Generalized Hypergeometric Functions Associated with $k$-Uniformly Convex Functions

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Abstract—For a certain linear operator which is defined by means of the Hadamard product (or convolution) with a generalized hypergeometric function, the authors aim at investigating various mapping and inclusion properties involving such subclasses of analytic and univalent functions as (for example) $k$-uniformly convex functions and $k$-starlike functions. Relevant connections of the definitions and results presented in this paper with those in several earlier works on the subject are also pointed out. © 2002 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION AND DEFINITIONS

Let $A$ be the class of functions $f$ normalized by
\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \] (1)
which are analytic in the open unit disk
\[ U := \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \}. \]
As usual, we denote by $S$ the subclass of $A$ consisting of functions which are also univalent in $U$. Furthermore, we denote by $k$-$UCV$ and $k$-$ST$ two interesting subclasses of $S$ consisting, respectively, of functions which are $k$-uniformly convex and $k$-starlike in $U$. Thus, we have

$$k$-$UCV := \left\{ f \in S : \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) \, k \left| \frac{zf''(z)}{f'(z)} \right| (z \in U; \, 0 \leq k < \infty) \right\}$$

and

$$k$-$ST := \left\{ f \in S : \Re \left( \frac{zf'(z)}{f(z)} \right) \, k \left| \frac{zf'(z)}{f(z)} - 1 \right| (z \in U; \, 0 \leq k < \infty) \right\}.$$

The class $k$-$UCV$ was introduced by Kanas and Wiśniowska [1], where its geometric definition and connections with the conic domains were considered. The class $k$-$ST$ was investigated in [2]; in fact, it is related to the class $k$-$UCV$ by means of the well-known Alexander equivalence between the usual classes of convex and starlike functions (see also the work of Kanas and Srivastava [3] for further developments involving each of the classes $k$-$UCV$ and $k$-$ST$). In particular, when $k = 1$, we obtain

$$1$-$UCV \equiv UCV \quad \text{and} \quad 1$-$ST \equiv SP,$$

where $UCV$ and $SP$ are the familiar classes of uniformly convex functions and parabolic starlike functions in $U$, respectively (see, for details, [4–7]). In fact, by making use of a certain fractional calculus operator, Srivastava and Mishra [8] presented a systematic and unified study of the classes $UCV$ and $SP$.

A function $f \in A$ is said to be in the class $R'(A, B)$ if it satisfies the inequality

$$\left| \frac{f'(z) - 1}{(A - B) \tau - B [f'(z) - 1]} \right| < 1 \quad (z \in U; \, \tau \in \mathbb{C} \setminus \{0\}; \, -1 \leq B < A \leq 1).$$

The class $R'(A, B)$ was introduced earlier by Dixit and Pal [9]. Two of the many interesting subclasses of the class $R'(A, B)$ are worthy of mention here. First of all, by setting

$$\tau = e^{-i\eta} \cos \eta \left( -\frac{\pi}{2} < \eta < \frac{\pi}{2} \right), \quad A = 1 - 2\beta \quad (0 \leq \beta < 1) \quad \text{and} \quad B = -1,$$

the class $R'(A, B)$ reduces essentially to the class $R_\eta(\beta)$ studied recently by Ponnusamy and Renning [10], where

$$R_\eta(\beta) := \left\{ f \in A : \Re \left( e^{i\eta} (f'(z) - \beta) \right) > 0 \quad (z \in U; \, -\frac{\pi}{2} < \eta < \frac{\pi}{2}; \, 0 \leq \beta < 1) \right\}.$$

Second, if we put

$$\tau = 1, \quad A = \beta, \quad \text{and} \quad B = -\beta \quad (0 < \beta \leq 1),$$

we obtain the class of functions $f \in A$ satisfying the inequality

$$\left| \frac{f'(z) - 1}{f'(z) + 1} \right| < \beta \quad (z \in U; \, 0 < \beta \leq 1),$$

which was studied by (among others) Padmanabhan [11] and Caplinger and Causey [12].

Next, we introduce the classes $S^\star_\lambda$ and $C_\lambda$ by (cf., e.g., [10, p. 142] for the class $S^\star_\lambda$)

$$S^\star_\lambda := \left\{ f \in A : \left| \frac{zf'(z)}{f(z)} - 1 \right| < \lambda \quad (z \in U; \, \lambda > 0) \right\}$$

and

$$C_\lambda := \left\{ f \in A : \left| \frac{zf''(z)}{f'(z)} \right| < \lambda \quad (z \in U; \, \lambda > 0) \right\},$$
so that, obviously,

\[ f(z) \in C_\lambda \iff zf'(z) \in S^*_\lambda \quad (\lambda > 0) , \]

which is analogous to the aforementioned Alexander equivalence (see, for details, [13]).

Finally, we recall a sufficiently adequate special case of a convolution operator which was introduced earlier by Dziok and Srivastava [14] by means of the Hadamard product (or convolution) involving generalized hypergeometric functions. Indeed, by employing the Pochhammer symbol (or the shifted factorial, since \((1)_n = n!\) \((\lambda)_n\) given, in terms of the gamma functions, by

\[
(\lambda)_n := \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 
1, & n = 0, \\
\lambda (\lambda + 1) \cdots (\lambda + n - 1), & n \in \mathbb{N} := \{1, 2, 3, \ldots\} ,
\end{cases}
\]

(10)
a generalized hypergeometric function \(pF_q\) with \(p\) numerator parameters \(\alpha_j \in \mathbb{C}\) \((j = 1, \ldots, p)\) and \(q\) denominator parameters \(\beta_j \in \mathbb{C}\backslash \mathbb{Z}^-\) \((Z^- := \{0, -1, -2, \ldots\}; j = 1, \ldots, q)\) is defined by (cf., e.g., [15, p. 19 et seq.])

\[
pF_q(z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n \beta_1 \cdots \beta_q \gamma^n}{n!} ,
\]

(11)

where an empty product is to be interpreted as \(1\) and

\[
\omega := \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j ,
\]

(12)

Thus, we have [14, p. 3]

\[
\left( f^{\alpha_1, \ldots, \alpha_p \beta_1, \ldots, \beta_q} \right)(z) := z \cdot pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) * f(z)
\]

(13)

so that, for a function \(f\) of form (1),

\[
\left( f^{\alpha_1, \ldots, \alpha_p \beta_1, \ldots, \beta_q} \right)(z) = z + \sum_{n=2}^{\infty} \Gamma_n \alpha_n z^n ,
\]

(14)

where, for convenience,

\[
\Gamma_n := \frac{(\alpha_1)_{n-1} \cdots (\alpha_p)_{n-1} \beta_1 \cdots \beta_q}{(\beta_1)_{n-1} \cdots (\beta_q)_{n-1} (n-1)!} \quad (n \in \mathbb{N} \backslash \{1\}) .
\]

(15)

Just as it was observed by Dziok and Srivastava [14, pp. 3–4], the convolution operator defined by (13) includes, as its further special cases, various other linear operators which were considered in many earlier works. In particular, for \(p = 2\) and \(q = 1\), we obtain the linear operator \(\mathcal{F}(\alpha, \beta, \gamma)\) defined by

\[
\mathcal{F}(\alpha, \beta, \gamma) f)(z) := z \cdot 2F1(\alpha, \beta, \gamma; z) * f(z) = (f^\alpha, \beta, \gamma f)(z) ,
\]

(16)

which was investigated by Hohlov [16].
It should be remarked here that many univalence, starlikeness, and convexity properties of the hypergeometric functions
\[ z_2 F_1 (\alpha, \beta; \gamma; z) \]
and
\[ z_\mu F_p (\alpha_1, \ldots, \alpha_\mu; \beta_1, \ldots, \beta_q; z) \quad (p \leq q + 1) \]
were investigated in a number of earlier works (cf., e.g., [17–19]; see also [20]).

Our main objective in the present paper is to make use of the linear operator defined by (13) in order to establish a number of connections between the classes \( k-UCV \), \( k-ST \), \( \mathcal{R}^\tau (A, B) \), and various other subclasses of \( \mathcal{A} \) including (for example) the classes \( S_\alpha^* \) and \( C_\lambda \) defined by (7) and (8), respectively.

Each of the following lemmas will be required in our investigation.

**Lemma 1.** (See [9].) If \( f \in \mathcal{R}^\tau (A, B) \) is of form (1), then
\[
|a_n| \leq (A - B) \frac{|\tau|}{n} \quad (n \in \mathbb{N} \setminus \{1\}).
\] (17)

The estimate in (17) is sharp for the function
\[
f(z) = \int_0^1 \left( 1 + (A - B) \frac{\tau t^{n-1}}{1 + B t^{n-1}} \right) dt \quad (z \in \mathbb{U}; \ n \in \mathbb{N} \setminus \{1\}).
\] (18)

**Lemma 2.** (See [9].) Let \( f \in \mathcal{A} \) be of form (1). If
\[
\sum_{n=2}^{\infty} (1 + |B|) n |a_n| \leq (A - B) |\tau| \quad (-1 \leq B < A \leq 1; \ \tau \in \mathbb{C} \setminus \{0\}),
\] (19)

then \( f \in \mathcal{R}^\tau (A, B) \). The result is sharp for the function
\[
f(z) = z + \frac{(A - B) \tau}{(1 + |B|) n} z^n \quad (z \in \mathbb{U}; \ n \in \mathbb{N} \setminus \{1\}).
\] (20)

**Lemma 3.** (See [11].) Let \( f \in \mathcal{A} \) be of form (1). If, for some \( k \) \( (0 \leq k < \infty) \), the following inequality:
\[
\sum_{n=2}^{\infty} n (n - 1) |a_n| \leq \frac{1}{k + 2}
\] (21)

holds true, then \( f \in k-UCV \). The number \( 1/(k + 2) \) cannot be increased.

**Lemma 4.** (See [2].) Let \( f \in \mathcal{A} \) be of form (1). If, for some \( k \) \( (0 \leq k < \infty) \), the following inequality:
\[
\sum_{n=2}^{\infty} \{n + (n - 1) k\} |a_n| < 1
\] (22)

holds true, then \( f \in k-ST \).

### 2. MAPPING AND INCLUSION PROPERTIES INVOLVING THE CLASSES \( k-UCV \) AND \( k-ST \)

We begin by proving a mapping and inclusion property of the convolution operator defined by (13) involving the class \( k-UCV \).
THEOREM 1. Suppose that
\[ \alpha_j \in \mathbb{C} \setminus \{0\} \quad (j = 1, \ldots, p), \quad \Re (\beta_j) > 0 \quad (j = 1, \ldots, q), \]
and (in the case when \( p = q + 1 \))
\[ \Re \left( \sum_{j=1}^{q} \beta_j \right) > 1 + \sum_{j=1}^{p} |\alpha_j|. \]

If \( f \in \mathcal{R}^\sim(A, B) \) and, for some \( k \) (\( 0 \leq k < \infty \)), the hypergeometric inequality
\[ pF_q ([\alpha_1] + 1, \ldots, [\alpha_p] + 1; \Re (\beta_1) + 1, \ldots, \Re (\beta_q) + 1; 1) \]
\[ \leq \frac{\Re (\beta_1) \cdots \Re (\beta_q)}{(k + 2) (A - B) |\tau| \cdot |\alpha_1 \cdots \alpha_p|} \quad (0 \leq k < \infty) \tag{23} \]
holds true, then
\[ \int_{\beta_1, \ldots, \beta_q} \alpha_1^{\alpha_1} \cdots \alpha_p^{\alpha_p} \, f \in k-U \mathcal{CV}. \]

PROOF. First of all, under the first two parametric constraints stated in Theorem 1, it is easily seen from (15) and (10) that
\[ |\Gamma_n| = \frac{|(\alpha_1)_{n-1} \cdots (\alpha_p)_{n-1}| \cdot 1}{|(\beta_1)_{n-1} \cdots (\beta_q)_{n-1}| \cdot (n-1)!} \]
\[ \leq \frac{|(\alpha_1)_{n-1} \cdots (\alpha_p)_{n-1}|}{\Re (\beta_1) \cdots \Re (\beta_q) \cdot |\alpha_1 \cdots \alpha_p|} \cdot \frac{1}{(n-1)!} \]
\[ = \frac{1}{n-1} \cdot \frac{(\Re (\beta_1) \cdots \Re (\beta_q))^{n-2} \cdots (\Re (\beta_1) + 1)_{n-2} \cdots (\Re (\beta_q) + 1)_{n-2}}{(n-2)!} \cdot 1 \quad (n \in \mathbb{N} \setminus \{1\}). \tag{24} \]

Thus, for \( f \in \mathcal{R}^\sim(A, B) \) of form (1), by applying Lemma 1 in conjunction with (24), we have
\[ \sum_{n=2}^{\infty} n (n-1) |\Gamma_n| \cdot |a_n| \leq \frac{(A - B) |\tau| \cdot |\alpha_1 \cdots \alpha_p|}{\Re (\beta_1) \cdots \Re (\beta_q)} \]
\[ \cdot \sum_{n=2}^{\infty} \frac{|(\alpha_1)_{n-1} \cdots (\alpha_p)_{n-2} \cdots (\alpha_p)_{n-2}|}{(\Re (\beta_1) + 1)_{n-2} \cdots (\Re (\beta_q) + 1)_{n-2}} \cdot \frac{1}{(n-2)!} \]
\[ = \frac{(A - B) |\tau| \cdot |\alpha_1 \cdots \alpha_p|}{\Re (\beta_1) \cdots \Re (\beta_q)} \cdot pF_q ([\alpha_1] + 1, \ldots, [\alpha_p] + 1; \Re (\beta_1) + 1, \ldots, \Re (\beta_q) + 1; 1), \tag{25} \]
where the convergence of the \( pF_q(1) \) series is guaranteed (when \( p = q + 1 \)) by the third parametric constraint stated in Theorem 1 by analogy with inequality (12).

Finally, if we make use of hypothesis (23) in (25), we find that
\[ \sum_{n=2}^{\infty} n (n-1) |\Gamma_n| \cdot |a_n| \leq \frac{1}{k + 2} \quad (0 \leq k < \infty), \tag{26} \]
which, in view of (14) and Lemma 3, immediately proves the mapping and inclusion property asserted by Theorem 1.
Theorem 1 can be applied to deduce the corresponding mapping and inclusion properties, involving the class $k\text{-UCV}$, for all those linear operators (listed by Dziok and Srivastava [14, pp. 3–4]), which happen to be further special cases of the convolution operator defined by (13). In particular, for the Hohlov operator $\mathcal{F}(\alpha, \beta, \gamma)$ defined by (16), by appealing to the Gauss summation theorem [15, p. 9, equation 1.2 (20)]

$$
\binom{2}{1} (a, b; c; 1) = \frac{\Gamma(c) \Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)} \quad (\Re(c - a - b) > 0; \ c \in \mathbb{C} \setminus \mathbb{Z}_0^-). \tag{27}
$$

Theorem 1 yields the following.

**Corollary 1.** Let $\gamma$ be a real number such that

$$
\gamma > |\alpha| + |\beta| + 1 \quad (\alpha, \beta \in \mathbb{C} \setminus \{0\}).
$$

If $f \in \mathcal{R}^+(A, B)$ and, for some $k$ ($0 \leq k < \infty$), the inequality

$$
\frac{\Gamma(\gamma) \Gamma(\gamma - |\alpha| - |\beta| - 1)}{\Gamma(\gamma - |\alpha|) \Gamma(\gamma - |\beta|)} \leq \frac{1}{(k + 2) (A - B) |\tau| \cdot |\alpha\beta|} \quad (0 \leq k < \infty) \tag{28}
$$

holds true, then

$$
\mathcal{F}(\alpha, \beta, \gamma) f \in k\text{-UCV}.
$$

In a similar manner, by applying Lemma 1 and Lemma 4 (instead of Lemma 3), we can prove the following mapping and inclusion property, involving the class $k\text{-ST}$, for the convolution operator defined by (13).

**Theorem 2.** Suppose that

$$
\alpha_j \in \mathbb{C} \setminus \{0\} \quad (j = 1, \ldots, p), \quad \Re(\beta_j) > 0 \quad (j = 1, \ldots, q),
$$

and (in the case when $p = q + 1$)

$$
\Re \left( \sum_{j=1}^{q} \beta_j \right) > \sum_{j=1}^{p} |\alpha_j|.
$$

If $f \in \mathcal{R}^+(A, B)$ and, for some $k$ ($0 \leq k < \infty$), the hypergeometric inequality

$$
\binom{k+1}{p} \binom{k+1}{p+1} (|\alpha_1|, \ldots, |\alpha_p|; \Re(\beta_1), \ldots, \Re(\beta_p); 1)
$$

$$
- k \binom{k+1}{p+1} (|\alpha_1|, \ldots, |\alpha_p|, 1; \Re(\beta_1), \ldots, \Re(\beta_q), 2; 1)
$$

$$
< 1 + 2k + \frac{1}{(A - B) |\tau|} \quad (0 \leq k < \infty) \tag{29}
$$

holds true, then

$$
\mathcal{I}^{\alpha_1, \ldots, \alpha_p}_{\beta_1, \ldots, \beta_q} f \in k\text{-ST}.
$$

For $p = 2$ and $q = 1$, Theorem 2 readily yields the following.

**Corollary 2.** Let $\gamma$ be a real number such that

$$
\gamma > |\alpha| + |\beta| \quad (\alpha, \beta \in \mathbb{C} \setminus \{0\}).
$$

If $f \in \mathcal{R}^+(A, B)$ and, for some $k$ ($0 \leq k < \infty$), the hypergeometric inequality

$$
k \binom{k+1}{3} \binom{k+1}{2} (|\alpha|, |\beta|, 1; \gamma, 2; 1) > (k + 1) \frac{\Gamma(\gamma) \Gamma(\gamma - |\alpha| - |\beta|)}{\Gamma(\gamma - |\alpha|) \Gamma(\gamma - |\beta|)} - 2k - 1 - \frac{1}{(A - B) |\tau|} \quad (0 \leq k < \infty) \tag{30}
$$
holds true, then
\[ F(\alpha, \beta, \gamma) f \in k-ST. \]

Next, for a function \( f \) of form (1) and belonging to the class \( k \mathcal{UCV} \), the following coefficient inequalities hold true (cf., [1]):
\[ |a_n| \leq \left( \frac{P_1}{n!} \right)^{n-1} \quad (n \in \mathbb{N} \setminus \{1\}), \tag{31} \]
where \( P_1 = P_1(k) \) is the coefficient of \( z \) in the function
\[ p_k(z) = 1 + \sum_{n=1}^{\infty} P_n(k) z^n, \tag{32} \]
which is the extremal function for the class \( \mathcal{P}(p_k) \) related to the class \( k-\mathcal{UCV} \) by the range of the expression
\[ 1 + \frac{zf''(z)}{f'(z)} \quad (z \in \mathbb{U}). \]
Similarly, if \( f \) of form (1) belongs to the class \( k-ST \), then (cf., [2])
\[ |a_n| \leq \left( \frac{P_1}{(n-1)!} \right)^{n-1} \quad (n \in \mathbb{N} \setminus \{1\}), \tag{33} \]
where \( P_1 = P_1(k) \) is given, as above, by (32).

Making use of the coefficient inequalities (31) and (33), in place of the coefficient inequality (17) asserted by Lemma 1, we can establish each of the following results (Theorems 3 and 4 below) by appealing appropriately to Lemmas 3 and 4, respectively.

**THEOREM 3.** Suppose that
\[ \alpha_j \in \mathbb{C} \setminus \{0\} \quad (j = 1, \ldots, p), \quad \Re(\beta_j) > 0 \quad (j = 1, \ldots, q), \]
and (in the case when \( p = q + 1 \))
\[ \Re \left( \sum_{j=1}^{q} \beta_j \right) > P_1 + \sum_{j=1}^{p} |\alpha_j|, \]
where \( P_1 = P_1(k) \) is given, as before, by (32). If, for some \( k \) (\( 0 \leq k < \infty \)), \( f \in k-\mathcal{UCV} \) and the hypergeometric inequality
\[ _{p+1}F_{q+1} \left( |\alpha_1| + 1, \ldots, |\alpha_p| + 1; P_1 + 1; \Re(\beta_1) + 1, \ldots, \Re(\beta_q) + 1, 2; 1 \right) \leq \frac{\Re(\beta_1) \cdots \Re(\beta_q)}{(k + 2) |\alpha_1 \cdots \alpha_p| P_1} \quad (0 \leq k < \infty) \tag{34} \]
holds true, then
\[ f_{\beta_1, \ldots, \beta_q}^{|\alpha_1, \ldots, \alpha_p|} f \in k-\mathcal{UCV}. \]

**THEOREM 4.** Suppose that
\[ \alpha_j \in \mathbb{C} \setminus \{0\} \quad (j = 1, \ldots, p), \quad \Re(\beta_j) > 0 \quad (j = 1, \ldots, q), \]
and (in the case when \( p = q + 1 \))
\[ \Re \left( \sum_{j=1}^{q} \beta_j \right) > P_1 + \sum_{j=1}^{p} |\alpha_j|, \]
where \( P_1 = P_1(k) \) is given, as before, by (32). If, for some \( k (0 \leq k < \infty) \), \( f \in k-ST \) and the hypergeometric inequality
\[
\frac{(k+1)|\alpha_1 \cdots \alpha_p|P_1}{\Re(\beta_1) \cdots \Re(\beta_q)} p+1 F_{q+1} (|\alpha_1| + 1, \ldots, |\alpha_p| + 1, P_1 + 1; \Re(\beta_1) + 1, \ldots, \Re(\beta_q) + 1, 2; 1) \\
+ p+1 F_{q+1} (|\alpha_1|, \ldots, |\alpha_p|, P_1; \Re(\beta_1), \ldots, \Re(\beta_q), 1; 1) < 2 \quad (0 \leq k < \infty)
\]
holds true, then
\[
I_{\beta_1, \ldots, \beta_q}^\alpha f \in k-ST.
\]

The following (seemingly interesting) variants of Theorems 3 and 4 can also be proven similarly, and we omit the details involved.

**Theorem 5.** Suppose that
\[
\alpha_j \in \mathbb{C} \setminus \{0\} \quad (j = 1, \ldots, p), \quad \Re(\beta_j) > 0 \quad (j = 1, \ldots, q),
\]
and (in the case when \( p = q + 1 \))
\[
\Re \left( \sum_{j=1}^q \beta_j \right) > P_1 - 1 + \sum_{j=1}^p |\alpha_j|,
\]
where \( P_1 = P_1(k) \) is given, as before, by (32). If, for some \( k (0 \leq k < \infty) \), \( f \in k-UCV \) and the hypergeometric inequality
\[
\frac{(k+1)|\alpha_1 \cdots \alpha_p|P_1}{2\Re(\beta_1) \cdots \Re(\beta_q)} p+2 F_{q+2} (|\alpha_1| + 1, \ldots, |\alpha_p| + 1, P_1 + 1; \Re(\beta_1) + 1, \ldots, \Re(\beta_q) + 1, 3; 1) \\
+ p+2 F_{q+2} (|\alpha_1|, \ldots, |\alpha_p|, P_1; \Re(\beta_1), \ldots, \Re(\beta_q), 2; 1) < 2 \quad (0 \leq k < \infty)
\]
holds true, then
\[
I_{\beta_1, \ldots, \beta_q}^\alpha f \in k-ST.
\]

**Theorem 6.** Suppose that
\[
\alpha_j \in \mathbb{C} \setminus \{0\} \quad (j = 1, \ldots, p), \quad \Re(\beta_j) > 0 \quad (j = 1, \ldots, q),
\]
and (in the case when \( p = q + 1 \))
\[
\Re \left( \sum_{j=1}^q \beta_j \right) > P_1 + 1 + \sum_{j=1}^p |\alpha_j|,
\]
where \( P_1 = P_1(k) \) is given, as before, by (32). If, for some \( k (0 \leq k < \infty) \), \( f \in k-ST \) and the hypergeometric inequality
\[
p+2 F_{q+2} (|\alpha_1| + 1, \ldots, |\alpha_p| + 1, P_1 + 1, 3; \Re(\beta_1) + 1, \ldots, \Re(\beta_q) + 1, 2, 2; 1) \\
\leq \frac{\Re(\beta_1) \cdots \Re(\beta_q)}{2(k+2)|\alpha_1 \cdots \alpha_p| P_1} \quad (0 \leq k < \infty)
\]
holds true, then
\[
I_{\beta_1, \ldots, \beta_q}^\alpha f \in k-UCV.
\]

In its special case when \( p = 2 \) and \( q = 1 \), Theorem 3 reduces at once to the following known result.
COROLLARY 3. (See [3, p. 128, Theorem 2.5].) Let $\gamma$ be a real number such that
\[ \gamma = |\alpha| + |\beta| + P_1 \quad (\alpha, \beta \in \mathbb{C} \setminus \{0\}), \]
where $P_1 = P_1(k)$ is given, as before, by (32). If, for some $k$ ($0 \leq k < \infty$), $f \in k-\mathcal{UCV}$ and the hypergeometric inequality
\[ \binom{\gamma}{3}F_2 (|\alpha| + 1, |\beta| + 1, P_1 + 1; \gamma + 1, 2; 1) \leq \frac{\gamma}{(k + 2)|\alpha\beta| P_1} \quad (0 \leq k < \infty) \] (38)
holds true, then
\[ \mathcal{F}(\alpha, \beta, \gamma) f \in k-\mathcal{UCV}. \]

For $p = 2$ and $q = 1$, Theorem 4 immediately yields the following corrected version of another known result.

COROLLARY 4. (See [3, p. 130, Theorem 3.5].) Let $\gamma$ be a real number such that
\[ \gamma > |\alpha| + |\beta| + P_1 \quad (\alpha, \beta \in \mathbb{C} \setminus \{0\}), \]
where $P_1 = P_1(k)$ is given, as before, by (32). If, for some $k$ ($0 \leq k < \infty$), $f \in k-ST$ and the hypergeometric inequality
\[ \binom{k + 1}{\gamma} \frac{|\alpha\beta| P_1}{\gamma} \binom{3}F_2 (|\alpha| + 1, |\beta| + 1, P_1 + 1; \gamma + 1, 2; 1) + 3F_2 (|\alpha|, |\beta|, P_1; \gamma, 2; 1) < 2 \quad (0 \leq k < \infty) \] (39)
holds true, then
\[ \mathcal{F}(\alpha, \beta, \gamma) f \in k-ST. \]

Similar consequences of Theorems 5 and 6 would lead us, respectively, to Corollaries 5 and 6 below.

COROLLARY 5. Let $\gamma$ be a real number such that
\[ \gamma > P_1 - 1 + |\alpha| + |\beta| \quad (\alpha, \beta \in \mathbb{C} \setminus \{0\}), \]
where $P_1 = P_1(k)$ is given, as before, by (32). If, for some $k$ ($0 \leq k < \infty$), $f \in k-\mathcal{UCV}$ and the hypergeometric inequality
\[ \frac{(k + 1)|\alpha\beta| P_1}{2\gamma} \binom{3}F_2 (|\alpha| + 1, |\beta| + 1, P_1 + 1; \gamma + 1, 3; 1) + 3F_2 (|\alpha|, |\beta|, P_1; \gamma, 2; 1) < 2 \quad (0 \leq k < \infty) \] (40)
holds true, then
\[ \mathcal{F}(\alpha, \beta, \gamma) f \in k-ST. \]

COROLLARY 6. Let $\gamma$ be a real number such that
\[ \gamma > P_1 + 1 + |\alpha| + |\beta| \quad (\alpha, \beta \in \mathbb{C} \setminus \{0\}), \]
where $P_1 = P_1(k)$ is given, as before, by (32). If, for some $k$ ($0 \leq k < \infty$), $f \in k-ST$ and the hypergeometric inequality
\[ 4F_3 (|\alpha| + 1, |\beta| + 1, P_1 + 1, 3; \gamma + 1, 2, 2; 1) \leq \frac{\gamma}{2(k + 2)|\alpha\beta| P_1} \quad (0 \leq k < \infty) \] (41)
holds true, then
\[ \mathcal{F}(\alpha, \beta, \gamma) f \in k-\mathcal{UCV}. \]
3. MAPPING AND INCLUSION PROPERTIES INVOLVING THE CLASSES $S^*_\lambda$ AND $C_\lambda$

Just as in the work of Silverman [21, p. 110] on the familiar classes of starlike and convex functions of order $\mu$ ($0 \leq \mu < 1$), it is fairly straightforward to derive Lemmas 5 and 6 involving the function classes $S^*_\lambda$ and $C_\lambda$ defined by (7) and (8), respectively.

**Lemma 5.** Let $f \in A$ be of form (1). If

$$\sum_{n=2}^{\infty} (\lambda + n - 1) |a_n| \leq \lambda \quad (\lambda > 0),$$

then $f \in S^*_\lambda$.

**Lemma 6.** Let $f \in A$ be of form (1). If

$$\sum_{n=2}^{\infty} n (\lambda + n - 1) |a_n| \leq \lambda \quad (\lambda > 0),$$

then $f \in C_\lambda$.

Making use of Lemmas 5 and 6, in conjunction with the coefficient inequalities (31) and (33), we now prove several mapping and inclusion properties for the convolution operator defined by (13), which involve the function classes $S^*_\lambda$ and $C_\lambda$.

**Theorem 7.** Suppose that

$$\alpha_j \in \mathbb{C}\setminus \{0\} \quad (j = 1, \ldots, p), \quad \Re(\beta_j) > 0 \quad (j = 1, \ldots, q),$$

and (in the case when $p = q + 1$)

$$\Re\left(\sum_{j=1}^{q} \beta_j\right) > P_1 - 1 + \sum_{j=1}^{p} |\alpha_j|,$$

where $P_1 = P_1(k)$ is given, as before, by (32). If, for some $k$ ($0 \leq k < \infty$), $f \in k-UCV$ and the hypergeometric inequality

$$f_{p+2}F_{q+2} (|\alpha_1|, \ldots, |\alpha_p|, P_1, \lambda + 1; \Re(\beta_1), \ldots, \Re(\beta_q), \lambda, 2; 1) < 2 \quad (\lambda > 0)$$

holds true, then

$$I_{\beta_1, \ldots, \beta_q}^{|\alpha_1, \ldots, |\alpha_p|} f \in S^*_\lambda.$$

**Proof.** In view of Lemma 5, it suffices to show, for $f \in k-UCV$ of form (1), that

$$\sum_{n=2}^{\infty} (\lambda + n - 1) |a_n| \cdot |\Gamma_n| \leq \lambda \quad (\lambda > 0),$$

where $\Gamma_n$ is defined by (15). Indeed, by applying the coefficient inequalities (31), we observe that

$$\sum_{n=2}^{\infty} (\lambda + n - 1) |a_n| \cdot |\Gamma_n| \leq \sum_{n=2}^{\infty} (\lambda + n - 1) \frac{(P_1)_{n-1}}{n!} \cdot \frac{(|\alpha_1|)_{n-1} \cdots (|\alpha_p|)_{n-1}}{(|\Re(\beta_1)|)_{n-1} \cdots (|\Re(\beta_q)|)_{n-1}} \cdot \frac{1}{(n-1)!}$$

$$= \sum_{n=1}^{\infty} (\lambda + n) \frac{(P_1)_{n}}{(n+1)!} \cdot \frac{(|\alpha_1|)_{n} \cdots (|\alpha_p|)_{n}}{(\Re(\beta_1))_{n} \cdots (\Re(\beta_q))_{n}} \cdot \frac{1}{n!}$$

$$= \lambda \{ f_{p+2}F_{q+2} (|\alpha_1|, \ldots, |\alpha_p|, P_1, \lambda + 1; \Re(\beta_1), \ldots, \Re(\beta_q), \lambda, 2; 1) - 1 \}$$

$$< \lambda \quad (\lambda > 0),$$

by virtue of hypothesis (44). This evidently completes the proof of Theorem 7.

Similarly, we can prove the following.
**Theorem 8.** Suppose that
\[ \alpha_j \in \mathbb{C} \setminus \{0\} \ (j = 1, \ldots, p), \quad \Re (\beta_j) > 0 \ (j = 1, \ldots, q), \]
and (in the case when \( p = q + 1 \))
\[ \Re \left( \sum_{j=1}^{q} \beta_j \right) > P_1 + \sum_{j=1}^{p} |\alpha_j|, \]
where \( P_1 = P_1(k) \) is given, as before, by (32). If, for some \( k \ (0 \leq k < \infty) \), \( f \in k-ST \) and the hypergeometric inequality
\[ \sum_{j=1}^{p} F_{q+2} (\{\alpha_1, \ldots, |\alpha_p|, P_1, \lambda + 1; \ Re (\beta_1), \ldots, Re (\beta_q), \lambda, 1; 1) < 2 \quad (\lambda > 0) \]
holds true, then
\[ \int_{\beta_1, \ldots, \beta_q} \alpha_1, \ldots, \alpha_p f \in \mathcal{C}_\lambda. \]

In an analogous manner, Lemma 6 and the coefficient inequalities (31) and (33) would lead us to Theorems 9 and 10, respectively.

**Theorem 9.** Suppose that
\[ \alpha_j \in \mathbb{C} \setminus \{0\} \ (j = 1, \ldots, p), \quad \Re (\beta_j) > 0 \ (j = 1, \ldots, q), \]
and (in the case when \( p = q + 1 \))
\[ \Re \left( \sum_{j=1}^{q} \beta_j \right) > P_1 + \sum_{j=1}^{p} |\alpha_j|, \]
where \( P_1 = P_1(k) \) is given, as before, by (32). If, for some \( k \ (0 \leq k < \infty) \), \( f \in k-ST \) and the hypergeometric inequality (45) holds true, then
\[ \int_{\beta_1, \ldots, \beta_q} \alpha_1, \ldots, \alpha_p f \in \mathcal{C}_\lambda. \]

**Theorem 10.** Suppose that
\[ \alpha_j \in \mathbb{C} \setminus \{0\} \ (j = 1, \ldots, p), \quad \Re (\beta_j) > 0 \ (j = 1, \ldots, q), \]
and (in the case when \( p = q + 1 \))
\[ \Re \left( \sum_{j=1}^{q} \beta_j \right) > P_1 + 1 + \sum_{j=1}^{p} |\alpha_j|, \]
where \( P_1 = P_1(k) \) is given, as before, by (32). If, for some \( k \ (0 \leq k < \infty) \), \( f \in k-ST \) and the hypergeometric inequality
\[ \sum_{j=1}^{p} F_{q+3} (\{\alpha_1, \ldots, |\alpha_p|, P_1, \lambda + 1, 2; \ Re (\beta_1), \ldots, Re (\beta_q), \lambda, 1, 1; 1) < 2 \quad (\lambda > 0) \]
holds true, then
\[ \int_{\beta_1, \ldots, \beta_q} \alpha_1, \ldots, \alpha_p f \in \mathcal{C}_\lambda. \]
For \( f \in S^*_k \) of form (1), Lemma 5 immediately yields the coefficient inequalities

\[
|a_n| \leq \frac{\lambda}{\lambda + n - 1} \quad (n \in \mathbb{N} \setminus \{1\}; \lambda > 0).
\]  

(47)

Similarly, for \( f \in C_\lambda \) of form (1), we have the coefficient inequalities

\[
|a_n| \leq \frac{\lambda}{n(\lambda + n - 1)} \quad (n \in \mathbb{N} \setminus \{1\}; \lambda > 0).
\]  

(48)

By applying the coefficient inequalities (47) and (48), in conjunction with Lemmas 3 and 4, we can deduce further mapping and inclusion properties for the convolution operator defined by (13), which are associated with the function classes \( k-UCV \) and \( k-ST \). The details involved in the derivation of these mapping and inclusion properties are being left as an exercise for the interested reader.

Finally, we remark that each of our results in this section (Theorems 7-10) can easily be restated, for \( p = 2 \) and \( q = 1 \), in terms of the Hohlov operator \( F(\alpha, \beta, \gamma) \) defined by (16).

REFERENCES