On Distance Transitive Graphs Whose Automorphism Groups Are Affine

KAZUHIRO YOKOYAMA

International Institute for Advanced Study of Social Information Science, Fujitsu Laboratories Ltd, 140 Miyamoto, Numazu-shi, Shizuoka 410-03, Japan

Communicated by the Managing Editors
Received June 9, 1989

By Praeger, Saxl, and Yokoyama, the classification problem of finite primitive distance transitive graphs is reduced to the case where the automorphism group is either almost simple or affine. Here we study graphs in the affine case, that is, we classify some classes of distance transitive graphs whose automorphism groups are affine.


1. Introduction

By a graph we shall mean a finite, undirected graph without loops and multiple edges. Let $\Gamma$ be a connected graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. For any two vertices $x, y$ in $\Gamma$, the distance $\delta(x, y)$ between $x$ and $y$ is the length of a shortest path joining $x$ and $y$. Then $\delta(x, x) = 0$ for all $x$, and $\delta(x, y) = 1$ if and only if $x$ and $y$ are adjacent. The diameter $d(\Gamma)$ of $\Gamma$ is the maximal distance between two vertices in $\Gamma$, i.e., $d(\Gamma) = \max\{\delta(x, y) | x, y \in V(\Gamma)\}$. Usually we use $d$ instead of $d(\Gamma)$.

We shall assume that $d > 2$. For $x \in V(\Gamma)$, we write as usual $\Gamma_i(x) = \{y \in V(\Gamma) | \delta(x, y) = i\}$ for $1 \leq i \leq d$.

Let $G$ be some subgroup of the automorphism group $\text{Aut}(\Gamma)$. $\Gamma$ is said to be $G$-distance transitive, if for each $i$, $1 \leq i \leq d$, $G$ acts transitively on the set of ordered pairs $(x, y)$ of vertices such that $\delta(x, y) = i$. Then for each vertex $x$ in $\Gamma$, the $\Gamma_i(x)$ is precisely the $G_x$-orbit in $V(\Gamma)$, where $G_x$ denotes the stabilizer of $x$ in $G$, and $|\Gamma_i(x)|$ does not depend on the choice of $x$ for $1 \leq i \leq d$. So $|\Gamma_i(x)|$ is denoted by $k_i$ for $1 \leq i \leq d$. In particular, $k_1$ is called the valency of $\Gamma$.

A $G$-distance transitive graph $\Gamma$ is primitive if the group $G$ acts primitively on the set $V(\Gamma)$ of vertices. Otherwise, it is imprimitive. Since imprimitive distance-transitive graphs are either bipartite or antipodal, (see [1]), primitivity depends only on the graph $\Gamma$ and not on the choice of the group $G$. 

190
Praeger, Saxl, and Yokoyama [18] presented the following theorem as the first step in the classification of finite primitive distance transitive graphs.

**Theorem A** (Praeger–Saxl–Yokoyama). *Let $\Gamma$ be a finite primitive $G$-distance transitive graph of valency $k \geq 2$ and diameter $d \geq 2$. Then one of the following is true:

1. $\Gamma$ is a Hamming graph or its complement ($d=2$), and $G$ is a wreath product group.
2. $G$ is almost simple: that is, $T \leq G \leq \text{Aut}(T)$ for some non-abelian simple group $T$.
3. $G$ is affine: that is, $G$ has a regular normal elementary abelian subgroup and $G \leq AGL(m, p)$, the affine group of dimension $m$ over the field $GF(p)$.

There are several studies on the problem of the classification of primitive distance transitive graphs. (See [2, 5, 14].)

The almost simple case is in the process of being settled by van Bon, Cohen, Cuypers, Inglis, Ivanov, Liebeck, Praeger, Saxl, and others. For example, the case for $T = A_n$ is dealt with and classified in [13, 17, 20], and the case for $T = PSL(n, q)$ is dealt with in [11, 12] for $n \geq 8$ and also in [4] for $n \geq 2$.

As for the classification of the affine case, the case of $d - 1$, where $d$ is the diameter, is nothing but Hering's classification of doubly transitive permutation groups with regular normal abelian subgroups, (see [9]), and the case of $d = 2$ was settled by Liebeck [16]. Recently, in his Ph.D. thesis [3], van Bon showed that if a primitive $G$-distance transitive graph $\Gamma$ in the affine case of diameter $\geq 3$ and valency $\geq 3$ is neither a Hamming graph nor a generalized Hamming graph, then either a one point stabilizer $H$ of $G$ is solvable or the generalized Fitting subgroup of $H/\text{Z}(H)$ is simple, where $\text{Z}(H)$ is the center of $H$. By his result, the classification of the affine case of $d \geq 3$ is reduced to the case where a one point stabilizer is either solvable or "nearly simple."

In this paper, we also deal with graphs in the affine case with diameters $d \geq 3$ and classify them, in another view point, under some additional conditions but not assuming their primitivity. In our view point, the structures of graphs can be determined by their local (geometric) properties elementarily, and so this can be called a "local approach" or a "geometric classification." On the other hand, the approach used in the result of Praeger, Saxl, and Yokoyama and the result of van Bon, where the structures of graphs are determined by global properties of their automorphism groups, can be called a "global approach" or a "group theoretic classification."
In what follows, we shall deal with the "affine case" of distance transitive graphs independently of the primitive case of those. (We note that the "affine case" is defined without primitivity.) Let $\Gamma$ be a finite $G$-distance transitive graph in the affine case. We assume the following.

**Assumption A.** There exists a 2-claw but there does not exist a 3-claw in $\Gamma_1(x)$ for all $x \in V(\Gamma)$.

An $n$-claw is an $(n + 1)$-set $\{x_0, x_1, \ldots, x_n\}$ such that $x_i$ is adjacent to $x_0$ for $1 \leq i \leq n$ and $x_i$, $x_j$ are not adjacent to each other for $1 \leq i < j \leq n$. Moreover, $x_0$ is called the center of $\{x_0, x_1, \ldots, x_n\}$. An ordered 2-claw is an ordered 3-set $\{x_0, x_1, x_2\}$ which is a 2-claw with its center $x_0$.

**Assumption B.** $G$ acts transitively on the set of all ordered 2-claws.

Then we have the following theorem.

**Main Theorem.** Let $\Gamma$ be a finite $G$-distance transitive graph with diameter $d \geq 3$ such that

1. $G$ is affine and its regular normal subgroup is not a 2-group,
2. $\Gamma$ and $G$ satisfy Assumption A and Assumption B.

Then $\Gamma$ is isomorphic to the generalized Hamming graph $H_r(n, d)$, where $n \geq d$ and $r$ is a power of a prime number $p$.

The generalized Hamming graph $H_r(n, d)$ is the graph with vertex set $M_{d \times n}(r)$, the set of all $d \times n$ matrices over the field $GF(r)$, with $A, B \in M_{d \times n}(r)$ being adjacent if and only if the rank of $A - B$ is 1. It admits

$$G = N \cdot (GL(d, r) \times GL(n, r)),$$

where $N$ is $M_{d \times n}(r)$ under addition, $N$ acts on $M_{d \times n}(r)$ by translation, and $A(g_1, g_2) = g_1^{-1}Ag_2$ for $A \in M_{d \times n}(r)$, $g_1 \in GL(d, r)$ and $g_2 \in GL(n, r)$. Then $H_r(n, d)$ is $G$-distance transitive and $H_r(n, d)$ and $G$ satisfy Assumption A and Assumption B.

**Remark.** Concerning Assumption A, we note the following. Three known families of graphs with large diameter in the affine case are listed in [1, 5]. Each family consists of graphs derived from the association schemes of linear forms, and linear forms are bilinear forms, alternating bilinear forms or Hermitian forms. Generalized Hamming graphs are the graphs derived from the association schemes of bilinear forms. For the case where a graph $\Gamma$ is derived from the association scheme of alternating bilinear forms, there are 3-claws in $\Gamma_1(x)$ for all $x \in V(\Gamma)$. And for the case where a graph $\Gamma$ is derived from the association scheme of Hermitian forms, there is no 2-claw in $N_1(x)$ for all $x \in V(\Gamma)$.
In order to prove the main theorem, we use the concept of distance regular graphs and the concept of incidence structures.

First we define a distance regular graph \( G \). For any two vertices \( x, y \) of a connected graph \( G \) and for any \( i, j \) in \( \{0, 1, \ldots, d(G)\} \), \( S_{i,j,x,y} \) is defined as the number of vertices \( z \) of \( G \) such that \( d(x, z) = i \) and \( d(y, z) = j \). If \( S_{i,j,x,y} \) depends on only \( i, j \) and \( k = d(x, y) \), then \( G \) is said to be distance regular and the above number \( S_{i,j,x,y} \) is denoted by \( S_{i,j,k} \). Moreover we set \( c_i = S_{i-1,1,i}, \quad a_i = S_{i,1,i} \) and \( b_i = S_{i+1,1,i} \). The constants \( a_i, b_i, \) and \( c_i \) are called the intersection numbers or the parameters of \( G \). Clearly distance transitivity implies distance regularity.

Next we define incidence structures and several classes of them. (See [22].)

An incidence structure \( \Pi \) is a triple \( (\mathcal{P}, \mathcal{L}, \mathcal{I}) \), where \( \mathcal{P} \) and \( \mathcal{L} \) are non-empty, disjoint finite sets and \( \mathcal{I} \subseteq \mathcal{P} \times \mathcal{L} \). Elements of \( \mathcal{P} \) and \( \mathcal{L} \) are called points and lines, respectively. An incidence structure is semilinear if at most one line contains two points, where “contain” means “be incident with.” The adjacency graph of the incidence structure \( \Pi = (\mathcal{P}, \mathcal{L}, \mathcal{I}) \) is the graph \( \Gamma_\Pi \) having \( \mathcal{P} \) as vertex set, and two points adjacent if same line contains both. An incidence structure \( \Pi = (\mathcal{P}, \mathcal{L}, \mathcal{I}) \) is connected, if the adjacency graph \( \Gamma_\Pi \) is connected.

A net is a semilinear connected incidence structure \( \Pi = (\mathcal{P}, \mathcal{L}, \mathcal{I}) \) which satisfies the following conditions;

(B1) \(|\mathcal{P}| > 1\), and

(B2) \( \mathcal{L} \) is partitioned into at least three non-empty classes such that

(i) the lines of each class partition \( \mathcal{P} \), and

(ii) the lines of different classes intersect.

A d-net is a semilinear connected incidence structure \( \Pi = (\mathcal{P}, \mathcal{L}, \mathcal{I}) \) with dimension \( d \) such that

(D1) each 2-subspace of \( \Pi \) is a net,

(D2) the intersection of any two 2-subspaces in a 3-subspace of \( \Pi \) is either \( \emptyset \) or a line, and

(D3) the intersection of any two subspaces of \( \Pi \) is a subspace.

As for the definition of subspaces and their dimensions, see Definition 4.1.

There is a d-net whose adjacency graph is \( H_c(n, d) \) and which is called a \((d, GF(r), n)\)-attenuated space. A \((d, GF(r), n)\)-attenuated space is defined as follows.

Let \( U \) and \( W \) be finite vector spaces over \( GF(r) \), where \( r \) is a power of a prime number \( p \), and let \( d \) and \( n \), \( d \leq n \), be the respective dimensions of \( U \) and \( W \). Let \( V = U \oplus W \). For \( i \leq d \), let \( \mathcal{U}_i \) be the set of all \( i \)-dimensional
subspaces $U'$ of $V$ so that $U' \cap W = \{0\}$. Then the incidence structure $(\mathcal{U}_d, \mathcal{U}_{d-1}, \equiv)$ is semilinear. Any incidence structure isomorphic to $(\mathcal{U}_d, \mathcal{U}_{d-1}, \equiv)$ is called a $(d, GF(r), n)$-attenuated space.

It is well known that the adjacency graph of a $(d, GF(r), n)$-attenuated space is isomorphic to the generalized Hamming graph $H_r(n, d)$. (See [19] or [10].) Sprague [22] showed the following.

**Theorem B (Sprague's Characterization).** Every finite $d$-net, where $d \geq 3$ is an integer, is a $(d, GF(r), n)$-attenuated space for some prime power $r$ and positive integer $n$.

In the remainder of this paper, we will construct a $d$-net from the given graph. Once we show this, by Sprague's characterization we can prove the main theorem.

**Remark.** On the characterization problem of distance regular graphs by their parameters, Huang [10] dealt with generalized Hamming graphs, and characterized them by their parameters under the "weak 4-vertex condition." In his work, he constructed a $d$-net from the given graph, and used Sprague's characterization.

Finally, we use the following notation in this paper.

Let $\Gamma$ be a graph, and let $\partial$ be the distance function on the set $V(\Gamma)$ of vertices. For two subsets $U$ and $V$ of $V(\Gamma)$, the distance $\partial(U, W)$ between $U$ and $V$, is defined as the minimum of $\partial(u, v)$ for all $u \in U$ and $v \in V$. Moreover $U \setminus V$ is defined to be the set $\{u \in U | \partial(u, V) = \partial(U, V)\}$.

Let $N$ be an elementary abelian $p$-group, where $p$ is a prime number. Then for two elements $x$ and $y$ in $N$, $x + y$ is defined as the product of $x$ and $y$. And for a subset $U$ and an element $x$, $U + x$ is defined to be the set $\{u + x | u \in U\}$. Moreover, for a subset $U$ of $N$, $\langle U \rangle$ is defined to be the subgroup generated by $U$.

2. **Elementary Properties**

From now on, we assume that a graph $\Gamma$ and its automorphism group $G$ satisfy the condition of the main theorem. Since $G$ has a regular normal $p$-subgroup $N$, we can identify $V(\Gamma)$ with $N$. In this identification, 0, the unit element of $N$, corresponds to a fixed vertex $\alpha_0$ and $x \in N$ corresponds to $x_0^\alpha$. And for elements $x, y$ in $N$, the distance $\partial(x, y)$ between $x$ and $y$ is defined as $\partial(x_0^\alpha, x_0^\beta)$. Moreover the action of $G_{\alpha_0}$ on $\Gamma$ is isomorphic to the natural action of $G_{\alpha_0}$ on $N$. We write $H$ or $G_0$ for $G_{\alpha_0}$. So we treat $N$ as the vertex set of the distance transitive graph $\Gamma$. We also use the following notation instead of $\Gamma$'s.
For $x$ in $N$, let $N_i(x) = \{ y \mid y \in N \text{ and } \delta(x, y) = i \}$, and we write $N_i = N_i(0)$ for simplicity.

Then $H$ acts transitively on each $N_i$. From this, we have the following lemmas.

**Lemma 2.1.** For all vertices $x \in N$ and all $i \in \{1, 2, \ldots, d\}$, $N_i(x) = N_i + x$.

**Proof.** Let $y$ be a vertex of $N$ and the distance between $x$ and $y$ be $i$. Then the distance between corresponding vertices $x_0$ and $x_i$ is $i$. Then by the action of the inverse element $-x$ of $x$, it follows that the distance between $x_0$ and $x_i$ is $i$ and so $y - x$ lies in $N_i$. Therefore we have $N_i(x) = N_i + x$. Q.E.D.

**Lemma 2.2.** $c_2 \geq 2$.

**Proof.** Assume, to the contrary, that $c_2 = 1$. Let $x$ be a vertex in $N_2$. Then there is a vertex $y$ which is adjacent to 0 and $x$. Then $z = x - y$ lies in $N_1$. If $y \neq z$, then two distinct vertices $y, z$ lie in $N_1 \cap N_1(x)$ and so $c_2 \geq 2$. This is a contradiction. So we have $x = 2y$. This implies that $H_x = H_y$ and so $|N_1| = |H : H_x| = |H : H_y| = |N_2|$. Therefore $b_1 = c_2 = 1$. By Sprague [21], the fact that “edge valence” $a_1$ equals to $b_0 - 1 - 1$, implies that the graph $\Gamma$ is isomorphic to the complement of $rK_2$ for some $r$ or $b_0 = 2$. If $b_0 = 2$, then $\Gamma$ is a circuit and it does not satisfy Assumption A. If $\Gamma$ is isomorphic to the complement of $rK_2$, then it follows easily that $c_2 \geq 2$ and this contradicts the assumption. Q.E.D.

**Lemma 2.3.** Let $x$ be a vertex in $N_i$, $1 \leq i \leq d$. Then $kx \in N_i$ for all $k \in \{1, 2, \ldots, p - 1\}$.

**Proof.** For the case $p = 3$, it is clear that $-x \in N_i$ for $x \in N_i$ by the commutativity of adjacency. So we can assume that $p \geq 5$. First we consider the case $i = 1$ and suppose, to the contrary, that there is some $x \in N_1$ and some $k \in \{2, \ldots, p - 1\}$ such that $kx$ does not lie in $N_1$. Then by the same argument as in the proof of Lemma 2.2, we have $|N_1| = |N_2|$. By [5, Lemma 5.1.2], $b_0 = 2$ or $\Gamma$ is an antipodal double cover. But $b_0 > c_2 \geq 2$ and $|V(\Gamma)|$ is odd by the condition of the main theorem. So $k_1 \neq 2$ and $\Gamma$ cannot be an antipodal double cover. This is a contradiction. Thus we proved the lemma for the case $i = 1$.

Next consider the case $i \geq 2$. Let $x$ be a vertex in $N_i$. Then there is a path $\{x_0 = 0, x_1, \ldots, x_i = x\}$ from 0 to $x$. Let $y_j = x_j - x_{j-1}$ for $1 \leq j \leq i$. Then each $y_j$ lies in $N_1$. Since the lemma holds for $i = 1$, $kx_j \in N_1$ for $1 \leq k \leq p - 1$ and $1 \leq j \leq i$. From this, $kx_j$ is adjacent to $kx_{j-1}$ for $1 \leq j \leq i$. This implies that there is a path $\{0, kx_1, \ldots, kx_i\}$ from 0 to $kx$. Therefore the distance $\delta(0, kx)$ between 0 and $kx$ is at most $i$, i.e., $\delta(0, kx) \leq \delta(0, x)$. Changing the
roles of \( x \) and \( kx \), we can also show that \( \partial(0, x) \leq \partial(0, kx) \). Hence we have \( \partial(0, kx) = \partial(0, x) = t \).

Q.E.D.

**Definition 2.1.** Let \( N^*_1 = N_1 \cup \{0\} \). For each \( x \) in \( N_1 \), we define \( E_x \) and \( E^*_x \) as follows.

\[
E_x = \{ y \in N_1 | N^*_1 \cap (N^*_1 + x) = N^*_1 \cap (N^*_1 + y) \} \quad \text{and} \quad E^*_x = E_x \cup \{0\}.
\]

By Lemma 2.3, \( N_1 \cap (N_1 + x) \neq \emptyset \) for all \( x \in N_1 \), and thus \( |N^*_1 \cap (N^*_1 + x)| \geq 3 \).

**Lemma 2.4.** Let \( x \) be a vertex in \( N_1 \). Then, for all \( k \in \{1, 2, \ldots, p-1\} \), \( kx \) lies in \( E_x \).

**Proof.** First we show that \(-x \in E_x\). Suppose, to the contrary, that \(-x\) does not lie in \( E_x \). Then by the definition of \( E_x \), there is a vertex \( y \in N_1 \) which is adjacent to \( x \), but is not adjacent to \(-x\). Since \( y \) is adjacent to \( x \), \( y-x \) is adjacent to \( 0 \) and \( y \).

Now we show that \( y-tx, ty-x \in N_1 \) for \( 1 \leq t \leq p-1 \) by the induction on \( t \). For the case \( t=1 \), the above claim is true. So we assume that the claim is true for \( t \leq p-2 \), i.e., \( iy-x, y-ix \in N_1 \) for \( i=1, \ldots, t \). Consider \( (t+1)y-x \) and \( y-(t+1)x \). If \( y-(t+1)x \notin N_1 \), then a diagram \( \{y, x, y-tx\} \) is a 2-claw in \( N_1 \) whose center is \( y \). Take an integer \( s \) in \( \{1, \ldots, p-1\} \) such that \( (t+1)s \equiv 1 \pmod{p} \). Since \( sy \) is adjacent to \( 0 \) and \( y \), a diagram \( \{y, x, y-tx, sy\} \) is contained in \( N_1 \) and \( y \) is adjacent to every other vertex in that diagram. Since \( N_1 \) contains no 3-claws, \( sy \) is adjacent to at least one of two vertices \( x, y-tx \). So it follows that \( sy-x \in N_1 \) or \( sy-(y-tx) = (s-1)y+tx \in N_1 \). As \((t+1)s \equiv 1 \pmod{p}\), we have \( sy-x = -(1/t)((s-1)y+tx) \). This implies that \( sy-x \) and \( (s-1)y+tx \) have the same distance from \( 0 \), and so both of them lie in \( N_1 \). By multiplying \( sy-x \) by \( t+1 \), we also have \( y-(t+1)x \in N_1 \). This is a contradiction. Hence \( y-(t+1)x \notin N_1 \). By changing the roles of \( x \) and \( y \), we can also show that \( (t+1)y-x \notin N_1 \). Therefore, by the induction, we have proved the claim. Then for the case \( k = p-1 \equiv -1 \pmod{p} \), \(-y-x \in N_1 \). This implies that \(-x \) is adjacent to \( y \), and a contradiction. Hence \(-x \in E_x \).

From the above, the claim of the lemma is true for the case \( p = 3 \). So we can assume that \( p \geq 5 \).

Next we show that \( 2x \in E_x \). Since \(-x \in E_x \), it follows that \( (N^*_1 + x) \cap (N^*_1 + 2x) = (N^*_1 \cap (N^*_1 + x)) + x = (N^*_1 \cap (N^*_1 - x)) + x = N^*_1 \cap (N^*_1 + x) \). So \( N^*_1 + 2x \) contains \( N^*_1 \cap (N^*_1 + x) \). By the distance-regularity, \( |N^*_1 \cap (N^*_1 + x)| = |N^*_1 \cap (N^*_1 + 2x)| = a_1 + 2 \). From this we obtain \( N^*_1 \cap (N^*_1 + 2x) = N^*_1 \cap (N^*_1 + x) \). Therefore \( 2x \in E_x \).

Finally we show the lemma by the induction on \( k \). Assume that the claim of the lemma is true for any \( t \leq k \) and \( k \leq p-2 \). Consider the case \( t = k + 1 \).
Then \((N^*_i + x) \cap (N^*_i + (k + 1)x) = (N^*_i \cap (N^*_i + kx)) + x\). By the assumption of the induction and the fact that \(-x \in E_x\), \(N^*_i \cap (N^*_i + kx) = N^*_i \cap (N^*_i + x) = N^*_i \cap (N^*_i - x)\). So \((N^*_i + x) \cap (N^*_i + (k + 1)x) = (N^*_i \cap (N^*_i + kx)) + x = (N^*_i \cap (N^*_i - x)) + x = N^*_i \cap (N^*_i + x)\). Hence the claim of the lemma is true.

Q.E.D.

**Lemma 2.5.** For all vertices \(x\) in \(N_1\), \(E^*_x\) is a group. Especially, the order of \(E^*_x\) is a power of \(p\).

**Proof.** We note that by the definition of \(E_x\), \(E_y = E_x\) for any \(y \in E_x\). Since \(-x \in E_x\), to show that \(E^*_x\) is a group, we have only to show that \(y + z \in E^*_x\) for \(y, z \in E_x\). Consider \(y + z\), where \(y, z \in E_x\). If \(y + z = 0\) then \(y + z \in E^*_x\). So we can assume that \(y + z \neq 0\). Since \(-z \in E_x = E_y\), it follows that \(y + z \in N_1\) and \(N^*_i \cap (N^*_i + y) = N^*_i \cap (N^*_i - z)\). Therefore \((N^*_i + y) \cap (N^*_i - z) = N^*_i \cap (N^*_i + y) \cap (N^*_i - z)\). By the distance-regularity, \(N^*_i \cap (N^*_i + y + z) \subseteq N^*_i\). So we have \(N^*_i \cap (N^*_i + y + z) = N^*_i \cap (N^*_i + z)\). Hence \(y + z \in E_x = E^*_x\).

Q.E.D.

By the transitivity of \(H\) on \(N_1\), the order of the group \(E^*_x\) does not depend on the choice of the vertex \(x\). We denote this order by \(r\).

**Definition 2.2.** Let \(x, y\) be vertices in \(N_1\). We define that \(x\) is equivalent to \(y\), denoted by \(x \equiv y\), if \(x \in E_y\), i.e., \(y \in E_x\). Moreover we call \(E_x\) an equivalence set in \(N_1\).

**Lemma 2.6.** Let \(x\) be a vertex adjacent to 0. If \(y, z \in N^*_i \cap (N^*_i + x)\) and \(y\) is adjacent to \(z\), then \(y + z \in N^*_i \cap (N^*_i + x)\).

**Proof.** By Lemma 2.4, \(ky, kz\) are adjacent to \(x\) for \(1 \leq k \leq p - 1\). It can be shown that \(x + 2y\) is adjacent to \(0\) and \(2y + z\), i.e., \(x + 2y \in N^*_i \cap (N^*_i + 2y + z)\). Since \(y\) is adjacent to \(z\) and \(2y \approx -y\), we have \(2y + z \in N_1\). Since \(2y + z\) is equivalent to \(4y + 2z\) by Lemma 2.4, we have \(x + 2y \in N^*_i \cap (N^*_i + 4y + 2z)\). By subtracting \(2y\), we also have \(x \in (N^*_i + 2y + 2z)\), i.e., \(x\) is adjacent to \(2y + 2z\). On the other hand, \(y + z \in N^*_i\) and so \(2y + 2z \in N^*_i\) by the fact \(z \approx -z\). By Lemma 2.4, \(2y + 2z\) is equivalent to \(y + z\) and this implies that \(x\) is also adjacent to \(y + z\), i.e., \(y + z \in (N^*_i + x)\). Hence we have \(y + z \in N^*_i \cap (N^*_i + x)\).

Q.E.D.

By Lemma 2.4, \(iy \approx jz \approx z\) for \(y, z \in N_1\) and all \(i, j \in \{1, 2, ..., p - 1\}\). From this, it follows that if \(y, z \in N^*_i \cap (N^*_i + x)\), then \(iy, jz \in N^*_i \cap (N^*_i + x)\) for all \(i, j \in \{1, 2, ..., p - 1\}\). Thus, if \(y\) and \(z\) satisfy the condition of Lemma 2.6, then \(iy\) and \(jz\) also satisfy the condition of Lemma 2.6, and by replacing \(y, z\) by \(iy, jz\) in Lemma 2.6, we have the following corollary.
**Corollary 2.7.** Let $x$ be a vertex adjacent to $0$. If $y, z \in N_1^* \cap (N_1^* + x)$ and $y$ is adjacent to $z$, then $iy + jz \in N_1^* \cap (N_1^* + x)$ for all $i, j \in \{0, 1, ..., p - 1\}$.

Let $x$ be a vertex in $N_1$. By Assumption A and the distance-transitivity, there is a 2-claw in $N_1$ whose center is $x$. From this, it can be shown that $N_1^* \cap (N_1^* + x) \neq E_1^*$. So there is some vertex $y$ in $N_1 \cap (N_1 + x)$ which is not equivalent to $x$.

**Lemma 2.8.** Let $x$ be a vertex in $N_1$ and let $y$ be a vertex in $N_1 \cap (N_1 + x)$ which is not equivalent to $x$. Then $N_1^* \cap (N_1^* + x) \cap (N_1^* + y)$ is a maximal clique. Moreover $N_1^* \cap (N_1^* + x) \cap (N_1^* + y)$ is a group.

**Proof.** First we show that there is a vertex $z$ in $N_1 \cap (N_1 + x)$ such that \{x, y, z\} is a 2-claw in $N_1$ whose center is $x$. If there does not exist such a vertex $z$, then any vertex $w$ in $N_1 \cap (N_1 + x) \setminus \{y\}$ is adjacent to $y$. This implies that $y$ is equivalent to $x$. This contradicts the assumption about $y$. So there exists such a vertex $z$ and we fix $z$.

Now we consider two distinct vertices $u, v$ in $(N_1 \cap (N_1 + x) \cap (N_1 + y)) \setminus (E_1 \cup E_1)$. By Lemma 2.6, $x + u$ and $x + v$ lie in $N_1 \cap (N_1 + x)$. If $u$ and $v$ are not adjacent, then $[x, u, v]$ is a 2-claw in $N_1$ whose center is $x$. Then by Assumption A, $z$ is adjacent to $u$ or $v$. By changing $u$ and $v$, if necessary, we can assume that $z$ is adjacent to $u$. Since $u$ and $v$ are not adjacent, $y + u$ and $v$ are not adjacent by Corollary 2.7. So \{x, y + u, v\} is also a 2-claw in $N_1$ whose center is $x$. Then $z$ is adjacent to $y + u$ or $v$. If $z$ is adjacent to $y + u$, then $z$ is adjacent to $y$. This is a contradiction. So $z$ is adjacent to $u$ and $v$. On the other hand, \{x, y + u, y + v\} is a 2-claw in $N_1$ whose center is $x$. And so $z$ is adjacent to $y + u$ or $y + v$. Since $z$ is adjacent to $u$ and $v$, it follows that $z$ is adjacent to $y$ by Corollary 2.7. This is a contradiction. Hence $u$ and $v$ are adjacent. So $N_1^* \cap (N_1^* + x) \cap (N_1^* + u)$ contains $N_1^* \cap (N_1^* + x) \cap (N_1^* + y)$. Conversely, replacing $y$ by $u$, $N_1^* \cap (N_1^* + x) \cap (N_1^* + y)$ contains $N_1^* \cap (N_1^* + x) \cap (N_1^* + u)$. From this, $N_1^* \cap (N_1^* + x) \cap (N_1^* + y) = N_1^* \cap (N_1^* + x) \cap (N_1^* + u)$. Hence $N_1^* \cap (N_1^* + x) \cap (N_1^* + y)$ is a maximal clique. Directly from the above argument it can also be shown that $N_1^* \cap (N_1^* + x) \cap (N_1^* + y)$ is a group. Q.E.D.

From Lemma 2.8, a maximal clique $M$ which contains three adjacent vertices $0, x,$ and $y$ with $x \neq y$, is uniquely determined. We denote this maximal clique $M$ by $M(x, y)$.

**Proposition 2.9.** Let $x$ and $y$ be adjacent vertices. Then there are exactly two maximal cliques which contain $x$ and $y$. Moreover there are vertices $z, z'$ in $N_1$ such that two maximal cliques are $M(y - x, z) + x$ and $M(y - x, z') + x$. 
Proof. Let $M$ be a maximal clique which contains $x$ and $y$. Then $M - x$ is a maximal clique which contains 0 and $y - x$. From this, we have only to show the lemma for the case $x = 0$. So assume that $x = 0$ and $y \in N_1$. Consider a maximal clique $M$ which contains 0 and $y$. Since $N^*_1 \cap (N^*_1 + y) \neq E^*_1$, there is a vertex $z \in M$ not equivalent to $y$. Then by Lemma 2.8, $M = M(y, z)$. By Assumption A and the argument in the proof of Lemma 2.8, there is a vertex $u \in N^*_1 \cap (N^*_1 + y)$ which is not adjacent to $z$. Then there is a maximal clique $M'$ which contains 0, $y$, and $u$. Also by Lemma 2.8, $M' = M(y, u)$. So there are at least two maximal cliques which contain 0 and $y$. Assume that there is another maximal clique $M''$ which contains 0 and $y$. Take a vertex $w$ in $M'' \setminus E^*_1$. Then $M'' = M(y, w)$. Since $z$ and $u$ are not adjacent, \{ $y, z, u$ \} is a 2-claw in $N_1$ whose center is $y$. So $w$ is adjacent to $u$ or $z$ by Assumption A. If $w$ is adjacent to $z$, it follows that $M'' = M$ by the argument in the proof of Lemma 2.8, and if $w$ is adjacent to $u$, it follows that $M'' = M'$. This is a contradiction. Hence there are exactly two maximal cliques which contain 0 and $y$. Q.E.D.

3. LINES AND ASSEMBLIES

In this section, we define lines and assemblies. By Proposition 2.9, for each pair $(x, y)$ of adjacent vertices there are exactly two maximal cliques which contain $x$ and $y$. We will call one of two maximal cliques a line, and the other an assembly. First we consider the case where the sizes of two maximal cliques are different.

Case where the Sizes of Two Maximal Cliques Are Different

We consider the case where the sizes of two maximal cliques are different. In this case, we define lines and assemblies as follows.

Definition 3.1. Let $x$ and $y$ be adjacent vertices in $N$. Then there are two maximal cliques $M$ and $M'$ which contain $x$ and $y$. Assume that $|M| > |M'|$. We call $M$ a line containing $x$ and $y$, and $M'$ an assembly containing $x$ and $y$.

We denote the line containing $x$ and $y$ by $\ell(x, y)$, and also denote the assembly containing $x$ and $y$ by $A(x, y)$. $|\ell(x, y)|$ and $|A(x, y)|$ do not depend on the choice of the pair of adjacent vertices $(x, y)$. So we denote $|\ell(x, y)|$ and $|A(x, y)|$ by $q$ and $q'$, respectively.

We have the following lemma by Proposition 2.9.

Lemma 3.1. Let $x$ and $y$ be adjacent vertices. Then, $\ell(x, y) \cup A(x, y) = (N^*_1 + x) \cap (N^*_1 + y)$ and $\ell(x, y) \cap A(x, y) = E^*_1 - y + y$.

From Lemma 3.1, it follows directly that $a_1 = q + q' - r - 2$. 
Lemma 3.2. Let \( M \) be a line or an assembly, and let \( x \) be a vertex with \( \partial(x, M) = 1 \). If \( |M[x]| \geq 2 \), then \( |M[x]| = r \), where \( M[x] = M \cap (N_1 + x) \). Moreover, if \( M[x] \) contains 0, then \( M[x] \) is the union \( E^* \) of an equivalence set \( E \) in \( N_1 \) and \( \{0\} \).

Proof. Let \( z \) be a vertex in \( M[x] = M \cap (N_1^* + x) \). Since \( M \) is a line or an assembly, without loss of generality we can assume that \( M \) contains 0 and \( x \notin M \) is adjacent to 0. Let \( y \) be another vertex in \( M[x] \). Then \( M \) contains \( E^*_y \) and so \( M[x] \supseteq E^*_y \). If \( M[x] \neq E^*_y \), there is a vertex \( w \) in \( M[x] \setminus E^*_y \). Then \( x \) lies in \( N_1^* \cap (N_1^* + y) \cap (N_1^* + w) \). By Lemma 2.8, \( M = M(y, w) = N_1^* \cap (N_1^* + y) \cap (N_1^* + w) \). This is a contradiction. Hence \( M[x] = E^*_y \) and this implies \( |M[x]| = r \). Q.E.D.

By Assumption B, we have the following.

Lemma 3.3. There exists a diagram \( \{u_1, u_2, u_3, u_4\} \) in \( N_1 \) such that \( \partial(u_1, u_2) = \partial(u_2, u_3) = \partial(u_3, u_4) = \partial(u_4, u_1) = 1 \), and \( \partial(u_1, u_3) = \partial(u_2, u_4) = 2 \).

Proof. By Assumption A, there is a 2-claw \( \{u_2, u_1, u_3\} \) with its center \( u_2 \) in \( N_1 \). Then \( N_1^* \cap (N_1^* + u_1) \cap (N_1^* + u_2) \) is a line or an assembly. Since \( \{0, u_1, u_3\} \) is a 2-claw with its center 0, there is an element \( g \) in \( H \) such that \( u_3 = u_1 \), \( u_3 = u_4 \). We consider \( u_3^g \). Since \( N_1^* \cap (N_1^* + u_1) \cap (N_1^* + u_2) \neq \emptyset \), \( N_1^* \cap (N_1^* + u_2) \) and \( N_1^* \cap (N_1^* + u_1) \) do not lie in \( N_1^* \cap (N_1^* + u_2) \). This implies that \( u_3^g \) is not adjacent to \( u_2 \). Hence the diagram \( \{u_1, u_2, u_3, u_4\} \) is a desired diagram. Q.E.D.

We call that diagram a 4-gon in \( N_1 \).

Lemma 3.4. Let \( M \) be a line or an assembly, and let \( x \) be a vertex with \( \partial(x, M) = 1 \). Then \( |M[x]| = r \).

Proof. Let \( D = \{u_1, u_2, u_3, u_4\} \) be a 4-gon obtained in Lemma 3.3. By the same argument as in the proof of Lemma 3.2, we can assume that \( M \) contains 0 and \( x \notin M \) is adjacent to 0. If \( M \) contains another vertex \( z \neq 0 \) adjacent to \( x \), we have \( |M[x]| = r \) by Lemma 3.2. So we assume, to the contrary, that \( M[x] = \{0\} \). Let \( y \) be a vertex in \( M \setminus \{0\} \). Then \( \partial(x, y) = 2 \). By Assumption B, there is an element \( g \) in \( H \) such that \( u_1^g = x \), \( u_3^g = y \). Then \( u_2^g \) and \( u_4^g \) lie in \( (N_1 + x) \cap (N_1 + y) \cap N_1 \). Since \( u_2^g \) and \( u_4^g \) are not adjacent, \( N_1^* \cap (N_1^* + y) = M(y, u_2^g) \cup M(y, u_4^g) \). Hence \( M = M(y, u_2^g) \) or \( M(y, u_4^g) \). This implies that \( M[x] \neq \{0\} \), and a contradiction. Q.E.D.

By Lemma 3.4, we obtain the parameters \( b_0 \) and \( b_1 \).

Lemma 3.5. There are exactly \( (q' - 1)/(r - 1) \) lines which contain 0. Moreover \( b_0 = (q - 1)(q' - 1)/(r - 1) \) and \( b_1 = (q - r)(q' - r)/(r - 1) \).
Proof. Let \( x \) be a vertex in \( N_1 \) and fix it. Consider lines which contain 0. Let \( \ell \) be a line which contains 0 but does not contain \( x \). By Lemma 3.4, \(|\ell[x]| = r \) and \( \ell[x] = E_x^* \) for \( y \) in \( \ell[x] \setminus \{0\} \). Then \( E_y \) is contained in \((N_1 \cap (N_1 + x)) \setminus \ell(0, x)\). Conversely consider vertices in \((N_1 \cap (N_1 + x)) \setminus \ell(0, x)\). Then, for each \( E_y \subseteq (N_1 \cap (N_1 + x)) \setminus \ell(0, x) \), there is a line \( \ell(0, y) \) which contains 0 but does not contain \( x \). Since \(|(N_1 \cap (N_1 + x)) \setminus \ell(0, x)| = a_1 - (q - 2) = q' - r \), there are exactly \((q' - r)/(r - 1)\) lines which contain 0 but do not contain \( x \). So there are exactly \((q' - 1)/(r - 1)\) lines which contain 0. Since each line has \((q - 1)\) vertices except 0, we have \( b_0 = (q - 1) \). As \( b_1 = b_0 - a_1 - c_1 \), we also have \( b_1 = (q - 2)(q' - r)/(r - 1) \). Q.E.D.

Case where Maximal Cliques Have the Same Size

Next we consider the case where maximal cliques have the same size. Let \( x \) be a vertex in \( N_1 \) and fix it. Then there are two maximal cliques \( M \) and \( M' \) which contain 0 and \( x \). In this case, \(|M| = |M'|\). So we denote \(|M| \) by \( q \). Then we have the following by the same arguments as in Lemmas 3.1 and 3.2. Moreover we also have \( a_1 = 2q - r - 2 \).

Lemma 3.1'. \( M \cap M' = N_1^* \cap (N_1^* + x) \) and \( M \cap M' = E_x^* \).

Lemma 3.2'. Let \( y \) be a vertex in \( N_1 \) which is not adjacent to \( x \). If \(|M[y]| \geq 2\), then \(|M'[y]| = r\). If \(|M'[y]| \geq 2\), then \(|M'[y]| = r\).

Now we show that Lemma 3.3 also holds in this case.

Proof of Lemma 3.3. Assume, to the contrary, that there does not exist a 4-gon in \( N_1 \). Consider a 2-claw \( \{u_2, u_1, u_3\} \) with its center \( u_2 \) in \( N_1 \). Then, from the above assumption, any vertex \( v \) which is adjacent to \( u_1 \) and \( u_3 \), is also adjacent to \( u_2 \). From this, \( N_1^* \cap (N_1^* + u_1) \cap (N_1^* + u_3) \subseteq N_1^* \cap (N_1^* + u_2) \). So it follows that \( N_1^* \cap (N_1^* + u_1) \cap (N_1^* + u_3) \subseteq N_1^* \cap (N_1^* + u_2) \cap (N_1^* + u_2) \cap (N_1^* + u_3) = M(u_1, u_2) \cap M(u_2, u_3) = E_\pi^* \). So \( E_\pi^* \) is determined uniquely by a pair of non-adjacent vertices \( u_1 \) and \( u_3 \). On the other hand, for each pair \((y, z)\) of non-adjacent vertices in \( N_1 \), there also exists a vertex \( u \) in \( N_1 \) which is adjacent to \( y \) and \( z \) by Assumption B. Therefore each pair \((y, z)\) of non-adjacent vertices in \( N_1 \) determines \( E_u \) in \( N_1 \) uniquely. Consider a pair \((y, z)\) of non-adjacent vertices in \( N_1 \) and the equivalence set \( E_u \) determined by \((y, z)\). Then, for a pair \((y', z')\) of vertices in \( N_1 \), \( y' \in M(u, y) \setminus E_\pi^* \) and \( z' \in M(u, z) \setminus E_\pi^* \) if and only if the equivalence set determined by \((y', z')\) is \( E_u \).

Now we count the number \( \beta \) of pairs \((y, z)\) of non-adjacent vertices in \( N_1 \). Then, it follows directly that \( \beta = b_0 b_1 \). On the other hand, since there are \( b_0/(r - 1) \) equivalence sets in \( N_1 \) and each of them is determined by
2(q - r)(q - r) pairs of vertices in \( N_1 \), we have \( \beta = 2b_0(q - r)^2/(r - 1) \). Therefore \( b_1 = 2(q - r)^2/(r - 1) \) and so \( b_0 = 2(q - r)^2/(r - 1) + 2q - r - 1 = (q - 1)^2/(r - 1) + (q - r)^2/(r - 1) \). Let the number of maximal cliques containing 0 be \( \gamma \). Then \( b_0 = \gamma(q - 1)/2 \). From this and the fact that \( r - 1 \) is even, it follows that \( (q - r)^2/(q - 1) \) must be an integer. Since \( (q - r)^2 = (q - 1)(q - 2r + 1) + (r - 1)^2 \), we have \( (r - 1)^2 \geq (q - 1) \).

Now we consider \( q \) and \( r \). Then \( q \) is the order of a subgroup \( M \), and \( r \) is also the order of a subgroup \( E^* \), where \( u \in M \). Moreover \( M \) contains a subgroup \( T = \langle E_u^*, E_v^* \rangle \), where \( u, v \in M \setminus \{0\} \) and \( u \neq v \). Since \( E_u^* \cap E_v^* = \{0\} \), we have \( |T| = r^2 \), and so \( q - 1 = |M| - 1 \geq r^2 - 1 > (r - 1)^2 \). This is a contradiction. Thus, we proved that Lemma 3.3 holds in this case.

By Lemma 3.3, we can also prove the following corresponding to Lemma 3.4.

**Lemma 3.4'.** Let \( y \) be a vertex in \( N_1 \) not adjacent to \( x \). Then, \( |M[y]| = |M'[y]| = r \).

Moreover we also have the following.

**Lemma 3.5'.** \( b_0 = (q - 1)^2/(r - 1) \) and \( b_1 = (q - r)^2/(r - 1) \).

**Proof.** Let \( y \) be a vertex in \( N_1 \) not adjacent to \( x \). By Lemma 3.4', there is an equivalence set, say \( E \), in \( M \) such that \( E^* - M[y] = (M \cap (N_1 + y)) \). Conversely, for each equivalence set \( E' \) in \( M \), there is a vertex \( z \) such that \( (E')^* = (M \cap (N_1 + z)) \), i.e., \( z \in N_1^* \cap (N_1^* + e) \setminus M \), where \( e \in E' \). Moreover \( \{z \in N_1 \setminus M | M[z] = (E')^* \} \cup (E')^* \) is a maximal clique. So the number of vertices in \( N_1 \) not adjacent to \( x \), is \((q - r)/(r - 1))(q - r) = (q - r)^2/(r - 1)\). This number is equal to \( b_1 \). Since the number of vertices in \( N_1 \) adjacent to \( x \) is \( 2q - r - 2 \), the number of vertices in \( N_1 \) is \( (q - 1)^2/(r - 1) \). Q.E.D.

Let the number of maximal cliques which contain 0 be \( \beta \). Since each vertex in \( N_1 \) is contained in two maximal cliques and each maximal clique has \( q - 1 \) vertices in \( N_1 \), we have \( b_0 \times 2 = \beta \times (q - 1) \). So \( \beta = 2(q - 1)/(r - 1) \). By the argument in the proof of Lemma 3.5', there are exactly \( (q - 1)/(r - 1) \) maximal cliques which intersect \( M \) at 0 and other \( r - 1 \) vertices. Let these maximal cliques be \( M'_1, ..., M'_{(q - 1)/(r - 1)} \), where \( M'_1 = M' \). Moreover there are \( (q - 1)/(r - 1) - 1 \) maximal cliques which intersect \( M \) only at 0. Let all equivalence sets in \( M' \) be \( E_1, ..., E_{(q - 1)/(r - 1)} \), where \( E_1 = E_x \). For each equivalence set \( E_i \), there is a maximal clique, say \( M_i \), which contains \( E_i \) and is not \( M' \). If \( M_i = M'_j \) for some \( j \), then there is a vertex \( z \) in \( M \cap M_i \) not equivalent to \( x \). This implies that \( M_i = M = N_1^* \cap (N_1^* + x) \cap (N_1^* + z) \). This is a contradiction. So \( M_i \neq M'_j \) for \( 1 \leq i, j \leq (q - 1)/(r - 1) \). By counting the numbers of maximal cliques, the set of all maximal cliques
containing 0 is \( \{M'_1, ..., M'_{(q-1)/(r-1)}\} \cup \{M_1, ..., M_{(q-1)/(r-1)}\} \). Set \( \mathcal{L} = \{M_1, ..., M_{(q-1)/(r-1)}\} \) and \( \mathcal{A} = \{M'_1, ..., M'_{(q-1)/(r-1)}\} \). Now we show the following.

**Lemma 3.6.** \( M_i \cap M_j = \{0\} \) for \( 1 \leq i \neq j < (q-1)/(r-1) \), \( M_i' \cap M_j' = \{0\} \) for \( 1 \leq i \neq j < (q-1)/(r-1) \) and \( |M_i \cap M_j'| = r \) for \( 1 \leq i, j \leq (q-1)/(r-1) \).

**Proof.** Consider two distinct maximal cliques \( M_i \) and \( M_j \). Assume, to the contrary, that \( M_i \cap M_j \neq \{0\} \). Let \( y \) be a vertex in \( M_i \cap M_j \). Then \( E_i \) is contained in \( N_1^* \cap (N_1^* + e_i) \cap (N_1^* + y) \), where \( e_i \in E_i \). So \( E_j \subseteq M_j \) and this implies \( M_i = N_1^* \cap (N_1^* + e_i) \cap (N_1^* + e_j) \), where \( e_i \in E_i \) and \( e_j \in E_j \). So \( M_i = M' \) and this is a contradiction. Thus \( M_i \cap M_j = \{0\} \). By the same argument as in the above, it can be shown that \( M_i' \cap M_j' = \{0\} \). Next we consider \( M_i \) and \( \mathcal{A} \). For each equivalence set \( E \) in \( M_j \), there is a maximal clique which contains \( E \) and is not \( M_i \). So there are \( (q-1)/(r-1) \) maximal cliques containing 0 whose intersections with \( M_i \) are not \( \{0\} \). From this and the fact that \( M_i \cap M_j = \{0\} \) for \( i \neq j \), it follows that \( |M_i \cap M_j'| = r \) for \( 1 \leq i, j \leq (q-1)/(r-1) \). Q.E.D.

By using the property obtained in Lemma 3.6, we define lines and assemblies.

**Definition 3.2.** Let \( \ell \) be a maximal clique in \( \mathcal{M}_0 \) and fix it, where \( \mathcal{M}_0 \) is the set of all maximal cliques containing 0. Let \( \mathcal{L}_0 \) be the set which consists of \( \ell \) and all maximal cliques in \( \mathcal{M}_0 \) which intersect \( \ell \) only at 0. We call maximal cliques in \( \mathcal{L}_0 \) lines containing 0. Moreover let \( \mathcal{A}_0 = \mathcal{M}_0 \setminus \mathcal{L}_0 \). We call maximal cliques in \( \mathcal{A}_0 \) assemblies containing 0.

From the above definition, each vertex in \( N_1 \) is contained in one line and one assembly. We denote the line in \( \mathcal{L}_0 \) containing \( x \in N_1 \) by \( \ell(0, x) \), and denote the assembly in \( \mathcal{A}_0 \) containing \( x \in N_1 \) by \( A(0, x) \).

Consider a pair \( (x, y) \) of two adjacent vertices. Then \( \ell(0, x-y) + y \) and \( A(0, x-y) + y \) are maximal cliques containing \( x \) and \( y \). Moreover \( \ell(0, y-x) + x \) and \( A(0, y-x) + x \) are also maximal cliques containing \( x \) and \( y \). For each \( z \in \ell(0, y-x) + x \), \( z-x \) and \( y-x \) lie in \( \ell(0, y-x) \). So \( (z-x) - (y-x) - y \) lies in \( \ell(0, y-x) \). On the other hand, by the definition, \( \ell(0, x-y) = \ell(0, y-x) \). From this, it follows that \( z-y \in \ell(0, x-y) \) and so \( z \in \ell(0, x-y) + y \). Hence we have \( \ell(0, x-y) + y = \ell(0, y-x) + x \). Moreover we also have \( A(0, x-y) + y = A(0, y-x) + x \).

**Definition 3.3.** For each pair \( (x, y) \) of adjacent vertices, we define the line containing \( x \) and \( y \) by \( \ell(0, x-y) + y \), and the assembly containing \( x \)
and \( y \) by \( A(0, x - y) + y \). We denote \( \ell(0, x - y) + y \) and \( A(0, x - y) + y \) by \( \ell(x, y) \) and \( A(x, y) \), respectively.

The following lemma asserts that lines and assemblies are well defined.

**Lemma 3.7.** Let \( x \) and \( y \) be adjacent vertices in \( N \). Then, for any pair \((u, v)\) of vertices in \( \ell(x, y) \), \( \ell(x, y) = \ell(u, v) \). Moreover, for any pair \((u, v)\) of vertices in \( A(x, y) \), \( A(x, y) = A(u, v) \).

**Proof.** Consider \( \ell(0, x - y) \) and \( \ell(0, u - v) \). Since \( u - y \) and \( v - y \) lie in \( \ell(0, x - y) \), \( u - v \) also lies in \( \ell(0, x - y) \). This implies that \( \ell(0, x - y) = \ell(0, u - v) \). Take a vertex \( w \) in \( \ell(x, y) \). Then \( w - y \) and \( v - y \) lie in \( \ell(0, x - y) \). So \( (w - y) - (v - y) = w - v \) lies in \( \ell(0, x - y) = \ell(0, u - v) \). Hence \( w \) lies in \( \ell(0, u - v) + v \). This implies \( \ell(x, y) = \ell(u, v) \). As for assemblies, the same argument is available. Q.E.D.

From now on, we use the following notation.

We denote the set of all lines in \( N \) by \( \mathcal{L} \), and the set of all assemblies in \( N \) by \( \mathcal{A} \). Moreover for each vertex \( x \) in \( N \), we denote the set of all lines containing \( x \) by \( \mathcal{L}_x \), and the set of all assemblies containing \( x \) by \( \mathcal{A}_x \). We denote the sizes of lines and assemblies by \( q \) and \( q' \), respectively.

### 4. Definition of Subspaces and the Structures of Assemblies and Lines

In this section, first we give the definition of subspaces of semilinear incidence structures, and next we determine the structures of assemblies and lines by introducing subspaces to them.

**Definition of Subspaces**

Let \( N = (\mathcal{P}, \mathcal{L}, \mathcal{A}) \) be a semilinear incidence structure, where \( \mathcal{P} \) is the set of all points and \( \mathcal{L} \) is the set of all lines.

**Definition 4.1.** (1) A subset \( S \) of \( \mathcal{P} \) is said to be **line-closed**, if whenever a line \( \ell \) intersects \( S \) in at least two vertices, \( \ell \) is contained in \( U \).

(2) A subset \( S \) of \( \mathcal{P} \) is called a **subspace**, if \( S \) is line-closed and connected.

(3) For a connected subset \( F \) of \( N \), the intersection \( S \) of all subspaces containing \( F \) is a subspace. Then we say that \( F \) generates \( S \) and \( F \) is a generating set and we denote \( S \) by \( \langle F \rangle \).

(4) For a subspace \( S \), the **dimension**, written \( \dim(S) \), is the number \( f \) such that the minimal cardinality of any generating set for \( S \) is \( f + 1 \). We call a generating set with the minimal cardinality a **minimal generating set**.
1-dimensional subspaces are lines. 2-dimensional subspaces are called planes, and \( j \)-dimensional subspaces are called \( j \)-subspaces for \( j \geq 3 \).

From Definition 4.1, we can introduce the notion of subspaces into \( N \) and its assemblies and lines as follows.

For each pair \((x, y)\) of adjacent vertices, there is one line \( \ell(x, y) \) containing \( x \) and \( y \). Therefore the incidence structure \( \Pi = (N, \mathcal{L}, \mathcal{E}) \) is semilinear, where \( \mathcal{L} \) is the set of all lines in \( N \). So we can define subspaces of \( N \) from Definition 4.1.

Similarly we can define another incidence structure \( \Pi_{\mathcal{A}} \) using assemblies instead of lines. Since for each pair \((x, y)\) of adjacent vertices there is one assembly \( A(x, y) \) containing \( x \) and \( y \), the incidence structure \( \Pi_{\mathcal{A}} = (N, \mathcal{A}, \mathcal{E}) \) is also semilinear. Then replacing lines with assemblies in Definition 4.1, we can also define subspaces of \( (N, \mathcal{A}, \mathcal{E}) \). To distinguish subspaces of \( \Pi \) and those of \( \Pi_{\mathcal{A}} \), we call subspaces of \( \Pi_{\mathcal{A}} \) \( \mathcal{A} \)-subspaces.

For each assembly \( A \), we can define an induced incidence structure \( \Pi(A) = (A, \mathcal{L}(A), \mathcal{E}) \), where \( \mathcal{L}(A) \) is defined to be the set \( \{ \ell \cap A | \ell \in \mathcal{L} \text{ and } |\ell \cap A| \geq 2 \} \). Since \( A \) is a clique, \( \Pi(A) \) is a linear incidence structure. Moreover, for each line \( \ell \), we can also define an induced incidence structure \( \Pi_{\mathcal{A}}(\ell) = (\ell, \mathcal{A}(\ell), \mathcal{E}) \), where \( \mathcal{A}(\ell) = \{ A \cap \ell | A \in \mathcal{A} \text{ and } |A \cap \ell| \geq 2 \} \).

**The Structures of Assemblies and Lines**

Here, we determine the structures of assemblies as linear incidence structures. Moreover, we also determine the setwise stabilizers of assemblies. Let \( A \) be an assembly and fix it. Without loss of generality, we can assume that \( A \) contains 0. Then \( A = A(0, x) \) for any \( x \neq 0 \) in \( A \). By Lemma 3.2 and Lemma 3.2', \( |\ell \cap A| = r \) for \( \ell \in \mathcal{L}(A) \). Thus, every line of \( \Pi(A) \) has exactly \( r \) vertices. Moreover, from Lemma 3.2, each line containing 0 coincides with the union \( E^* \) of some equivalence set \( E \) and \( \{0\} \).

Now we consider the setwise stabilizer \( G_A \) of \( A \) in \( G \). Since \( A \) is a group, \( G_A \) contains \( A \) as a normal subgroup which acts regularly on \( A \). Therefore \( G_A \) is of form \( A \cdot H_A \), where \( H_A \) is the stabilizer of 0 in \( G_A \). We denote by \( G_A^A \) and \( H_A^A \) the automorphism group of \( A \) induced by the action of \( G_A \) on \( A \) and that of \( H_A \) on \( A \) respectively, i.e., \( G_A^A = G_A/K \) and \( H_A^A = H_A/(K \cap H_A) \), where \( K \) is the kernel of the action of \( G_A \) on \( A \). Since \( A \cap K = 1 \), \( A \) is also a regular normal subgroup of \( G_A^A \) and \( G_A^A \) is of form \( A \cdot H_A^A \).

**Lemma 4.1.** \( G_A \) acts doubly transitively on the set of all vertices of \( A \).

**Proof.** Since \( G_A \) contains a regular normal subgroup \( A \), it suffices to show that the stabilizer \( (G_A)_0 = H_A \) of the vertex 0 acts transitively on \( A \setminus \{0\} \).
Consider a pair \((x, y)\) of distinct vertices in \(A\) and we show that there is an element in \(H_A\) which transforms \(x\) to \(y\). Take another vertex \(z\) in \(N_1 \setminus A\). Then \(\{0, x, z\}\) and \(\{0, y, z\}\) are 2-claws. By Assumption B, there is an element \(g\) in \(H\) such that \(z^g = z\) and \(x^g = y\). Since \(A = A(0, x) = A(0, y)\), \(A^g = A(0, y) = A\) or \(A^g = \ell'(0, y)\). From this, if \(q \neq q'\), then \(A^g = A\) and \(g\) lies in \(H_A\). Thus, we have only to deal with the case where \(q = q'\) and \(A^g = \ell'(0, y)\) for every \(g\) in \(G_{0,z} = H_z\) such that \(x^g = y\). If there is some element \(h\) in \(G_{0,y} = H_y\) such that \(\ell'(0, y)^h = A(0, y)\), then \(gh\) transforms \(x\) to \(y\) and it stabilizes \(A\). This implies that the element \(gh\) in \(H_A\) transforms \(x\) to \(y\). If there does not exist such an element \(h\) in \(G_{0,y}\), then \(G_{0,y}\) stabilizes \(\ell'(0, y)\) and \(A(0, y)\). By Lemma 3.6, this implies that \(G_{0,y}\) stabilizes \(L_0\) and \(\mathcal{S}_0\). By the distance transitivity, \(G_0\) also stabilizes \(L_0\) and \(\mathcal{S}_0\). Since \(g\) lies in \(G_{0,z}\), \(g\) stabilizes \(L_0\) and \(\mathcal{S}_0\). Therefore, \(g\) also stabilizes \(A = A(0, y)\). This is a contradiction. Thus, we have proved that there exists an element in \(H_A\) which transforms \(x\) to \(y\). Q.E.D.

**Lemma 4.2.** Let \(E\) be an equivalence set contained in \(A\), i.e., \(E^* = E \cup \{0\}\) is a line of \(\Pi(A)\). Then, the action of the setwise stabilizer \((H_A)_{E^*}\) of \(E^*\) on \(A \setminus E^*\) is one of the following:

1. \((H_A)_{E^*}\) acts transitively on \(A \setminus E^*\), or
2. \((H_A)_{E^*}\) has only two orbits \(O_1\) and \(O_2\) in \(A \setminus E^*\), and \(|O_1| = |O_2|\).

**Proof.** First we deal with the case where \(q \neq q'\). Let \(\ell\) be the unique line which contains \(E^*\). Consider a pair \((y, y')\) of vertices in \(A \setminus E^*\). Take a vertex \(x\) in \(\ell \setminus A\) and fix it. Then, by Assumption B, there is an element \(g\) in \(H\) which transforms \(y\) to \(y'\) and fixes \(x\). Since \(q \neq q'\) and \(A = A(0, y) = A(0, y')\), \(g\) stabilizes \(A\) and so \(g\) lies in \((H_A)_{E^*}\). This implies that \((H_A)_{E^*}\) acts transitively on \(A \setminus E^*\).

Next we deal with the case where \(q = q'\). Assume that \((H_A)_{E^*}\) does not act transitively on \(A \setminus E^*\). Take a vertex \(x\) in \(\ell \setminus E^*\) and a vertex \(y\) in \(A \setminus E^*\) and fix them. Let \(O_1\) be the orbit of \((H_A)_{E^*}\) containing \(y\) and \(O_2 = (A \setminus E^*) \setminus O_1\).

Now we show that \(O_2\) is also an orbit of \((H_A)_{E^*}\). Consider a pair \((z_1, z_2)\) of vertices in \(O_2\). Then, by the argument in the case where \(q \neq q'\), there are elements \(g_1\) and \(g_2\) in \(H\) such that \(x^g_1 = x\) and \(y^g_1 = z_1\) for \(i = 1, 2\). By the definition of \(O_2\), neither \(g_1\) nor \(g_2\) stabilizes \(A\), i.e., they exchange \(L_0\) and \(\mathcal{S}_0\). But from this, it follows that \(g_1^{-1}g_2\) stabilizes \(\mathcal{S}_0\). Since \(g_1^{-1}g_2\) fixes \(x\), \(g_1^{-1}g_2\) stabilizes \(E^* = \ell \cap A\) and so it stabilizes \(\ell\) and \(A\), that is, it lies in \((H_A)_{E^*}\). Thus, the element \(g_1^{-1}g_2\) in \((H_A)_{E^*}\) transforms \(z_1\) to \(z_2\). This implies that \(O_2\) is an orbit of \((H_A)_{E^*}\).

Finally, we show that \(|O_1| = |O_2|\). The subgroup \(H_{x,E^*} = G_{0,x,E^*}\), which fixes \(0\) and \(x\) and stabilizes \(E^*\), also stabilizes \(\ell(0, x)\), since \(E\) is not con-
tained in $A(0, x)$. Therefore, $G_{0,x,E^*}$ stabilizes $A$. Moreover, it can be shown that $G_{0,x,E^*}$ have two orbits $O_1$ and $O_2$ by the argument in the previous paragraph. Therefore, we have $|O_i| = |G_{0,x,E^*} : G_{0,x,E^*,y_i}|$, where $y_i$ lies in $O_i$. On the other hand, we can show that for each $y$ in $A \setminus E^* G_{0,x,y}$ does not exchange $\mathcal{L}_0$ and $\mathcal{A}_0$ by using the same argument as in the proof of Lemma 4.1. From this, $G_{0,x,y}$ stabilizes $A$ and so $G_{0,x,y}$ also stabilizes $E^*$, that is, $G_{0,x,y_i} = G_{0,x,E^*,y_i}$ for $i = 1, 2$. By Assumption B, $|G_{0,x,y_1}| = |G_{0,x,y_2}|$. From this, we have

$$|O_i| = |G_{0,x,E^*} : G_{0,x,E^*,y_i}| = |G_{0,x,E^*} : G_{0,x,y_i}|$$

Thus, two orbits have the same size. Q.E.D.

Now we determine the structure of the assembly $A$. Since a linear incidence structure whose lines contain same number of vertices can be considered as a 2-design with $\lambda = 1$, where lines are considered as blocks, $\Pi(A)$ is a 2-design with $v = q^e$, $k = r$, and $\lambda = 1$ admitting a doubly transitive automorphism group, where $v$ is the number of points and $k$ is the number of points incident with same block. We state Kantor's classification of 2-designs with $\lambda = 1$ admitting doubly transitive groups in [15].

**Theorem C (Kantor's Classification).** Let $\mathcal{D}$ be a 2-design $(v, k, \lambda)$ with $v > k > 2$ admitting an automorphism group 2-transitive on points. Then, $\mathcal{D}$ is one of the following designs:

1. $\text{PG}(d^*, q^*)$;
2. $\text{AG}(d^*, q^*)$;
3. The design with $v = (q^e)^3 + 1$ and $k = q^e + 1$ associated with $\text{PSU}(3, q^e)$ or $2G_2(q^e)$;
4. One of two affine planes having $3^4$ or $3^6$ points;
5. One of two designs having $v = 3^6$ and $k = 3^2$.

**Proposition 4.3.** $\Pi(A)$ is isomorphic to the affine space $\text{AG}(\bar{a}, r)$ of dimension $\bar{d}$ over $GF(r)$, where $r^2 = q^e$. Moreover, $A$ coincides with the translation group of $\text{AG}(\bar{a}, r)$ as groups.

**Proof.** We show that only the case (ii) in Theorem C occurs. By the fact that both of $v = q^e$ and $k = r$ are powers of the prime number $p$ and $G_A^4$ has a regular normal subgroup $A$, neither the case (i) nor the case (iii) occur.

For the case (v), $H_A^4 = SL(2, 13)$ by the proof of Theorem C in [15]. By seeing the order of $SL(2, 13)$, we can show that $H_A$ does not satisfy Lemma 4.2. Therefore, this case does not occur.
For the case (iv), it follows that $A = E_1^* \oplus E_2^*$ for any two distinct equivalence sets $E_1$ and $E_2$ in $A$ by seeing their orders. From this, by letting $\ell_i$ be the line containing $E_i$, we can show that $N_1$ is contained in $\ell_1 \oplus \ell_2$. On the other hand, for every vertex $w$, $w$ can be written as $w = w_1 + \cdots + w_s$, where $w_i \in N_1$. Thus, every vertex $w$ is contained in $\ell_1 \oplus \ell_2$ and so $w$ can be written as $w = x_1 + x_2$, where $x_i \in \ell_i$. This implies that the diameter $d$ of the graph $\Gamma$ is 2. This contradicts the assumption $d \geq 3$. Thus, the case (iv) does not occur, and we have proved that $\Pi(A) \cong AG(\tilde{A}, r)$.

Since every doubly transitive group has a unique minimal normal subgroup, $A$ coincides with the translation group of $AG(\tilde{A}, r)$. Q.E.D.

Now we denote the dimension of $A$ as an affine space by $\tilde{d}$. Then, by the distance transitivity of the graph $\Gamma$, $\tilde{d}$ does not depend on the choice of the assembly $A$.

**Corollary 4.4.** The dimension $\tilde{d}$ of assemblies is greater than 2.

**Proof.** Assume, to the contrary, that $\tilde{d} = 2$. By the same argument as in the case (iv) of the proof of Proposition 4.3, we can show that the diameter $d$ of the graph is equal to 2. This is a contradiction. Q.E.D.

Next we determine the structure of $H^*_A$ by using Kantor’s classification and Delandtsheer’s result. As for automorphism groups which act doubly transitively on affine spaces, Kantor classified as follows. (See [15] or [6].)

**Theorem C’ (Kantor’s Classification).** Let $S$ be an affine space $AG(d^* q^*)$ with $d^* \geq 3$ and $q^* \geq 3$ admitting a doubly transitive automorphism group $G^*$. Then, the one point stabilizer $G^*_0$ is one of the following, where $0$ is a point in $S$.

1. $G^*_0 \cong GL(1, (q^*)^d)$,
2. $G^*_0 \cong SL(d^*/\delta, (q^*)^\delta)$,
3. $G^*_0 \cong Sp(d^*/\delta, (q^*)^\delta)$, $d^* \geq 4\delta$,
4. $G^*_0 \cong G_2((q^*)^\delta)'$, $q^*$ even, $d^* = 6\delta$,
5. $(d^*, q^*) = (4, 3)$ and $G^*_0 \cong SL(2, 5)$,
6. $(d^*, q^*) = (4, 3)$ and $G^*_0$ has a normal extraspecial subgroup $M$ of order $2^5$, and $G^*_0/M$ is isomorphic to a subgroup of $S_5$, or
7. $(d^*, q^*) = (6, 3)$ and $G^*_0 = SL(2, 13)$.

In [8], Delandtsheer classified the following.

**Theorem D (Delandtsheer).** Let $S$ be a finite linear space containing a proper linear subspace and a line of size $> 2$. $S$ has an automorphism group
G* which is transitive on the unordered pairs of intersecting lines if and only if one of the following occurs:

(I) \( S = PG(d^*, q^*) \) and \( G^* \cong PSL(d^* + 1, q^*) \) \((d^* \geq 3 \text{ and } q^* \geq 2)\) or \( G^* \cong A_7 \) inside \( PSL(4, 2) \); or

(II) \( S = AG(d^*, q^*) \) and \( G^* \cong ASL(d^*, q^*) \) \((d^* \geq 3, q^* \geq 3)\).

By using Theorem C' and Theorem D, we have the following.

**Proposition 4.5.** \( H_A \) contains \( SL(\bar{d}, r) \) which acts naturally on the affine space \( \Pi(A) \cong AG(\bar{d}, r) \).

**Proof.** By Lemma 4.2, \((H_A)_{E^*}\) has one or two orbits in \( A \setminus \{E^*\} \). If \((H_A)_{E^*}\) acts transitively on \( A \setminus \{E^*\} \), \( G_A \) acts transitively on the ordered pairs of intersecting lines. Since \( A \) has a structure of an affine space whose dimension is greater than 2, it follows that \( H_A \) contains \( SL(\bar{d}, r) \) which acts naturally on \( \Pi(A) \cong AG(\bar{d}, r) \) by Theorem D. Thus, we consider the remaining case, i.e., the case where \((H_A)_{E^*}\) has exactly two orbits of same size in \( A \setminus E^* \) and \((H_A)_{E^*}\) does not act transitively on lines in \( L_0(A) \setminus \{E^*\} \), and show that this case does not occur.

By Theorem C', there are seven cases and so we will check those seven cases. The case (4) does not occur, since \( r \) is an odd number.

For the cases (6) and (7), we can estimate the order of \((H_A)_{E^*}\), and from this we can show that \((H_A)_{E^*}\) does not have exactly two orbits of same size in \( A \setminus E^* \).

For the case (5), \((H_A)_{E^*}\) has two orbits whose sizes are 39. Therefore, \( |(H_A)_{E^*}| \) is divisible by 13. But from the complete list of all maximal subgroups of \( PSL(4, 3) \), it follows that the one point stabilizer \( IGL(4, 3) \) of the full automorphism \( AGL(4, 3) \) does not have a subgroup which contains \( SL(2, 5) \) and whose order is divisible by 13. (See [7].) Thus, the case (5) does not occur.

For the case (1), by seeing the order of \( IGL(1, r^3) \), \( |(H_A)_{E^*}| \leq (r - 1) \bar{d}a \), where \( r = p^a \). Since \((H_A)_{E^*}\) has exactly two orbits of size \((r^3 - r)/2\), we have \((r - 1) \bar{d}a \geq (r^3 - r)/2\). From this, we also have

\[
2\bar{d}a > p^{(a-1)/a} \geq 3^{a-1}a.
\]

But it is impossible, since \( \bar{d} \geq 3 \).

For the case (2), we can assume that \( \bar{d} > \delta \) by the proof for the case (1). We consider orbits of \((H_A)_{x} = G_{0,E^*} = (G_A)_{0,x} \) on \( A \setminus E^* \), where \( x \in E \). Then, \((H_A)_{x}\) has an orbit whose size is \( r^3 - r^\delta \). Since \((H_A)_{E^*}\) contains \((H_A)_{x}\), the size \((r^3 - r)/2 = (r^3 - r)/2\) of two orbits of \((H_A)_{E^*}\) is not smaller than \( r^3 - r^\delta \). Thus, we have the following inequality

\[
r^3 - r^\delta \leq (r^3 - r)/2.
\]
From this inequality, we have $\delta \leq \bar{d} < \delta + 1$ and so $\bar{d} = \delta$. This is a contradiction.

As for the case (3), we consider the set $\mathcal{L}_0(A)$ of all lines of $\Pi(A)$ containing 0 as a projective space $PG(\bar{d}-1, r)$ derived from the affine space $A$ and its point 0. We denote the induced automorphism group of $H'_A$ on $\mathcal{L}_0(A)$ by $H'$. Since each point of $PG(\bar{d}-1, r)$ is a line of $A$, the one point stabilizer $H'_{E^*}$ of $H'$ has exactly two orbits of size $(r^\delta - r)/2(r-1)$ on $\mathcal{L}(A) \setminus E^*$. On the other hand, the action of $Sp(\bar{d}/\delta, r^\delta)$ on $A$ introduces a structure of another space with symplectic metric to $A$, and each “line” in that space contains $(r^\delta - 1)/(r-1)$ lines of $\Pi(A)$. From this, a projective space $PG(\bar{d}/\delta - 1, r^\delta)$ can be constructed in $PG(\bar{d}-1, r)$. Then, the induced automorphism group of $H'$ on it contains $PSp(\bar{d}/\delta, r^\delta)$. Each one point stabilizer of $PSp(\bar{d}/\delta, r^\delta)$ has three orbits on $PG(\bar{d}/\delta - 1, r^\delta)$, and those sizes are $1$, $r^\delta - \delta$, and $r^\delta(r^{\delta-2\delta} - 1)/(r^\delta - 1)$. Let $F$ be a “line” in the symplectic metric space containing the line $E^*$ of $\Pi(A)$, and let $P_1$ and $P_2$ be the remaining orbits of the stabilizer of $F$ of $PSp(\bar{d}/\delta, r^\delta)$ on $PG(\bar{d}/\delta - 1, r^\delta)$. If $(H'_A)_{E^*}$ transforms a line of $\Pi(A)$ which is contained in a “line” in $P_1$ to a line of $\Pi(A)$ which is contained in a “line” in $P_2$, then the induced action of $H'$ on $PG(\bar{d}/\delta - 1, r^\delta)$ is doubly transitive. This implies that $H'_A$ contains $SL(\bar{d}/\delta, r^\delta)$ by Theorem D. By the argument for the case (2), this case does not occur. Therefore, since $(H'_A)_{E^*}$ has exactly two orbits in $\mathcal{L}_0(A) \setminus \{E^*\}$, all lines of $\Pi(A)$ which are contained in “lines” in $P$, are contained in same orbit for each $i$. From this, the size of one orbit exceeds $r^{\delta-\delta}(r^\delta - 1)/(r-1) = (r^\delta - r^\delta)/(r-1)$. Thus, we have the following inequality.

$$(r^{\delta-\delta}/(r-1) \leq (r^\delta - r)/2(r-1).$$

From this inequality, we have $\bar{d} = \delta$. This is a contradiction. Q.E.D.

As for each line $\ell$, we can use the same arguments as were used in Lemma 4.1, Lemma 4.2, Proposition 4.3, Corollary 4.4, and Proposition 4.5. Thus, we have the following.

**Proposition 4.6.** For each line $\ell$, $\Pi_{af}(\ell)$ is isomorphic to the affine space $AG(n, r)$ of dimension $n$ over $GF(r)$, where $r^n = q$ and $n \geq \bar{d} \geq 3$. Moreover, for the setwise stabilizer $H_{\ell}$ of $\ell$ in $H$ its induced automorphism group $H'_{\ell}$ on $\ell$ contains $SL(n, r)$ which acts naturally on the affine space $\Pi_{af}(\ell) \cong AG(n, r)$, and if $\ell$ contains 0, then $\ell$ coincides with the translation group of $AG(n, r)$ as groups.

5. Planes Are Nets

In this section, we show that subgraphs defined as planes are nets. By Section 4, the incidence structures $\Pi = (N, \ell, \varepsilon)$ and $\Pi_{af} = (N, \mathcal{A}, \varepsilon)$ are
semilinear. First we consider subspaces of \( \Pi \). Then from Definition 4.1, we have the following.

**Proposition 5.1.** Let \( U \) be a subspace of \( N \). Then, for any vertex \( x \) in \( N \), \( U + x \) is also a subspace. Moreover \( \dim(U) = \dim(U + x) \).

**Proof.** For each line \( \ell \) in \( U \), \( \ell + x \) is also a line in \( U + x \). So it follows easily that \( U + x \) is a subspace. Let \( F = \{f_1, \ldots, f_{k+1}\} \) be a minimal generating set for \( U \). We show that \( F + x = \{f_1 + x, \ldots, f_{k+1} + x\} \) is also a generating set for \( U + x \). \( V \) is a subspace containing \( F \) if and only if \( V' = V + x \) is a subspace containing \( F + x \). So we have the following.

\[
\bigcap_{V' \supseteq F + x} V' = \bigcap_{V + x \supseteq F + x} (V + x) = \left( \bigcap_{V \supseteq F} V \right) + x = U + x.
\]

This implies that \( F + x \) is a generating set for \( U + x \). Conversely, for a minimal generating set \( F' \) for \( U + x \), we can show that \( F' - x \) is a generating set for \( U \). From this, we have \( \dim(U) = \dim(U + x) \). Q.E.D.

Now we consider a plane \( U \). By the definition of planes, there is a generating set \( F = \{x, y, z\} \) which is connected. We can assume that \( x \) is adjacent to \( y \) and \( z \). Then \( U \) is generated by two distinct lines \( \ell(x, y) \) and \( \ell(x, z) \), i.e., \( U = \langle \ell(x, y), \ell(x, z) \rangle \). Consider \( U - x \). Then \( U - x \) is also a plane which is generated by two lines \( \ell(x, y) - x \) and \( \ell(x, z) - x \). Since \( \ell(x, y) - x = \ell(0, y - x) \) and \( \ell(x, z) - x = \ell(0, z - x) \), we can see that \( U - x \) is generated by two lines in \( \mathcal{L}_0 \). So first we consider planes generated by two distinct lines in \( \mathcal{L}_0 \).

Let \( T \) be a subspace (plane) generated by two lines \( \ell_1 \) and \( \ell_2 \) in \( \mathcal{L}_0 \), i.e., \( T = \langle \ell_1, \ell_2 \rangle \), and let \( R \) be the group generated by two groups \( \ell_1 \) and \( \ell_2 \), i.e., \( R = \langle \ell_1, \ell_2 \rangle = \langle x_1 + x_2 | x_1 \in \ell_1, x_2 \in \ell_2 \rangle \). Since \( \ell_1 \cap \ell_2 = \{0\} \), \( R \simeq \ell_1 \oplus \ell_2 \) and \( |R| = q^2 \). We will show \( R = T \).

**Lemma 5.2.** \( R \) is line-closed.

**Proof.** First we show that for each vertex \( u \in R \) adjacent to 0, the line \( \ell(0, u) \) is contained in \( R \). Let \( V \) be the set of all vertices in \( R \) adjacent to 0. Let \( u \) be a vertex in \( V \). Since \( u \) lies in \( R \), \( u \) can be written as \( u = u_1 + u_2 \), where \( u_1 \in \ell_1 \) and \( u_2 \in \ell_2 \). Moreover, since \( u \) is adjacent to 0, i.e., \( u_1 \) and \( -u_2 \) are adjacent, it follows that \( u_1 \) and \( u_2 \) are adjacent by Lemma 2.4. Conversely, for adjacent vertices \( v_1 \) and \( v_2 \), where \( v_1 \in \ell_1 \) and \( v_2 \in \ell_2 \), a vertex \( v_1 + v_2 \) in \( R \) is adjacent to 0. Therefore \( V = \{v_1 + v_2 | v_1 \in \ell_1, v_2 \in \ell_2[v_1]\} \) and \( v_1 + v_2 \neq 0 \). By Lemmas 3.2 and 3.4, for a vertex \( v_1 \neq 0 \) in \( \ell_1[v_1] \}, \{0\} \) is an equivalence set \( E_{v_2} \), where \( v_2 \in \ell_2[v_1] \} \{0\} \) and \( \ell_2[v_2] = \ell_2 \cap A(0, v_1) \). So we have \( |V| = (q-1)(r-1) + 2(q-1) = (q-1)(r+1) \).
and \( |V \setminus (\ell_1 \cup \ell_2)| = (q-1)(r-1) \). Moreover, for each \( v_i \neq 0 \in \ell_1 \) and \( v_i' \in \ell_1, \ i = 1, 2 \), \( v_i' + v_2' \) belongs to \( A(0, v_1) \) if and only if \( v_2' \) belongs to \( E_{v_i'}^{A} \), where \( v_2 \in \ell_2 \cap A(0, v_1) \setminus \{0\} \), and \( v_i' - v_1 \) is adjacent to \( v_2 \). From this, it follows that \( v_i' + v_2' \) belongs to \( A(0, v_1) \) if and only if \( v_i' \) belongs to \( E_{v_i'}^{A} \), where \( v_i' \in A(O, \nu_i) \setminus \{0\} \) and \( v_i' - v_i \) is adjacent to \( 0_{2} \). From this, it follows that \( u_i + v_i \) belongs to \( A(0, v_i) \) if and only if \( u_i \) belongs to \( E_{u_i}^{A} \), and \( v_i \) belongs to \( E_{v_i}^{A} \), where \( v_i \in A(O, \nu_i) \setminus \{0\} \) and \( v_i - v_i \) is adjacent to \( 0_{2} \).

From this, it follows that \( u_i + v_i \) belongs to \( A(0, v_i) \) if and only if \( u_i \) belongs to \( E_{u_i}^{A} \), and \( v_i \) belongs to \( E_{v_i}^{A} \), where \( v_i \in A(O, \nu_i) \setminus \{0\} \) and \( v_i - v_i \) is adjacent to \( 0_{2} \).

Now consider a vertex \( u \) in \( V \setminus (\ell_1 \cup \ell_2) \), and fix it. Then as mentioned in the above, \( u \) can be written as \( u = u_1 + u_2 \), where \( u_1 \in \ell_1 \) and \( u_2 \in \ell_2 \). We show that \( \ell(0, u) \) is contained in \( R = \langle \ell_1, \ell_2 \rangle \).

Assume, to the contrary, that \( \ell(0, u) \) is not contained in \( R \). Then, there is a vertex \( v \) in \( \ell(0, u) \) which does not lie in \( \langle \ell_1 \cap A(0, v), \ell_2 \cap A(0, v) \rangle \). This implies that \( v \) does not lie in \( \langle \ell_1 \cap A(0, v), \ell_2 \cap A(0, v) \rangle \), which is an affine plane of \( \Pi(A) \) containing \( 0 \) and \( \Pi(A) \) is an affine space \( AG(\tilde{d}, r) \) by Proposition 4.3. \( A \cap V \) is an affine plane containing \( 0 \) in the affine space \( \Pi(A) \).

Next we show that \( R \) contains only two lines \( \ell_1 \) and \( \ell_2 \) among all lines in \( \mathcal{L}_0 \). Assume, to the contrary, that there is a line \( m \) in \( \mathcal{L}_0 \) such that \( m \neq \ell_1, \ell_2 \) but \( m \) is contained in \( R \). Since \( H_{A(0,v)}^{A(0,v)} \) contains \( SL(\tilde{d}, r) \) by Proposition 4.5, where \( H_{A(0,v)}^{A(0,v)} \) is the setwise stabilizer of the assembly \( A(0, v) \) in \( \mathcal{H} \), the stabilizer \( H_{A(0,v)}^{A(0,v)} \) of two “lines” \( \ell_1 \cap A(0, v) \) and \( \ell_2 \cap A(0, v) \) of the affine space \( \Pi(A(0, v)) \) acts transitively on all “lines” containing \( 0 \) of the affine plane \( \langle \ell_1 \cap A(0, v), \ell_2 \cap A(0, v) \rangle \) except two “lines” \( \ell_1 \cap A(0, v) \) and \( \ell_2 \cap A(0, v) \). Therefore, for each line \( m' \) in \( \mathcal{L}_0 \) such that \( m' \cap A(0, v) \) is contained in \( \langle \ell_1 \cap A(0, v), \ell_2 \cap A(0, v) \rangle \), there is an element \( g \) in \( H_{A(0,v)}^{A(0,v)} \) such that \( (\ell_1 \cap A(0, v)) g = \ell_2 \cap A(0, v) \), and \( (\ell_1 \cap A(0, v)) g = \ell_1 \cap A(0, v) \). Moreover, since \( g \) stabilizes the assembly \( A(0, v) \), \( g \) also stabilizes \( \mathcal{L}_0 \) and \( \mathcal{A}_0 \). Therefore, it follows that \( g \) fixes two lines \( \ell_1 \) and \( \ell_2 \) and \( g \) transforms \( m \) to \( m' \). Then, it follows that \( \ell_i \cap A(0, v) \) \((m \cap A(0, v)) g = \ell_i \cap A(0, v) \) for \( i = 1, 2 \) and \( (m \cap A(0, v)) g = m' \cap A(0, v) \). Since the affine plane \( (\ell_1 \cap A(0, v)) \oplus (\ell_2 \cap A(0, v)) \) has \( r + 1 \) “lines” containing \( 0 \), \( R \) contains \( r + 1 \) lines in \( \mathcal{L}_0 \). By seeing the number of vertices of those lines, it follows that every line in \( \mathcal{L}_0 \) which intersects \( R \) at least two vertices is contained in \( R \). This is a contradiction.

Next we show that every line in \( \mathcal{L}_0 \) intersects \( R \) in at least two vertices. Suppose that there is a line \( m \) in \( \mathcal{L}_0 \) which intersects \( R \) only at the vertex \( 0 \), i.e., \( m \cap R = \{0\} \). Then, since \( H_{A(0,v)}^{A(0,v)} \) contains \( SL(\tilde{d}, r) \) by Proposition 4.5, \( \ell(0, u) \cap A(0, v) \) can be transformed to any “line” of the affine space \( \Pi(A) \) containing \( 0 \) which intersects the affine plane \( (\ell_1 \cap A(0, v)) \oplus (\ell_2 \cap A(0, v)) \) only at \( 0 \) by the action of the stabilizer of two “lines” \( \ell_1 \cap A(0, v) \) and \( \ell_2 \cap A(0, v) \) in \( H_{A(0,v)}^{A(0,v)} \). Therefore, there is an element \( g \) in
the setwise stabilizer $H_{A(0,v)}$ of $A(0,v)$ such that $(\ell_1 \cap A(0,v))^g = \ell_1 \cap A(0,v)$, $(\ell_2 \cap A(0,v))^g = \ell_2 \cap A(0,v)$, and $(\ell(0,v) \cap A(0,v))^g = m \cap A(0,v)$. Since $g$ stabilizes $A(0,v)$, $g$ stabilizes $\mathcal{L}_0$ and $\mathcal{A}_0$. Therefore, it follows that $\ell_1^g = \ell_1$, $\ell_2^g = \ell_2$, and $\ell(0,v)^g = m$. From this, we have $(\ell(0,v) \cap R)^g = \ell(0,v)^g \cap R = m \cap R$ and so $m \cap R \neq \{0\}$. This is a contradiction. Thus, we conclude that every line in $\mathcal{L}_0$ intersects $R$ at least two vertices. Moreover, by the same argument as in the above and the fact that $|\{\ell \in \mathcal{L}_0 | (\ell \cap A) \cap \langle \ell_1 \cap A, \ell_2 \cap A \rangle = \{0\}\}| > |\{\ell \in \mathcal{L}_0 | (\ell \cap A) \cap \langle \ell_1 \cap A, \ell_2 \cap A \rangle = \{0\}\}|$, for every $A$ in $\mathcal{A}_0$, for every pair $(m, m')$ of distinct lines in $\mathcal{L}_0 \\setminus \{\ell_1, \ell_2\}$, there is an element $g$ in the setwise stabilizer $H_R$ of $R$ in $H$ such that $m^g = m'$.

Now we consider the number of vertices in the intersection of $R$ and each line $m$ in $\mathcal{L}_0 \\setminus \{\ell_1, \ell_2\}$. Then, that number does not depend on the choice of the line $m$ by the transitivity of $H_R$ on them. We denote by $k$ that number, i.e., $k = |m \cap R|$, where $m \in \mathcal{L}_0 \\setminus \{\ell_1, \ell_2\}$. Then, we have the following.

$$(k - 1)((q' - 1)/(r - 1) - 2) = (q - 1)(r - 1).$$

Since $q' = r^d$ and $q = r^n$, we also have

$$(k - 1)(r^d - 2r + 1) = (r^n - 1)(r - 1)^2.$$

On the other hand, for each line $m$ in $\mathcal{L}_0$ and its vertex $x$ in $m \cap R$, the equivalence set $E_x$ is also contained in $m \cap R$, since the line $E_x = m \cap A(0,x)$ is contained in the affine plane $(\ell_1 \cap A(0,x)) \oplus (\ell_2 \cap A(0,x))$. Moreover, as $R$ and $m$ are groups, $m \cap R$ is also a group. Since $m$ has a structure of an affine space $AG(n, r)$ and each “line” of $m$ is the union of an equivalence class and the vertex 0, $m \cap R$ has a structure of an affine subspace of $AG(n, r)$. Thus, there is a positive integer $e$ smaller than $n$ such that $k = |m \cap R| = r^e$.

Then, we also have the following equation.

$$(r^e - 1)(r^d - 2r + 1) = (r^n - 1)(r - 1)^2. \quad (5.1)$$

By considering (5.1) modulo $r^2$ and using the fact $n \geq d \geq 3$, we have $e \geq 2$ and

$$r^{d+e} - r^{d} - 2r^{e+1} + r^{e} = r^{n+2} - 2r^{n+1} + r^{n} - r^2. \quad (5.2)$$

Also by considering (5.2) modulo $r^3$ and using the fact $n \geq d \geq 3$, we have

$$r^e \equiv -r^2 \pmod {r^3}.$$

But it is impossible. Hence we obtain a contradiction.

Finally we consider lines which intersect $R$ in at least two vertices. Let $\ell$ be a line which contains two distinct vertices, say $x$ and $y$, in $R$. Then
\( \ell' = \ell - x \) contains 0 and \( y - x \in R \). So \( R \) contains \( \ell' \). Since \( R \) is a group, \( R + x = R \) and so \( R \) contains \( \ell' + x = \ell \). From this, we conclude that \( R \) is line-closed. Q.E.D.

**Lemma 5.3.** \( R = \langle \ell_1, \ell_2 \rangle \) is connected.

**Proof.** Let \( u \) be a vertex in \( R \). Then \( u \) can be written as \( u = u_1 + u_2 \), where \( u_1 \in \ell_1 \) and \( u_2 \in \ell_2 \). From this, \( u \) is adjacent to \( u_1 \) and \( u_1 \) is adjacent to 0. So \( u \) is connected to 0. Thus, any vertex in \( R \) is connected to 0. Therefore \( R \) is connected. Q.E.D.

From Lemmas 5.2 and 5.3, \( R \) is a subspace of \( N \). So \( R \) contains \( T \).

**Proposition 5.4.** \( T = R \).

**Proof.** Consider the following sets.

\[
X = \{ \{u_1, u_2\} \mid u_1 \in \ell_1 \setminus \{0\}, u_2 \in \ell_2\{u_1\} \setminus \{0\} \},
\]

\[
Y = \{ \ell(u_1, u_2) \mid \{u_1, u_2\} \in X \},
\]

\[
Z = \bigcup_{m \in Y} m.
\]

Since \( T \) is line-closed and \( T \) contains all pairs in \( X \), \( T \) also contains all lines in \( Y \), i.e., \( T \supseteq Z \). We will count the number of vertices in \( Z \) and the number of vertices in \( T \).

First we show that \( \ell(u_1, u_2) \neq \ell(u_1', u_2') \) for \( \{u_1, u_2\} \neq \{u_1', u_2'\} \in X \). Suppose that \( \ell(u_1, u_2) = \ell(u_1', u_2') \). If \( u_1 \neq u_1' \), then \( \ell(u_1, u_2) = \ell(u_1, u_1') = \ell_1 \). So \( u_2 \) lies in \( \ell_1 \) and this is a contradiction. For the case where \( u_2 \neq u_2' \), we also obtain a contradiction. Hence we conclude that \( \ell(u_1, u_2) \neq \ell(u_1', u_2') \) for \( \{u_1, u_2\} \neq \{u_1', u_2'\} \in X \). From this, we have \( |X| - |Y| = (q - 1)(r - 1) \).

Next consider the number of lines in \( Y \) containing a fixed vertex in \( Z \). For each vertex \( z \) in \( Z \), let \( Y_z \) be the set of all lines in \( Y \) containing \( z \). We show that \( |Y_z| \leq r - 1 \) for \( z \in Z \setminus (\ell_1', \ell_2') \). Consider the case where the vertex \( z \in N_1 \). Let \( \ell(u_1, u_2) \in Y_z \). Then \( u_1 \) lies in \( \ell_1[z] \setminus \{0\} \). If there is another vertex \( u_2' \in \ell_2 \) such that \( \ell(u_1, u_2') \in Y_z \), then \( z \in N_1^* \cap (N_1^* + u_2') \cap (N_1^* + u_2') = \ell_2 \). So the number of lines in \( Y_z \) is not greater than the number of vertices in \( \ell_1[z] \setminus \{0\} \). By Lemma 3.4, \( |\ell_1[z] \setminus \{0\}| = r - 1 \). So \( |Y_z| \leq r - 1 \). Next consider the case where the vertex \( z \notin N_1 \). By the same argument as in the case where \( z \in N_1 \), we have \( |Y_z| \leq |\ell_1[z]| = r \). Suppose now that \( |Y_z| = r \). Then for any vertex \( u_1 \in \ell_1[z] \), there is a vertex \( u_2 \) in \( \ell_2 \) such that \( z \in \ell(u_1, u_2) \). As \( z \) is a vertex in \( Z \subseteq R \), \( z \) can be written uniquely as \( z = z_1 + z_2 \), where \( z_1 \in \ell_1[z] \setminus \{0\} \) and \( z_2 \in \ell_2 \setminus \{0\} \). Then \( z_1 \) lies in \( \ell_1[z] \). So there is a vertex \( w_2 \neq 0 \) in \( \ell_2 \) such that \( z \in \ell(z_1, w_2) \). Since \( \ell(z_1, w_2) = \ell(0, w_2 - z_1) + z_1 \), \( z_2 \) lies in \( \ell(0, w_2 - z_1) \) and so \( \ell_2 = \ell' \).
ON DISTANCE TRANSITIVE GRAPHS

215

Then $z_1$ lies in $\ell_2$. This is a contradiction. Thus we have $|Y_z| \leq r - 1$.

Now we count the number of vertices in $T$. We note that each line in $Y$ contains one vertex in $\ell_1$ and one vertex in $\ell_2$, but does not contain 0. From the fact that $|Y| = (q - 1)(r - 1)$ and $|Y_z| \leq r - 1$ for $z \in Z \setminus (\ell_1 \cup \ell_2)$, it follows that $|Z \setminus (\ell_1 \cup \ell_2)| \geq (q - 1)(r - 1)(q - 2)/(r - 1) = (q - 1)(q - 2)$. So $|Z| \geq (q - 1)(q - 2) + 2(q - 1) + 1 = q(q - 1) + 1$. Thus $|T| \geq |Z| \geq q(q - 1) + 1$ and so $|R \setminus T| \leq q - 1$.

Finally we show $R = T$. Assume the contrary. Then there is a vertex $z$ in $R$ which does not lie in $T$. As mentioned above, $z$ is written uniquely as $z = z_1 + z_2$, where $z_1 \in \ell_1 \setminus \{0\}$ and $z_2 \in \ell_2 \setminus \{0\}$. So $z$ is adjacent to $z_1$ and $z_1 + 2z_2$, i.e., $z \in \ell(z_1, z_1 + 2z_2)$. Since $|R \setminus T| \leq q - 1$, $|\ell(z_1, z_1 + 2z_2) \cap T| \geq 1$. Moreover, since $T$ is line-closed, it follows that $|\ell(z_1, z_1 + 2z_2) \cap T| = 1$ and $R = T \cup \ell(z_1, z_1 + 2z_2)$. On the other hand, $z$ is also adjacent to $z_2$ and $z_2 + 2z_1$, i.e., $z \in \ell(z_2, z_2 + 2z_1)$. It can be shown easily that $\ell(z_1, z_1 + 2z_2) \not\subseteq \ell(z_2, z_2 + 2z_1)$. So $|\ell(z_1, z_1 + 2z_2) \cap \ell(z_2, z_2 + 2z_1)| = 1$ and $|\ell(z_2, z_2 + 2z_1) \cap T| \geq q - 1$. By the line-closedness of $T$, $T$ contains $\ell(z_2, z_2 + 2z_1)$ and so $T$ contains $z$. This contradicts the assumption. Hence we conclude that $R = T$. Q.E.D.

From Proposition 5.4, every subspace generated by two lines in $S_0$ is a group of order $q^2$. Therefore, for two distinct intersecting lines $m_1$ and $m_2$ in $T$ which contain 0, it follows that $\langle m_1, m_2 \rangle = \langle \ell_1, \ell_2 \rangle = T$ by seeing the orders. Moreover, for any two distinct intersecting lines $m_1$ and $m_2$ in $T$, $\langle m_1 - z, m_2 - z \rangle$ is contained in $T$, where $\{z\} = m_1 \cap m_2$. Also by seeing the orders, we have $\langle m_1 - z, m_2, m_2 - z \rangle = T$. Since $\langle m_1, m_2 \rangle - z = \langle m_1 - z, m_2 - z \rangle$, we have $\langle m_1, m_2 \rangle = T + z = T$. So any pair of two intersecting lines in $T$ generates $T$. From this and Proposition 5.4, we have the following.

**Corollary 5.5.** Every plane is generated by any pair of its intersecting lines, and has $q^2$ vertices. Moreover, if a plane contains 0, then it is a group.

Next we show that every plane is a net. Let $U$ be a plane generated by two intersecting lines $\ell_1$ and $\ell_2$. Consider the incidence structure $\Pi(U) = (U, L(U), \in)$, where $L(U)$ is the set of all lines in $U$. Then $\Pi(U)$ is semilinear.

**Lemma 5.6.** If $U$ contains 0, then $\Pi(U)$ is a net.

**Proof.** We examine the conditions of nets in Section 1 for $U$. It is clear that $|U| = q^2 > 1$ and $\Pi(U)$ satisfies (B1). So we consider (B2). We define classes of $L(U)$ as follows. For each line $\ell$, let $L_{\ell} = \{\ell + x | x \in U\}$, which
will be the set of all "parallel" lines of $\ell$ in $U$. We note that since $U$ is a group, every $\ell + x$ is contained in $U$. If $m \in \mathcal{L}(U)$ lies in $\mathcal{L}_1$, then $\mathcal{L}_m = \mathcal{L}_\ell$. Let $\mathcal{L}_1, \ldots, \mathcal{L}_r$ be all distinct classes of $\mathcal{L}(U)$. First we show the number of distinct classes is not smaller than $3$, i.e., $r \geq 3$. By the proof of Lemma 5.4, there are $r + 1$ lines in $U$ which contain $0$. Let $m$ and $m'$ be distinct lines in $\mathcal{L}(U)$ which contain $0$. If $m$ and $m'$ belong to same class, there is some vertex $v$ in $U$ such that $m' = m + v$. Then $m' = \ell(0, v)$. From this, $m'$ also contains $-v$. On the other hand, $m = m' - v$ contains $-v$ and so $m = \ell(0, -v)$. Since $m'$ contains $0$ and $-v$, we have $m' = m$. This is a contradiction. Hence it follows that there are at least $r + 1$ distinct classes in $\mathcal{L}(U)$ and so $r \geq r + 1 > 3$.

Next we show that $\Pi(U)$ satisfies (B2(i)), i.e., the lines of each class partition $U$. Consider a class $\mathcal{L}_l$. Suppose that there are distinct lines $\ell$ and $\ell + x$ in $\mathcal{L}_l$ such that $\ell \cap (\ell + x) \neq \emptyset$, where $x \in U$. Let $u$ be a vertex in $\ell \cap (\ell + x)$. Then $\ell - u$ and $\ell + x - u$ contain $0$ and so they are groups. From this, $-x + u$ lies in $\ell$ and so $-x$ lies in $\ell - u$. Since $\ell - u$ is a group, we have $\ell - u + x = \ell - u$. This implies that $\ell = \ell + x$, and a contradiction. Thus lines in $\mathcal{L}_l$ do not intersect each other.

Now we take a line $\ell$ in $\mathcal{L}_l$ and fix it. Then there is a line $m$ in $\mathcal{L}(U)$ such that $m$ intersects $\ell$ and $\langle \ell, m \rangle = U$ by Corollary 5.5. Let $\{x\} = \ell \cap m$. Then $U - x = \langle \ell - x, m - x \rangle$ by Proposition 5.4. So $U - x = \{u + v | u \in \ell - x, v \in m - x\}$. From this, it follows that $U - x = \cup_{y \in m - x}(\ell - x + y)$, and so $U = \cup_{y \in m - x}(\ell + y)$. Since every $\ell + y$ lies in $\mathcal{L}_l$, we have $U = \cup_{\ell \in \mathcal{L}_l} \ell'$. Thus $\Pi(U)$ satisfies (B2(ii)).

Finally we show that $\Pi(U)$ satisfies (B2(ii)), i.e., the lines of different classes intersect. Let $\ell$ and $m$ be lines which do not belong to same class. Let $u$ be a vertex in $\ell$. Since $\Pi(U)$ satisfies (B2(i)), there is a line $m' = m + v$ such that $m'$ contains $u$, where $v \in U$. By Proposition 5.4 and Corollary 5.5, $U = \langle \ell, m' \rangle = \langle \ell - u, m' - u \rangle + u$ and so $U = \langle \ell - u, m' - u \rangle$. Therefore $v$ and $u$ are written as $v = v_1 + v_2$ and $u = u_1 + u_2$, where $v_1, u_1 \in \ell - u$ and $v_2, u_2 \in m' - u$. Then $m = m' - u + v = m' - u + u_1 - v_1$ and $m' = m' - u + u = m' - u + u_1$. So $m = m' - v_1$. Since $u$ lies in $m'$, $u - v_1$ also lies in $m$. On the other hand, $u - v_1$ lies in $\ell - u + u = \ell$. From this, $m$ intersects $\ell$ at $u - v_1$. Hence $\Pi(U)$ satisfies (B2(ii)) and so $\Pi(U)$ is a net. Q.E.D.

**Proposition 5.7.** For every plane $U$ in $N$, $\Pi(U)$ is a net.

**Proof.** It is clear that $|U| \geq 3$. So we consider the condition (B2). Let $u$ be a vertex in $U$. Then $U - u$ is a plane which contains $0$. By Lemma 5.6, $\Pi(U - u)$ is a net. For each class $\mathcal{L}_l$ of $\mathcal{L}(U - u)$, set $\mathcal{L}_l' = \{\ell + u | \ell \in \mathcal{L}_l\}$. Since the lines of $\mathcal{L}_l$ partition $U - u$, the lines of $\mathcal{L}_l'$ also partition $U$. Thus $\Pi(U)$ satisfies (B1(i)). Let $\ell$ and $m$ be lines in $\mathcal{L}(U)$ which do not belong to same class. Then $\ell - u$ and $m - u$ do not belong to same class of
\( \ell(U-u) \). So by Lemma 5.6, \( \ell - u \) intersects \( m - u \). This implies that \( \ell \) intersects \( m \). Hence \( \Pi(U) \) satisfies (B2(ii)). Q.E.D.

The following lemma gives additional properties of planes, and it makes their structures clear.

**Lemma 5.8.** Let \( U \) be a plane.

1. For each vertex \( u \) in \( U \), there are exactly \( r+1 \) lines in \( U \) which contain \( u \). Moreover the number of classes of \( \mathcal{L}(U) \) is \( r+1 \).
2. Let \( x \) and \( y \) be non-adjacent vertices in \( U \). Then
   \[ |U \cap (N_1+x) \cap (N_1+y)| = r(r+1). \]
3. Let \( A \) be an assembly which intersects \( U \) in at least 2 vertices. Then \( |A \cap U| = r^2 \). So for any pair \( (x, y) \) of adjacent vertices in \( U \), \( |A(x, y) \cap U| = r^2 \).

**Proof.**

1. Let \( x \) be vertex in \( U \). Then \( U-x \) is a plane and so there is a pair \( (\ell_1, \ell_2) \) of two lines in \( \mathcal{L}_0 \) which generates \( U \). Then by Lemma 5.2, there are exactly \( (r+1)(q-1) \) vertices in \( U-x \) which are adjacent to 0. From this, there are exactly \( r+1 \) lines in \( U-x \) which contain 0. Hence there are exactly \( r+1 \) lines in \( U \) which contain \( x \). By the definition of nets, each class has one line which contains \( x \). Therefore the number of classes is \( r+1 \).

2. Let \( x \) and \( y \) be vertices in \( U \) with \( \partial(x, y) = 2 \). Consider \( U-x \). From (1), there are \( r+1 \) lines \( \ell_1, \ldots, \ell_{r+1} \) in \( U-x \) which contain 0. By Proposition 5.4, \( U-x = \langle \ell_1, \ell_i \rangle \) for any \( i \neq 1 \). So \( y-x \in U-x \) can be written as \( y-x = t_{1,i} + t_{2,i} \), where \( t_{1,i} \in \ell_1 \) and \( t_{2,i} \in \ell_i \). From this, it follows that \( y-x \) is adjacent to all \( \ell_i \)'s. By Lemma 3.4, \( |\ell_i \cap (N_1 + (y-x))| = r \) for any \( i \). Hence \( |(U-x) \cap N_1 \cap (N_1 + (y-x))| = r(r+1) \) and so \( |U \cap (N_1+x) \cap (N_1+y)| = r(r+1) \).

3. Take a vertex \( x \) in \( A \cap U \), and consider \( U-x \). Then \( U-x \) is a plane which contains 0. Let \( \ell_1 \) and \( \ell_2 \) be two distinct lines in \( U-x \) which intersect at 0. Then \( U-x = \langle \ell_1, \ell_2 \rangle \). From the proof of Lemma 5.2, \( (A-x) \cap (U-x) = \langle \ell_1 \cap (A-x), \ell_2 \cap (A-x) \rangle \). Hence we have \( (A \cap U) - x = (A-x) \cap (U-x) = \langle \ell \cap (A-x), m \cap (A-x) \rangle \), and \( |A \cap U| = r^2 \). Q.E.D.

As for planes of \( \Pi_\mathcal{A} \), say \( \mathcal{A} \)-planes, we have the following proposition corresponding to Proposition 5.4, Corollary 5.5, Proposition 5.7, and Lemma 5.8.
PROPOSITION 5.9. (1) Every $A$-plane is generated by any pair of its intersecting assemblies, and has $q^2$ vertices. Moreover, if a $A$-plane contains 0, then it is a group.

(2) For every $A$-plane $W$ in $N$, $\Pi_A(W) = (W, A(W), \varepsilon)$ is a net, where $A(W)$ is the set of all assemblies in $W$.

(3) All statements in Lemma 5.8 hold by changing the roles of lines and assemblies.

Proposition 5.9 can be shown by changing the roles of lines and assemblies in the corresponding lemmas and propositions.

LEMMA 5.10. Let $U$ be a plane. For any pair $(x, y)$ of non-adjacent vertices in $U$, $(N_1 + x) \cap (N_1 + y) \subseteq U$. Therefore, $c_2 = r(r + 1)$.

Proof. Let $x$ and $y$ be non-adjacent vertices in $U$. By subtracting $x$, we can assume $x = 0$ without loss of generality. Then $U$ is a plane containing 0. Take distinct lines $e_i$ and $e_j$ in $U$ which contain 0. Then $e_i \cap e_j = U$, and $y$ can be written as $y = t_1 + t_2$, where $t_1 \in e_i$ and $t_2 \in e_j$. Assume, to the contrary, that there is a vertex $u$ which is adjacent to 0 and $y$, but does not lie in $U$. Then, by the line-closedness of $U$, $e(0, u) \cap U = \{0\}$. Let $v = t_1 + t_2 - u$. Then, $v$ is adjacent to 0. Let $V = \langle e_i, e_j, e(0, u) \rangle$ and $A = A(0, v)$. Since $e(0, u) \cap U = \{0\}$, $V = U \oplus e(0, u)$ and so $|V| = q^3$.

CLAIM 1. $N = V = \langle e_i, e_j, e(0, v) \rangle$. Consequently, $|N| = q^3$.

Proof of Claim 1. Consider the assembly $A$. Then, $\langle e_i \cap A, e_j \cap A, e(0, u) \cap A \rangle$ generates a 3-subspace in the affine space $H(A) \cong AG(2, r)$. We denote this 3-subspace by $T$. If $e(0, v) \cap A$ is contained in $T$, then $v$ can be written as $v = t_1' + t_2' + u'$, where $t_i' \in e_i \cap A$ for $i = 1, 2$ and $u' \in e(0, u) \cap A$. From this, we have $(t_1' - t_1) + (t_2' - t_2) - (u - u') = 0$. Since $e(0, u)$ is not contained in $U$, it follows that $t_i' = t_i$ for $i = 1, 2$ and $u = u'$. This contradicts the fact that $t_i'$ is not adjacent to $t_i$. Thus, $e(0, v) \cap A$ is not contained in $T$ and $\langle T, e(0, v) \cap A \rangle$ is a 4-subspace.

Now we show that $V$ contains $e(0, v)$. Since $v = t_1 + t_2 - u$, $v$ is contained in $V$. Assume, to the contrary, that there is a vertex $v'$ in $e(0, v) \setminus V$. Then, there is an element $g$ in $H$ such that $v^g = v'$ and $g$ fixes $L_0$ and $A_0$ by Lemma 4.1. Then, $v' = t_1' + t_2' - u^g$. By the line-closedness of $U$ and $U^g$, $\{e_i \cap A(0, v'), e_j \cap A(0, v'), e(0, u) \cap A(0, v') \}$ generates a 3-subspace in $A(0, v')$ and $\{e_i^g \cap A(0, v'), e_j^g \cap A(0, v'), e(0, u)^g \cap A(0, v') \}$ also generates a 3-subspace in $A(0, v')$. Moreover, by the same argument as in the previous paragraph, it follows easily that $\{e_i^g \cap A(0, v'), e_j^g \cap A(0, v'), e(0, u)^g \cap A(0, v'), e(0, v')^g \cap A(0, v') \}$ generates a 4-subspace. On the other hand, since $v'$ is not contained in $V$, $\{e_i \cap A(0, v'), e_j \cap A(0, v')$,
ON DISTANCE TRANSITIVE GRAPHS

\[ \ell(0, u) \cap A(0, v'), \ell(0, v') \cap A(0, u') \] also generates a 4-subspace. Since
\[ H_{A(0, v')} \] contains \( SL(\beta, r) \), there is an element \( g' \) in \( H_{A(0, v')} \) such that
\[ (\ell_i \cap A(0, v'))^{g'} = \ell_i \cap A(0, v') \] for \( i = 1, 2 \) and
\[ (\ell(0, u)^g \cap A(0, v'))^{g'} = \ell(0, u) \cap A(0, v') \] and \( v'^g = v' \). Therefore, since \( g \) and \( g' \) stabilize \( \mathcal{L}_0 \) and \( A_0 \), it follows that
\[ t_i^{g'} + t_i^{g'} - u_i^{g'} = v', \quad t_i^{g'} \in \ell_i \] for \( i = 1, 2 \) and \( u_i^{g'} \in \ell(0, u) \).
This implies that \( V \) contains \( v' \) and a contradiction. Hence \( V \) contains
\( \ell(0, v) \).

Next we show that every lines in \( \mathcal{L}_0 \) is contained in \( T \). Consider an arbitrary line \( m \) containing \( 0 \). We can assume that \( m \cap A \) is contained in \( T = \langle \ell_1 \cap A, \ell_2 \cap A, \ell(0, u) \cap A \rangle \). Then an element \( w \in m \cap A \) can be written as
\[ w = t_i^{u} + t_2^{u} + u^{u'}, \quad \text{where} \quad t_i^{u} \in \ell_i \] for \( i = 1, 2 \) and \( u^{u'} \in \ell(0, u) \). Then, \( m \) is contained in \( \langle \ell(0, t_1^{u} + t_2^{u}), \ell(0, u') \rangle \) by Proposition 5.4. Since \( \ell(0, t_1^{u} + t_2^{u}) \) is contained in \( U \), \( m \) is also contained in \( \langle U, \ell(0, u') \rangle = V \). Thus, we have only to consider the case where \( m \cap A \) is not contained in \( T \). Since \( H_A \) contains \( SL(d, r) \), the stabilizer \( (H_A)_{\ell_1 \cap A, \ell_2 \cap A, \ell(0, u) \cap A} \) of three “lines” \( \ell_1 \cap A, \ell_2 \cap A \) and \( \ell(0, u) \cap A \) acts transitively on all “lines” in \( A \) which intersect \( T \) only at \( \{0\} \). Therefore, there is an element \( g \) in \( H_A \) such that \( g \) stabilizes three lines \( \ell_1, \ell_2 \), and \( \ell(0, u) \) and transforms \( \ell(0, v) \) to \( m \). Since \( g \) stabilizes \( V \) and \( \ell(0, v) \) is contained in \( V \), \( m \) is also contained in \( V \). Thus, we have proved that every line in \( \mathcal{L}_0 \) is contained in \( V \). From this and the fact that \( V \) is a group, we have \( N = V \).

Q.E.D. of Claim 1

By changing the roles of lines and assemblies in Claim 1, we can show the following.

CLAIM 2. \( q = q' \).

Proof of Claim 2. We consider \( \langle A(0, t_1), A(0, t_2) \rangle \) which is an \( A \)-plane containing 0. If \( N_1 \cap (N_1 + t_1 + t_2) \) is contained in \( \langle A(0, t_1), A(0, t_2) \rangle \), then
\[ c_2 = \|N_1 \cap (N_1 + t_1 + t_2)\| = r(r+1) \] by Proposition 5.9. This implies that \( N_1 \cap (N_1 + t_1 + t_2) \) is also contained in \( U \). This contradicts the assumption that \( u \in N_1 \cap (N_1 + t_1 + t_2) \) does not lie in \( U \).
Therefore, there is a vertex \( u' \) in \( N_1 \cap (N_1 + t_1 + t_2) \) which does not lie in \( \langle A(0, t_1), A(0, t_2) \rangle \). Hence, we can apply the same argument as in the Proof of Claim 1 to show that \( N = \langle A(0, t_1), A(0, t_2), A(0, u') \rangle \). From this, we have
\[ q^3 = |N| = q'^3 \quad \text{and} \quad q = q'. \]

Q.E.D. of Claim 2

Now we consider vertices in \( U \) which are not adjacent to 0, that is, vertices in \( U \cap N_2 \). Then, the number of those vertices is \( (q-1)(q-r) \), since there are \( (q-1)(r+1) \) vertices adjacent to 0 in \( U \). For each vertex \( w \) in \( U \), let \( S(w) \) be the set of all planes containing 0 and \( w \), and let \( S = \bigcup_{w \in U \cap N_2} S(w) \). By the distance transitivity of \( N \), \( |S(w)| \) does not
depend on the choice of the vertex \( w \). So we denote \(|S(w)|\) by \( s \). Since \( \langle \ell(0, u), \ell(0, v) \rangle \) is a plane and it contains \( u + v = t_1 + t_2 \), we have \( s \geq 2 \).

**Claim 3.** Let \( W \) be a plane containing \( 0 \) and \( W \neq U \). Then, \( W \) contains a vertex \( w \) in \( U \cap N_2 \) if and only if \( U \cap W \cap A = \{0\} \) for the assembly \( A \) containing \( 0 \). Moreover, if \( W \) contains a vertex \( w \) in \( U \cap N_2 \), then \( W \cap U \subseteq \{0\} \cup N_2 \) and \( |W \cap U| = q \).

**Proof of Claim 3.** Since \(|N| = q^3\) and \( U \) is line-closed, it follows that \( \langle W, U \rangle = N \) and so \( U \cap W \) contains \( q \) vertices. If \( U \cap W \) contains a vertex \( w' \) adjacent to \( 0 \), then \( U \cap W \) contains the line \( \ell(0, w') \). Since \(|U \cap W| = q\), \( U \cap W \) coincides with \( \ell(0, w') \). This shows the only if part.

Conversely, if \( U \cap W \cap A = \{0\} \) for the assembly \( A \) containing \( 0 \), \( U \cap W \) contains no line containing \( 0 \). Thus, \( U \cap W \) is contained in \( \{0\} \cup N_2 \).

Q.E.D. of Claim 3

By the fact that every plane containing \( 0 \) is generated by its two lines intersecting at \( 0 \) and Proposition 5.8, there is a one to one correspondence between the set of all planes containing \( 0 \) in \( N \) and the set of all “planes” containing \( 0 \) of the affine space \( \Pi(A) \). From this and Claim 3, \(|S \setminus \{U\}|\) is the number of all “planes” which intersect \( A \cap U \) only at \( \{0\} \). Also from Claim 3, we have \(|S \setminus \{U\}| \times (q - 1) = (s - 1) |U \cap N_2| \). As \( \Pi(A) \) is an affine space, the number of all “planes” of \( \Pi(A) \) which intersect \( A \cap U \) only at \( 0 \) is \((q - r^2)(q - r^3)/(r^2 - 1)(r^2 - r)\). Hence we have the equation

\[
k(q - 1)(q - r) = (q - 1)(q - r^2)(q - r^3)/(r^2 - 1)(r^2 - r),
\]

where \( k = s - 1 \) is a positive integer. Therefore,

\[
k = (q - r^2)(q - r^3)/(q - r)(r^2 - 1)(r^2 - r).
\]

Since \( q = r^n \), the following number \( k' \) is also a positive integer.

\[
k'(r^{n-1} - 1) = (r^{n-2} - 1)(r^{n-3} - 1).
\]

By considering (5.3) modulo powers of \( r \), we also have

\[
k' \equiv r^{n-3} - 1 \pmod{r^{n-2}}.
\]

Therefore, it follows that \( k' > r^{n-3} - 1 \) and this implies

\[
(r^{n-2} - 1)/(r^{n-1} - 1) > 1.
\]

But it is impossible. Hence we obtain a contradiction. Q.E.D.

We have the following directly from Lemma 5.10.
COROLLARY 5.11. For each pair \((x, y)\) of vertices in \(N\) with \(d(x, y) = 2\), there exists a unique plane \(U\) which contains \(x\) and \(y\).

6. EVERY PROPER SUBSPACE IS AN ATTENUATED SPACE

In this section, we show that every proper subspace is an attenuated space. To prove this, we provide the following definition and proposition which will give a sufficient condition.

**DEFINITION 6.1.** Let \(S\) be a \(k\)-claw \(\{v_0 = 0, v_1, ..., v_k\}\) with its center 0. \(S\) is a non-degenerate \(k\)-claw with its center 0 if the subspace generated by \(S\) is not contained in any subspace generated by a \((k-1)\)-claw with its center 0.

**PROPOSITION 6.1.** Let \(k\) be a positive integer such that \(3 < k < 2^n\). Assume that

1. there is a non-degenerate \(k\)-claw with its center 0, and
2. for any positive integer \(i\) in \(\{2, 3, ..., k\}\) and any non-degenerate \(i\)-claw \(\{0, v_1, ..., v_i\}\) with its center 0, \(v_1 + \cdots + v_i\) is not adjacent to 0.

Then, for every \(i\) in \(\{3, ..., k\}\), each non-degenerate \(i\)-claw with its center 0 generates an \(i\)-subspace and each \(i\)-subspace is an \((i, GF(r), n)\)-attenuated space. Consequently, each plane is a \((2, GF(r), n)\)-attenuated space, since it is a 2-subspace of some \((3, GF(r), n)\)-attenuated space.

We note that the conditions of Proposition 6.1 hold for \(k = 3\) by Lemma 5.10 and the fact that \(2^n > 3\).

To prove Proposition 6.1, we use the induction argument. Let \(k\) be a positive integer which satisfies the conditions (1) and (2) of Proposition 6.1. We assume the following for a positive integer \(i \leq k - 1\):

(i) every \(j\)-subspace containing 0 is a group of order \(q^j\) for \(j \leq i\), and
(ii) every \(j\)-subspace is a \((j, GF(r), n)\)-attenuated space for \(3 \leq j \leq i\).

And we will show that the above assumptions (i) and (ii) also hold for \(i + 1\). Once we show this, we will complete the proof of Proposition 6.1 by the induction argument. By Section 5, we can assume \(i \geq 2\). From the above assumptions (i), (ii) and the definition of non-degenerate claws, each non-degenerate \((i + 1)\)-claw with its center 0 generates an \((i + 1)\)-subspace. Also from the assumptions (i), (ii) and the conditions (1), (2) of Proposition 6.1, we have the following.

For any \(j \leq i\), a \(j\)-subspace \(\langle \ell_1, ..., \ell_j \rangle\) which is generated by \(j\) lines \(\ell_1, ..., \ell_j\) in \(\mathcal{L}_0\) coincides with the subgroup which is generated by those lines, i.e.,
\[ \langle \ell_1, ..., \ell_j \rangle = \langle \ell_1, ..., \ell_j \rangle. \] Moreover, \( \langle \ell_1, ..., \ell_j \rangle = \langle \ell_1, ..., \ell_j \rangle \cong \ell_1 \oplus \cdots \oplus \ell_j \) and so it is a group of order \( q^j \). From this, for an \((i + 1)\)-subspace \( \langle \ell_1, ..., \ell_{i+1} \rangle \) which is generated by \( i + 1 \) lines \( \ell_1, ..., \ell_{i+1} \) in \( \mathcal{L}_0 \), the number of its vertices exceeds \( q^i \). Moreover, since \( \ell_{i+1} \) is not contained in \( \langle \ell_1, ..., \ell_i \rangle \), \( \langle \ell_1, ..., \ell_i \rangle \cap \ell_{i+1} = \{0\} \) by the line-closedness of \( \langle \ell_1, ..., \ell_i \rangle \) and so the order of the subgroup \( \langle \ell_1, ..., \ell_{i+1} \rangle \) is \( q^{i+1} \).

**Lemma 6.2.** Let \( \ell_1, ..., \ell_{i+1} \) be lines containing 0 such that \( \langle \ell_1, ..., \ell_{i+1} \rangle \) is an \((i + 1)\)-subspace. If \( x_1 + \cdots + x_{i+1} \) is adjacent to 0 for some \( x_j \in \ell_j, 1 \leq j \leq i + 1 \), then all non-zero vertices in \( \{x_1, ..., x_{i+1}\} \) are adjacent to each other.

**Proof.** We prove Lemma 6.2 by the induction argument. Since Lemma 6.2 holds for \( i = 2 \), we can assume that Lemma 6.2 is true for \( i \geq 2 \).

By the assumption of the induction, we can also assume that every vertex \( x_j \in \ell_j \) is not 0. If there is a subset \( \{x_{j_1}, ..., x_{j_k}\} \) of \( \{x_1, ..., x_{i+1}\} \) such that \( x_{j_1} + \cdots + x_{j_k} = 0 \), then \( \ell_{j_k} \) is contained in the subspace generated by \( 0, x_{j_1}, ..., x_{j_{k-1}} \) and this is a contradiction. Therefore, \( x_{j_1} + \cdots + x_{j_k} \) is not 0 for any subset \( \{x_{j_1}, ..., x_{j_k}\} \) of \( \{x_1, ..., x_{i+1}\} \). Now we construct a set of vertices \( \{y_{j_1}, ..., y_{j_k}\} \) from \( \{x_1, ..., x_{i+1}\} \) as follows. Each vertex \( y_{j_k} \) is the sum of all vertices \( x_{j_1}, ..., x_{j_k} \) in \( \{x_1, ..., x_{i+1}\} \) which belong to same assembly containing 0, i.e., \( y_{j_k} = x_{j_1} + \cdots + x_{j_k} \). Then, each \( y_{j_k} \) is not 0 and adjacent to 0. Moreover, it follows that the lines \( \ell(0, y_{j_1}), ..., \ell(0, y_{j_k}) \) generate an \( s \)-subspace by seeing the order of the group generated by them. If \( s = 1 \), Lemma 6.2 holds. So we assume, to the contrary, that \( s \geq 2 \). Since there is no pair of distinct vertices \( y_{j_k} \) and \( y_{j_{k'}} \) which belong to same assembly containing 0, it follows that \( \{0, y_{j_1}, ..., y_{j_k}\} \) is a non-degenerate \( s \)-claw with its center 0. Since \( s \leq i + 1 \leq k \), \( y_{j_1} + \cdots + y_{j_k} = x_1 + \cdots + x_{i+1} \) is not adjacent to 0 by the condition (2) of Proposition 6.1. This is a contradiction. Thus we complete the proof of Lemma 6.2. Q.E.D.

Now we consider an \((i + 1)\)-subspace \( U \) generated by \( i + 1 \) lines \( \ell_1, ..., \ell_{i+1} \) in \( \mathcal{L}_0 \). Let \( T \) be the subgroup which is generated by those lines, i.e., \( T = \langle \ell_1, ..., \ell_{i+1} \rangle \). Then, \( T \cong \ell_1 \oplus \cdots \oplus \ell_{i+1} \) and \( |T| = q^{i+1} \). Moreover \( T \) is connected by the definition of \( T \).

**Lemma 6.3.** \( U = T \).

**Proof.** First we show that \( T \) is a subspace. To prove this, it suffices to show that \( T \) is line-closed. Let \( x \) and \( y \) be adjacent vertices in \( T \). Since \( T \) is a group, we can assume \( x = 0 \) without loss of generality. By the assumptions (1) and (ii) of the induction, it follows that if \( y \) is contained in some \( i \)-subspace generated by \( i \) distinct lines among \( \ell_1, ..., \ell_{i+1} \), then \( \ell(0, y) \) is contained in \( T \). So we can assume that \( y \) is not contained in any \( i \)-subspace
generated by \(i\) distinct lines among \(\ell_1, \ldots, \ell_{i+1}\). Since \(T = \langle \ell_1, \ldots, \ell_{i+1} \rangle \cong \ell_1 \oplus \cdots \oplus \ell_{i+1}\), \(y\) can be written uniquely as \(y = y_1 + \cdots + y_{i+1}\), where \(y_j \neq 0 \in \ell_j\) for \(1 \leq j \leq i+1\). Since \(y\) is adjacent to \(0\), there are adjacent vertices in \(\{y_1, \ldots, y_{i+1}\}\) by the condition (2) of Proposition 6.1. Without loss of generality, we can assume that \(y_i\) is adjacent to \(y_{i+1}\). Then, by the assumptions (i) and (ii) of the induction, \(\langle \ell_1, \ldots, \ell_{i-1}, \ell(0, y_i+y_{i+1}) \rangle = \langle \ell_1, \ldots, \ell_{i-1}, \ell(0, y_i+y_{i+1}) \rangle\) and it contains \(y = y_1 + \cdots + y_{i-1} + (y_i + y_{i+1})\) and \(\ell(0, y)\). As \(T\) contains \(\langle \ell_1, \ell_{i+1} \rangle\), \(T\) also contains \(\ell(0, y_i+y_{i+1})\). Thus, \(T\) contains \(\langle \ell_1, \ldots, \ell_{i-1}, \ell(0, y_i+y_{i+1}) \rangle\). From this, \(T\) contains \(\langle \ell(0, y) \rangle\). Hence \(T\) is line-closed and so \(T\) is a subspace.

Next we show that \(U = T\). Since \(T\) is a subspace and contains \(\ell_1, \ldots, \ell_{i+1}\), \(T\) contains \(U\). So we have only to prove that \(U\) contains \(T\). Consider a vertex \(x\) in \(T\). Then \(x\) can be written uniquely as \(x = x_1 + \cdots + x_{i+1}\), where \(x_j \in \ell_j\) for \(1 \leq j \leq i+1\). By the assumptions (i) and (ii) of the induction, \(x_1 + \cdots + x_i\) lies in \(\langle \ell_1, \ldots, \ell_i \rangle\) and so \(x_1 + \cdots + x_i\) lies in \(U\). Similarly, \(x_2 + \cdots + x_{i+1}\) and \(x_2 + \cdots + x_i\) lie in \(U\). So \(\langle x_1 + \cdots + x_i, x_2 + \cdots + x_{i+1}, x_2 + \cdots + x_i \rangle\) is contained in \(U\). By Section 5, \(\langle x_1 + \cdots + x_i, x_2 + \cdots + x_{i+1}, x_2 + \cdots + x_i \rangle = \langle \ell_1, \ell_{i+1} \rangle + x_2 + \cdots + x_i\). Since \(x_1 + x_{i+1}\) lies in \(\langle \ell_1, \ell_{i+1} \rangle\), \(x = x_1 + x_{i+1} + x_2 + \cdots + x_i\) lies in \(\langle x_1 + \cdots + x_i, x_2 + \cdots + x_{i+1}, x_2 + \cdots + x_i \rangle\). Hence \(x\) lies in \(U\). This implies that \(U\) contains \(T\).

By Lemma 6.3, every \((i+1)\)-subspace generated by \(i+1\) lines in \(\mathcal{L}_0\) is a group of order \(q^{i+1}\). Moreover, it follows that every \((i+1)\)-subspace containing \(0\) is a group of order \(q^{i+1}\). As for arbitrary \((i+1)\)-subspace \(U\) of \(N\), \(U - x\) is also an \((i+1)\)-subspace containing \(0\), where \(x \in U\). From this, we have the following.

**Lemma 6.4.** Let \(U\) be an \((i+1)\)-subspace of \(N\). Then \(|U| = q^{i+1}\), and \(U = \langle \ell_1, \ldots, \ell_{i+1} \rangle\) for any \((i+1)\) lines \(\ell_1, \ldots, \ell_{i+1}\) in \(U\) which intersect at one vertex, and the whole of which is not contained in an \(i\)-subspace. Moreover, if \(U\) contains \(0\), then \(U\) is a group.

Next, we show that every \((i+1)\)-subspace is an \((i+1, GF(r), n)\)-attenuated space. And this completes the proof of Proposition 6.1.

**Lemma 6.5.** Every \((i+1)\)-subspace \(U\) is an \((i+1, GF(r), n)\)-attenuated space.

**Proof.** Let \(U\) be an \((i+1)\)-subspace of \(N\). To prove that \(U\) is an \((i+1, GF(r), n)\)-attenuated space, it suffices to show that the incidence structure \(\Pi(U) = (U, \mathcal{P}(U), \varepsilon)\) is an \((i+1)\)-net. So we verify the conditions (D1), (D2), and (D3) for \(\Pi(U)\). Since every plane is a net, \(\Pi(U)\) satisfies the condition (D1). Thus we begin with verifying the condition (D2).
Let $V$ be a 3-subspace in $U$, and let $T$ and $R$ be distinct intersecting planes in $V$. Take a vertex $x$ in $T \cap R$. Then $T - x$ and $R - x$ are groups of order $q^3$, and those groups are contained in a group $U - x$ of order $q^3$. As $T \neq R$, $\langle T - x, R - x \rangle$ is a 3-subspace, and so $\langle T - x, R - x \rangle = U - x$. From this, $|\langle T - x \rangle \cap (R - x)| = |T - x| |R - x|/|U - x| = q$ and $|T \cap R| = q$. So there is another vertex $y$ in $T \cap R$. Since $x$ and $y$ are contained in a 2-subspace $T$, $\delta(x, y) \leq 2$. If $\delta(x, y) = 2$, it follows that $T = R$ by Corollary 5.11. This is a contradiction. Therefore, $x$ and $y$ are adjacent, and so the intersection $T \cap R$ is contained in the line $\ell(x, y)$. By counting the numbers of vertices, we have $T \cap R = \ell(x, y)$. From this, $U$ satisfies the condition (D2).

Finally, we verify the condition (D3) for $II(U)$. If $U$ is a 3-subspace, i.e., $i = 2$, then the condition (D3) holds from the line-closedness of planes and (D2). Therefore, each 3-subspace is a $(3, GF(r), n)$-attenuated space and so each plane is also a $(2, GF(r), n)$-attenuated space. Thus, we have only to consider the case where $i \geq 3$. Let $V, W$ be proper intersecting subspaces in $U$, and the dimensions of $V$ and $W$ be $h$ and $j$, respectively. We can assume that $V \not\subset W$ and $W \not\subset V$. If $\langle V, W \rangle \neq U$, then the dimension of $\langle V, W \rangle$ as a subspace is not greater than $i$. By the assumption of the induction, $\langle V, W \rangle$ is an attenuated space. This implies that the intersection $V \cap W$ is also a subspace. Therefore we can assume $\langle V, W \rangle = U$. Take a vertex $x$ in $V \cap W$. By subtracting $x$, we can assume that $x = 0$ without loss of generality. Then, $U, V,$ and $W$ are groups whose orders are powers of $q$. From this, the order $|V \cap W|$ of the intersection $V \cap W$ is also a power of $q$. If $|V \cap W| = 1$, then $V \cap W$ consists of a vertex and it is a subspace of 0-dimension. So we only have to consider the case where $|V \cap W| \geq q$. Since $\langle V, W \rangle = U$, $h + j \geq i + 2$ and so $j \geq i + 2 - h$. Let $W'$ be a subspace of the minimal dimension in $W$ so that $\langle V, W' \rangle = U$ and $W'$ contains 0. Then $\dim(W') \geq i + 1 - h$. We show that $\dim(W') = i + 1 - h$. Suppose the contrary. Take an $(i + 1 - h)$-subspace $W''$ containing 0 in $W'$. Then $\langle V, W'' \rangle \neq U$ by the choice of $W'$. So $\langle V, W'' \rangle$ is an attenuated space by the assumption of the induction. Therefore $V \cap W''$ is a subspace, and $\dim(V \cap W'') \geq 1$ by seeing the orders of $U$, $V$, and $W''$. Since $W''$ is also an attenuated space, there is a subspace $\tilde{W}$ containing 0 in $W''$ such that $\langle V \cap W'' \rangle = \tilde{W}$ and $V \cap W'' \cap \tilde{W} = \{0\}$. Then $\langle V, \tilde{W} \rangle = U$ and $\dim(\tilde{W}) < \dim(W')$. This is a contradiction. Thus $\dim(W') = i + 1 - h$. From this and the fact that $W$ is an attenuated space, there are $j$ lines $m_1, \ldots, m_j$ containing 0 such that $\langle m_1, \ldots, m_{i-1} \rangle = W'$ and $\langle m_1, \ldots, m_j \rangle = W$. Consider subspaces $W'_s = \langle W', m_s \rangle$ for $i + 2 - h \leq s \leq j$. Then $\langle V, W'_s \rangle = U$. So we have $|V \cap W'_s| = |V||W'_s|/|U| = q$. Now, we will show that $V \cap W'_s$ is a line for each $s$. Take a vertex $y \neq 0$ in $V \cap W'_s$. Let $h'$ be the distance between $y$ and 0 in $V$, and $j'$ be the distance between $y$ and 0 in $W'_s$. Then, there are $h'$ lines $\ell_1, \ldots, \ell_{h'}$ in $V$ containing 0 such that
\( \langle \ell_1, \ldots, \ell_{h'} \rangle \) is an \( h' \)-subspace and \( y \) is written as \( y_1 + \cdots + y_{h'} \), where \( y_t \in \ell_t \setminus \{0\} \) for \( 1 \leq t \leq h' \). Similarly, there are also \( j' \) lines \( \ell'_1, \ldots, \ell'_{j'} \) in \( W'_r \) containing 0 such that \( \langle \ell'_1, \ldots, \ell'_{j'} \rangle \) is a \( j' \)-subspace and \( y \) is written as \( y'_1 + \cdots + y'_{j'} \), where \( y'_t \in \ell'_t \setminus \{0\} \) for \( 1 \leq t \leq j' \). Let \( V' = \langle \ell_1, \ldots, \ell_{h'} \rangle, W'' = \langle \ell'_1, \ldots, \ell'_{j'} \rangle \) and \( U' = \langle V', W'' \rangle \). If the dimension of \( U' \) is not equal to \( i + 1 \), i.e., \( U' \neq U \), then \( U' \) is an attenuated space and so the intersection \( V' \cap W'' \) of its subspaces \( V' \) and \( W'' \) is a subspace by the assumption of the induction. Since the subspace \( V' \cap W'' \) contains distinct vertices 0 and \( y \), we have \( |V' \cap W''| \geq q \). But we also have \( |V' \cap W''| \leq q \) by the fact that \( V' \cap W'' \) is contained in \( V \cap W'_r \) and \( |V \cap W'_r| = q \). From this, we obtain \( V' \cap W'' = V \cap W'_r \). Therefore \( V \cap W'_r \) is a subspace of order \( q \), that is, a line. As for the case where \( U = U' \), we have \( V' = V \) and \( W'' = W'_r \). So \( h' = h \) and \( j' = i + 2 - h \). Then, \( y = y_1 + \cdots + y_{h'} = y'_1 + \cdots + y'_{i+2-h} \). This implies that \( y'_{i+2-h} = y_{i+1} + \cdots + y_{h} - y'_1 - \cdots - y'_{i+1-h} \) is adjacent to 0. By Lemma 6.2, all of \( i+1 \) vertices \( y_1, \ldots, y_h, y'_1, \ldots, y'_{i+2-h} \) are adjacent to each other. But at least \( y_1 \) and \( y_2 \) are not adjacent if \( h \geq 2 \). Therefore we have \( h = 1 \) and also \( i+2-h = 1 \). This is a contradiction. Thus, we proved that \( V \cap W' \) is a line, say \( \ell'_{i' + 2} \), containing 0. Moreover, it follows directly that \( W'_r = \langle W', \ell'_{i' + 2} \rangle \) and \( W = \langle W', \ell'_{i+2-h}, \ldots, \ell'_{i'} \rangle \). This implies that \( \ell'_{i+2-h}, \ldots, \ell'_{i'} \) generate a \( (j + h - i - 1) \)-subspace. By seeing the orders, we obtain \( V \cap W = \langle \ell'_{i+2-h}, \ldots, \ell'_{i'} \rangle = \langle \ell'_{i+2-h}, \ldots, \ell'_{i'} \rangle \). Hence we proved that \( V \cap W \) is a subspace. Q.E.D.

Next we consider the conditions of Proposition 6.1. Let \( d_0 \) be the dimension of \( N \) as a semilinear space.

**Proposition 6.6.** The conditions (1) and (2) of Proposition 6.1 hold for k = \( d_0 - 1 \).

**Proof.** We prove Proposition 6.6 by the induction argument. So we assume that the claim is true for \( k = 2, \ldots, i < d_0 - 1 \), and we will show that the claim is also true for \( k = i + 1 \). So there is a non-degenerate \( i \)-claw \( C \) with its center 0. And by the assumption of the induction and Proposition 6.1, the subspace \( \langle C \rangle \) is an \( (i, GF(r), n) \)-attenuated space.

Now we show that the condition (1) of Proposition 6.1 holds for \( k = i + 1 \). Since the dimension of \( N \) is greater than \( i \), there is an \( (i+1) \)-subspace. Let a minimal generating set of an \( (i+1) \)-subspace be \( \{u_0, \ldots, u_i \} \). Without loss of generality, we can assume that \( \{u_0, \ldots, u_i \} \) is connected and \( u_{i+1} \) is adjacent to \( u_0 \). Then, by Proposition 5.1, \( \{0, u_1 - u_0, \ldots, u_i - u_0, u_{i+1} - u_0 \} \) generates an \( (i+1) \)-subspace and \( \{0, u_1 - u_0, \ldots, u_i - u_0 \} \) generates an \( i \)-subspace. By the assumption of the induction, there is an \( i \)-claw \( \{0, v_1, \ldots, v_i\} \) which generates the subspace \( \langle 0, u_1 - u_0, \ldots, u_i - u_0 \rangle \).
So \( \{0, v_1, ..., v_i, u_{i+1} - u_0\} \) generates an \((i + 1)\)-subspace. Since \( \langle 0, v_1, ..., v_i \rangle \) does not contain \( u_{i+1} - u_0 \), the number of lines containing 0 in \( N \) is greater than the number of lines containing 0 in \( \langle 0, v_1, ..., v_i \rangle \). From this, we have \( (q' - 1)/(r - 1) = (r^2 - 1)/(r - 1) > (r' - 1)/(r - 1) \) and so \( q' > r' \). By \( q' > r' \), there is a vertex \( v_{i+1} \) in \( \ell'(0, u_{i+1} - u_0) \) such that \( v_{i+1} \) is not adjacent to any \( v_j \) for \( j = 1, ..., i \). Hence we have an \((i + 1)\)-claw \( \{0, v_1, ..., v_{i+1}\} \) which generates an \((i + 1)\)-subspace, and so we proved that there exists a non-degenerate \((i + 1)\)-claw with its center 0.

Next we show that the condition (2) of Proposition 6.1 also holds for \( k = i + 1 \). Assume, to the contrary, that there is a non-degenerate \((i + 1)\)-claw \( \{0, v_1, ..., v_{i+1}\} \) in \( N \) such that \( v_1 + \cdots + v_{i+1} \) is adjacent to 0. Let \( V = \langle \ell'(0, v_1), ..., \ell'(0, v_{i+1}) \rangle \), \( v = v_1 + \cdots + v_{i+1} \), and \( A = A(0, v) \). We show the following claim by using the same argument as was used for the Proof of Claim 1 of Lemma 5.10.

**Claim.** \( V = N \).

**Proof of Claim.** Since \( \langle \ell'(0, v_1), ..., \ell'(0, v_i) \rangle = \langle \ell'(0, v_1), ..., \ell'(0, v_j) \rangle \) is line-closed by the assumption of the induction, \( \langle \ell'(0, v_1), ..., \ell'(0, v_i) \rangle \cap \ell'(0, v_{i+1}) = \{0\} \) and so \( \langle \ell'(0, v_1), ..., \ell'(0, v_{i+1}) \rangle \cong \ell_1 \oplus \cdots \oplus \ell_{i+1} \). If there is a vertex \( v_j \) adjacent to 0, then it holds for a non-degenerate \( i \)-claw \( \{0, v_1, ..., v_{j-1}, v_{j+1}, ..., v_{i+1}\} \) that \( v_1 + \cdots + v_{j-1} + v_{j+1} + \cdots + v_{i+1} \) is adjacent to 0 or equal to 0. This contradicts the assumption of the induction. So any \( v_j \) is not adjacent to 0. Since \( \langle \ell'(0, v_1), ..., \ell'(0, v_{i+1}) \rangle \) is an attenuated space, \( \{\ell'(0, v_1) \cap A, ..., \ell'(0, v_{i+1}) \cap A\} \) generates an \( i \)-subspace isomorphic to \( AG(i, r) \) in \( \Pi(A) \), where each \( \ell_j \cap A, j = 1, ..., i \), is regarded as a line in \( \Pi(A) \). Since \( \ell'(0, v_{i+1}) \cap \ell'(0, v_1), ..., \ell'(0, v_j) \cong \{0\} \), \( \{\ell'(0, v_1) \cap A, ..., \ell'(0, v_{i+1}) \cap A\} \) also generates an \((i + 1)\)-subspace isomorphic to \( AG(i + 1, r) \) in \( \Pi(A) \), and so \( \ell'(0, v_1) \cap A, ..., \ell'(0, v_{i+1}) \cap A \). If \( \ell'(0, v) \cap A \) is contained in the \((i + 1)\)-subspace \( \ell'(0, v_1) \cap A, ..., \ell'(0, v_{i+1}) \cap A \), then \( v \) can be written as \( v = v' + \cdots + v'_{i+1} \), where \( v_j \in \ell'(0, v_j) \cap A \). From this, \( v_1 + \cdots + v_{j-1} + v_{j+1} + \cdots + v_{i+1} \) and so \( (v_1 - v_1') + \cdots + (v_{i+1} - v'_{i+1}) = 0 \). Let \( w_j = v_j - v' \) for \( j = 1, ..., i + 1 \). Since \( v_j \) is adjacent to 0 and \( v_j \) is not adjacent to \( v \), we have \( w_j \neq 0 \). Since \( \{0, v_1, ..., v_i\} \) is a non-degenerate \( i \)-claw and the conditions (1) and (2) hold for \( i \), it follows that all \( w_j \)'s are adjacent by Lemma 6.2. From this, an \( (i, GF(r), n) \)-attenuated space \( \langle \ell'(0, w_1), ..., \ell'(0, w_i) \rangle = \ell'(0, v_1), ..., \ell'(0, v_i) \rangle \) contains \( \ell'(0, w_{i+1}) \cap A \). This contradicts the assumption that \( \{0, v_1, ..., v_{i+1}\} \) is a non-degenerate \((i + 1)\)-claw. Thus, \( \langle \ell'(0, v_1) \cap A, ..., \ell'(0, v_{i+1}) \cap A \rangle \) does not contain \( \ell'(0, v) \cap A \) and so \( \ell'(0, v_1) \cap A, ..., \ell'(0, v_{i+1}) \cap A, \ell'(0, v) \cap A \rangle \) is an \((i + 2)\)-subspace isomorphic to \( AG(i + 2, r) \).

Next we show that the group \( V = \langle \ell'(0, v_1), ..., \ell'(0, v_{i+1}) \rangle \) contains
ON DISTANCE TRANSITIVE GRAPHS

227

\[ t(0, u). \] Assume, to the contrary, that there is a vertex \( v' \) in \( t(0, u) \setminus V. \) By Proposition 4.6, there is an element \( g \) in \( H \) such that \( v^g = v' \) and \( g \) stabilizes \( t(0, u). \) Then, \( \{v_1^g, \ldots, v_{i+1}^g\} \) is a non-degenerate \((i+1)\)-claw and \( v_1^g + \cdots + v_{i+1}^g = v'. \) Therefore, \( \{\ell(0, v_1)^g \cap A(0, v'), \ldots, \ell(0, v_{i+1})^g \cap A(0, v'), \ell(0, v) \cap A(0, v')) \} \) generates an \((i+2)\)-subspace isomorphic to \( AG(i+2, r) \) in \( \Pi(A(0, v')). \) On the other hand, since \( v' \) is not contained in \( V, \) \( \{\ell(0, v_1) \cap A(0, v'), \ldots, \ell(0, v_{i+1}) \cap A(0, v'), \ell(0, v) \cap A(0, v')) \} \) also generates an \((i+2)\)-subspace isomorphic to \( AG(i+2, r) \) in \( \Pi(A(0, v')). \)

Then, by Proposition 4.5, there exists an element \( g' \) in \( H_{A(0, v')} \) such that \( v_j^g \cap A(0, v') = v_j^g \cap A(0, v') = v_j^g \cap A(0, v') \) for \( j = 1, \ldots, i+1 \) and \( g' \) fixes \( v'. \) Therefore, since \( g \) and \( g' \) stabilize \( L_0 \) and \( A_0, \) it follows that \( v_1^g + \cdots + v_{i+1}^g = v' \) and each \( v_j^g \) lies in \( \ell(0, v_j) \) for \( j = 1, \ldots, i+1. \) This implies that \( V \) contains \( v' \) and a contradiction. Hence, \( \langle \ell(0, v_1), \ldots, \ell(0, v_{i+1}) \rangle \) contains \( \ell(0, v). \)

Finally, we complete the Proof of Claim by showing that \( V \) also contains every line containing \( 0. \) Consider an arbitrary line \( m \) containing \( 0. \) We can assume that \( m \neq \ell(0, v), \) \( \ell(0, v), \) for \( j = 1, \ldots, i+1. \) First we deal with the case where \( m \cap A \) is contained in \( \langle \ell(0, v_1) \cap A, \ldots, \ell(0, v_{i+1}) \cap A \rangle. \) For this case, an element \( w \) in \( m \cap A \) is written as \( w = w_1 + \cdots + w_{i+1}, \) where \( w_j \in \ell(0, v_j) \cap A \) for \( j = 1, \ldots, i+1. \) Choose some element \( w_j \neq 0, \) and let \( w' = w_j \) and \( w'' = w_1 + \cdots + w_{j-1} + w_{j+1} + \cdots + w_{i+1}. \) Then \( w = w' + w'', \) where \( w' \neq 0 \) and \( w'' \neq 0. \) By the assumption of the induction, \( \ell(0, w') \) is contained in \( \langle \ell(0, v_1), \ldots, \ell(0, v_{j-1}), \ell(0, v_{j+1}), \ldots, \ell(0, v_{i+1}) \rangle. \) Moreover, \( m = \ell(0, w) \) is contained in the plane \( \langle \ell(0, w'), \ell(0, w'') \rangle. \) Therefore, \( m \) is contained in \( V = \langle \ell(0, v_1), \ldots, \ell(0, v_{i+1}) \rangle. \) Next we deal with the remaining case, i.e., the case where \( m \cap A \) is not contained in \( \langle \ell(0, v_1) \cap A, \ldots, \ell(0, v_{i+1}) \cap A \rangle. \) Then, by the same argument as was used for proving that \( V \) contains \( \ell(0, v), \) it follows that there is an element \( g' \) such that \( (m \cap A)^g = \ell(0, v) \cap A \) and \( \ell(0, v_j)^g = \ell(0, v_j) \) for \( j = 1, \ldots, i+1. \) Since \( \ell(0, v) \) is contained in \( V \) and \( V^g = V, m \) is contained in \( V. \) Hence we have \( V = N \) and we have proved Claim. Q.E.D. of Claim.

Finally, we show that \( V = N \) is an \((i+1)\)-subspace, i.e., \( d_0 = i+1. \) Let \( W \) be the subspace generated by \( \{0, v_1, \ldots, v_{i+1}\}. \) Assume, to the contrary, that \( V \neq W. \) Since \( \langle 0, v_1, \ldots, v_j \rangle = \langle \ell(0, v_1), \ldots, \ell(0, v_j) \rangle, \) \( W \) contains \( \langle \ell(0, v_1), \ldots, \ell(0, v_j) \rangle. \) By the line-closedness of \( W, W \) also contains \( \ell(0, v_{i+1}). \) Therefore, each vertex \( v \) in \( V \setminus W \) is written as \( v = v_1' + \cdots + v_j' + v_{i+1}', \) where \( v_j' \in \ell(0, v_j) \) for \( j = 1, \ldots, i+1 \) and \( v_1' + \cdots + v_{i+1}' \neq 0 \) and \( v_{i+1}' \neq 0. \) If some \( v_j' \) is 0, then \( v \) is contained in an \( i\)-subspace \( \langle v_1, \ldots, v_j, v_{j+1}, \ldots, v_{i+1} \rangle = \langle \ell(0, v_1), \ldots, \ell(0, v_j), \ell(0, v_{j+1}), \ldots, \ell(0, v_{i+1}) \rangle. \) Since this subspace is contained in \( W, v \) is also contained in \( W. \) This is a contradiction. So any vertex \( v_j' \) is not 0. Consider three vertices \( v_1' + \cdots + v_{i+1}', v_1' + \cdots + v_j' \) and \( v_1' + \cdots + v_{i+1}'. \) Since each of them is contained in an
Let $U$ be the subspace generated by those three vertices. Then $U$ is a plane and $U$ is contained in $W$. By Section 5, $U = \langle (0, v_1), (0, v_{i+1}) \rangle + v'_2 + \cdots + v'_i$. From this, $U$ contains $v = v'_1 + (v'_2 + \cdots + v'_i) + v'_{i+1}$ and so $W$ also contains $v$. This is a contradiction. Thus we completed the proof of Proposition 6.6.

By the arguments in the proof of Propositions 6.6 and 6.1, we also have the following.

**Proposition 6.7.** If the conditions (1) and (2) of Proposition 6.1 also hold for $k = d_0$, then $N$ is a $(d_0, GF(r), n)$-attenuated space and $d_0 = d$.

Finally in this section, we consider differences between the distances in a subspace and those in $N$. Let $s$ be the largest integer which does not exceed $d_0/2$, i.e., $s = \lfloor d_0/2 \rfloor$. Then, the statement of [10, Proposition 5.3] holds for $i \leq s$.

**Proposition 6.8.** Let $i$ be a positive integer not greater than $s$, and let $U$ be an $i$-subspace of $N$. Then the distance function of $U$ coincides with that of $N$. Moreover, for distinct vertices $x, y$ in $U$ with $\partial(x, y) = t$, $t$ is not greater than $i$ and $(N_{t-1} + x) \cap (N_1 + y) \subseteq U$.

**Proof:** We write $\partial_U$ for the distance function on $U$. We can assume that $i \geq 2$. Since each $i$-subspace is an $(i, GF(r), n)$-attenuated space, the distance $\partial(x, y)$ between $x$ and $y$ in $U$ does not exceed $U$'s diameter $i$.

First we show that the distance function of $U$ coincides with that of $N$. Assume the contrary. So there is a pair $(x, y)$ of vertices in $U$ such that $\partial_U(x, y) \neq \partial(x, y)$. Let $t = \partial(x, y)$ and $t' = \partial_U(x, y)$. Then $i \geq t' > t$. Without loss of generality, we can assume that $x = 0$. Then, there exist $t'$ vertices $y_1', \ldots, y_{t'}$ in $N_1 \cap U$ such that $y = y_1' + \cdots + y_{t'}'$ and $\{0, y_1', y_1' + y_2', \ldots, y_1' + \cdots + y_{t'}'\}$ is a shortest path from 0 to $y$ in $U$. Moreover, $\{0, y_1', \ldots, y_{t'}'\}$ is a non-degenerate $t'$-claw with its center 0.

Similarly, there also exist $t$ vertices $y_1, \ldots, y_t$ in $N_1$ such that $y = y_1 + \cdots + y_t$ and $\{0, y_1, \ldots, y_t\}$ is a non-degenerate $t$-claw with its center 0. Let $V = \langle 0, y_1', \ldots, y_{t'}' \rangle$, $W = \langle 0, y_1, \ldots, y_t \rangle$ and $X = \langle V, W \rangle$. Then $V$ is a $t'$-subspace and $W$ is a $t$-subspace. Moreover, $V$ is a group of order $q't'$ and $W$ is a group of order $q't'$. Since $X$ is generated by $t + t' + 1$ vertices, the dimension $t''$ of $X$ is at most $t + t'$ and so $t'' \leq [d_0/2] + [d_0/2] - 1 = d_0 - 1$. By Propositions 6.1 and 6.6, $X$ is an attenuated space. Therefore, the intersection $V \cap W$ of two subspaces $V$ and $W$ is also a subspace (an attenuated space). Since the subspace $V \cap W$ contains $y$ and 0, $V \cap W$ is a group and contains a path from 0 to $y$. The length of a shortest path from 0 to $y$ in $V \cap W$ is at least $t$. Since $V \cap W$ contains the subspace generated by that
path, $V \cap W$ contains at least $q'$ vertices. From this, $V \cap W$ contains $W$, and so $U$ contains $W$. Hence we have $\partial_U(0, y) = \partial(0, y)$, and this is a contradiction.

Next we show that for distinct vertices $x, y$ with $\partial(x, y) = t$ in $U$, $(N_{i-1} + x) \cap (N_1 + y) \subseteq U$. Here we can also assume that $x = 0$. Suppose the contrary. Then, there are two minimal paths $P_1, P_2$ from 0 to $y$ such that $U$ contains $P_1$ but does not contain $P_2$. Let $V = \langle P_1 \rangle, W = \langle P_2 \rangle$, and $X = \langle P_1, P_2 \rangle$. By the same argument as in the proof for the coincidence of distance functions, if $\dim(X) < d_0$, then $V = W$ and this contradicts the assumption. Thus, $X$ is $N$ and $\dim V = \dim W = d_0/2$. (So $d_0$ is even.) From this, $y$ can be written as $y = y_1 + \cdots + y_{d_0/2} = y'_1 + \cdots + y'_{d_0/2}$, where $y_j, y'_j \in N_1$ and $V = \langle y_1, \ldots, y_{d_0/2} \rangle, W = \langle y'_1, \ldots, y'_{d_0/2} \rangle$. Since $X = N$, $\{0, y_1, \ldots, y_{d_0/2}, y'_1, \ldots, y'_{d_0/2} - 1\}$ generates a $(d_0 - 1)$-subspace. But $y_1 + \cdots + y_{d_0/2} - y'_1 - \cdots - y'_{d_0/2} - 1$ is adjacent to 0. By Lemma 6.2, this implies that all of those vertices $y_1, \ldots, y_{d_0/2}, y'_1, \ldots, y'_{d_0/2} - 1$ are adjacent to each other. So $y$ is also adjacent to 0, and this is a contradiction. Q.E.D.

7. $N$ is a $(d, GF(r), n)$-attenuated space

In this section, we complete the proof of the main theorem. Therefore, we deal with the case where not both of the conditions of Proposition 6.1 hold for $k = d_0$, and we show that such a case does not occur. To prove this, we generalize the argument in the proof of Lemma 5.10. The condition (1) of Proposition 6.1 holds even for $k = d_0$ by the proof of Proposition 6.6. So we can assume that there is a non-degenerate $d_0$-claw $\{0, v_1, \ldots, v_{d_0}\}$ which does not satisfy the condition (2) of Proposition 6.1. Then, by the proof of Proposition 6.6, $N = \langle \ell(0, v_1), \ldots, \ell(0, v_{d_0}) \rangle \cong \ell(0, v_1) \oplus \cdots \oplus \ell(0, v_{d_0})$. From this, each vertex $w$ can be written as $w = w_1 + \cdots + w_{d_0}$, where $w_i \in \ell(0, v_i)$ for $i = 1, \ldots, d_0$. So we have $\partial(0, w) \leq d_0$. Moreover, we have $d \leq d_0 < \tilde{d} \leq n$ by the proof of Proposition 6.6.

Now let $W_i$ be the $\mathcal{A}$-subspace generated by $\{0, v_1, \ldots, v_i\}$ for $i = 2, \ldots, d_0$.

**Proposition 7.1.** For each $i \in \{2, \ldots, d_0\}$, $W_i$ is an $i$-$\mathcal{A}$-subspace and $W_i = \langle A(0, v_1), \ldots, A(0, v_i) \rangle$.

Before proving Proposition 7.1, we provide necessary notions and propositions.

As for $\mathcal{A}$-subspaces, we have the similar result as Proposition 6.1. Replacing subspaces by $\mathcal{A}$-subspaces, we can also define a non-degenerate $k$-claw with its center 0 for $\Pi_{\mathcal{A}}$. Moreover, since the dimension $\tilde{d}$ of every assembly as an affine space exceeds $d_0$ by the proof of Proposition 6.6, we can employ the same argument as in Section 6 and consequently we obtain the following.
**Proposition 7.2.** Let \( k \) be a positive integer such that \( 3 \leq k \leq d_0 \). Assume that

1. there is a non-degenerate \( k \)-claw with its center 0 for \( \Pi, \) and
2. for any positive integer \( i \) in \( \{2, \ldots, k\} \) and any non-degenerate \( i \)-claw \( \{0, w_1, \ldots, w_i\} \) with its center 0 for \( \Pi, w_1 + \cdots + w_i \) is not adjacent to 0.

Then, each non-degenerate \( i \)-claw with its center 0 generates an \( i, \mathcal{A} \)-subspace and each \( i, \mathcal{A} \)-subspace is an \( (i, GF(r), d) \)-attenuated space.

We also note that the conditions of Proposition 7.2 hold for \( k = 3 \) by Lemma 5.10 and the fact that \( n \geq d \geq 3 \).

**Proposition 7.3.** The conditions (1) and (2) of Proposition 7.2 hold for \( k = d_0 - 1 \). Moreover, if not both of the conditions of Proposition 7.2 hold for \( k = d_0 \), then \( N = q^{d_0} \). (This implies that \( q = q' \) and \( d = d_0 \).)

Now we prove Proposition 7.1.

**Proof of Proposition 7.1.** To prove Proposition 7.1, it suffices to show that \( W_{d_0} \) is a \( d_0, \mathcal{A} \)-subspace. Assume, to the contrary, that \( W_{d_0} \) is not a \( d_0, \mathcal{A} \)-subspace. Then, by Proposition 7.3, \( W_{d_0} \) is an attenuated space as an \( \mathcal{A} \)-space. For simplicity, we write \( W = W_{d_0} \). Let \( s = [d_0/2] \) and let \( t \) be the dimension of \( W \) as an \( \mathcal{A} \)-space. First we consider the distance between 0 and \( v_1 + \cdots + v_s \) and the distance between 0 and \( v_1 + \cdots + v_{s+1} \) in \( W \).

Since \( \langle 0, v_1, \ldots, v_s \rangle \) is an \( s \)-subspace, \( \partial(0, v_1 + \cdots + v_s) = s \) by Proposition 6.8. Moreover, it can be shown that \( \partial(0, v_1 + \cdots + v_s + v_{s+1}) = s \) or \( s + 1 \) by using the fact that \( N_1 + v_1 + \cdots + v_s \cap N_{s-1} \subseteq \langle 0, v_1, \ldots, v_s \rangle \). On the other hand, since there is an \( s \)-path \( \{0, v_1, v_1 + v_2, \ldots, v_1 + \cdots + v_s \} \) from 0 to \( v_1 + \cdots + v_s \) in \( W \), \( \partial_W(0, v_1 + \cdots + v_s) \) does not exceed \( s \). So we have \( \partial_W(0, v_1 + \cdots + v_s) = s \). Similarly, we also have \( \partial_W(0, v_1 + \cdots + v_s + v_{s+1}) = s \) or \( s + 1 \).

Next we consider a cycle \( C = \{0, v_1, v_1 + v_2, \ldots, v_1 + \cdots + v_{d_0}\} \) and a subgroup \( \langle \partial(0, v_1) \cap W, \partial(0, v_{d_0}) \cap W \rangle \). Since \( W \) is a \( t \)-dimensional attenuated space, we have \( |\partial(0, v_j) \cap W| = r' \) for \( j = 1, \ldots, d_0 \), where each \( \partial(0, v_j) \cap W \) is regarded as an assembly in \( W \). Moreover, by the properties of an attenuated space, the fact that \( \partial_W(0, v_1 + \cdots + v_s) = s = [d_0/2], \partial_W(0, v_1 + \cdots + v_s + v_{s+1}) = s \) or \( s + 1 \) and \( C = \{0, v_1, v_1 + v_2, \ldots, v_1 + \cdots + v_s, v_1 + \cdots + v_{s+1}, \ldots, v_1 + \cdots + v_{d_0}\} \) is a cycle, implies that the order of a group \( \langle \partial(0, v_1) \cap W, \partial(0, v_{d_0}) \cap W \rangle \) cannot exceed \( (r')^{s+1} \). But \( \langle \partial(0, v_1), \ldots, \partial(0, v_{d_0}) \rangle \cap \partial(0, v_{s+1}) = \{0\} \) for \( i = 1, \ldots, d_0 - 1 \). So we have \( |\partial(0, v_i) \cap W| = (r')^{d_0} \) and this is a contradiction. Q.E.D.

By Proposition 7.1, \( \{0, v_1, \ldots, v_{d_0}\} \) is a non-degenerate \( d_0 \)-claw for \( \Pi, \mathcal{A} \).
Since \( v_1 + \cdots + v_{d_0} \) is adjacent to 0, \( \{0, v_1, \ldots, v_{d_0}\} \) is a non-degenerate \( d_0 \)-claw for \( \Pi_\alpha \) which does not satisfy the condition (2) of Proposition 7.2. This implies that \( q = q' \) by Proposition 7.3. Therefore we conclude the following.

**Proposition 7.4.** If \( q \neq q' \), then \( N \) is a \((d, GF(r), n)\)-attenuated space.

Thus, we have only to deal with the case \( q = q' \). So from now on, we assume that \( q = q' \), i.e., \( n = d_0 \). And let \( s = [d_0/2] \).

First we deal with the case where \( d_0 \) is odd, i.e., \( d_0 = 2s + 1 \). Let \( V \) be the subspace generated by \( \{0, v_1, \ldots, v_{s+1}\} \), and \( V' \) be the subspace generated by \( \{0, v_{s+2}, \ldots, v_{d_0}, v_{d_0+1}\} \), where \( v_{d_0+1} = -(v_1 + \cdots + v_{d_0}) \). Then both are \((s + 1)\)-subspaces. By similar arguments as in the proof of Proposition 6.8, we have the following lemma.

**Lemma 7.5.** Both the distance functions of \( V \) and \( V' \) coincide with that of \( N \).

*Proof.* First consider the distance function of \( V \). Assume, to the contrary, that there is a vertex \( w \) in \( V \) such that \( \partial(0, w) < \partial_\nu(0, w) \). Let \( t = \partial(0, w) \) and \( t' = \partial_\nu(0, w) \). By the same argument as in the proof of Proposition 6.8, it follows that \( t + t' \geq d_0 = 2s + 1 \). Since \( t' < s + 1 \), we have \( t = s \) and \( t' = s + 1 \). Therefore, there are vertices \( w_i \in \ell(0, v_i) \) such that \( w = w_1 + \cdots + w_{s+1} \) and \( \{0, w_1, \ldots, w_{s+1}\} \) generates \( V \), and there are vertices \( w'_i \) in \( N_1 \) such that \( w'_i = w'_1 + \cdots + w'_s \) and \( \{0, w'_1, \ldots, w'_s\} \) generates an \( s \)-subspace, which we denote by \( \bar{V} \). Also by the same argument as in the proof of Proposition 6.8, \( \langle V, \bar{V} \rangle \) is the whole space \( N \). But by counting numbers of vertices in those spaces, \( |V \cap \bar{V}| = 1 \). This contradicts the fact that \( V \cap \bar{V} \) contains two distinct vertices 0 and \( w \). As for the distance function of \( V' \), the same argument is available. Q.E.D.

Now we consider vertices in \( V \) whose distances from 0 are \( s + 1 \). The number of those vertices is \((q-1)(q-r)\cdots(q-r')\), i.e., \( |V \cap N_{s+1}| = (q-1)(q-r)\cdots(q-r') \), since \( V \) is an \((s + 1, GF(r), n)\)-attenuated space. For each vertex \( x \) in \( V \cap N_{s+1} \), let \( S(x) \) be the set of all \((s + 1)\)-subspaces containing 0 and \( x \), and let \( S = \bigcup_{x \in V \cap N_{s+1}} S(x) \). Then, by the distance transitivity of \( N \), \(|S(x)|\) does not depend on the choice of the vertex \( x \), i.e., \(|S(x)|\) is constant. As \( v_1 + \cdots + v_{s+1} = -(v_{s+2} + \cdots + v_{2s+1} + v_{2s+2}) \), \( v_1 + \cdots + v_{s+1} \) is contained in \( V \) and \( V' \). So \(|S(x)| \geq 2 \). Moreover, those two subspaces \( V \) and \( V' \) intersect at \( q \)-vertices, all of which lie in \( N_{s+1} \cup \{0\} \). By using the fact that each \((s + 1)\)-subspace is generated by some \( s + 1 \) lines containing 0, we have the following.

**Lemma 7.6.** Let \( X \) be an \((s + 1)\)-subspace containing 0 and \( X \neq V \). Then, \( X \) contains a vertex \( x \) in \( V \cap N_{s+1} \) if and only if \( V \cap X \cap A = \{0\} \) for an
assembly $A$ containing 0. Moreover, if $X$ contains a vertex $x$ in $V \cap N_{s+1}$, then $X \cap V \subseteq N_{s+1} \cup \{0\}$ and $|X \cap V| = q$.

**Proof.** First we show the only if part. So let $X$ contain a vertex in $V \cap N_{s+1}$. Consider the subspace $Y$ which is generated by $V$ and $X$. If $Y \neq N$, then $Y$ is an attenuated space by Proposition 6.1. By using the properties of an attenuated space, we have $V = X$ and a contradiction. Therefore, we have $N = Y$. Since $|N| = |X| |V|/|V \cap X|$, we have $|V \cap X| = q$. Now we show that $X \cap V$ is contained in $N_{s+1} \cup \{0\}$. Assume, to the contrary, that there is a vertex $x'$ in $V \cap X$ such that $\delta(0, x') < s + 1$. By Proposition 6.8, $(N_1 + x') \cap N_{s-1}$ is contained in $V$ and $X$, where $s'$ denotes the distance $\delta(0, x')$. So it follows that a path of the minimal length from 0 to $x'$ is contained in $V \cap X$. Since $V$ and $X$ are line-closed, $V \cap X$ contains a line. As $|V \cap X| = q$, $V \cap X$ is a line. This is a contradiction. Thus, we proved that $V \cap X$ is contained in $N_{s+1} \cup \{0\}$. Since each assembly $A$ containing 0 is contained in $N \cap \{0\}$, we have $V \cap X \cap A = \{0\}$.

Finally we show the if part. Let $X$ be an $(s+1)$-subspace such that $V \cap X \cap A = \{0\}$ for an assembly $A$ containing 0. Take an $(s+1)$-subspace $X' \neq V$ such that $X'$ contains a vertex in $V \cap N_{s+1}$. Then $X' \cap A$ intersects $V \cap A$ only at 0. Since $V \cap A$, $X \cap A$ and $X' \cap A$ are $(s+1)$-subspaces of $\Pi(A)$ and $V \cap X = V \cap X' = \{0\}$, there is an element $g$ in $H_A$ such that $(V \cap A)^g = V \cap A$ and $(X' \cap A)^g = X \cap A$ by Proposition 4.5. Since $V$, $X$ and $X'$ are attenuated spaces, $(V \cap A) = V$, $(X' \cap A) = X$, and $(V \cap X') = X'$. Therefore, by the action of $g$, $V^g = V$, $(X')^g = X$, and $(V \cap X')^g = V \cap X$. Moreover, $g$ does not change the distances from 0. Hence $V \cap X$ contains vertices whose distances from 0 are $s+1$. Q.E.D.

Since every $(s+1)$-subspace is an $(s+1, GF(r), n)$-attenuated space, there is a one to one correspondence between the set of all $(s+1)$-subspaces containing 0 in $N$ and the set of all $(s-1)$-subspaces containing 0 of $\Pi(A)$. By Lemma 7.6, $|S \setminus \{V\}|$ is the number of all $(s+1)$-subspaces of $\Pi(A)$ which intersect $A \cap V$ only at 0, and $|S \setminus \{V\}| \times (q-1)$ is a multiple of $|V \cap N_{s+1}| = (q-1)(q-r) \cdots (q-r^s)$. As $A$ is an affine space, the number of all $(s+1)$-subspaces of $\Pi(A)$ which intersect $A \cap V$ only at 0 is $(q-r^{s+1}) \cdots (q-r^{2s+1})/(r^{s+1}-1) \cdots (r^{s+1}-r^s)$. Hence we have the following equation.

$$k(q-1)(q-r) \cdots (q-r^s)$$

$$= (q-1)(q-r^{s+1}) \cdots (q-r^{2s+1})/(r^{s+1}-1) \cdots (r^{s+1}-r^s),$$

where $k$ is a positive integer.

Therefore,

$$k = (q-r^{s+1}) \cdots (q-r^{2s+1})/(q-r) \cdots (q-r^s)(r^{s+1}-1) \cdots (r^{s+1}-r^s).$$
Since \( q \) and \( r \) are powers of the prime \( p \), the following number \( k' \) is also a positive integer.

\[
k' = (r^{n-s-1} - 1) \cdots (r^{n-2s-1} - 1)/(r^{n-1} - 1) \cdots (r^{n-s} - 1).
\] (7.1)

Then, by considering (7.1) modulo powers of \( r \), we also have

\[
k' \equiv r^{n-2s-1} - 1 \pmod{r^{n-2s}}.
\] (7.2)

From (7.2), it follows that \( k' > (r^{n-2s-1} - 1) \), and this implies

\[
(r^{n-s-1} - 1) \cdots (r^{n-2s} - 1)/(r^{n-1} - 1) \cdots (r^{n-s} - 1) > 1.
\]

But it is impossible. Hence we conclude that the case where \( d_0 \) is odd cannot occur.

Finally we deal with the case where \( d_0 \) is even, i.e., \( d_0 = 2s \). Let \( V \) be the subspace generated by \( \{0, v_1, \ldots, v_s\} \) and \( V' \) be the subspace generated by \( \{0, v_{s+1}, \ldots, v_{d_0}, v_{d_0+1}\} \), where \( v_{d_0+1} = -(v_1 + \cdots + v_{d_0}) \). Then, \( V \) is an \( s \)-subspace and \( V' \) is an \((s+1)\)-subspace. Moreover, \( V' \) does not contain \( V \), but both \( V \) and \( V' \) contain a vertex \( v_1 + \cdots + v_s \) whose distance from 0 is \( s \). Now we consider how many \((s+1)\)-subspaces do not contain \( V \), but contain a vertex in \( V \cap N_s \). For each vertex \( x \) in \( V \cap N_s \), let \( S(x) \) be the set of all \((s+1)\)-subspaces which contain 0 and \( x \) but do not contain \( V \), and let \( S = \bigcup_{x \in V \cap N_s} S(x) \). By the distance transitivity of \( N_s \), \( |S(x)| \) does not depend on the choice of the vertex \( x \). Since \( S(v_1 + \cdots + v_s) \) contains \( V' \), we have \( |S(x)| \geq 1 \) for \( x \in V \cap N_s \). Moreover, we can show that Lemma 7.6 also holds in this case by employing the same argument as in the proof of Lemma 7.6.

**Lemma 7.7.** Let \( X \) be an \((s+1)\)-subspace containing 0 which does not contain \( V \). Then, \( X \) contains a vertex \( x \) in \( V \cap N_s \) if and only if \( V \cap X \cap A = \{0\} \) for an assembly \( A \) containing 0. Moreover, if \( X \) contains a vertex in \( V \cap N_s \), then \( X \cap V \subseteq N_s \cap \{0\} \) and \( |X \cap V| = q \).

By Lemma 7.7 and the fact that \( A \) is an affine space, we have

\[
|S| = (q - r^s) \cdots (q - r^{2s})/(r^{s+1} - 1) \cdots (r^{s+1} - r^s).
\]

As in the case where \( d_0 \) is odd, \(|S| \times (q - 1)\) is a multiple of the number \(|V \cap N_s|\). Since \( V \) is an \((s, GF(r), n)\)-attenuated space, \(|V \cap N_s| = (q-1)(q-r) \cdots (q-r^{s-1})\). Hence we have the following equation.

\[
k(q - 1)(q - r) \cdots (q - r^{s-1}) = (q - 1)(q - r^s) \cdots (q - r^{2s})/(r^{s+1} - 1) \cdots (r^{s+1} - r^s),
\]

where \( k \) is a positive integer.
Therefore,
\[ k - (q - r^s) \cdots (q - r^{2s})/(q - r) \cdots (q - r^{s-1})(r^{s+1} - 1) \cdots (r^{s+1} - r^s). \]

and so the following \( k' \) is also a positive integer.

\[ k' = (r^{n-s} - 1) \cdots (r^{n-2s} - 1)/(r^{n-1} - 1) \cdots (r^{n-s+1} - 1). \quad (7.3) \]

By considering (7.3) modulo powers of \( r \), it can be shown that

\[ k' \geq (r^{n-2s} - 1)(r^{n-2s+1} - 1) \quad (7.4) \]

From (7.4), we obtain

\[ (r^{n-s} - 1) \cdots (r^{n-2s+2} - 1)/(r^{n-1} - 1) \cdots (r^{n-s+1} - 1)(r^{s+1} - 1) \cdots (r - 1) > 1. \]

But it is impossible. Hence we conclude that \( k \) cannot be an integer, and we obtain a final contradiction. Thus, we have completed the proof of the main theorem.

**Acknowledgments**

The author expresses his appreciation to Professor Tatsuro Ito, Osaka Kyoiku University and Professor Eiichi Bannai, Kyushu University, for their helpful suggestions and encouragement.

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