Stability criteria for certain high even order delay differential equations

Baruch Cahlon ∗, Darrell Schmidt

Department of Mathematics and Statistics, Oakland University, Rochester, MI 48309-4401, USA

Received 24 March 2006
Available online 4 January 2007
Submitted by William F. Ames

Abstract

In this paper we study the asymptotic stability of the zero solution of even order linear delay differential equations of the form

\[ y^{(2m)}(t) = \sum_{j=0}^{2m-1} a_j y^{(j)}(t) + \sum_{j=0}^{2m-1} b_j y^{(j)}(t - \tau), \]

where \( a_j \) and \( b_j \) are certain constants and \( m \geq 1 \). Here \( \tau > 0 \) is a constant delay. In proving our results we make use of Pontryagin’s theory for quasi-polynomials.

© 2007 Elsevier Inc. All rights reserved.

Keywords: Asymptotic stability; Stability criteria; Sufficient conditions; Delay; Characteristic functions; Stability regions

1. Introduction

The aim of this paper is to study the asymptotic stability of the zero solution of the delay differential equation

\[ y^{(2m)}(t) = \sum_{j=0}^{2m-1} a_j y^{(j)}(t) + \sum_{j=0}^{2m-1} b_j y^{(j)}(t - \tau), \quad (1.1) \]

* Corresponding author.
E-mail address: cahlon@oakland.edu (B. Cahlon).

0022-247X/$ – see front matter © 2007 Elsevier Inc. All rights reserved.
where $\tau > 0$, $a_j$ and $b_j$ are constants and $m \geq 1$. In previous papers [1,2], we considered Eq. (1.1) with $m = 1$. See the text [3] for stability of second order delay equations and applications. Odd higher order equations were considered in [4]. The current paper and [4] will provide a complete treatment for higher order delay differential equations with constant coefficients and constant delay. There are no practical stability robust criteria of the zero solution of (1.1) for $m \geq 2$. For stability and oscillation of certain fourth order equations see [5–12]. See [13–16] for studies of systems that may shed light on (1.1). The study on systems does not, however, yield complete practical stability criteria of (1.1). It is clear that with $4m$ independent parameters in (1.1) one cannot expect to get regions of stability. Our goal is to derive algorithmic type stability criteria.

We take the view that part of the $j$th derivative term of the equation

$$y^{(2m)}(t) = \sum_{j=0}^{2m-1} p_j y^{(j)}(t)$$

(1.2)

is delayed and the remaining part is not. Note that with $\tau = 0$ the zero solution of (1.1) or (1.2) is asymptotically stable if and only if all the characteristic roots of a real polynomial

$$x^{2m} - p_{2m-1}x^{2m-1} - p_{2m-2}x^{2m-2} - p_{2m-3}x^{2m-3} - \cdots - p_0 = 0$$

(1.3)

are in complex left half plane. Relative to (1.1), our view is that

$$p_j = a_j + b_j, \quad j = 0, 1, \ldots, 2m - 1.$$  

(1.4)

By Hurwitz Criterion [17] all roots have negative real parts if and only if

$$\delta_j > 0, \quad j = 1, 2, \ldots, 2m,$$

(1.5)

where the $\delta_j$ are the following determinants:

$$\begin{align*}
\delta_1 &= -p_{2m-1}, \\
\delta_2 &= \begin{vmatrix}
-p_{2m-1} & -p_{2m-3} \\
1 & -p_{2m-2}
\end{vmatrix}, \\
\delta_k &= \begin{vmatrix}
-p_{2m-1} & -p_{2m-3} & -p_{2m-5} & \cdots & -p_{2m-2k+1} \\
1 & -p_{2m-2} & -p_{2m-4} & \cdots & -p_{2m-2k+2} \\
0 & -p_{2m-1} & -p_{2m-3} & \cdots & -p_{2(m-k)+3} \\
0 & 1 & -p_{2m-2} & \cdots & -p_{(m-k)+4} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & -p_{2m-k}
\end{vmatrix}, \quad k = 3, \ldots, 2m,
\end{align*}$$

with $-p_{2m-j} = 0$ for $j > 2m + 1$. For delay equations, there is no simple criterion as the Routh–Hurwitz criterion.

This paper is organized as follows. In Section 2, we present the tools used in our asymptotic stability analysis. In Section 3 we give our main results and some special cases. In Section 4 we present some examples.

Some of the results and techniques in this paper are different than the results of the odd case [4]. We have also sharpened stopping criteria of all algorithms.

2. Background

In this section, we identify the characteristic function of (1.1) in order to study the asymptotic stability of the zero solution. We also cite the main results of Pontryagin related to asymptotic stability [18] and the applications of Pontryagin’s results [19, §13.7–13.9].
The characteristic function of (1.1) is given by
\[
\hat{H}(s) = s^{2m} - \sum_{j=0}^{2m-1} a_j s^j - \sum_{j=0}^{2m-1} b_j e^{-s\tau} s^j .
\] (2.1)

Multiplying (2.1) by \( e^{s\tau} \) yields
\[
e^{s\tau} \hat{H}(s) = e^{s\tau} s^{2m} - \sum_{j=0}^{2m-1} a_j s^j e^{s\tau} - \sum_{j=0}^{2m-1} b_j s^j .
\] (2.2)

Letting \( s = \frac{z}{\tau} \), we examine the zeros of
\[
H(z) = \tau^{2m} e^{z} \hat{H}\left(\frac{z}{\tau}\right) = z^{2m} e^{z} - \sum_{j=0}^{2m-1} A_j z^j e^{z} - \sum_{j=0}^{2m-1} B_j z^j ,
\] (2.3)

where
\[
A_j = a_j \tau^{2m-j} \quad \text{and} \quad B_j = b_j \tau^{2m-j}, \quad j = 0, \ldots, 2m - 1.
\] (2.4)

**Theorem 2.1.** In order that all solutions of (1.1) approach zero as \( t \to \infty \) it is necessary and sufficient that all zeros of (2.1), or equivalently (2.3), have negative real parts.

See [20]. The function (2.3) is a special function, usually called an exponential polynomial or a quasi-polynomial. The problem of analyzing the distribution of the zeros in the complex plane of such functions has received a great deal of attention.

**Definition 2.1.** Let \( h(z, w) \) be a polynomial in the two variables \( z \) and \( w \) (with complex coefficients),
\[
h(z, w) = \sum_{m,n} a_{mn} z^m w^n \quad (m, n: \text{nonnegative integers}).
\]

We call the term \( a_{rs} z^r w^s \) the principal term of \( h(z, w) \) if \( a_{rs} \neq 0 \), and for each term \( a_{mn} z^m w^n \) with \( a_{mn} \neq 0 \), we have \( r \geq m \) and \( s \geq n \).

Note that \( H(z) = h(z, e^z) \) where
\[
h(z, w) = z^{2m} w - \sum_{j=0}^{2m-1} A_j z^j w - \sum_{j=0}^{2m-1} B_j z^j .
\] (2.5)

From Definition 2.1 the function \( h(z, w) \) given in (2.5) has principal term \( z^{2m} w \). We now cite two theorems of Pontryagin, see [19,20].

**Theorem 2.2.** Let \( H(z) = h(z, e^z) \), where \( h(z, w) \) is a polynomial with a principal term. The function \( H(iy) \) is now separated into real and imaginary parts; that is, we set \( H(iy) = F(y) + i G(y) \). If all the zeros of the function \( H(z) \) lie in the open left half plane, then the zeros of the functions \( F(y) \) and \( G(y) \) are real, are interlacing, and
\[
D(y) = G'(y) F(y) - G(y) F'(y) > 0
\] (2.6)
for all real \(y\). Moreover, in order that all the zeros of the function \(H(z)\) lie in the open left half plane, it is sufficient that any one of the following conditions be satisfied:

(a) All the zeros of the functions \(F(y)\) and \(G(y)\) are real and interlace, and the inequality (2.6) is satisfied for at least one value of \(y\).

(b) All the zeros of the function \(F(y)\) are real and for each of these zeros \(y = y_0\) condition (2.6) is satisfied; that is, \(F'(y_0)G(y_0) < 0\).

(c) All the zeros of the function \(G(y)\) are real and for each of these zeros the inequality (2.6) is satisfied; that is, \(G'(y_0)F(y_0) > 0\).

In our case,

\[
H(iy) = (iy)^{2m} e^{iy} - \sum_{j=0}^{2m-1} A_j(iy)^j e^{iy} - \sum_{j=0}^{2m-1} B_j(iy)^j = F(y) + iG(y) \tag{2.7}
\]

equivalently

\[
H(iy) = (iy)^{2m} e^{iy} - \sum_{j=0}^{m-1} A_{2j}(iy)^j e^{iy} - \sum_{j=0}^{m-1} A_{2j+1}(iy)^{2j+1} e^{iy} - \sum_{j=0}^{m-1} B_{2j}(iy)^{2j} - \sum_{j=0}^{m-1} B_{2j+1}(iy)^{2j+1}, \tag{2.8}
\]

where

\[
F(y) = (-1)^m y^{2m} \cos y - \sum_{j=0}^{m-1} A_{2j} (-1)^j y^{2j} \cos y - \sum_{j=0}^{m-1} A_{2j+1} (-1)^{j+1} y^{2j+1} \cos y \tag{2.9}
\]

and

\[
G(y) = (-1)^m y^{2m} \sin y - \sum_{j=0}^{m-1} A_{2j} (-1)^j y^{2j} \sin y - \sum_{j=0}^{m-1} A_{2j+1} (-1)^{j+1} y^{2j+1} \sin y \tag{2.10}
\]

In order to study the location of the zeros of \(H(z)\) one has to study the zeros of \(F\) and \(G\). To do so, we need the following result which is useful in determining whether all roots of \(F\) and \(G\) are real. Let \(f(z, u, v)\) be a polynomial in \(z, u,\) and \(v\), which we write in the form

\[
f(z, u, v) = \sum_{m,n} \varepsilon^m \phi^{(n)}_m(u, v), \tag{2.11}
\]
where $\phi^{(n)}_m(u, v)$ is a polynomial of degree $n$, homogeneous in $u$ and $v$, and let $z^r \phi^{(s)}_r(u, v)$ be the principal term of $f(z, u, v)$, and let $\phi^{(s)}_r(u, v)$ denote the coefficient of $z^r$ in $f(z, u, v)$, so that

$$\phi^{(s)}_r(u, v) = \sum_{n \leq s} \phi^{(n)}_r(u, v).$$

(The principal term for the polynomials of the form (2.11) is analogous to that defined in Definition 2.1, see [19, pp. 440–443].)

Also we let

$$\Phi^{(s)}(z) = \phi^{(s)}(\cos z, \sin z).$$

**Theorem 2.3.** Let $f(z, u, v)$ be a polynomial with principal term $z^r \phi^{(s)}_r(u, v)$. If $\epsilon$ is such that $\Phi^{(s)}(\epsilon + iy) \neq 0$ for all real $y$, then in the strip $-2\pi k + \epsilon \leq x \leq 2\pi k + \epsilon$, the function $F(z) = f(z, \cos z, \sin z)$ will have, for all sufficiently large values of $k$, exactly $4sk + r$ zeros. Thus, in order for the function $F(z)$ to have only real roots, it is necessary and sufficient that in the interval $-2\pi k + \epsilon \leq x \leq 2\pi k + \epsilon$, it has exactly $4sk + r$ real roots for all sufficiently large $k$.

Note that the functions $F(y)$ and $G(y)$ in (2.10) and (2.11) have principal terms $(-1)^m y^{2m} \cos y$ and $(-1)^m y^{2m} \sin y$, respectively. Thus $s = 1$ and $r = 2m$. Therefore $F$ has all real zeros if and only if $F$ has $4k + 2m$ zeros in $(-2k\pi, 2k\pi)$ for $k$ sufficiently large, and the same holds for $G$ with $(-2k\pi, 2k\pi)$ replaced by $(-2k\pi + \epsilon, 2k\pi + \epsilon)$, where $0 < \epsilon < \pi$.

We will use Theorems 2.2 and 2.3 to study the asymptotic stability of (1.1). In the next section we will present the main results of this paper.

### 3. Main results

In this section we present the main results of this paper. We first describe the asymptotic behavior of the zeros of $F$. Throughout this paper for $x$ real and $a > 0$, $[x]_a$ denotes the unique real number in the interval $[0, a)$ for which $x - [x]_a$ is an integer multiple of $a$. We will use $a = \pi$ and $a = 2\pi$.

**Lemma 3.1.** For $n$ sufficiently large, the interval $(n\pi - \pi/2, n\pi + \pi/2)$ contains exactly one zero $r_n$ of $G$ and $\lim_{n \to \infty} (r_n - n\pi) = 0$.

**Proof.** From (2.10), $y = 0$ is zero of $G$, and if $u = n\pi + \pi/2$, then

$$G(u) = (-1)^{m+n} u^{2m} - \sum_{j=0}^{m-1} A_{2j} (-1)^{n+j} u^{2j+1} - \sum_{j=0}^{m-1} B_{2j+1} (-1)^j u^{2j+1}. \quad (3.1)$$

So $G(n\pi + \pi/2)$ is a polynomial of degree $2m - 1$ or less in $n\pi + \pi/2$, and thus there can be at most $2m - 1$ zeros of $G$ that are of the form $n\pi + \pi/2$. All other zeros of $G$ are the roots of the equation

$$w(y) = \xi(y) \quad (3.2)$$

where
zeros of $F$


A and so Eqs. (2.10) and (2.11)

Proof. Assume the zero solution of (1.1) is asymptotically stable. From Theorems 2.1 and 2.2

$$w(y) = \left( y^2 + A_{2m-2} - \sum_{j=0}^{m-2} \frac{A_{2j}(-1)^{m+j}}{y^{2(m-j)-2}} \right) \tan y$$

$$+ \left( B_{2m-1}y - \sum_{j=0}^{m-2} \frac{B_{2j+1}(-1)^{m+j}}{y^{2(m-j)-3}} \right) \sec y$$

(3.3)

and

$$\xi(y) = A_{2m-1}y + \sum_{j=0}^{m-2} \frac{A_{2j+1}(-1)^{m+j}}{y^{2(m-j)-3}}.$$  (3.4)

For $n$ sufficiently large, $w$ resembles the tangent function on $(n\pi - \pi/2, n\pi + \pi/2)$ in that $w$ has limits $-\infty$ and $\infty$ at $n\pi - \pi/2$ and $n\pi + \pi/2$ when the limits are taken from inside the interval $(n\pi - \pi/2, n\pi + \pi/2)$ for $n$ sufficiently large. Also, $w$ is increasing in this interval for $n$ sufficiently large (see Fig. 1). Now (3.2) yields

$$\sin y = \frac{\sum_{j=0}^{m-1} \frac{A_{2j+1}(-1)^{m+j}}{y^{2(m-j)-1}} \cos y + \sum_{j=0}^{m-1} \frac{B_{2j+1}(-1)^{m+j}}{y^{2(m-j)-1}}}{1 + \sum_{j=0}^{m-1} \frac{A_{2j}(-1)^{m+j}}{y^{2(m-j)}}}.$$  (3.5)

It follows from (3.5) that

$$\lim_{G(y)\to 0} \sin y = 0,$$

and so

$$\lim_{G(y)\to 0} [y - \pi/2]_\pi = \pi/2. \quad \square$$

The following is a useful necessary condition for the asymptotic stability of the zero solution of (1.1).

**Theorem 3.1.** If the zero solution of (1.1) is asymptotically stable, then $A_0 + A_1 + B_1 < 0$ and $A_0 + B_0 < 0$.

**Proof.** Assume the zero solution of (1.1) is asymptotically stable. From Theorems 2.1 and 2.2 and Eqs. (2.10) and (2.11)

$$\mathbb{D}(0) = (A_0 + A_1 + B_1)(A_0 + B_0) > 0.$$  (3.6)

It follows from Theorems 2.1–2.3, that $G$ has all real zeros and for $k$ sufficiently large $(-2k\pi + \epsilon, 2k\pi + \epsilon)$ contains precisely $4k + 2m$ zeros of $G$. Since $y = 0$ is a zero of $G$ and $G$ is odd, $(0, 2k\pi + \epsilon)$ contains precisely $2k + m$ zeros $r_1 < r_2 < \cdots < r_{2k+m}$ of $G$ where $k$ is sufficiently large. By Lemma 3.1, $r_{2k+m} \in ((2k)\pi - \epsilon, 2k\pi + \epsilon)$ and $r_{2k+m} - 2\pi \to 0$ as $k \to \infty$. From (2.10) it follows that $F(r_{2k+m})$ has sign $(-1)^m$ for $k$ sufficiently large. By Theorems 2.1 and 2.2, the zeros of $F$ and $G$ interlace and thus the $F(r_j)$ must strictly alternate in sign (where $r_0 = 0$). Thus $(-1)^m F(0) F(r_{2k+m}) > 0$, and since $(-1)^m F(r_{2k+m}) > 0$, $F(0) = -(A_0 + B_0) > 0$. Thus $A_0 + B_0 < 0$, and by (3.6) $A_1 + A_0 + B_1 < 0$. The proof is complete.  \( \square \)
Evidently if \( A_1 + A_0 + B_1 \geq 0 \) or \( A_0 + B_0 \geq 0 \), then the zero solution of (1.1) is not asymptotically stable.

In this paper \( \mathbb{Z}^+ \) denotes the set of all nonnegative integers. We first consider some special cases of (1.1). We denote

\[
P_{2m}(y) = (-1)^m y^{2m} - \sum_{j=0}^{m-1} A_{2j} (-1)^j y^{2j}.
\]  

**Theorem 3.2.** Assume that \( A_{2j+1} = 0 \) and \( B_{2j+1} = 0 \), \( j = 0, 1, \ldots, m - 1 \). The zero solution of (1.1) is asymptotically stable if and only if

1. \( P_{2m} \) has \( m \) distinct positive (real) zeros \( r_j \), \( j = 1, 2, \ldots, m \), and \( r_j \neq n\pi \) for all \( j = 1, 2, \ldots, m \) and \( n \in \mathbb{Z}^+ \),
2. \( D(r_j) > 0 \), \( j = 1, 2, \ldots, m \),
3. \( D(n\pi) > 0 \) \((n \in \mathbb{Z}^+)\), and
4. \( D(0) > 0 \).

**Theorem 3.3 (Algorithmic Stability Test I).** Assume that \( A_{2j+1} = 0 \) and \( B_{2j+1} = 0 \), \( j = 0, 1, \ldots, m - 1 \). Assume further that \( P_{2m} \) has \( m \) distinct positive zeros \( 0 < r_1 < r_2 < \cdots < r_m \) none of which coincide with any \( n\pi \) \((n \in \mathbb{Z}^+)\). Let \( N \) be the smallest positive integer such that

\[
\sum_{j=0}^{m-1} \frac{|B_{2j}|}{(N\pi)^{2(m-j)}} + \sum_{j=0}^{m-1} \frac{|A_{2j}|}{(N\pi)^{2(m-j)}} < 1.
\]  

If \( D(n\pi) > 0 \) for \( n = 0, 1, 2, \ldots, N - 1 \), \( D(r_j) > 0 \), for \( j = 1, 2, \ldots, m \), and \( D(0) > 0 \), then the zero solution of (1.3) is asymptotically stable.

The proofs of Theorems 3.2 and 3.3 are similar to the proofs of corresponding theorems given in [1] when the differential equation is of odd order, and we omit them.

**Remark 3.1.** When \( m = 1 \) and \( A_1 = B_1 = 0 \), \( P_2(y) \) is an even quadratic polynomial. In [2] and [3], we used this fact and essentially the second order analog of Theorem 3.2 to obtain elegant asymptotic stability criteria. When \( m = 2 \) and \( A_1 = A_3 = B_1 = B_3 = 0 \), \( P_4(y) \) is a quadratic polynomial in \( y^2 \). We develop asymptotic stability criteria in this case. From (3.7),

![Fig. 1. The function \( w \).](image-url)
\[ P_4(y) = y^4 + A_2 y^2 - A_0. \] (3.9)

The zeros of \( P_4 \) are

\[ r_1 = \sqrt{-\frac{A_2 - \sqrt{A_2^2 + 4A_0}}{2}}, \] (3.10)

\[ r_2 = \sqrt{-\frac{A_2 + \sqrt{A_2^2 + 4A_0}}{2}}. \] (3.11)

\(-r_1\), and \(-r_2\). It is easy to see that \( P_4 \) has two distinct positive zeros precisely when \( A_2 < 0 \) and \(-A_2^2/4 < A_0 < 0\). From (2.9) and (2.10), \( G(y) = P_4(y) \sin y \) and \( F(y) = P_4(y) \cos y - (B_0 - B_2 y^2) \). From Theorem 3.2, we require that \( r_1, r_2 \neq n\pi \) for all \( n = 1, 2, \ldots \). This condition will be redundant on subsequent conditions. We examine the determinant \( \mathbb{D} \) at 0, \( r_1, r_2 \) and \( n\pi \) \((n = 1, 2, \ldots)\). Certainly \( \mathbb{D}(0) = -(A_0 + B_0) > 0 \) only when \( A_0 + B_0 < 0 \). As well,

\[ \mathbb{D}(r_1) = F(r_1)G'(r_1) = P_4'(r_1) \sin r_1 (B_2 r_1^2 - B_0). \]

Since \( P_4'(r_1) < 0, \mathbb{D}(r_1) > 0 \) when \( \sin r_1 (B_2 r_1^2 - B_0) < 0 \). Now \( P_4'(r_2) > 0, \mathbb{D}(r_2) > 0 \) if and only if \( \sin r_2 (B_2 r_2^2 - B_0) > 0 \). Define

\[ Q(y) = y^4 + (A_2 + B_2) y^2 - (A_0 + B_0) \] (3.12)

and

\[ R(y) = y^4 + (A_2 - B_2) y^2 - (A_0 - B_0). \] (3.13)

If \( n \) is even, then

\[ \mathbb{D}(n\pi) = P_4(n\pi)Q(n\pi) = ((n\pi)^2 - r_1^2)((n\pi)^2 - r_2^2) Q(n\pi). \] (3.14)

If \( A_0 + B_0 < 0 \), it can be seen that \( Q \) has either zero or two positive zeros (counting multiplicity). Consider the open intervals in \((0, \infty)\) determined by \( r_1, r_2 \) and the positive zeros of \( Q \), and number them in reverse order. In case of coincident roots, an interval can be empty. If \( Q \) has two positive zeros \( s_1, s_2 \) and \( 0 < r_1 < s_1 < r_2 < s_2 \), the first interval is \((s_2, \infty)\), the second interval is \((r_2, s_2)\), and the third, fourth and fifth are \((s_1, r_2), (r_1, s_1)\) and \((0, r_1)\), respectively. If \( 0 < r_1 < s_1 = r_2 = s_2 \), then the intervals are \((s_2, \infty), (r_2, s_2) = \phi, (s_1, r_2) = \phi, (r_1, s_1)\), and \((0, r_1)\) in the order listed. From (3.14), \( \mathbb{D}(n\pi) > 0 \) for all even \( n \) if and only if the even numbered intervals contain no even multiples of \( \pi \). The polynomial \( R \) may have zero, one, or two positive zeros. As above, we consider the intervals in \((0, \infty)\) determined by \( r_1, r_2 \) and the positive zeros of \( R \), and we number these intervals in reverse order. Then \( \mathbb{D}(n\pi) > 0 \) for all \( n \) odd if and only if the even intervals contain no odd multiples of \( \pi \). The following theorem collects the criteria obtained in this remark.

**Theorem 3.4.** Suppose that \( m = 2 \) and \( A_1 = A_3 = B_1 = B_3 = 0 \). Then the zero solution of (1.1) is asymptotically stable if and only if

(i) \( A_0 + B_0 < 0, A_2 < 0, \) and \(-A_2^2/4 < A_0 < 0,\)

(ii) \( \sin r_1 (B_2 r_1^2 - B_0) < 0 < \sin r_2 (B_2 r_2^2 - B_0) \) where \( r_1 \) and \( r_2 \) are given in (3.9) and (3.10), respectively,
(iii) the even numbered intervals in $(0, \infty)$ determined by $r_1$, $r_2$ and the positive zeros of $Q$ in (3.11) (numbered in reversed order, as above) contain no even multiples of $\pi$, and
(iv) the even numbered intervals in $(0, \infty)$ determined by $r_1$, $r_2$ and the positive zeros of $R$ in (3.12) (numbered in reverse order) contain no odd multiples of $\pi$.

It is of interest to note that only the quadratic formula is needed in Theorem 3.4 to find the zeros of $P_4$, $Q$ and $R$. Example 4.1 will illustrate this case and will show a contrast between Theorems 3.3 and 3.4.

Analogous results could be obtained for $m$ larger. However, these would require the cubic or quadratic forms on numerical methods to find all positive zeros of certain polynomials.

**Remark 3.2.** We now consider the special case of pure delay, i.e. $A_j = 0$, $j = 0, 1, \ldots, 2m - 1$. In this case,

$$G(y) = (-1)^{m} y^{2m} \sin y - \sum_{j=0}^{m-1} B_{2j+1} (-1)^j y^{2j+1}$$

and

$$F(y) = (-1)^{m} y^{2m} \cos y - \sum_{j=0}^{m-1} B_{2j} (-1)^j y^{2j}.$$  

The nonzero zeros of $G$ are the roots of

$$\sin y = \eta(y),$$

where

$$\eta(y) = \sum_{j=0}^{m-1} \frac{B_{2j+1} (-1)^{j+m}}{y^{2(m-j)-1}}.$$  

By Theorem 3.1, $B_1 < 0$ is necessary for the zero solution of (1.3) to be asymptotically stable, and so we assume $B_1 < 0$. Observe that $\lim_{y \to 0^+} \eta(y) = \infty (-1)^{m+1}$. Let $\ell$ be the largest index so that $B_{2\ell+1} \neq 0$, and thus $\ell \leq m - 1$ and

$$\eta(y) = \sum_{j=0}^{\ell} \frac{B_{2j+1} (-1)^{j+m}}{y^{2(m-j)-1}}.$$  

If $B_{2\ell+1} (-1)^{\ell+m} > 0$, then $\eta$ is eventually decreasing and convex, and if $B_{2\ell+1} (-1)^{\ell+m} < 0$, then $\eta$ is eventually increasing and concave. In either case, $\lim_{y \to \infty} \eta(y) = 0$. We obtain a value $Y_1$ so that $\eta$ is either decreasing and convex or increasing and concave on $[Y_1, \infty)$. It can be seen that if $\eta'' > 0$ (respectively $\eta'' < 0$) in an interval $[Y_1, \infty)$, then $\eta' < 0$ (respectively $\zeta' > 0$) on $[Y_1, \infty)$. We have

$$\eta''(y) = \sum_{j=0}^{\ell} \frac{2(m-j)(2m-2j-1)B_{2j+1} (-1)^{j+m}}{y^{2m-2j+1}} ,$$

and $\eta''$ is of constant sign on $[Y_1, \infty)$ if
Lemma 3.2. Suppose that \( A_j = 0, \quad j = 0, 1, \ldots, 2m - 1, \) and \( B_1 \neq 0. \) The function \( G \) has all real zeros if and only if \( G \) has \( 2\theta + m - 1 \) zeros in the interval \((0, 2(\theta + 1)\pi)\) if \( B_{2\ell+1}(-1)^{\ell+m} > 0 \) and \( 2\theta + m \) zeros in \((0, 2(\theta + 1)\pi)\) if \( B_{2\ell+1}(-1)^{\ell+m} < 0. \)

Proof. From Remark 3.2, it can be seen that (3.18) has precisely two roots in \((2(n - 1)\pi, 2n\pi)\) when \( n \geq \theta + 1. \) If \( B_{2\ell+1}(-1)^{\ell+m} > 0, \) then \( \eta \) is eventually decreasing and convex. For \( k \) sufficiently large, \( G \) has one zero in \([2\pi k, 2\pi k + \epsilon]\) and no zeros in \([-2\pi k, -2\pi k + \epsilon]. \) Also \( y = 0 \) is a zero of \( G, \) and thus for \( k \) sufficiently large \( G \) has \( 2[2\theta + m - 1 + 2(k - \theta)] + 1 + 1 = 4k + 2m \) zeros in \((-2k\pi + \epsilon, 2k\pi + \epsilon). \) If \( B_{2\ell+1}(-1)^{\ell+m} < 0, \) then \( \eta \) is eventually increasing and concave. In this case \( G \) has no zeros in \([2\pi k, 2\pi k + \epsilon]\) and one zero in \([-2\pi k, -2\pi k + \epsilon]. \) Also \( y = 0 \) is a zero of \( G, \) and so for \( k \) sufficiently large \( G \) has \( 2[2\theta + m + 2(k - \theta)] + 1 - 1 = 4k + 2m \) zeros in \((-2k\pi + \epsilon, 2k\pi + \epsilon). \) The proof of sufficiency now follows from Theorem 2.3 since \( G \) has precisely \( 4k + 2m \) zeros in \((-2k\pi + \epsilon, 2k\pi + \epsilon) \) for \( k \) sufficiently large. Necessity follows from Remark 3.2 and essentially same argument.

The pure delay case for even order equations is similar to that of odd order equations considered in [1]. However, Lemma 3.2 constitutes a marked difference between the even and odd order cases.

The following theorem applies to the general case (as well as the pure delay case). It gives infinitely many conditions for asymptotic stability. The proof is essentially the same as the proof of the corresponding theorem in the odd case [1], and we omit it.

Theorem 3.5. The zero solution of (1.1) is asymptotically stable if and only if

1. \( A_0 + B_0 < 0, \) \( A_1 + A_0 + B_1 < 0, \)
2. \( G \) has all real zeros, and
3. \((-1)^n F(r_n) > 0 \ (n = 1, 2, \ldots)\)

where \(r_1 < r_2 < r_3 < \cdots\) are the positive zeros of \(G\).

**Remark 3.3.** Suppose that \(A_j = 0, \ j = 0, 1, \ldots, 2m - 1\). Remark 3.2 and the proof of Lemma 3.2 yield that if \(m\) is even, \([r_{2n}]_2\pi \) approaches 0 or \(2\pi\) and thus \((-1)^m \cos r_{2n} \to 1\). If \(m\) is odd, then \([r_{2n}]_2\pi \to \pi\) and thus \((-1)^m \cos r_{2n} \to 1\). Recall that \(\sin r_n \to 0\) as \(n \to \infty\). Select a positive integer \(N\) so that \(r_N > 2(\theta + 1)\pi, |\cos r_N| > \frac{1}{2}\), and

\[
\frac{|B_{2j+1}|}{r_{2(m-j)}^2} < \frac{1}{2m}, \quad j = 0, 1, \ldots, m - 1.
\]

Since \(\eta\) is monotonic on \([2\theta \pi, \infty)\), (3.17) yields that if \(n \geq N\), then \(r_n > 2(\theta + 1)\pi, (-1)^{n+m} \cos r_n > \frac{1}{2}\), and

\[
\frac{|B_{2j+1}|}{r_{2(m-j)}^2} < \frac{1}{2m}, \quad j = 0, 1, 2, \ldots, m - 1.
\]

**Theorem 3.6 (Algorithmic Stability Test II).** Suppose that \(A_j = 0, \ j = 0, 1, \ldots, 2m - 1\). Moreover, assume that the necessary conditions of Lemma 3.2 are satisfied. If

1. \(B_0 < 0, B_1 < 0, \) and
2. \((-1)^n F(r_n) > 0, \ n = 0, 1, 2, \ldots, N - 1, \) where \(N\) is given in Remark 3.3,

then the zero solution of (1.1) is asymptotically stable.

**Proof.** The proof is contained in Remark 3.3, Theorems 3.5, and (3.16). \(\Box\)

Theorem 3.6 is a slight improvement of the corresponding one given in [1] for the odd case. Our next discussion results in a robust algorithmic stability test that applies to all cases of (1.1). It comes at a cost in that it is not as sharp as the development of Algorithmic Stability Tests I and II. Particularly, the condition for \(G\) to have all real zeros is not as straightforward as in Lemma 3.2. In addition, stopping criteria is not as sharp. Never the less, it can be implemented and applied to all cases.

Lemma 3.1 reveals that for \(n\) sufficiently large \((n\pi - \pi/2, n\pi + \pi/2)\) contains exactly one zero of \(G\) and these zeros “tend to \(n\pi\.” The next lemma give conditions under which there is exactly one zero \(r\) of \(G\) in this interval and \(n\pi - \pi/4 < r < n\pi + 3\pi/4\).

**Lemma 3.3.** Let \(n \in \mathbb{Z}^+\). If

\[n \geq \max(M_1, M_2, M_3, M_4),\]

where \(M_1, M_2, M_3\) and \(M_4\) are smallest positive integers satisfying (3.24), (3.26), (3.29) and (3.31), respectively, then the interval \([n\pi - \pi/2, n\pi + \pi/2)\) contains exactly one zero \(r\) of \(G\) and \(n\pi - \pi/4 < r < n\pi + \pi/4\).

**Proof.** Let \(M_1\) be a positive integer such that all zeros of \(G\) of the form \(n\pi + \pi/2\) are in \((0, M_1\pi)\). From (3.1), it suffices to choose \(M_1\) to be the smallest positive integer for which
\[
\sum_{j=0}^{m-1} \frac{|A_{2j}| + |B_{2j+1}|}{(M_1 \pi + \pi/2)^{2(m-j)+1}} < 1. \tag{3.24}
\]

(We could get an explicit formula for \(M_1\) as in [1], but this would sacrifice sharpness even more.)

We now use (3.5) to obtain a value of \(M_2\) so that if \(n \geq M_2\) and \(r \in (n\pi - \pi/2, n\pi + \pi/2)\) is a zero of \(G\), then \(n\pi - \pi/4 < r < n\pi + \pi/4\). This is guaranteed if the absolute value of the right side of (3.5) is less than \(\frac{1}{\sqrt{2}}\) which in turns holds if

\[
\frac{1}{\sqrt{2}} > \sum_{j=0}^{m-1} \frac{|A_{2j}| + |B_{2j+1}|}{(M_2 \pi - \pi/2)^{2(m-j)+1}} \left(1 - \sum_{j=0}^{m-1} \frac{|A_{2j}|}{r^{(m-j)-1}}\right) \tag{3.25}
\]

and the denominator of (3.25) is positive. This is guaranteed by choosing \(M_2\) to be the smallest positive integer for which

\[
\sum_{j=0}^{m-1} \frac{|A_{2j}|}{(M_2 \pi - \pi/2)^{2(m-j)+1}} + \sqrt{2} \sum_{j=0}^{m-1} \frac{|A_{2j+1}| + |B_{2j+1}|}{(M_2 \pi - \pi/2)^{2(m-j)-1}} < 1. \tag{3.26}
\]

Now we determine \(M_3\) so that if \(n \geq M_3\), then \(w - \zeta\) is strictly increasing on \((n\pi - \pi/4, n\pi + \pi/4)\). Thus if \(n \geq \max(M_1, M_2, M_3)\), then \((n\pi - \pi/2, n\pi + \pi/2)\) contains at most one zero of \(G\). Using (3.3) and (3.4),

\[
w'(y) - \zeta'(y) = \left(2y + \sum_{j=0}^{m-2} \frac{A_{2j}(-1)^{m+j}(2(m-j) - 2)}{y^{2(m-j)-1}}\right) \tan y
\]

\[
+ \left(y^2 - \sum_{j=0}^{m-1} \frac{A_{2j}(-1)^{m+j}}{y^{2(m-j)-2}}\right) \sec^2 y - \sum_{j=0}^{m-1} \frac{B_{2j+1}(-1)^{m+j}}{y^{2(m-j)-3}} \sec y \tan y
\]

\[
+ \sum_{j=0}^{m-1} \frac{B_{2j+1}(-1)^{m+j}(2(m-j) - 3)}{y^{2(m-j)-2}} \sec y
\]

\[
- \sum_{j=0}^{m-1} \frac{A_{2j+1}(-1)^{m+j}(2(m-j) - 3)}{y^{2(m-j)-2}}. \tag{3.27}
\]

If \(n\pi - \pi/4 < y < n\pi + \pi/4\), then \(|\tan y| < 1, 1 < |\sec y| < \sqrt{2}\), and thus by (3.27)

\[
\frac{w'(y) - \zeta'(y)}{y^2} > 1 - \frac{2}{y} - 2 \sum_{j=0}^{m-1} \frac{|A_{2j}|}{y^{2(m-j)} - \sum_{j=0}^{m-2} \frac{|A_{2j}|(2(m-j) - 2)}{y^{2(m-j)+1}}}
\]

\[
- \sum_{j=0}^{m-1} \frac{|A_{2j+1}|(2(m-j) - 3)}{y^{2(m-j)}} - \sqrt{2} \sum_{j=0}^{m-1} \frac{|B_{2j+1}|}{y^{2(m-j)-1}}
\]

\[
- \sqrt{2} \sum_{j=0}^{m-1} \frac{|B_{2j+1}|(2(m-j) - 3)}{y^{2(m-j)}} - \sum_{j=0}^{m-1} \frac{|A_{2j+1}|(2(m-j) - 3)}{y^{2(m-j)}} \tag{3.28}
\]

Since the right side of (3.28) is an increasing function of \(y\) on \((0, \infty)\), we select \(M_3\) to be the smallest positive integer for which
\[
\frac{2}{(M_3\pi - \pi/4)} + 2\sum_{j=0}^{m-2} \frac{(m-j-1)|A_{2j}|}{(M_3\pi - \pi/4)^{2(m-j)+1}}
\]
\[
+ 2\sum_{j=0}^{m-1} \frac{|A_{2j}|}{(M_3\pi - \pi/4)^{2(m-j)}} + \sqrt{2}\sum_{j=0}^{m-1} \frac{|B_{2j+1}|}{(M_3\pi - \pi/4)^{2(m-j)-1}}
\]
\[
+ \sum_{j=0}^{m-1} \frac{3(|A_{2j+1}| + \sqrt{2}|B_{2j+1}|)}{(M_3\pi - \pi/4)^{2(m-j)}} < 1. \tag{3.29}
\]

For \( n \) a positive integer, let \( s_n = n\pi - \pi/4 \) and \( v_n = n\pi + \pi/4 \). We obtain \( M_4 \) so that if \( n \geq M_4 \), then \( w(s_n) < \xi(s_n) \) and \( w(v_n) > \xi(v_n) \). Thus if \( n \geq \max(M_1, M_2, M_3, M_4) \), then \( G \) has precisely one zero \( r \in (n\pi - \pi/2, n\pi + \pi/2) \) and \( n\pi - \pi/4 < r < n\pi + \pi/4 \). Using (3.3) and (3.4)
\[
\frac{w(s_n) - \xi(s_n)}{s_n^2} < -1 + \frac{1}{2} \sum_{j=0}^{m-1} \frac{|A_{2j}|}{s_n} + \sqrt{2}\sum_{j=0}^{m-1} \frac{|B_{2j+1}|}{s_n} + \sum_{j=0}^{m-1} \frac{|A_{2j+1}|}{s_n}, \tag{3.30}
\]
and we choose \( M_4 \) to be the smallest positive integer for which
\[
\sum_{j=0}^{m-1} \frac{|A_{2j}|}{(M_4\pi - \pi/4)^{2(m-j)}} + \sum_{j=0}^{m-1} \frac{|A_{2j+1}| + \sqrt{2}|B_{2j+1}|}{(M_4\pi - \pi/4)^{2(m-j)-1}} < 1. \tag{3.31}
\]
It is easy to see that this choice of \( M_4 \) also yields \( w(v_n) > \xi(v_n) \) for \( n \geq M_4 \).

The proof is now complete. \( \square \)

**Remark 3.4.** Recall that \( G \) has all real zeros if and only if \( G \) has \( 4k + 2m \) zeros in \((-2k\pi + \epsilon, 2k\pi + \epsilon)\) (or, equivalently, \( 2k + m \) zeros in \((0, 2k\pi + \epsilon)\)) for all sufficiently large \( k \). From Lemma 3.1, it follows that if \( G \) has all real zeros, then \( r_{2k+m} - 2k\pi \to 0 \) and \( r_{2j+m-1} - (2k-1)\pi \to 0 \). Here \( r_1 < r_2 < \cdots \) represent all positive zeros of \( G \). If \( m \) is even, (2.10) yields
\[
F(r_{2j+1}) = r_{2j+1}^{2m} \left( \cos r_{2j+1} - \sum_{\ell=0}^{m-1} \frac{A_{2\ell}(-1)^\ell \cos r_{2j+1}}{r_{2j+1}^{2m-\ell}} - \sum_{\ell=0}^{m-1} \frac{A_{2\ell+1}(-1)^\ell \sin r_{2j+1}}{r_{2j+1}^{2m-\ell-1}} - \sum_{\ell=0}^{m-1} \frac{B_{2\ell}(-1)^\ell}{r_{2j+1}^{2m-\ell}} \right). \tag{3.32}
\]

If \( r_{2j+1} \in (n\pi - \pi/2, n\pi + \pi/2) \) where \( n > M := \max(M_1, M_2, M_3, M_4) \), then
\[
F(r_{2j+1}) < r_{2j+1}^{2m} \left( -\frac{1}{\sqrt{2}} + \sum_{\ell=0}^{m-1} \frac{|A_{2\ell}|}{r_{2j+1}^{2m-\ell}} + \sum_{\ell=0}^{m-1} \frac{|A_{2\ell+1}|}{r_{2j+1}^{2m-\ell-1}} + \sum_{\ell=0}^{m-1} \frac{|B_{2\ell}|}{r_{2j+1}^{2m-\ell}} \right). \tag{3.33}
\]
By (3.33), \( F(r_{2j+1}) < 0 \) if \( n > N_1 \) where \( N_1 \) is the smallest positive integer for which
\[
\sum_{\ell=0}^{m-1} \frac{|A_{2\ell}| + |B_{2\ell}|}{(N_1\pi - \pi/4)^{2m-\ell}} + \sum_{\ell=0}^{m-1} \frac{|A_{2\ell+1}|}{(N_1\pi - \pi/4)^{2m-\ell-1}} < \frac{1}{\sqrt{2}}. \tag{3.34}
\]
It is easy to see that if \( r_{2j} \in (n\pi - \pi/2, n\pi + \pi/2) \) when \( n > \max(M, N_1) \), then \( F(r_{2j}) > 0 \). When \( m \) is odd, the same choice of \( N \) yields \((-1)^n F(r_n) > 0 \) when \( r_n \in (n\pi - \pi/2, n\pi + \pi/2) \) and \( n > \max(M, N_1) \).
Theorem 3.7 (General Algorithmic Stability Test). Let $2N$ be the smallest even integer greater than or equal to $\max\{N_1, M_1, M_2, M_3, M_4\}$. The zero solution of (1.1) is asymptotically stable if and only if

1. $A_0 + B_0 < 0, A_1 + A_0 + B_1 < 0,$
2. $G$ has $2N + m$ distinct zeros $r_1 < r_2 < \cdots < r_{2N+m}$ in $(0, 2N\pi + \pi/2)$, and
3. $(-1)^n F(r_n) > 0$ ($n = 1, \ldots, 2N + m$).

The proof is contained in Theorem 3.5 and Remarks 3.3 and 3.4. □

4. Examples

Example 4.1. Consider (1.1) with $m = 2, a_1 = a_3 = 0$, and $b_1 = b_3 = 0$, i.e.

$$y^{(4)}(t) = a_0 y'(t) + a_2 y''(t) + b_0 y(t - \tau) + b_2 y''(t - \tau),$$

(4.1)

where

$$A_0 = a_0 \tau^4 = -(20\pi + 1)^2(20\pi + 2)^2, \quad A_2 = a_2 \tau^2 = -(20\pi + 1)^2 - (20\pi + 2)^2, \quad B_0 = b_0 \tau^4 = 100(20\pi + 2), \quad B_2 = b_2 \tau^2 = 102.$$ 

(4.2)

Here $r_1 = 20\pi + 1$, and $r_2 = 20\pi + 2$. By Theorem 3.4 the zero solution of (4.1) is asymptotically stable if and only if (i) $A_0 + B_0 = -1.67 \times 10^7 < 0, A_2 < 0, -A_2^2/4 - A_0 = -4138.59 < 0$. For item (ii) $\sin r_1(B_2 r_1^2 - B_0) = -3969.52 < 0, \sin r_2(-B_0 + B_2 r_2^2) = 7643.86 > 0$. For item (iii), the positive zeros of $Q(y)$ are $s_2 = 64.48, s_1 = 63.39$ and the even intervals determined by $r_1, r_2, s_1, s_2$ contain no even multiples of $\pi$. For item (iv), the positive zeros of $R(y)$ are $v_2 = 65.45$ and $v_1 = 64.01$, and the even intervals determined by $r_1, r_2, v_1, v_2$ contain no odd multiples of $\pi$. Therefore the zero solution of (4.1) is asymptotically stable. Using Algorithmic Stability Test I, we would need to examine $\mathbb{D}(n\pi)$ for $n = 1, 2, \ldots, 36.$

Example 4.2. Consider (1.1) with $m = 2$ and $a_0 = a_1 = a_2 = a_3 = 0$, i.e.

$$y^{(4)}(t) = b_0 y(t - \tau) + b_1 y'(t - \tau) + b_2 y''(y - \tau) + b_3 y^{(3)}(t - \tau) + b_4 y^{(4)}(t - \tau),$$

(4.3)

where $B_0 = b_0 \tau^5 = -0.1, B_1 = b_1 \tau^4 = -0.45, B_2 = b_2 \tau^3 = -0.76, B_3 = b_3 \tau^2 = -1.35, B_4 = b_4 \tau = -0.18$. In this example we use Algorithmic Stability Test II. Here $\theta = 1$, and the function $G$ has 5 real zeros in $(0, 4\pi]$, $r_1 = 0.7142173215, r_2 = 1.096865918, r_3 = 2.631411230, r_4 = 6.491012628$, and $r_5 = 9.279344614$, and by Lemma 3.2, $G$ has all real zeros. See Fig. 2 for the roots of $\zeta(y) = w(y)$ in the interval $[0, 4\pi + \epsilon]$.

By Lemma 3.2, $G$ has all real zeros. In addition $\mathbb{D}(0) = B_0 B_1 = 0.045 > 0$. In this example $F(0) = 0.1, F(r_1) = -0.0442281584, F(r_2) = 0.1067935711, F(r_3) = -38.37276034, F(r_4) = 2024.627098, F(r_5) = -6066.782455$, and by the Algorithmic Stability Test II the zero solution of (4.3) is asymptotically stable.

In Table 1 we listed several of the zeros of $G$ and the values of $F$ and $\sin y$ at those zeros. This gives a glimpse into the behavior of the zeros of $G$. Calculations were done in high precision although Tables 1 and 2 only reports values to four decimal points.
Table 1

<table>
<thead>
<tr>
<th>( r_1 )</th>
<th>( F(r_1) )</th>
<th>( \sin(r_1) )</th>
<th>( \eta(y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.7142</td>
<td>-0.04423</td>
<td>0.6550</td>
<td>0.1098</td>
</tr>
<tr>
<td>1.0969</td>
<td>0.1068</td>
<td>0.8898</td>
<td>0.2063</td>
</tr>
<tr>
<td>2.6314</td>
<td>-38.3728</td>
<td>0.4883</td>
<td>0.2063</td>
</tr>
<tr>
<td>4.6910</td>
<td>2024.6271</td>
<td>0.2063</td>
<td>0.2063</td>
</tr>
<tr>
<td>9.2793</td>
<td>-6066.7825</td>
<td>0.1449</td>
<td>0.2063</td>
</tr>
<tr>
<td>12.6729</td>
<td>30167.5620</td>
<td>0.1063</td>
<td>0.2063</td>
</tr>
<tr>
<td>15.6216</td>
<td>-48795.8982</td>
<td>0.08630</td>
<td>0.2063</td>
</tr>
<tr>
<td>18.9209</td>
<td>1.5064 \times 10^5</td>
<td>0.07128</td>
<td>0.2063</td>
</tr>
<tr>
<td>21.9296</td>
<td>-1.8957 \times 10^5</td>
<td>0.06152</td>
<td>0.2063</td>
</tr>
<tr>
<td>25.1863</td>
<td>4.7377 \times 10^5</td>
<td>0.05357</td>
<td>0.2063</td>
</tr>
<tr>
<td>28.2265</td>
<td>-5.2041 \times 10^5</td>
<td>0.04781</td>
<td>0.2063</td>
</tr>
<tr>
<td>31.4588</td>
<td>1.1541 \times 10^6</td>
<td>0.04280</td>
<td>0.2063</td>
</tr>
</tbody>
</table>

Table 2

<table>
<thead>
<tr>
<th>( r_1 )</th>
<th>( F(r_1) )</th>
<th>( \sin(r_1) )</th>
<th>( \eta(y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.7689</td>
<td>-9856.5564</td>
<td>-0.5869</td>
<td>0.2063</td>
</tr>
<tr>
<td>5.0658</td>
<td>15537.1859</td>
<td>-0.9382</td>
<td>0.2063</td>
</tr>
<tr>
<td>9.5516</td>
<td>-3.5418 \times 10^{17}</td>
<td>0.6458</td>
<td>0.2063</td>
</tr>
<tr>
<td>21.3890</td>
<td>6.2809 \times 10^7</td>
<td>0.5664</td>
<td>0.2063</td>
</tr>
<tr>
<td>25.5935</td>
<td>-2.2544 \times 10^8</td>
<td>0.4447</td>
<td>0.2063</td>
</tr>
<tr>
<td>27.8627</td>
<td>3.7855 \times 10^8</td>
<td>0.4001</td>
<td>0.2063</td>
</tr>
<tr>
<td>31.7647</td>
<td>-8.9888 \times 10^8</td>
<td>0.3417</td>
<td>0.2063</td>
</tr>
<tr>
<td>34.2387</td>
<td>1.4121 \times 10^9</td>
<td>0.3134</td>
<td>0.2063</td>
</tr>
<tr>
<td>37.9815</td>
<td>-2.7433 \times 10^9</td>
<td>0.2786</td>
<td>0.2063</td>
</tr>
<tr>
<td>40.5787</td>
<td>4.0771 \times 10^9</td>
<td>0.2590</td>
<td>0.2063</td>
</tr>
<tr>
<td>44.2203</td>
<td>-7.0051 \times 10^{19}</td>
<td>0.2357</td>
<td>0.2063</td>
</tr>
<tr>
<td>46.9008</td>
<td>9.9569 \times 10^9</td>
<td>0.2213</td>
<td>0.2063</td>
</tr>
<tr>
<td>50.4771</td>
<td>-1.5733 \times 10^{10}</td>
<td>0.2045</td>
<td>0.2063</td>
</tr>
</tbody>
</table>

Fig. 2. \( G \) has all real zeros: \( \sin y = \eta(y) \).
Example 4.3. Consider (1.1) with \( m = 3 \) and \( a_2 = a_4 = 0 \), and \( b_3 = b_5 = 0 \), i.e.

\[
y^{(6)}(t) = a_1 y'(t) + a_3 y'''(t) + a_5 y^{(5)}(t) + b_0 y(t - \tau) + b_1 y'(t - \tau) + b_2 y''(t - \tau) + b_4 y^{(4)}(t - \tau),
\]

where

\[
A_0 = a_0 \tau^6 = -2, \quad A_1 = a_1 \tau^4 = -1, \quad A_3 = a_3 \tau^2 = -3, \quad A_5 = a_5 \tau = -0.01,
\]

\[
B_0 = b_0 \tau^5 = -0.5, \quad B_1 = b_1 \tau^4 = -0.1, \quad B_2 = b_2 \tau^3 = -2, \quad B_4 = b_4 \tau = -1.
\]

In this example we apply the General Algorithmic Stability Test. Here \( N = 2 \) since \( M_1 = 1, M_2 = M_3 = 2, M_4 = 1, N_1 = 1, \) and so \( 2N + m = 7 \). Condition 1 of the test is satisfied since \( A_0 + B_0 = -2.5, A_1 + A_0 + B_1 = -3.1 \). For Condition 2, \( G \) has six zeros in \( (0, 2N + \pi/2) \) which are \( r_1 = 0.9483739533, r_2 = 3.039055398, r_3 = 6.271113988, r_4 = 9.421205204, r_5 = 12.56486115, r_6 = 15.70719027 \), therefore Condition 2 fails. The zero solution is not asymptotically stable. In this example Condition 3 also fails since \( F(r_1) = 1.55429108 > 0 \). One can use Condition 3, which is easier to apply in this example. In the next example we will see that is not always the case.

Example 4.4. Consider (1.1) with \( m = 3 \), i.e.

\[
y^{(6)}(t) = a_0 y(t) + a_1 y'(t) + a_2 y''(t) + a_3 y'''(t) + a_4 y^{(4)}(t) + a_5 y^{(5)}(t) + b_0 y(t - \tau) + b_1 y'(t - \tau) + b_2 y''(t - \tau) + b_3 y'''(t - \tau) + b_4 y^{(4)}(t - \tau) + b_5 y^{(5)}(t - \tau),
\]

where

\[
A_0 = a_0 \tau^6 = -2, \quad A_1 = a_1 \tau^5 = -10, \quad A_2 = a_2 \tau^4 = -30, \quad A_3 = a_3 \tau^3 = -3,
\]

\[
A_4 = a_4 \tau^2 = -80, \quad A_5 = a_5 \tau = -1, \quad B_0 = b_0 \tau^6 = -0.5, \quad B_1 = b_1 \tau^5 = -0.1,
\]

\[
B_2 = b_2 \tau^4 = -2, \quad B_3 = b_3 \tau^3 = -3, \quad B_4 = b_4 \tau^2 = -1, \quad B_5 = b_5 \tau = -10.
\]

In this example we use the General Algorithmic Stability Test. Here \( A_0 + B_0 = -2.5, A_1 + A_0 + B_1 = -12.1, \) and Condition 1 is satisfied. For Condition 2 we have \( M_1 = 1, M_2 = 7, M_3 = 8, M_4 = 8 \) and \( N_1 = 8 \), and thus \( N = 8 \) and \( 2N + m = 19 \). In the interval \( (0, 16\pi + \pi/2) \) we have the following zeros of \( G \):

\[
r_1 = 3.768860258, \quad r_2 = 5.065769241, \quad r_3 = 19.55159629, \quad r_4 = 21.38895983,
\]

\[
r_5 = 25.59353754, \quad r_6 = 27.86270734, \quad r_7 = 31.76467037, \quad r_8 = 34.23869862,
\]

\[
r_9 = 37.98147546, \quad r_{10} = 40, \quad r_{11} = 44.22025035, \quad r_{12} = 46.90078254,
\]

\[
r_{13} = 50.477144280,
\]

and Condition 2 fails since we have only 13 real positive zeros in \( (0, 16\pi + \pi/2) \), while the required number is 19. It is of interest to check the sign changes of \( F \) at the zeros of \( G \) and the values of \( \sin y \) at these points. In Table 2 we listed some of the zeros of \( G, r_j, j = 1, 2, \ldots, 13 \), and the values of \( F(r_j) \) and \( \sin r_j \).

This sign behavior does not reveal that \( G \) does not have all real zeros (see Remark 3.3).

Notice that \( F(r_1) < 0 \) and \( F \) changes sign at the zeros of \( G \). Thus the instability is revealed by the zero count of \( G \) rather than Condition 3. Also the convergence of \( r_n \) to the odd multiples of \( \pi \) appear to be rather slow.
It appears to the authors that asymptotic stability is very rare in higher order delay equations whether the order is even or odd. It would be interesting to find a physical interpretation to this phenomenon.

References