An algorithm for approximate solving of differential equations with “maxima”

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An algorithm for constructing two sequences of successive approximations of the solution of the initial value problem for nonlinear differential equations with “maxima” is given. This algorithm is based on the monotone iterative technique. It is proved that both sequences are monotonically convergent. Each term of the constructed sequences is a solution of an initial value problem for linear differential equations with “maxima” and it is a lower/upper solution of the given problem. Both the scalar case and the multidimensional case are studied. An example, solved by computer realization of the suggested algorithm, illustrates the practical application of the method.

1. Introduction

Differential equations with “maxima” find wide applications in the theory of automatic regulation. As a simple example of mathematical simulation by means of such equations, we shall consider a system for the regulation of voltage of a generator of constant current [1]. The object of regulation is a generator of constant current with parallel stimulation, and the quantity regulated is the voltage on the clamps of the generator feeding an electric circuit with different loads. A differential equation with “maxima” is used if the regulator is constructed such that the maximal deviation of the quantity regulated is on the segment [t − r, t]. The equation describing the work of the regulator has the form

\[ Tu'(t) + u(t) + q \max_{s \in [t-r,t]} u(s) = f(t), \]

where \( T \) and \( q \) are constants characterizing the object, \( u(t) \) is the voltage regulated, and \( f(t) \) is the perturbing effect.

In most cases the solutions of differential equations with “maxima” are not possible to be obtained in a closed form and that requires the application of different approximate methods. In recent years several effective approximate methods, based on the upper and lower solutions of the given problem, was proven for various problems of differential equations [2–15].

In the current paper, using the monotone iterative technique, based on the method of lower and upper solutions, the solution of an initial value problem for nonlinear differential equations with “maxima” is theoretically obtained. Both multidimensional and scalar cases are considered. A procedure for obtaining the terms of two sequences is given and their monotonic convergence to the solution of the considered initial value problem is proven. The main advantage of this method is that practically it is easy to find the successive approximations of the unknown solution. At the same time these terms are lower/upper solutions of the given problem. In order to illustrate the practical application of the proved procedure, a computer realization on a particular example is given and the algorithm for solving integrals, involving the maximum of the unknown function is suggested.

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2. Preliminary notes and definitions

Consider the following initial value problem for the system of differential equations with “maxima”

\[
\begin{align*}
    x'(t) &= f(t, x(t), \max_{s \in [t-r, t]} x(s)), \quad t \in [0, T], \\
    x(t) &= \varphi(t), \quad t \in [-r, 0],
\end{align*}
\]

where \( x \in \mathbb{R}^n, f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, T, r = \text{const} > 0. \)

Note that for \( x : [-r, t] \to \mathbb{R}^n, x = (x_1, x_2, \ldots, x_n) \) we denote

\[
    \sup_{s \in [t-r, t]} x(s) = \left( \sup_{s \in [t-r, t]} x_1(s), \sup_{s \in [t-r, t]} x_2(s), \ldots, \sup_{s \in [t-r, t]} x_n(s) \right).
\]

We will use notations that are analogous to those used in [12] for systems of ordinary differential equations. These notations play an important role in the definitions of different types of lower and upper solutions for systems of differential equations with “maxima”.

For each natural number \( j : 1 \leq j \leq n \) we consider two nonnegative integers \( p_j \) and \( q_j \) such that \( p_j + q_j = n - 1 \) and for the points \( x, y, z \in \mathbb{R}^n \) we introduce the notation

\[
    (x_j, [z]_{p_j}, [y]_{q_j}) = \begin{cases}
        (z_1, \ldots, z_{j-1}, x_j, z_{j+1}, \ldots, z_n, y_{p_j+1}, \ldots, y_n) & \text{for } p_j > j, \\
        (z_1, z_2, \ldots, z_{p_j+1}, y_{p_j+1}, \ldots, y_n, x_{j+p_j+1}, \ldots, x_n) & \text{for } p_j \leq j.
    \end{cases}
\]

For example, let \( n = 3 \). Choose \( p_1 = 2, q_1 = 0, p_2 = 1, q_2 = 1 \) and \( p_3 = 1, q_3 = 1 \). Then \((x_1, [z]_{p_1}, [y]_{q_1}) = (x_1, z_2, z_3), (x_2, [z]_{p_2}, [y]_{q_2}) = (z_1, x_2, y_3), (x_3, [z]_{p_3}, [y]_{q_3}) = (z_1, y_2, x_3)\).

According to the introduced notation (3) the initial value problem (1), (2) can be rewritten in the form

\[
\begin{align*}
    x_j'(t) &= f_j(t, x_j(t), [x(t)]_{p_j}, [x(t)]_{q_j}), \quad \max_{s \in [t-r, t]} x_j(s), \left[ \max_{s \in [t-r, t]} x(s) \right]_{p_j}, \left[ \max_{s \in [t-r, t]} x(s) \right]_{q_j}, \quad t \in [0, T], \\
    x_j(t) &= \varphi_j(t), \quad t \in [-r, 0], \quad j = 1, 2, \ldots, n.
\end{align*}
\]

Let \( x = (x_1, x_2, \ldots, x_n) \) and \( y = (y_1, y_2, \ldots, y_n) \). We will say that the inequality \( x \leq (\geq) y \) holds, if for all natural numbers \( j : 1 \leq j \leq n \) the inequalities \( x_j \leq (\geq) y_j \) hold.

**Definition 1.** The pair of functions \( v, w \in C([-r, T], \mathbb{R}^n), v = (v_1, v_2, \ldots, v_n), w = (w_1, w_2, \ldots, w_n) \) is called a pair of lower and upper quasisolutions of the initial value problem for the system of differential equations with “maxima” (1), (2) if

\[
\begin{align*}
    v_j' &\leq f_j(t, v_j(t), [v(t)]_{p_j}, [v(t)]_{q_j}), \quad \max_{s \in [t-r, t]} v_j(s), \left[ \max_{s \in [t-r, t]} v(s) \right]_{p_j}, \left[ \max_{s \in [t-r, t]} v(s) \right]_{q_j}, \\
    w_j' &\geq f_j(t, w_j(t), [w(t)]_{p_j}, [w(t)]_{q_j}), \quad \max_{s \in [t-r, t]} w_j(s), \left[ \max_{s \in [t-r, t]} w(s) \right]_{p_j}, \left[ \max_{s \in [t-r, t]} w(s) \right]_{q_j}, \quad t \in [0, T], \\
    v_j(t) &\leq \varphi_j(t), \quad w_j(t) \geq \varphi_j(t), \quad t \in [-r, 0], \quad j = 1, 2, \ldots, n.
\end{align*}
\]

**Remark 1.** Note that in Eq. (6) the notation

\[
    \left( \max_{s \in [t-r, t]} v_j(s), \left[ \max_{s \in [t-r, t]} v(s) \right]_{p_j}, \left[ \max_{s \in [t-r, t]} w(s) \right]_{q_j} \right) = \begin{cases}
        \left( \max_{s \in [t-r, t]} v_1(s), \ldots, \max_{s \in [t-r, t]} v_{p_j+1}(s), \left[ \max_{s \in [t-r, t]} v_{p_j+2}(s), \ldots, \max_{s \in [t-r, t]} v_n(s) \right]_{p_j > j}, \\
        \left( \max_{s \in [t-r, t]} w_1(s), \ldots, \max_{s \in [t-r, t]} w_{p_j+1}(s), \left[ \max_{s \in [t-r, t]} w_{p_j+2}(s), \ldots, \max_{s \in [t-r, t]} w_n(s) \right]_{p_j \leq j} \right) \end{cases}
\]

is used.

**Remark 2.** We will note that the pair of lower and upper quasisolutions is generalization of the lower and upper solutions in the scalar case \((n = 1, p_1 = q_1 = 0)\).

**Definition 2.** The pair of functions \( v, w \in C([-r, T], \mathbb{R}^n), v = (v_1, v_2, \ldots, v_n), w = (w_1, w_2, \ldots, w_n) \) is called a pair of quasisolutions of the initial value problem for the system of differential equations with “maxima” (1), (2) if (6), (7) are satisfied only for equalities.

**Definition 3.** The pair of functions \( v, w \in C([-r, T], \mathbb{R}^n), v = (v_1, v_2, \ldots, v_n), w = (w_1, w_2, \ldots, w_n) \) is called a pair of minimal and maximal quasisolutions of the initial value problem for the system of differential equations with “maxima” (1), (2) if it is a pair of quasisolutions of the same problem, \( v(t) \leq w(t) \) and for any other pair \((\mu, \nu)\) of quasisolutions of (1), (2), the inequalities \( v(t) \leq \mu(t) \leq w(t), v(t) \leq \nu(t) \leq w(t) \) hold for \( t \in [-r, T] \).
Remark 3. We will note that if the pair of functions \( v, w \in C([-r, T], \mathbb{R}^n) \) is a pair of minimal and maximal quasisolutions, then the inequality \( v(t) \leq w(t) \) holds. Also, for any pair of quasisolutions this inequality could not be satisfied.

Remark 4. We will note that for all natural numbers \( j : 1 \leq j \leq n \) the equalities \( p_j = n - 1 \) and \( q_j = 0 \) hold and the pair \( v, w \in PC([-r, T], \mathbb{R}^n) \) is a pair of quasisolutions of (1), (2). In this case the functions \( v \) and \( w \) are also solutions of the same problem. If the initial value problem (1), (2) has an unique solution \( u(t) \), then the pair of minimal and maximal quasisolutions is \( (u, u) \).

For all pairs of functions \( v, w \in C([-r, T], \mathbb{R}^n) \) such that \( v(t) \leq w(t) \) for \( t \in [-r, T] \), we define the set
\[
S(v, w) = \{ u \in C([-r, T], \mathbb{R}^n) : v(t) \leq u(t) \leq w(t), \ t \in [-r, T] \}.
\]

**Lemma 2.1.** The scalar function \( m \in C([-r, T], \mathbb{R}) \cap C^1([0, T], \mathbb{R}) \) satisfies the inequalities
\[
m'(t) \leq -Mm(t) - N \min_{s \in [t-h,t]} m(s) \quad \text{for} \quad t \in [0, T],
\]
\[
m(t) = m(0), \quad t \in [-r, 0],
\]
\[
m(0) \leq 0,
\]
where \( M, N \) are positive constants and
\[
(M + N)T < 1.
\]

Then the inequality \( m(t) \leq 0 \) holds for \( t \in [-r, T] \).

**Proof.** Assume the contrary, i.e. there exists a point \( \xi \in (0, T) \) such that \( m(\xi) > 0 \). Consider the following three cases:

Case 1. Let \( m(0) = 0, m(t) \geq 0, m(t) \neq 0 \) for \( t \in [0, b] \), where \( b \in (0, T) \) is a small enough constant. Then from equality (11) it follows that \( m(t) \equiv 0 \) for \( t \in [-r, 0] \). Therefore there exist points \( \xi_1, \xi_2 \in [0, T], \xi_1 < \xi_2 \) such that \( m(t) = 0 \) for \( t \in [-r, \xi_1] \), and \( m(t) > 0 \) for \( t \in (\xi_1, \xi_2) \). From inequality (10) it follows that \( m'(t) \leq 0 \) for \( t \in (\xi_1, \xi_2) \). Therefore, the function \( m(t) \) is a continuous nonincreasing function on \([\xi_1, \xi_2] \), i.e. \( m(t) \leq m(\xi_1) = 0 \) for \( t \in (\xi_2, \xi_2) \). The last inequality contradicts the assumption.

Case 2. Let \( m(0) < 0 \). According to the assumptions there exists a point \( \eta \in (0, T) \) such that \( m(t) \leq 0 \) for \( t \in [-r, \eta] \), \( m(\eta) = 0, m(t) > 0 \) for \( t \in (\eta, \eta + \varepsilon) \), where \( \varepsilon > 0 \) is a small enough constant. Denote
\[
\inf\{m(t) : t \in [-r, \eta]\} = -\lambda < 0.
\]

Let \( \zeta \in (0, \eta) \) be such that \( m(\zeta) = -\lambda \). According to the mean value theorem there exists a point \( \xi_0 \in (\zeta, \eta) \) such that \( m(\xi_0) = m(\zeta) = m'(\xi_0)(\eta - \zeta) \). Therefore, from inequality (10) and \(-\lambda \leq \min_{s \in [-r, 0]} m(s) \) we get
\[
\lambda = m(\eta) - m(\xi_0) = m'(\xi_0)(\eta - \zeta) \leq (M + N)\lambda T.
\]

Inequality (14) contradicts inequality (13).

Case 3. Let \( m(0) = 0 \), and \( m(t) \leq 0 \), and \( m(t) \neq 0 \) for \( t \in [0, b] \), where \( b > 0 \) is a small enough constant. As in the proof of case 2, we obtain a contradiction, that proves Lemma 2.1. \( \square \)

3. Main results

We will give an algorithm for constructing a sequence of successive approximations and we will prove the application of monotone iterative technique to the initial value problem for a system of nonlinear differential equations with “maxima”.

3.1. Multidimensional case

**Theorem 3.1.** Let the following conditions be fulfilled:
1. The function \( \varphi \in C([-r, 0], \mathbb{R}^n) \), \( \varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n) \).
2. The pair of functions \( \alpha, \beta \in C([0, T], \mathbb{R}^n) \), \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \), \( \beta = (\beta_1, \beta_2, \ldots, \beta_n) \) is a pair of lower and upper quasisolutions of the initial value problem (1), (2), such that \( \alpha(t) \leq \beta(t) \) for \( t \in [-r, T] \), and \( \alpha(0) - \varphi(0) \leq \alpha(t) - \varphi(t), \beta(0) - \varphi(0) \geq \beta(t) - \varphi(t) \) for \( t \in [-r, 0] \).
3. The function \( f \in C([0, T] \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n) \), \( f = (f_1, f_2, \ldots, f_n) \), where \( f_j(t, x, y) = f_j(t, x_j, [x]_y, [x]_y, [y]_y) \), is nondecreasing in \([x]_y \) and \([y]_y \), nonincreasing in \([x]_y \) and \([y]_y \), and for \( x, y, u, v \in \mathbb{R}^n \), \( y \leq x, u \leq v \) the inequality
\[
f_j(t, x_j, [x]_y, [y]_y, [v]_y, [v]_y) - f_j(t, y_j, [y]_y, [y]_y, [u]_y, [u]_y)
\]
holds, where \( M_j, N_j, j = 1, 2, \ldots, n \) are positive constants.
4. The inequalities \( M_j + N_j ) < 1, j = 1, 2, \ldots, n \) hold.
Then there exist two sequences of functions \( \{\alpha^k(t)\}^\infty_0 \) and \( \{\beta^k(t)\}^\infty_0 \) such that:

a) The sequences are increasing and decreasing correspondingly;

b) The pair of functions \( \alpha^k(t) \), \( \beta^k(t) \) is a pair of lower and upper quasisolutions of the initial value problem for the system of nonlinear differential equations with “maxima” (1), (2);

c) Both sequences uniformly converge on \([-r, T]\);

d) The limits \( V(t) = \lim_{k \to \infty} \alpha^k(t) \), \( W(t) = \lim_{k \to \infty} \beta^k(t) \) are a pair of minimal and maximal solutions of the initial value problem for the system of nonlinear differential equations with “maxima” (1), (2).

e) If \( u(t) \in S(\alpha, \beta) \) is a solution of the initial value problem for the system of nonlinear differential equations with “maxima” (1), (2), then \( V(t) \leq u(t) \leq W(t) \).

**Proof.** We fix two arbitrary functions \( \eta, \mu \in S(\alpha, \beta) \) and for all natural numbers \( j : 1 \leq j \leq n \) we consider the initial value problem for the scalar linear differential equation with “maxima”

\[
\begin{align*}
u'(t) + M_j u(t) + N_j \max_{s \in [−r,t]} u(s) &= \psi_j(t, \eta, \mu), \quad \text{for } t \in [0, T], \quad (15) \\
u(t) &= \varphi_j(t), \quad t \in [-r, 0], \quad (16)
\end{align*}
\]

where \( u \in \mathbb{R} \),

\[
\psi_j(t, \eta, \mu) = f_j(t, \eta_j(t), [\eta(t)]_{[t-r,t]}, [\mu(t)]_{[t-r,t]}, \max_{s \in [−r,t]} \eta_j(s), \max_{s \in [−r,t]} \eta(s)\big|_{t}, \max_{s \in [−r,t]} \mu(s)\big|_{t})
\]

\[
+M_j \eta_j(t) + N_j \max_{s \in [−r,t]} \eta_j(s).
\]

The initial value problem (15), (16) has an unique solution for the fixed pair of functions \( \eta, \mu \in S(\alpha, \beta) \).

For any two functions \( \eta, \mu \in S(\alpha, \beta) \) such that \( \eta(t) \leq \mu(t) \) for \( t \in [-r, T] \) we define the operator \( \Omega : S(\alpha, \beta) \times S(\alpha, \beta) \to S(\alpha, \beta) \) by \( \Omega(\eta, \mu) = x(t) \), where \( x(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \) and \( x_j(t) \) is the unique solution of the initial value problem for the scalar differential equation with “maxima” (15), (16) for the pair of functions \( \eta, \mu \).

The operator \( \Omega(\eta, \mu) \) possesses the following properties:

**(P1)** \( \alpha \leq \Omega(\alpha, \beta) \) and \( \beta \geq \Omega(\beta, \alpha) \);

**(P2)** For any functions \( \eta, \mu \in S(\alpha, \beta) \) such that \( \eta(t) \leq \mu(t) \) for \( t \in [-r, T] \) and \( (\eta, \mu) \) is a pair of lower and upper quasisolutions of the initial value problem (1), (2), the inequality \( \Omega(\eta, \mu) \leq \Omega(\mu, \eta) \) holds.

Indeed, we will prove property (P1). We denote \( m(t) = \alpha(t) - \alpha_j^{(1)}(t) \), where \( \alpha_j^{(1)}(t) = \Omega(\alpha, \beta) \).

Then from condition 2 and Eq. (15) for any \( j : 1 \leq j \leq n \) applying the inequality \( \min_{s \in [−r,t]}(\alpha_j(s) - \alpha_j^{(1)}(s)) \leq \max_{s \in [−r,t]} \alpha_j^{(1)}(s) - \max_{s \in [−r,t]} \alpha_j^{(1)}(s) \) we get

\[
\begin{align*}
m_j^*(t) &\leq M_j(\alpha_j^{(1)}(t) - \alpha_j(t)) + N_j(\max_{s \in [−r,t]} \alpha_j^{(1)}(s) - \max_{s \in [−r,t]} \alpha_j(s)) \\
&= -M_j m_j(t) - N_j(\max_{s \in [−r,t]} \alpha_j(s) - \max_{s \in [−r,t]} \alpha_j^{(1)}(s)) \\
&\leq -M_j m_j(t) - N_j(\min_{s \in [−r,t]} (\alpha_j(s) - \alpha_j^{(1)}(s))), \quad t \in [0, T]. \quad (17)
\end{align*}
\]

Therefore, the function \( m(t) \) satisfies the inequalities (10), (12). According to Lemma 2.1 the function \( m(t) \) is nonpositive, i.e. \( \alpha \leq \Omega(\beta, \alpha) \).

Analogously the validity of the inequality \( \beta \geq \Omega(\alpha, \beta) \) can be proved.

We will prove property (P2). Let \( \eta, \mu \in S(\alpha, \beta) \) form a pair of lower and upper quasisolutions of the initial value problem (1), (2) and \( \eta(t) \leq \mu(t) \) for \( t \in [-r, T] \). Consider functions \( x^1(t) \) and \( x^2(t) \) for \( t \in [-r, T] \), where \( x^1 = \Omega(\eta, \mu), x^2 = \Omega(\mu, \eta) \).

Then from condition 3 and Eq. (15) we get for any \( j : 1 \leq j \leq n \) and \( t \in [0, T] \)

\[
\begin{align*}
g_j^*(t) &\leq -M_j g_j(t) - N_j(\max_{s \in [−r,t]} x_j^1(s) - \max_{s \in [−r,t]} x_j^2(s)) + M_j(\eta_j(t) - \mu_j(t)) + N_j(\max_{s \in [−r,t]} \eta_j(s) - \max_{s \in [−r,t]} \mu_j(s)) \\
&+ f_j(t, \eta_j(t), [\eta_j(t)]_{[t-r,t]}, [\mu_j(t)]_{[t-r,t]}, \max_{s \in [−r,t]} \eta_j(s), \max_{s \in [−r,t]} \eta(s)\big|_{t}, \max_{s \in [−r,t]} \mu_j(s)\big|_{t}) \\
&- f_j(t, \mu_j(t), [\mu_j(t)]_{[t-r,t]}, [\eta_j(t)]_{[t-r,t]}, \max_{s \in [−r,t]} \mu_j(s), \max_{s \in [−r,t]} \mu(s)\big|_{t}, \max_{s \in [−r,t]} \eta_j(s)\big|_{t}) \\
&\leq -M_j g_j(t) - N_j(\max_{s \in [−r,t]} x_j^1(s) - \max_{s \in [−r,t]} x_j^2(s)) \\
&\leq -M_j g_j(t) - N_j(\min_{s \in [−r,t]} g_j(s)). \quad (18)
\end{align*}
\]

Inequality (18) proves the validity of conditions of Lemma 2.1. According to Lemma 2.1 functions \( g_j(t), j = 1, 2, \ldots, n \) are nonpositive, i.e. \( \Omega(\eta, \mu) \leq \Omega(\mu, \eta) \).
We define the sequences of functions \( \{\alpha^{(k)}(t)\}_{0}^{\infty} \) and \( \{\beta^{(k)}(t)\}_{0}^{\infty} \) by the equalities

\[
\alpha^{(0)} = \alpha, \quad \beta^{(0)} = \beta, \\
\alpha^{(k+1)} = \Omega(\alpha^{(k)}, \beta^{(k)}), \quad \beta^{(k+1)} = \Omega(\beta^{(k)}, \alpha^{(k)}).
\]

According to property (P1) of the operator \( \Omega(\eta, \mu) \) it follows that functions \( \alpha^{(k)}(t) \) and \( \beta^{(k)}(t) \) form a pair of lower and upper quasisolutions. According to property (P2) of the operator \( \Omega(\eta, \mu) \) it follows that for \( t \in [-h, T] \) the following inequalities

\[
\alpha^{(0)}(t) \leq \alpha^{(1)}(t) \leq \cdots \leq \alpha^{(k)}(t) \leq \beta^{(k)}(t) \leq \cdots \leq \beta^{(1)}(t) \leq \beta^{(0)}(t)
\]

hold.

Both sequences of functions \( \{\alpha^{(k)}(t)\}_{0}^{\infty} \) and \( \{\beta^{(k)}(t)\}_{0}^{\infty} \) are convergent on the interval \([-r, T]\). Let

\[
V_j(t) = \lim_{k \to \infty} v_j^{(k)}(t), \quad W_j(t) = \lim_{k \to \infty} w_j^{(k)}(t), \quad j = 1, 2, \ldots, n.
\]

We will prove that both functions \( V(t) \) and \( W(t) \), \( V = (V_1, V_2, \ldots, V_n) \) and \( W = (W_1, W_2, \ldots, W_n) \), form a pair of minimal and maximal quasisolutions of the initial value problem (1), (2). From the definition of functions \( \alpha^{(k)}(t) \) and \( \beta^{(k)}(t) \), where \( \alpha^{(k)} = (\alpha^{(k)}_1, \alpha^{(k)}_2, \ldots, \alpha^{(k)}_n) \), \( \beta^{(k)} = (\beta^{(k)}_1, \beta^{(k)}_2, \ldots, \beta^{(k)}_n) \), it follows that these functions satisfy the initial value problems \( j = 1, 2, \ldots, n \)

\[
(\alpha_j^{(k)}(t))' + M_j \alpha_j^{(k)}(t) + N_j \max_{s \in [-r, t]} \alpha_j^{(k)}(s) = \psi_j(t, \alpha^{(k-1)}, \beta^{(k-1)}),
\]

\[
(\beta_j^{(k)}(t))' + M_j \beta_j^{(k)}(t) + N_j \max_{s \in [-r, t]} \beta_j^{(k)}(s) = \psi_j(t, \beta^{(k-1)}, \alpha^{(k-1)}), \quad \text{for } t \in [0, T],
\]

\[
\alpha_j^{(k)}(t) = \alpha_j^{(k)}(0), \quad \beta_j^{(k)}(t) = \beta_j^{(k)}(0), \quad t \in [-r, 0].
\]

From Eqs. (20), (21) it follows that the pair of functions \( V(t) \) and \( W(t) \) is a pair of quasisolutions of the initial value problem (1), (2). Let \( u, z \in S(\alpha, \beta) \) be a pair of quasisolutions of the initial value problem (1), (2). From inequalities (19) it follows that there exists a natural number \( k \) such that \( \alpha^{(k)}(t) \leq u(t) \leq \beta^{(k)}(t) \) and \( \alpha^{(k)}(t) \leq z(t) \leq \beta^{(k)}(t) \) for \( t \in [-r, T] \). We introduce the notation \( g(t) = \alpha^{(k-1)}(t) - u(t), \gamma = (g_1, g_2, \ldots, g_n) \). According to Lemma 2.1 the inequalities \( g_j(t) \leq 0, \quad j = 1, 2, \ldots, n \) hold for \( t \in [-r, T] \), i.e. \( \alpha^{(k+1)}(t) \leq u(t) \).

Analogously the validity of inequalities \( \beta^{(k+1)}(t) \geq u(t) \) and \( \alpha^{(k+1)}(t) \leq z(t) \leq \beta^{(k+1)}(t) \) for \( t \in [-r, T] \) can be proved.

Let \( u(t) \in S(\alpha, \beta) \) be a solution of the initial value problem (1), (2). Consider the pair of functions \( (u, u) \) which is a pair of quasisolutions of the initial value problem (1), (2). According to the proof given above the inequality \( V(t) \leq u(t) \leq W(t) \) holds for \( t \in [-r, T] \). □

### 3.2. Scalar case

Note that in the scalar case \( n = 1 \) the problem (1), (2) reduces to an initial value problem for a scalar differential equation with “maxima”. In this case we use lower and upper solutions:

**Definition 4.** The function \( v \in C([-r, T], \mathbb{R}) \), is called a lower solution of the initial value problem for the differential equation with “maxima” (1), (2) \((n = 1)\) if

\[
v \leq f(t, v(t), \max_{s \in [-r, t]} v(s)) \quad (22)
\]

\[
v(t) \leq \varphi(t), \quad t \in [-r, 0]. \quad (23)
\]

Analogously the upper solution of the initial value problem for the differential equation with “maxima” (1), (2) \((n = 1)\) is defined.

Then the following result is a partial case of the theorem proved above:

**Theorem 3.2.** Let the following conditions be fulfilled:

1. The function \( \varphi \in C([-r, 0], \mathbb{R}) \).
2. The functions \( \alpha, \beta \in C([0, T], \mathbb{R}) \) are lower and upper solutions of the initial value problem (1), (2) for \( n = 1 \), correspondingly, and \( \alpha(t) \leq \beta(t) \) for \( t \in [-r, T] \), and \( \alpha(0) - \varphi(0) \leq \alpha(t) - \varphi(t), \beta(0) - \varphi(0) \geq \beta(t) - \varphi(t) \) for \( t \in [-r, 0] \).
3. The function \( f \in C([0, T] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}) \), and for \( x, y, u, v \in \mathbb{R}, y \leq x, v \leq u \) the inequality

\[
f(t, x, u) - f(t, y, v) \geq -M(x - y) - N(u - v), \quad t \in [0, T],
\]

holds, where \( M, N \) are positive constants.
4. The inequality \( (M + N)T < 1 \) holds.
Then there exist two sequences of functions \( \{\alpha^{(k)}(t)\}^\infty_0 \) and \( \{\beta^{(k)}(t)\}^\infty_0 \) such that:

a) the functions \( \alpha^{(k)}(t), \beta^{(k)}(t) \in C([-r, T], \mathbb{R}) \) are solutions of the initial value problems for the following scalar equations

\[
(\alpha^{(k)}(t))' + M\alpha^{(k)}(t) + N \max_{s \in [t-r, t]} \alpha^{(k)}(s) = \psi(t, \alpha^{(k-1)}), \quad \text{for } t \in [0, T],
\]

and

\[
(\beta^{(k)}(t))' + M\beta^{(k)}(t) + N \max_{s \in [t-r, t]} \beta^{(k)}(s) = \psi(t, \beta^{(k-1)}), \quad \text{for } t \in [0, T],
\]

with initial conditions

\[
\alpha^{(k)}(t) = \varphi(t), \quad \beta^{(k)}(t) = \varphi(t), \quad t \in [-r, 0],
\]

where

\[
\psi(t, \eta) = f(t, \eta, \max_{s \in [t-r, t]} \eta(s)) + M\eta(t) + N \max_{s \in [t-r, t]} \eta(s);
\]

b) Both sequences are increasing and decreasing correspondingly;

c) Each function \( \alpha^{(k)}(t) \) is a lower solution and each function \( \beta^{(k)}(t) \) is an upper solution of the initial value problem for the scalar nonlinear differential equation with “maxima” (1), (2) \((n = 1)\);

d) Both sequences uniformly converge on \([-r, T]\);

e) The limits \( V(t) = \lim_{k \to \infty} \alpha^{(k)}(t) \), \( W(t) = \lim_{k \to \infty} \beta^{(k)}(t) \) are minimal and maximal solutions, correspondingly, of the initial value problem for the nonlinear differential equation with “maxima” (1), (2) \((n = 1)\);

f) If \( u(t) \in S(\alpha, \beta) \) is a solution of the initial value problem for the nonlinear differential equation with “maxima” (1), (2), then \( V(t) \leq u(t) \leq W(t) \).

Remark 5. As partial cases of the above results, several results follow for the initial value problem for nonlinear differential equations obtained in monograph [11].

4. Computer realization of the method

Since differential equations with “maxima” are still a new branch in the theory of differential equations, there is no mathematical software for solving integrals, that involve the maximum of the unknown function. We will give an algorithm for numerically calculating the integral with “maxima” and it will be illustrated on a particular example, combining this algorithm with the above suggested procedure for approximate solving of the initial value problem for differential equations with “maxima”.

Consider the following scalar differential equation with “maxima”

\[
x' = \frac{1}{8}e^{-t}x(t) - \frac{1}{4} \max_{s \in [t-0.5, t]} x(s), \quad \text{for } t \in [0, 2],
\]

with initial condition

\[
x(t) = 0, \quad t \in [-0.5, 0].
\]

It is easy to check that problem (27), (28) has zero solution.

From the inequality \( \frac{1}{8}e^{-t} - \frac{1}{4} \leq \frac{1}{8} - \frac{1}{4} < 0 \) on \([0, 2]\) it follows that the function \( \alpha^{(0)}(t) \equiv -2 \) is a lower solution of the initial value problem (27), (28) and the function \( \beta^{(0)}(t) \equiv 2 \) is an upper solution of (27), (28), i.e. inequalities \((\alpha^{(0)}(t))' \leq \frac{1}{8}e^{-t}\alpha^{(0)}(t) - \frac{1}{4} \max_{s \in [t-0.5, t]} \alpha^{(0)}(s) \) and \((\beta^{(0)}(t))' \geq \frac{1}{8}e^{-t}\beta^{(0)}(t) - \frac{1}{4} \max_{s \in [t-0.5, t]} \beta^{(0)}(s) \) hold.

In this case \( f(t, u, v) = \frac{1}{8}e^{-t}u - \frac{1}{4} - \frac{1}{4}v \) and \( f(t, x, u) - f(t, y, v) = \frac{1}{8}e^{-t}(x - y) - \frac{1}{4} (u - v) \geq -M(x - y) - N(u - v) \) for \( x \geq y, \ u \geq v \), where \( M = \frac{1}{8} \) and \( N = \frac{1}{4} \). Then \( (M + N)T = (\frac{1}{8} + \frac{1}{4})2 = \frac{3}{4} < 1 \).

Then the successive approximations to zero solution of the initial value problem (27), (28) are solutions of linear differential equations with “maxima” (24), (25), which are reduced in this case to the following equations

\[
(\alpha^{(k)}(t))' = -\frac{1}{8}\alpha^{(k)}(t) - \frac{1}{4} \max_{s \in [t-0.5, t]} \alpha^{(k)}(s) + \frac{1}{8}(e^{-t} + 1)\alpha^{(k-1)}(t), \quad \text{for } t \in [0, 2],
\]

and

\[
(\beta^{(k)}(t))' = -\frac{1}{8}\beta^{(k)}(t) - \frac{1}{4} \max_{s \in [t-0.5, t]} \beta^{(k)}(s) + \frac{1}{8}(e^{-t} + 1)\beta^{(k-1)}(t), \quad \text{for } t \in [0, 2],
\]

with initial conditions

\[
\alpha^{(k)}(t) = 0, \quad \beta^{(k)}(t) = 0, \quad t \in [-0.5, 0].
\]
The solution of the initial value problem (29), (31) is given by the formula

\[
\alpha^{(k)}(t) = \left\{ \int_0^t \frac{1}{8} (e^{-t} + 1) \alpha^{(k-1)}(s) ds - 0.25 \int_0^t \max_{\xi \in [l-0.5, s]} \alpha^{(k)}(\xi) ds \right\} (e^{0.125t} - 1) \quad \text{for} \ t \in [0, 2] \quad (32)
\]

\[
\alpha^{(k)}(t) = 0 \quad \text{for} \ t \in [-0.5, 0]
\]

and the solution of the initial value problem (30), (31) is given by the formula

\[
\beta^{(k)}(t) = \left\{ \int_0^t \frac{1}{8} (e^{-t} + 1) \beta^{(k-1)}(s) ds - 0.25 \int_0^t \max_{\xi \in [l-0.5, s]} \beta^{(k)}(\xi) ds \right\} (e^{0.125t} - 1) \quad \text{for} \ t \in [0, 2] \quad (33)
\]

\[
\beta^{(k)}(t) = 0 \quad \text{for} \ t \in [-0.5, 0].
\]

Algorithm for calculating lower and upper solutions:

Choose a natural number \( m \) and points \( t_i \in [0, 2], i = 1, 2, \ldots, m \) such that \( t_{i+1} - t_i = \delta \).

We will describe the algorithm for obtaining \( \alpha^{(k)}(t_i) \) for \( k = 1, 2, \ldots, \), starting from \( \alpha^{(0)}(t) \equiv -2 \) for \( t \in [-0.5, 2] \).

Note that formula (32) for obtaining \( \alpha^{(k)}(t) \) involves the integral

\[
L_k(t) = \int_0^t \max_{\xi \in [l-0.5, s]} \alpha^{(k)}(\xi) ds,
\]

that contains the local maximum of the function \( \alpha^{(k)}(t) \) on a previous interval with a length \( r = 0.5 \).

We calculate numerically the integral \( L_k(t_i) \) and at the same time we obtain the value of the function \( \alpha^{k}(t_i) \). The value of the function \( \alpha^{(k)}(t_i) \) will be used for obtaining the maximum on the next step of calculating the integral \( I_k(t_{i+1}) \).

Now we will explain the algorithm for obtaining the value of the integral \( I_k(t_i) \), that contains the maximum of the unknown function (that algorithm will also be applied for calculating the value of \( I_k(t_i) \)). The algorithm is based on the application of the trapezoid’s method.

Let \( T_i \) be fixed. Then the algorithm for obtaining the value of the integral \( I = \int_0^{T_i} \max_{\xi \in [l-0.5, s]} f(\xi) ds \) is as follows:

Open a cycle by \( t \) from 0 to \( T_i \) with step \( h \), which is chosen by us. The body of the cycle consists of

- Calculating the value of the function \( f(t) \);
- Applying the trapezoid’s method, we add the term \( \frac{f(t-1)+f(t)}{2h} \) to the previous value of the integral \( I \);
- Calculating the value of the function \( \alpha^{(k)}(t) \) by formula (32) for \( k = 1 \);
- Searching the new maximum in the previous interval by using the function \( \text{find\_new\_max} \).

Now we will describe more precisely the function \( \text{find\_new\_max} \).

In the case of a monotonic function \( \alpha^{(k)}(t) \), obtaining the maximum is trivial.

Consider the general case:

To obtain \( \max_{\xi \in [l-r, t]} \alpha^{(k)}(\xi) \), i.e. the maximum of the function over the previous interval, we will use a doubly linked list \( \text{dlist} \) and a circular array \( \text{value\_a} \) which contains the pointers to the elements of the \( \text{dlist} \). The number of its elements depends on the step \( h \) for calculating the value of the integral \( I \). It is equal to \( \lceil \frac{r}{h} \rceil \).

The doubly linked list contains the sorted decreasing values of the function \( \alpha^{(k)}(t) \) on the previous interval of length \( r \).

The beginning of the doubly linked list contains the maximal element and the end contains the minimal element. We will use pointers to the beginning and to the end of the list, and a pointer to the last inserted element. The list is doubly linked, so we could search in both directions.

At any step of the cycle we will add as a new element a value of \( \alpha^{(k)}(t) \) and will remove the element at the left end of the interval.

To do this, we use the following operations:

- Insert the new element into the doubly linked list: Compare the new value of \( \alpha^{(k)}(t) \) and the value of the most recently inserted element. If the new value is bigger, then we search in the direction of bigger values in the list until we find the element of value bigger than or equal to the new value. Then we put this new element at this position into the doubly linked list and we save the pointer of the new element in the right end of the array \( \text{value\_a} \).
- If the new value is less than the most recent one we find the position of the element in the direction of smaller values in the list.
- Since the function \( \alpha^{(k)}(t) \) is continuous, the value of the function at the new point will be approximate to the most recent one and finding the position of the new element in the sorted list will be done in several steps in one of the two directions. The number of these steps depends on the number of local extrema in the interval.

- Removing the element which is out of the interval: From the left end of the array \( \text{value\_a} \) we take the pointer to the element in \( \text{dlist} \) and remove this element from the list. Element removal does not depend on the sorting of the list.
- At every step we can take the maximum element using the pointer to the beginning of the \( \text{dlist} \).
The order of the operations connected with finding of the position of the new element in the list is \(O(mn)\), where \(n\) is the number of steps of the main cycle, and \(m\) is the maximal number of the local extrema in the present interval.

In the case of \(h = 1.00 \times 10^{-7}\) the numerical values are given in Table 1.

**Table 1**

Numerical values of lower solutions \(\alpha^{(i)}(t)\) and upper solutions \(\beta^{(i)}(t)\).

<table>
<thead>
<tr>
<th>(t)</th>
<th>0.25</th>
<th>0.50</th>
<th>0.75</th>
<th>1.00</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\alpha_1 = -2.00)</td>
<td>(-3.74 \times 10^{-3})</td>
<td>(-1.44 \times 10^{-2})</td>
<td>(-3.14 \times 10^{-2})</td>
<td>(-5.42 \times 10^{-2})</td>
</tr>
<tr>
<td>(\alpha_2 = -3.74 \times 10^{-3})</td>
<td>(-6.99 \times 10^{-6})</td>
<td>(-2.69 \times 10^{-5})</td>
<td>(-5.87 \times 10^{-5})</td>
<td>(-1.01 \times 10^{-4})</td>
</tr>
<tr>
<td>(\alpha_3 = -6.99 \times 10^{-6})</td>
<td>(-0.01 \times 10^{-6})</td>
<td>(-0.05 \times 10^{-6})</td>
<td>(-0.11 \times 10^{-6})</td>
<td>(-0.19 \times 10^{-6})</td>
</tr>
<tr>
<td>(\alpha_4 = -0.01 \times 10^{-6})</td>
<td>(-0.00 \times 10^{-6})</td>
<td>(-0.00 \times 10^{-6})</td>
<td>(-0.00 \times 10^{-6})</td>
<td>(-0.00 \times 10^{-6})</td>
</tr>
<tr>
<td>(x(t) = 0.00)</td>
<td>(0.00 \times 10^{-6})</td>
<td>(0.00 \times 10^{-6})</td>
<td>(0.00 \times 10^{-6})</td>
<td>(0.00 \times 10^{-6})</td>
</tr>
<tr>
<td>(\beta_4 = 0.01 \times 10^{-6})</td>
<td>(0.00 \times 10^{-6})</td>
<td>(0.00 \times 10^{-6})</td>
<td>(0.00 \times 10^{-6})</td>
<td>(0.00 \times 10^{-6})</td>
</tr>
<tr>
<td>(\beta_3 = 6.98 \times 10^{-6})</td>
<td>(0.01 \times 10^{-6})</td>
<td>(0.05 \times 10^{-6})</td>
<td>(0.11 \times 10^{-6})</td>
<td>(0.19 \times 10^{-6})</td>
</tr>
<tr>
<td>(\beta_2 = 3.74 \times 10^{-6})</td>
<td>(6.98 \times 10^{-6})</td>
<td>(2.68 \times 10^{-5})</td>
<td>(5.83 \times 10^{-5})</td>
<td>(1.00 \times 10^{-4})</td>
</tr>
<tr>
<td>(\beta_1 = 2.00)</td>
<td>(3.74 \times 10^{-3})</td>
<td>(1.44 \times 10^{-2})</td>
<td>(3.12 \times 10^{-2})</td>
<td>(5.37 \times 10^{-2})</td>
</tr>
</tbody>
</table>

**Conclusions.** The results in Table 1 gives the values of four lower solutions \(\alpha^{(i)}(t)\) and four upper solutions \(\beta^{(i)}(t)\), \(i = 1, 2, 3, 4\), calculated on the interval \([0, 2]\) by a step 0.25. Calculations are based on formulas (32), (33). The numerical values of the lower and upper solutions prove numerically the convergence of both sequences to zero solution of the initial value problem (27), (28).

**Remark 6.** Note that in the algorithm, suggested above, the trapezoid’s method could be replaced by any other numerical method for integrals, which produces the desired level of error.

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**References**