{V}^* = \mathcal{U}_i^0 \qquad \text{for all } i,$$

$$S^*z_i \in S(S^{*-1}S)^{q-i-1}z_q \subseteq (SS^{*-1})^{q-i}\mathcal{V}^* = \mathcal{W}_{q-i}^0$$

$$\text{for all} \quad i<q,$$

and of course $S^*z_q \in \mathcal{V}^* = \mathcal{W}_{q-q}^0$, so $S^*z_i \in \mathcal{U}_i^0 \cap \mathcal{W}_{q-i}^0$ for all $i \leqslant q$. Hence $z_i^* Sx_j = (x_j^* S^* z_i)^* = 0$ for all $i \neq j$ (consider separately the two cases $i>j$ and $i<j$, using $x_j \in \mathcal{U}_{j+1} \cap \mathcal{W}_{q-j+1}$). Furthermore $z_i^* Sx_i = z_i^* S^* x_{i-1} = z_{i-1}^* Sx_{i-1} = \cdots = z_1^* Sx_1 = 1$; thus $z_i^* Sx_j = \delta_{ij}$ (= the Kronecker delta) for all i and j with $1 \leqslant i \leqslant q$ and $0 \leqslant j \leqslant q$.

Next, let $\alpha_i = z_q^* Sz_{q-i+1}$ for all $i \in \{1,2,\ldots,q\}$ and

$$y_k = z_k - \alpha_1 x_k - \alpha_2 x_{k+1} - \cdots - \alpha_{q-k+1} x_q$$

for each $k \in \{1,\ldots,q\}$. Then $y_1^* Sx_1 = z_1^* Sx_1 = 1$ (because $x_i^* Sx_1 = 0$ for all i), and $y_1^* S\mathcal{W}_{q-1} = z_1^* S\mathcal{W}_{q-1} - 0 = 0$ because $S^*x_i \in S^*\mathcal{W}_{q-i+1} \subseteq S\mathcal{W}_{q-i}$ [by Lemma 3.5(a)], which is $\subseteq S\mathcal{W}_{q-1} \subseteq \mathcal{W}_{q-1}^0$ if $i \geqslant 1$ [by Lemma 3.5(e)]. Also

$$S^*y_k = S^*z_k - \sum_{i=k}^{q} \alpha_{i-k+1} S^* x_i$$

$$= Sz_{k+1} - \sum_{i=k}^{q} \alpha_{i-k+1} Sx_{i+1} = Sy_{k+1}$$

for all $k \in \{1,2,\ldots,q-1\}$, because $x_{q+1} = 0$. Thus $y_1,\ldots,y_q$ satisfy all the properties of $z_1,\ldots,z_q$ used in the last paragraph to show that $z_i^* Sx_j = \delta_{ij}$, and hence $y_i^* Sx_j = \delta_{ij}$ for all i and j $(1 \leqslant i \leqslant q,\ 0 \leqslant j \leqslant q)$. Furthermore, we shall show that $y_i^* Sy_k = 0$ for all i and k. To show this, first note that $y_i^* Sy_k = y_{i+1}^* S^* y_k = (y_k^* Sy_{i+1})^* = (y^*{}_{i+1} Sy_{k+1})^{**} = y_{i+1}^* Sy_{k+1}$, so it suffices to show $y_q^* Sy_k = 0$ for all $k \geqslant 1$ (consider separately the three cases $i = q$, $q > i \geqslant k$, and $i<k$). Now, $z_q^* Sx_i = \delta_{qi}$ and $x_q^* S = (S^* x_q)^* = 0$, so, omitting all terms having factors of the form $x_q^* Sx_i$ (since these factors are 0), we have

$$y_q^* Sy_k = z_q^* Sz_k - \sum_{i=k}^{q} \alpha_{i-k+1} z_q^* Sx_i - \alpha_1 x_q^* Sz_k$$

$$= \alpha_{q-k+1} - \sum_{i=k}^{q} \alpha_{i-k+1} \delta_{qi} - 0 = 0.$$

Now, let $\mathcal{V}_{11}, \mathcal{V}_{10}, \mathcal{V}_1, \mathcal{V}_0$ be the subspaces of $\mathcal{V}$ given by

$$\mathcal{V}_{11} = \operatorname{span}(x_0, x_1, \dots, x_q), \qquad \mathcal{V}_{10} = \operatorname{span}(y_1, \dots, y_q),$$

$$\mathcal{V}_1 = \mathcal{V}_{11} + \mathcal{V}_{10},$$

$$\mathcal{V}_0 = \{Sx_1, Sx_2, \dots, Sx_q, Sy_1, Sy_2, \dots, Sy_q, S^*y_q\}^0.$$

(Exception: when $q=0$ here, we let $\mathcal{V}_0$ be any subspace complementary to $\mathcal{V}_1 = \mathcal{V}_{11}$ in $\mathcal{V}$.) Then $\dim \mathcal{V}_{11} = q+1$, $\dim \mathcal{V}_1 \leqslant 2q+1$, and $\dim \mathcal{V}_0 \geqslant \dim \mathcal{V} - (2q+1)$ because $\mathcal{V}_0$ is defined as the annihilator of $2q+1$ vectors. Thus, in order to show that $\dim \mathcal{V}_1 = 2q+1$ and $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_0$, it suffices (when $q \geqslant 1$ here) to suppose that $x \in \mathcal{V}_{11}$, $y = \beta_1 y_1 + \cdots + \beta_q y_q (\in \mathcal{V}_{10})$, and $x + y \in \mathcal{V}_0$ (and show that consequently we must have $x = 0 = y$ and every $\beta_i = 0$). We may put $x = \alpha_0 x_0 + \cdots + \alpha_q x_q$ here, and hence $x^* S x_i = 0$ for all i. Thus $0 = (x+y)^* S x_i = y^* S x_i = \sum_j \bar{\beta}_j y_j^* S x_i = \bar{\beta}_i$ for all $i \geqslant 1$, so $y = 0$. Also $0 = (Sy_i)^*(x+y) = y_i^* S^* x = y_i^* \sum_j \alpha_j S^* x_j = \sum_j \alpha_j y_i^* S x_{j+1} = \alpha_{i-1}$ for all $i \leqslant q$, and hence $x + y = x = \alpha_q x_q$ and $0 = (S^* y_q)^*(x+y) = y_q^* S(\alpha_q x_q) = \alpha_q$. Thus $x = 0$ (as well as all $\beta_i = 0$), so ($\dim \mathcal{V}_1 = 2q+1$ and) $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_0$.

Finally, there remains only to show that $\mathcal{V}_1$ and $\mathcal{V}_0$ are S-orthogonal and S^*-orthogonal, i.e., that $z \in \mathcal{V}_0$ implies $z^* S \mathcal{V}_1 = 0 = z^* S^* \mathcal{V}_1$. (The matrix of the $\mathcal{V}_1$-restricted *-bilinear form $u^* S v$ is clearly in the desired form S_0 if we choose the basis $(x_0, \dots, x_q, y_1, \dots, y_q)$ for $\mathcal{V}_1$.) These orthogonalities are trivial if $q = 0$, so assume $q \geqslant 1$. Then $z^* S x_i = 0$ for all $i \geqslant 1$ (by definition of $\mathcal{V}_0$) and for $i = 0$ (because $x_0 \in \mathcal{U}_1 = S^{-1}0$), and $z^* S y_i = 0$ for all $i \geqslant 1$ and $z^* S^* y_q = 0$ (by definition of $\mathcal{V}_0$). Also $z^* S^* x_q = 0$ because $x_q \in \mathcal{W}_1$. Lastly, for all $i < q$, we have $z^* S^* x_i = z^* S x_{i+1} = 0$ and $z^* S^* y_i = z^* S y_{i+1} = 0$ (by definition of $\mathcal{V}_0$). Thus indeed $z^* S \mathcal{V}_1 = 0 = z^* S^* \mathcal{V}_1$, and the lemma is proved. ■

THEOREM 4.4. *Each $n \times n$ matrix S over F is *-congruent to $L \oplus M$, where $M + tM^*$ is a nonsingular pencil (i.e., $\det(M + tM^*) \neq 0$ in $F[t]$) and the pencil $L + tL^*$ has no elementary divisors; in fact, L can be taken here as*

$$L = \bigoplus_{j=0}^{n} \bigoplus_{i=1}^{l(j)} \begin{bmatrix} 0 & \dot{J}_j \\ \dot{I}_j & 0 \end{bmatrix}$$

(see Notation 4.1). An alternate form for L here is $L = \bigoplus_{j=0}^{n} \bigoplus_{i=1}^{l(j)} J_{2j+1}$.

Proof. The main result follows by a routine induction on Lemma 4.3. The alternate form follows by noting that

$$P^* J_{2j+1} P = \begin{bmatrix} 0 & \dot{J} \\ \dot{I} & 0 \end{bmatrix}$$

if $P = P^{*-1}$ is the permutation matrix whose columns are (in order) $e_1, e_3, \ldots, e_{2j+1}, e_2, e_4, \ldots, e_{2j}$, where the columns (in order) of I_{2j+1} are $e_1, e_2, \ldots, e_{2j+1}$. ■

REMARK 4.5.

(1) For each j the number $l(j)$ in Theorem 4.4 is of course just the multiplicity of j as a column-minimal index (likewise as a row-minimal index) of the pencil $S + tS^*$ (see [6, Vol. II, p. 38]). Thus the F-equivalence class of the pencil $S + tS^*$ determines the $*$-congruence class of L, and hence so does the (F,E)-congruence class of S. In Corollary 4.7 (below) we shall show that the latter class determines the (F,E)-congruence class of M and give another way to see that it determines the numbers $l(j)$.

(2) If $C^*SC = T$ with C nonsingular, then (as in the proof of Corollary 3.12)

$$C^{-1}\mathfrak{U}_i = (T^{-1}T^*)^i 0, \qquad C^{-1}(S^*\mathfrak{U}_i)^0 = (T^{-1}T^*)^i \mathfrak{V},$$

$$C^{-1}\mathfrak{W}_i = (T^{*-1}T)^i 0, \qquad C^{-1}(S\mathfrak{W}_i)^0 = (T^{*-1}T)^i \mathfrak{V},$$

for all $i \geq 0$, and hence the $*$-congruence class of S determines, among other things, $\dim \mathfrak{U}_i, \dim \mathfrak{W}_j, \dim(\mathfrak{U}_i \cap \mathfrak{W}_j)$ (for all nonnegative i and j), and the $*$-congruence class of the matrix of the $[(S^*\mathfrak{U})^0 + (S\mathfrak{W})^0]$-restricted $*$-bilinear form x^*Sy (as in [3, p. 66, following (7)]).

(3) When $S = L \oplus M$ in Theorem 4.4, the subspaces $\mathfrak{U} \cap \mathfrak{W}$ and $(S^*\mathfrak{U})^0 + (S\mathfrak{W})^0$ turn out to be coordinate subspaces, but they can be more easily identified as such after a reordering of the blocks of L by means of a permutation $*$-congruence. To that end, we make the following definitions.

(4) Let L be a square matrix such that the pencil $L + tL^*$ has no e.d.'s. We say L is in *first standard form* provided L is a direct sum of neutrally partitioned matrices $\begin{bmatrix} 0 & \dot{J} \\ \dot{I} & 0 \end{bmatrix}$, in *second standard form* provided it is a direct sum of matrices J_{2j+1}, and in *third standard form* provided it is a neutrally partitioned matrix $\begin{bmatrix} 0 & Q \\ P & 0 \end{bmatrix}$ where $P + tQ^*$ has no nonzero row-minimal indices and $Q + tP^*$ has no nonzero column-minimal indices [6, Vol. II, p. 38].

(5) Thus Theorem 4.4 says that each such matrix L is *-congruent to matrices in the first and second standard form, and hence to a matrix in third standard form. For example we can take for P, in addition to some zero columns, the direct sum of the rectangular blocks $\dot{I}$ occurring in the first standard form, and for Q, in addition to some zero rows, the direct sum of the rectangular blocks $\dot{J}$.

LEMMA 4.6.

(a) *If S is a single block* $\begin{bmatrix} 0 & \dot{J} \\ \dot{I} & 0 \end{bmatrix}$ *of order* $2m+1$ (*with* $\dot{I}=\dot{I}_m$ *and* $\dot{J}=\dot{J}_m$) *and* $e_1,e_2,\ldots,e_{2m+1}$ *is the standard ordered basis for* $\mathcal{V}$, *then, for* $i\leqslant m+1$,

$$\mathcal{U}_i=\operatorname{span}(e_{m-i+2},\ldots,e_m,e_{m+1}),$$

$$\mathcal{W}_i=\operatorname{span}(e_1,e_2,\ldots,e_i),$$

$$\mathcal{U}=\mathcal{W}=(S^*\mathcal{U})^0=(S\mathcal{W})^0=\operatorname{span}(e_1,e_2,\ldots,e_m,e_{m+1}).$$

(b) *If* $S=M\oplus L$ (*with M and L as in Theorem* 4.4) *and*

$$L=\begin{bmatrix} 0 & Q \\ P & 0 \end{bmatrix}$$

is in third standard form (*Remark* 4.5(4)), *and if* $\mathcal{V}=\mathcal{V}_1\oplus\mathcal{V}_0$ *conforms to* $M\oplus L$ *and* $\mathcal{V}_0=\mathcal{V}_{01}\oplus\mathcal{V}_{02}$ *conforms to* $\begin{bmatrix} 0 & Q \\ P & 0 \end{bmatrix}$, *then* $\mathcal{U}\cap\mathcal{W}=\mathcal{V}_{01}$ *and* $(S^*\mathcal{U})^0+(S\mathcal{W})^0=\mathcal{V}_1\oplus\mathcal{V}_{01}$.

Proof. The proof of (a) is a routine computation using the special form of the blocks $\dot{I}_m$ and $\dot{J}_m$. One finds also that

$$(S^*\mathcal{U}_i)^0=\mathcal{U}_{m+1}+\operatorname{span}(e_{m+2},\ldots,e_{2m+1-i}),$$

$$(S\mathcal{W}_i)^0=\mathcal{W}_{m+1}+\operatorname{span}(e_{m+2+i},\ldots,e_{2m+1})$$

for $i<m$; and $(S^*\mathcal{U}_i)^0=(S\mathcal{W}_i)^0=\mathcal{U}_{m+1}=\mathcal{W}_{m+1}=\mathcal{U}=\mathcal{W}$ for $i\geqslant m$.

The proof of (b) is also routine: we start by assuming L is in first standard form and use [3, Lemma 1, p. 63] [and the corresponding result for $(S^{-1}T)^i0$] to reduce the computations to those in (a) (plus the obvious computations for the case $S=M$), and then rearrange the blocks in L [via a permutation *-congruence, as suggested in Remark 4.5(5)] to third standard form, keeping track of what happens to the coordinate subspaces in the process. ■

COROLLARY 4.7.

(a) *In Theorem* 4.4 *the* *-*congruence class of S determines the numbers* $l(j)$ *by the fact that*

$$l(j)=\dim(\mathfrak{U}_{j+1}\cap\mathfrak{W}_1)-\dim(\mathfrak{U}_j\cap\mathfrak{W}_1),$$

and determines the *-*congruence class of M by the fact that the matrix of the* $[(S^*\mathfrak{U})^0+(S\mathfrak{W})^0]$-*restricted* *-*bilinear form* x^*Sy *is* *-*congruent to* $M\oplus 0$, *in which the* 0 *direct summand has order* $=\dim(\mathfrak{U}\cap\mathfrak{W})=\Sigma_j(j+1)l(j)$.

(b) *Furthermore, if* $C^*SC=T$ *with C nonsingular and* $S=M\oplus L$, $T=N\oplus R$, *and* $M+tM^*$ *and* $N+tN^*$ *are nonsingular pencils and*

$$L=\begin{bmatrix}0 & Q\\ P & 0\end{bmatrix}\quad\textit{and}\quad R=\begin{bmatrix}0 & W\\ V & 0\end{bmatrix}$$

are in third standard form (and the pencils $L+tL^*$ *and* $R+tR^*$ *have no e.d.'s), then*

$$C=\begin{bmatrix}C_{11} & 0 & C_{13}\\ C_{21} & C_{22} & C_{23}\\ 0 & 0 & C_{33}\end{bmatrix}$$

conformably with

$$S=\begin{bmatrix}M & 0 & 0\\ 0 & 0 & Q\\ 0 & P & 0\end{bmatrix}\quad\textit{and}\quad T=\begin{bmatrix}N & 0 & 0\\ 0 & 0 & W\\ 0 & V & 0\end{bmatrix},$$

and hence C_{11} *is nonsingular,* $N=C_{11}^*MC_{11}$, *and M is* *-*congruent to the matrix of the restriction of the form* x^*Sy *to any subspace complementary to* $\mathfrak{U}\cap\mathfrak{W}$ *in* $(S^*\mathfrak{U})^0+(S\mathfrak{W})^0$.

(Note that $C=C^{-1}$ here implies $C_{11}=C_{11}^{-1}$, and that $C=\bar{C}^{-1}$ implies $C_{11}=\bar{C}_{11}^{-1}$.)

Proof. (a): By Lemma 4.6(a), if

$$S=\begin{bmatrix}0 & \dot{J}\\ \dot{I} & 0\end{bmatrix}$$

is a single block of order $2m+1$, then $\dim(\mathcal{U}_j \cap \mathcal{W}_1)=0$ for $j \leqslant m$ and $=1$ for $j>m$. Thus, if S is any square matrix in Theorem 4.4, then

$$\dim(\mathcal{U}_j \cap \mathcal{W}_1) = l(0) + l(1) + \cdots + l(j-1)$$

for all $j \geqslant 0$. (Recall that $\mathcal{U}_0 = 0$ and, for $S = M$ in Theorem 4.4, $\mathcal{U} \cap \mathcal{W} = 0$ by Lemma 3.3.) This proves the formula for $l(j)$ in (a). The rest of (a) follows from Lemma 4.6 and Remark 4.5(2). Part (b) follows routinely from Lemma 4.6(b) and Remark 4.5(2) in much the same way as the second proof of Corollary 3.12 was carried out. ■

REMARK (about Corollary 4.7). Notice that the uniqueness result in (b) of Corollary 4.7 is stronger than the uniqueness result for M in (a). It is clear that this stronger form of uniqueness must also hold for pencils $A+tB$ in the following three cases (namely, those mentioned in [3] as having the weaker form of uniqueness): (1) where (F,E) is admissible and $A=A^*$ and $B=B^*$, (2) where (F,E) is simplic and A is symmetric and B is alternating, and (3) where (F,E) is simplic and A and B are both alternating.

Next we summarize some of the existence and uniqueness results from this section and last section.

COROLLARY 4.8. *Let (F,E) be an admissible pair of fields, and let S be a square matrix over F. Then S is (F,E)-congruent to a direct sum $S_0 \oplus S_1$ with S_1 nonsingular and every e.d. of the pencil $S_0 \oplus tS_0^*$ at 0 or ∞. Moreover, S_0 here is (F,E)-congruent to $\bigoplus_j(\bigoplus_{i=1}^{m(j)} J_j)$, and the numbers $m(j)$, as well as the (F,E)-congruence class of S_1, are uniquely determined by the (F,E)-congruence class of S. Finally, if C is a nonsingular matrix over F partitioned as*

$$C = \begin{bmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{bmatrix},$$

conformably with $S_0 \oplus S_1$, and $C^*(S_0 \oplus S_1)C = T_0 \oplus T_1$, where T_1 is nonsingular and every e.d. of $T_0 + tT_0^*$ is at 0 or ∞, then $C_{11}^* S_1 C_{11} = T_1$ and S_0 is (F,E)-congruent to T_0; also $C_{11} = C_{11}^{-1}$ here if $C = C^{-1}$, and $C_{11} = \bar{C}_{11}^{-1}$ if $C = \bar{C}^{-1}$.

Proof. The existence part comes from Theorems 4.4 and 3.7 (plus Remark 3.13). The rest comes from Corollaries 4.7 and 3.12.

REMARK 4.9.

(a) Let S be $n \times n$ in Corollary 4.8 (thus $n = \dim \mathcal{V}$) and S_1 be $n_1 \times n_1$. Then from Theorems 4.4 and 3.7 one sees that

$$n_1 = \dim[(S^*\mathcal{U})^0 + (S\mathcal{W})^0] - \dim(\mathcal{U} + \mathcal{W})$$

$$= \dim(S^*\mathcal{U})^0 - \dim \mathcal{U} = \dim(S\mathcal{W})^0 - \dim \mathcal{W},$$

etc., and in fact $\dim \mathcal{U}_i = \dim \mathcal{W}_i$ and $\dim S^*\mathcal{U}_i = \dim S\mathcal{W}_i$ for every i. Also the order of S_0 is $\Sigma_j jm(j) = n - n_1 = \dim \mathcal{U} + \dim S^*\mathcal{U}$, etc.

(b) The (F,E)-congruence class of S_1 in Corollary 4.8 [as well as the number n_1 in (a)] can be characterized by the following maximality property: for every direct sum $R_0 \oplus R_1$ which is (F,E)-congruent to S with R_1 nonsingular, the order of R_1 is less than or equal to the order of S_1, with equality only if R_1 is (F,E)-congruent to S_1 and R_0 is (F,E)-congruent to S_0. [Proof: first apply Corollary 4.8 to R_0 in place of S, getting that R_0 is (F,E)-congruent to $R_{00} \oplus R_{01}$ with R_{01} nonsingular and every e.d. of $R_{00} + tR_{00}^*$ at 0 or ∞, so that S is (F,E)-congruent to $R_{00} \oplus (R_{01} \oplus R_1)$; then apply Corollary 4.8 to S, getting that S_1 is (F,E)-congruent to $R_{01} \oplus R_1$.]

REMARK 4.10. If B is a nonsingular matrix over F, and A is a matrix over F conforming to B, it is known [6, Vol. I, Theorem 6, p. 145] that the $F[t]$-equivalence class of the pencil $A + tB$ determines the F-equivalence class of $A + tB$ (i.e., the similarity class of $B^{-1}A$). This no longer holds if B is singular, even if the pencil $A + tB$ is still nonsingular, but it does hold for a nonsingular pencil of the form $S + tS^*$, since in the latter case the e.d.'s at ∞ are the same as those at 0 (and the e.d.'s at 0 are determined by the $F[t]$-equivalence class). Of course the F-equivalence class of $S + tS^*$ does not determine the (F,E)-congruence class of $S + tS^*$ (which is essentially that of S) in general, but does if S is (F,E)-neutral (as we shall see in Corollary 4.12).

Note that Corollary 4.8 reduces the study of (F,E)-congruence of arbitrary square matrices over F to the case of nonsingular matrices over F. (An analogous remark in [5, p. 67] failed to take into account the uniqueness aspect of this reduction.)

COROLLARY 4.11. *If S is an (F,E)-neutral matrix, then so is M in Theorem 4.4 and so is S_1 in Corollary 4.8.*

Proof. The proof is practically the same as that of Corollary 3.14, except that here we use [6, Vol. II, pp. 35–40] and the uniqueness results in Corollaries 4.7 and 4.8. ∎

COROLLARY 4.12. *If S is (F,E)-neutral, then the (F,E)-congruence class of $S+tS^*$ (which is essentially that of S) is determined by its F-equivalence class and hence, if nonsingular, by its $F[t]$-equivalence class.*

Proof. Let S be (F,E)-neutral. Then S is (F,E)-congruent by Corollary 4.8 to $S_0 \oplus S_1$, and hence $S+tS^*$ is correspondingly (F,E)-congruent to $(S_0+tS_0^*)\oplus(S_1+tS_1^*)$, with S_1 nonsingular and every e.d. of $S_0+tS_0^*$ at 0 or ∞. Here S_1 is (F,E)-neutral by Corollary 4.11, so by Theorem 2.11(d) its (F,E)-congruence class is determined by the similarity class of $S_1^{*-1}S_1$, which is essentially the same as the F-equivalence class of $S_1+tS_1^*$. Thus the (F,E)-congruence class of S_1 is determined by the F-equivalence class of $S+tS^*$ (in fact by the e.d.'s of $S+tS^*$ other than those at 0 or ∞). By Corollary 4.8 the (F,E)-congruence class of S_0 also is determined by the F-equivalence class of $S+(tS^*$ (in fact by the minimal indices and the e.d.'s at 0). ■

ADDENDUM

After this paper was prepared, the authors became aware of the paper by C. Riehm and M. A. Shrader-Frechette, The equivalence of sesquilinear forms, *J. Algebra* 42:495–530 (1976), which deals with the complic cases [over semisimple (Artinian) rings] but considers generally deeper questions than those considered here. In particular, it proves a generalization of part of the complic case of Theorem 2.4 (in its Proposition 25, p. 516) and "reduces" the singular complic cases to the nonsingular ones (in its Section 9, pp. 526–529) and, unlike [5], also considers the uniqueness aspect (in pp. 528–529); one of the referees assures us that the uniqueness is "very easy" to prove also in the context of [5].

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