Separating sets by Darboux quasi-continuous functions

Marcin Kowalewski a,1, Aleksander Maliszewski b,∗,2

a Mathematics Department, Casimir the Great University, pl. Weyssenhoffa 11, 85-072 Bydgoszcz, Poland
b Institute of Mathematics, Technical University of Łódź, Wólczańska 215, 93-005 Łódź, Poland

Received 30 October 2006; accepted 3 April 2007

Abstract
In this paper we characterize the pairs \( \langle A_0, A_1 \rangle \) of disjoint subsets of \( \mathbb{R} \) which can be separated by a Darboux quasi-continuous function.

© 2008 Elsevier B.V. All rights reserved.

MSC: primary 26A21; secondary 26A15

Keywords: Quasi-continuity; Darboux property; Urysohn Lemma; Separating sets

The classical Urysohn Lemma states that if \( X \) is a \( T_4 \)-space and the sets \( A_0, A_1 \subset X \) are disjoint and closed, then there exists a continuous function \( f : X \rightarrow \mathbb{R} \) such that \( f = 0 \) on \( A_0 \) and \( f = 1 \) on \( A_1 \), and if moreover \( A_0 \) and \( A_1 \) are \( G_\delta \) sets, then we can require that \( f(x) \in (0, 1) \) for each \( x \in X \setminus (A_0 \cup A_1) \). In [6] A. Maliszewski replaced the continuity of the function \( f \) by Darboux property. More precisely, he examined when, given two sets \( A_0, A_1 \subset \mathbb{R} \), we can find a Darboux function \( f \) such that \( f = 0 \) on \( A_0 \) and \( f = 1 \) on \( A_1 \). Moreover he investigated the pairs \( \langle A^-, A^+ \rangle \) for which there exists a Darboux function \( f \) such that \( f < 0 \) on \( A^- \) and \( f > 0 \) on \( A^+ \). Similar problems for the family of quasi-continuous functions were examined by Kowalewski in [2]. In this paper we deal with the family of Darboux quasi-continuous functions.

The letters \( \mathbb{N} \) and \( \mathbb{R} \) denote the set of positive integers and the real line, respectively. For all \( a, b \in \mathbb{R} \) we denote by \( \text{I}(a, b) \) (respectively \( \text{I}[a, b] \)) the open (respectively closed) interval with the end-points \( a \) and \( b \). For each set \( A \subset \mathbb{R} \), we use the symbols \( \text{cl} A, \text{int} A \), and \( \text{bd} A \) to denote the closure, the interior, and the boundary of \( A \), respectively. We say that a set \( A \subset \mathbb{R} \) is semi-open, if \( A \subset \text{cl int} A \), and it is semi-closed, if \( \text{int cl} A \subset A \). (See, e.g., [3].) Notice that a set \( A \subset \mathbb{R} \) is semi-open iff its complement is semi-closed.

Let \( f : \mathbb{R} \rightarrow \mathbb{R} \). For each \( a \in \mathbb{R} \) we define \( [f = a] := \{x \in \mathbb{R} : f(x) = a \} \). Similarly we define the sets \( [f > a] \) and \( [f < a] \). We say that \( f \) is Darboux, if it has the intermediate value property. We say that \( f \) is quasi-continuous in the sense of Kempisty [1] at a point \( x \in \mathbb{R} \), if for each open neighborhood \( V \) of \( x \) and each \( \varepsilon > 0 \), there is a nonempty open

---

1 Partially supported by Casimir the Great University in Bydgoszcz.
2 Partially supported by Technical University of Łódź.
set \( G \subset V \) such that \( |f(t) - f(x)| < \varepsilon \) for each \( t \in G \). If \( f \) is quasi-continuous at each point \( x \in \mathbb{R} \), then we say that \( f \) is quasi-continuous. We say that \( f \) is a strong Świątkowski function [4], if for all \( a < b \) and each \( y \in \text{Int}(f(a), f(b)) \) there is a \( t \in (a, b) \cap C_f \) such that \( f(t) = y \), where \( C_f \) stands for the set of all continuity points of \( f \). It is easy to see that strong Świątkowski functions are both Darboux and quasi-continuous.

The proof of the following lemma is immediate.

**Lemma 1.** Let \( U \) be a nonempty open subset of \( \mathbb{R} \). There is a continuous function \( \varphi : U \to (0, 1) \) such that for all \( x \in \text{bd} U \) and \( t \neq x \), if \( I(x, t) \cap U \neq \emptyset \), then

\[ \varphi[I(x, t) \cap U] = (0, 1). \]

First we consider the classical separation property.

**Theorem 2.** Let \( A_0, A_1 \subset \mathbb{R} \) be disjoint. The following are equivalent:

(i) there is a strong Świątkowski function \( f : \mathbb{R} \to \mathbb{R} \) such that

\[ A_0 = [f = 0] \quad \text{and} \quad A_1 = [f = 1]; \]

(ii) there is a Darboux quasi-continuous function \( f : \mathbb{R} \to \mathbb{R} \) such that condition (1) holds;

(iii) the sets \( A_0, A_1 \) are semi-closed, \( \mathbb{R} \setminus (A_0 \cup A_1) \) is bilaterally dense in itself, and

\[ (\forall \alpha \in A_0) \ (\forall \beta \in A_1) \quad \text{Int}(I(\alpha, \beta) \setminus (A_0 \cup A_1)) \neq \emptyset. \]

**Proof.** (i) ⇒ (ii). This implication is evident.

(ii) ⇒ (iii). Let \( f : \mathbb{R} \to \mathbb{R} \) be a Darboux quasi-continuous function which satisfies (1). Since the inverse images of open sets by quasi-continuous functions are semi-open [7], the sets \( A_0 \) and \( A_1 \) are semi-closed. By [6, Corollary 3.2], the set \( \mathbb{R} \setminus (A_0 \cup A_1) \) is bilaterally dense in itself.

Let \( \alpha \in A_0 \) and \( \beta \in A_1 \). Since \( f \) is Darboux, there is a \( t \in I(\alpha, \beta) \setminus (A_0 \cup A_1) \) such that \( f(t) = 2^{-1} \). Since \( g \) is quasi-continuous at \( t \), there is a nonempty open set \( V \subset I(\alpha, \beta) \) such that \( |g(x) - 2^{-1}| < 2^{-1} \) for each \( x \in V \). Hence \( 0 < f(x) < 1 \) for each \( x \in V \), and \( V \subset I(\alpha, \beta) \setminus (A_0 \cup A_1) \).

(iii) ⇒ (i). Put \( U \equiv \mathbb{R} \setminus \text{Cl}(A_0 \cup A_1) \). Let \( \varphi : U \to \mathbb{R} \) be a function constructed according to Lemma 1. Define \( f : \mathbb{R} \to \mathbb{R} \) by the formula

\[
\begin{align*}
f(x) &\equiv \begin{cases}
0 & \text{if } x \in A_0, \\
1 & \text{if } x \in A_1, \\
\varphi(x) & \text{if } x \in U, \\
2^{-1} & \text{otherwise}.
\end{cases}
\end{align*}
\]

Clearly condition (1) is fulfilled. We will show that \( f \) is strong Świątkowski.

Fix \( a < b \) and \( y \in \text{Int}(f(a), f(b)) \subset (0, 1) \). First suppose that

\( (a, b) \subset \text{Cl}(A_0 \cup A_1). \)  \hspace{1cm} (3)

If \( (a, b) \cap \text{Cl} A_0 \neq \emptyset \neq (a, b) \cap \text{Cl} A_1 \), then also \( (a, b) \cap A_0 \neq \emptyset \neq (a, b) \cap A_1 \). Hence by (2),

\[ \emptyset \neq \text{Int}(\{a, b\} \setminus (A_0 \cup A_1)) = (a, b) \setminus \text{Cl}(A_0 \cup A_1), \]

contrary to (3). So, \( (a, b) \cap \text{Cl} A_0 = \emptyset \) or \( (a, b) \cap \text{Cl} A_1 = \emptyset \).

Since \( A_0 \) and \( A_1 \) are semi-closed, we have \( (a, b) \subset A_0 \) or \( (a, b) \subset A_1 \). Let, e.g., \( (a, b) \subset A_0 \). (The other case is analogous.) If \( a \notin A_0 \), then either \( a \in A_1 \) contrary to (2), or \( a \in \mathbb{R} \setminus (A_0 \cup A_1) \) and \( a \) is not the bilateral accumulation point of the set \( \mathbb{R} \setminus (A_0 \cup A_1) \). Consequently \( a \in A_0 \). Analogously we can prove that \( b \in A_0 \). Hence \( \text{Int}(f(a), f(b)) = \emptyset \), which is impossible. So, condition (3) is not fulfilled.

It follows that \( (a, b) \cap U \neq \emptyset \). If \( [a, b] \cap \text{bd} U \neq \emptyset \), then by construction, there is a \( t \in (a, b) \cap C_f \) with \( f(t) = y \). In the opposite case \( [a, b] \subset U \), so \( f \) is continuous on \( [a, b] \). It follows that \( f \) is a strong Świątkowski function. \( \square \)

**Theorem 3.** Let \( A_0 \) and \( A_1 \) be disjoint subsets of \( \mathbb{R} \). The following conditions are equivalent:
(i) there is a strong Šwiątkowski function \( f : \mathbb{R} \to \mathbb{R} \) such that
\[
A_0 \subset [f = 0] \quad \text{and} \quad A_1 \subset [f = 1];
\]
(ii) there is a Darboux quasi-continuous function \( f : \mathbb{R} \to \mathbb{R} \) such that condition (4) holds;
(iii) the sets \( A_0 \) and \( A_1 \) satisfy (2).

**Proof.** (i) ⇒ (ii). This implication is evident.
(ii) ⇒ (iii). Let \( f : \mathbb{R} \to \mathbb{R} \) be a Darboux quasi-continuous function which satisfies (4). Let \( \alpha \in A_0 \) and \( \beta \in A_1 \). Then by Theorem 2,
\[
\text{int}(I(\alpha, \beta) \setminus (A_0 \cup A_1)) \supset \text{int}(I(\alpha, \beta) \setminus ([f = 0] \cup [f = 1])) \neq \emptyset.
\]
(iii) ⇒ (i). Define
\[
B_0 \equiv A_0 \cup \bigcup_{(a,b) \subset cl A_0} cl(a, b) \quad \text{and} \quad B_1 \equiv A_1 \cup \bigcup_{(a,b) \subset cl A_1} cl(a, b).
\]
We will prove that \( B_0 \) and \( B_1 \) fulfill the assumptions listed in Theorem 2(iii).

Observe that \( A_0 \cup cl A_0 \subset B_0 \subset cl A_0 \). So, \( cl B_0 = cl A_0 \) and the set \( B_0 \) is semi-closed. Similarly we can prove that \( cl B_1 = cl A_1 \) and the set \( B_1 \) is semi-closed.

Now we will show that
\[
(\forall \alpha \in B_0) \ (\forall \beta \in B_1) \ \text{int}(I(\alpha, \beta) \setminus (B_0 \cup B_1)) \neq \emptyset. \tag{5}
\]

Let \( \alpha \in B_0 \) and \( \beta \in B_1 \). Assume that, e.g.,
\[
I(\alpha, \beta) \cap A_0 \neq \emptyset \quad \text{and} \quad I(\alpha, \beta) \cap A_1 = \emptyset.
\]
(The other cases are similar.) Take any \( \alpha' \in I(\alpha, \beta) \cap A_0 \). By definition, \( I(\beta, \beta') \subset cl A_1 \) for some \( \beta' \notin I(\alpha, \beta) \). Using condition (2) we obtain
\[
\emptyset \neq \text{int}(I(\alpha', \beta') \setminus (A_0 \cup A_1)) = I(\alpha', \beta') \setminus cl(A_0 \cup A_1)
\]
\[
= I(\alpha', \beta') \setminus cl(B_0 \cup B_1) \subset I(\alpha, \beta) \setminus cl(B_0 \cup B_1) = \text{int}(I(\alpha, \beta) \setminus (B_0 \cup B_1)).
\]
So, condition (5) holds.

Now assume that \((x, t) \subset B_0 \cup B_1\). Then by (5), \((x, t) \subset B_0 \) or \((x, t) \subset B_1\). Assume that the first case holds. (The other case is analogous.) Then by definition, \( x, t \in B_0 \). It follows that the set \( \mathbb{R} \setminus (B_0 \cup B_1) \) is bilaterally dense in itself.

By Theorem 2, there is a strong Šwiątkowski function \( f : \mathbb{R} \to \mathbb{R} \) such that
\[
[f = 0] = B_0 \supset A_0 \quad \text{and} \quad [f = 1] = B_1 \supset A_1.
\]

Now we turn to another separation property.

**Theorem 4.** Let \( A^+ \) and \( A^- \) be disjoint subsets of \( \mathbb{R} \). The following conditions are equivalent:

(i) there is a Darboux quasi-continuous function \( f : \mathbb{R} \to \mathbb{R} \) such that
\[
A^- = [f < 0] \quad \text{and} \quad A^+ = [f > 0]; \tag{6}
\]
(ii) the sets \( A^- \) and \( A^+ \) are semi-open and bilaterally dense in themselves, and
\[
(\forall \alpha \in A^-) \ (\forall \beta \in A^+) \ I(\alpha, \beta) \setminus (A^- \cup A^+) \neq \emptyset. \tag{7}
\]

**Proof.** (i) ⇒ (ii). Let \( f : \mathbb{R} \to \mathbb{R} \) be a Darboux quasi-continuous function which satisfies (6). Since the inverse images of open sets by quasi-continuous functions are semi-open [7], the sets \( A^- \) and \( A^+ \) are semi-open. Since \( f \) is Darboux, the sets \( A^- \) and \( A^+ \) are bilaterally dense in themselves.

Let \( \alpha \in A^- \) and \( \beta \in A^+ \). Then \( f(\alpha) < 0 < f(\beta) \), so there is a \( t \in I(\alpha, \beta) \) such that \( f(t) = 0 \), whence \( t \in I(\alpha, \beta) \setminus (A^- \cup A^+) \).
(ii) \implies (i). Let \( \varphi^- : \text{int} A^- \to \mathbb{R} \) and \( \varphi^+ : \text{int} A^+ \to \mathbb{R} \) be functions constructed according to Lemma 1. Define the function \( f : \mathbb{R} \to \mathbb{R} \) by the formula:

\[
f(x) = \begin{cases} 
-\varphi^-(x) & \text{if } x \in \text{int} A^-, \\
-1 & \text{if } x \in A^- \setminus \text{int} A^-, \\
\varphi^+(x) & \text{if } x \in \text{int} A^+, \\
1 & \text{if } x \in A^+ \setminus \text{int} A^+, \\
0 & \text{if } x \in \mathbb{R} \setminus (A^- \cup A^+).
\end{cases}
\]

Clearly the condition (6) is fulfilled. First we will prove that

\[
\text{if } I[a, b] \not\subseteq A^- \text{ and } I[a, b] \cap A^- \neq \emptyset, \text{ then } f[I(a, b) \cap \text{int} A^-] = (-1, 0), \\
\text{if } I[a, b] \not\subseteq A^+ \text{ and } I[a, b] \cap A^+ \neq \emptyset, \text{ then } f[I(a, b) \cap \text{int} A^+] = (0, 1).
\]

(8)

(9)

Indeed, assume that \( I[a, b] \not\subseteq A^- \) and \( I[a, b] \cap A^- \neq \emptyset \). Since \( A^- \) is bilaterally dense in itself, \( I[a, b] \cap A^- \neq \emptyset \). But \( A^- \) is semi-open, so \( (a, b) \cap \text{int} A^- \neq \emptyset \).

Let \( (c, d) \) be a connected component of \( I[a, b] \cap A^- \). Then \( c \in \text{bd} \text{int} A^- \) or \( d \in \text{bd} \text{int} A^- \). By construction,

\[
f[I(a, b) \cap \text{int} A^-] = (-\varphi^-)((c, d)] = (-1, 0).
\]

The proof of (9) is analogous.

First we verify that \( f \) is Darboux. Let \( a < b \) and \( y \in I(f(a), f(b)) \subset (-1, 1) \). Observe that \( f \) is continuous on \( U \overset{df}{=} \text{int} A^- \cup \text{int} A^+ \cup \text{int}(\mathbb{R} \setminus (A^- \cup A^+)) \). So, we can assume that \( [a, b] \not\subseteq U \). From (8) and (9) we conclude that if \( y \neq 0 \), then there is a \( t \in (a, b) \) such that \( f(t) = y \). If \( y = 0 \), then \( a \in A^- \) and \( b \in A^+ \), or \( a \in A^+ \) and \( b \in A^- \). By (7), there is a \( t \in (a, b) \setminus (A^- \cup A^+) \). By definition, \( f(t) = 0 \). It follows that \( f \) is Darboux.

Now we will show that \( f \) is quasi-continuous. Fix an \( x \in \mathbb{R} \). If \( x \in U \), then \( f \) is continuous at \( x \). So, assume that \( x \not\in U \). Let \( V \) be an open neighborhood of \( x \) and \( \varepsilon > 0 \). We may assume that \( V \) is an interval. If \( f(x) < 0 \), then by (8), the set

\[
G \overset{df}{=} V \cap \text{int} A^- \cap \{|f - f(x)| < \varepsilon\}
\]

is nonempty and open, \( G \subset V \), and \( |f(t) - f(x)| < \varepsilon \) for each \( t \in G \), whence \( f \) is quasi-continuous at \( x \). Similarly we proceed if \( f(x) > 0 \).

So, let \( f(x) = 0 \). Since \( V \not\subseteq [f = 0] \), we have \( V \cap A^- \neq \emptyset \) or \( V \cap A^+ \neq \emptyset \). Hence by (8) or (9), the set

\[
G \overset{df}{=} V \cap (\text{int} A^- \cup \text{int} A^+) \cap \{|f| < \varepsilon\}
\]

is nonempty and open, \( G \subset V \), and \( |f(t) - f(x)| < \varepsilon \) for each \( t \in G \), whence \( f \) is quasi-continuous at \( x \). \( \square \)

**Theorem 5.** Let \( A^- \) and \( A^+ \) be disjoint subsets of \( \mathbb{R} \). The following conditions are equivalent:

(i) there is a Darboux quasi-continuous function \( f : \mathbb{R} \to \mathbb{R} \) such that

\[
A^- \subset [f < 0] \quad \text{and} \quad A^+ \subset [f > 0];
\]

(ii) the sets \( A^- \) and \( A^+ \) satisfy conditions:

\[
(\forall \alpha \in A^-) \ (\forall \beta \in A^+) \quad I(\alpha, \beta) \setminus (A^- \cup A^+) \neq \emptyset, \\
(\forall \alpha \in A^-) \ (\forall \beta \in A^+) \quad \text{int}(I(\alpha, \beta) \setminus A^-) \neq \emptyset \neq \text{int}(I(\alpha, \beta) \setminus A^+).
\]

(10)

(11)

(12)

**Proof.** (i) \implies (ii). Let \( f : \mathbb{R} \to \mathbb{R} \) be a Darboux quasi-continuous function which satisfies (10). Let \( \alpha \in A^- \) and \( \beta \in A^+ \). Since \( f \) is Darboux, there is an \( x \in I(\alpha, \beta) \) such that \( f(x) = 0 \). Then clearly \( x \not\in A^- \cup A^+ \).

Since \( f \) is Darboux, there is a \( \beta' \in I(\alpha, \beta) \) such that \( f(\beta') > 0 \). Since \( f \) is quasi-continuous at \( \beta' \), there is a nonempty open set \( G \subset I(\alpha, \beta) \) such that \( f(t) > 0 \) for each \( t \in G \). It follows that

\[
\emptyset \neq G \subset \text{int}(I(\alpha, \beta) \setminus A^-).
\]

Analogously we can show that \( \text{int}(I(\alpha, \beta) \setminus A^+) \neq \emptyset \).
(ii) $\Rightarrow$ (i). Let $\{I^{-}_n; n < N^{-}\}$, where $N^{-} \subseteq \mathbb{N} \cup \{\infty\}$, be the family of all connected components $I^{-}$ of int cl $A^{-}$ such that $I^{-} \not\subset A^{-}$. For each $n < N^{-}$ choose an isolated set $Z^{-}_n \subset I^{-}_n \setminus A^{-}$ such that
\[
\inf Z^{-}_n = \inf (I^{-}_n \setminus A^{-}) \text{ and } \sup Z^{-}_n = \sup (I^{-}_n \setminus A^{-}).
\] (13)

Similarly, let $\{I^{+}_n; n < N^{+}\}$, where $N^{+} \subseteq \mathbb{N} \cup \{\infty\}$, be the family of all connected components $I^{+}$ of int cl $A^{+}$ such that $I^{+} \not\subset A^{+}$. For each $n < N^{+}$ choose an isolated set $Z^{+}_n \subset I^{+}_n \setminus A^{+}$ such that
\[
\inf Z^{+}_n = \inf (I^{+}_n \setminus A^{+}) \text{ and } \sup Z^{+}_n = \sup (I^{+}_n \setminus A^{+}).
\]

Let $\{(a_n, b_n); n < N\}$, where $N \subseteq \mathbb{N} \cup \{\infty\}$, be the family of all connected components of $\mathbb{R} \setminus \text{cl}(A^{+} \cup A^{-})$. For each $n < N$, let $a_n < c_n < d_n < b_n$, and define the open sets $L^{-}_n$ and $L^{+}_n$ as follows:

- if $a_n \in A^{-}$ and $b_n \in A^{+}$, then $L^{-}_n \overset{\text{def}}{=} (a_n, c_n)$ and $L^{+}_n \overset{\text{def}}{=} (d_n, b_n)$,
- if $a_n \in A^{-}$ and $b_n \notin A^{+}$, then $L^{-}_n \overset{\text{def}}{=} (a_n, c_n) \cup (d_n, b_n)$ and $L^{+}_n \overset{\text{def}}{=} (c_n, d_n)$,
- if $a_n \notin A^{-}$ and $b_n \in A^{+}$, then $L^{-}_n \overset{\text{def}}{=} (a_n, d_n)$ and $L^{+}_n \overset{\text{def}}{=} (a_n, c_n) \cup (d_n, b_n)$,
- if $a_n \notin A^{-}$ and $b_n \notin A^{+}$, then $L^{-}_n \overset{\text{def}}{=} (d_n, b_n)$ and $L^{+}_n \overset{\text{def}}{=} (a_n, c_n)$.

Put
\[
B^{-} \overset{\text{def}}{=} \left( A^{-} \cup \text{int cl } A^{-} \cup \bigcup_{n < N} L^{-}_n \right) \setminus \bigcup_{n < N^{-}} Z^{-}_n,
\]
\[
B^{+} \overset{\text{def}}{=} \left( A^{+} \cup \text{int cl } A^{+} \cup \bigcup_{n < N} L^{+}_n \right) \setminus \bigcup_{n < N^{+}} Z^{+}_n.
\]

It is evident that $\bigcup_{n < N} L^{-}_n \cap B^{-} = \emptyset = B^{-} \cap \bigcup_{n < N} L^{+}_n$. Observe that by (12), $A^{-} \cap B^{+} = \emptyset = B^{-} \cap A^{+}$ and int cl $A^{-} \cap \text{int cl } A^{+} = \emptyset$. We conclude that $B^{-} \cap B^{+} = \emptyset$.

We will prove that $B^{-}$ and $B^{+}$ fulfill the assumptions listed in Theorem 4(ii).

First we will show that these sets are semi-open and bilaterally dense in themselves. Let $x \in B^{-}$. If $x \in \text{int cl } A^{-}$ or $x \in L^{-}_n$ for some $n < N$, then $(x', x) \subset B^{-}$ for some $x' < x$. So, assume that $x \in A^{-}$.

Let $t < x$. If $a_n \in (t, x)$ for some $n < N$, then either $a_n \in A^{-}$ or $x \geq b_n$. By definition, $(t, x) \cap L^{-}_n$ is a nonempty open subset of $(t, x) \cap B^{-}$. If $x = b_n$ for some $n < N$, then $(t, x) \cap L^{-}_n$ is a nonempty open subset of $(t, x) \cap B^{-}$.

Otherwise $(t', x) \subset \text{cl}(A^{-} \cup A^{+})$ for some $t' \in (t, x)$. If $(t', x) \cap A^{+} \neq \emptyset$, then by (12), there is an open interval $I \subset (t', x) \setminus A^{+}$. If $(t', x) \cap A^{-} = \emptyset$, then we can take $I \overset{\text{def}}{=} (t', x)$. Then $I \subset I^{-}_n$ for some $n < N^{-}$, and
\[
\emptyset \neq I \setminus Z^{-}_n \subset (t, x) \cap I^{-}_n \setminus Z^{-}_n \subset (t, x) \cap B^{-}.
\]

We have proved that int $((t, x) \cap B^{-}) \neq \emptyset$ for each $t < x$. Similarly we can show that int $((t, x) \cap B^{+}) \neq \emptyset$ for each $t \neq x$.

Now let $\alpha \in B^{-}$ and $\beta \in B^{+}$. We will prove that $I(\alpha, \beta) \setminus (B^{-} \cup B^{+}) \neq \emptyset$. First assume that we can find $\alpha' \in I(\alpha, \beta) \cap A^{-}$ and $\beta' \in I(\alpha, \beta) \cap A^{+}$. If $(a_n, b_n) \subset (I(\alpha', \beta') \cap (\text{cl} A^{-} \cup \text{cl} A^{+}))$ for some $n < N$, then $c_n \in I(\alpha, \beta) \setminus (B^{-} \cup B^{+})$. Otherwise $I(\alpha', \beta') \subset \text{cl}(A^{-} \cup A^{+})$.

By (11), there is an $x \in I(\alpha', \beta') \setminus (A^{-} \cup A^{+})$. If $x \notin \text{int cl } A^{-} \cup \text{int cl } A^{+}$, then $x \notin B^{-} \cup B^{+}$. If $x \in \text{int cl } A^{-}$, then $x \in I^{-}_n$ for some $n < N^{-}$. By (13), we obtain
\[
I(\alpha', \beta') \setminus (B^{-} \cup B^{+}) \supset I(\alpha', \beta') \cap Z^{-}_n \neq \emptyset.
\]

Similarly, if $x \in \text{int cl } A^{+}$, then $x \in I^{+}_n$ for some $n < N^{+}$, and by (13), we obtain
\[
I(\alpha', \beta') \setminus (B^{-} \cup B^{+}) \supset I(\alpha', \beta') \cap Z^{+}_n \neq \emptyset.
\]

Now assume that $I(\alpha, \beta) \cap A^{-} = \emptyset$. (The case $I(\alpha, \beta) \cap A^{+} = \emptyset$ is analogous.) Then in particular $\alpha \in L^{-}_n$ for some $n < N$. Observe that:

- if $\alpha \in (a_n, c_n)$, then by definition, $a_n \in A^{-}$; it follows that $\alpha < \beta$ and since $\beta \in B^{+}$, we conclude that $c_n \in I(\alpha, \beta) \setminus (B^{-} \cup B^{+})$,
• if $\alpha \in (c_n, d_n)$, then either $c_n \in (\beta, \alpha) \setminus (B^- \cup B^+)$ (in case $\alpha > \beta$) or $d_n \in (\alpha, \beta) \setminus (B^- \cup B^+)$ (in case $\alpha < \beta$),
• if $\alpha \in (d_n, b_n)$, then either $d_n \in (\beta, \alpha) \setminus (B^- \cup B^+)$ (in case $\alpha > \beta$) or $b_n \in (\alpha, \beta) \setminus (B^- \cup B^+)$ (in case $\alpha < \beta$; notice that by definition, $b_n \notin A^+$, and by our assumption, $b_n \notin A^-$).

By Theorem 4, there is a Darboux quasi-continuous function $f : \mathbb{R} \to \mathbb{R}$ such that

$$[f > 0] = B^+ \supset A^+ \quad \text{and} \quad [f < 0] = B^- \supset A^-.$$

The following example is patterned on [5, Proposition III.4.1].

**Example 6.** There exists a Darboux quasi-continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $f : \mathbb{R} \to \mathbb{R}$ is not a strong Świątkowski function whenever

$$[f < 0] \supset [g < 0] \quad \text{and} \quad [f > 0] \supset [g > 0].$$

**Proof.** Let $F$ be the Cantor ternary set, and let $I_1$ and $I_2$ be disjoint families of bounded components of the complement of $F$ such that $F \cup I_1 \cup I_2 = [0, 1]$ and $F = (\text{cl} I_1) \cap (\text{cl} I_2)$. Define

$$g(x) \begin{cases} (-1)^j \frac{|x-c_j|}{r} & \text{if } x \in [c-j, c+r] \text{ and } (c-j, c+r) \in I_j, j \in \{1, 2\}, \\ 1 & \text{otherwise.} \end{cases}$$

It can be readily verified that $f$ is Darboux and quasi-continuous.

Suppose that there exists a strong Świątkowski function $f : \mathbb{R} \to \mathbb{R}$ which satisfies (14). Observe that there is an open interval $J$ such that $J \cap F \neq \emptyset$ and either $f \geq 0$ on $J$ or $f \leq 0$ on $J$.

Indeed, consider the set $A \overset{df}{=} F \cap C_f$. If $A$ is uncountable, then there is an $x \in A$ such that $f(x) \neq 0$. Consequently, there is an open interval $J$ such that $J \cap F \neq \emptyset$ and either $f > 0$ on $J$ or $f < 0$ on $J$.

Otherwise $A$ is an at most countable $G_\delta$ set, so $A$ is nowhere dense in $F$. Hence there is an open interval $J$ such that $J \cap A = \emptyset \neq J \cap F$. Then $g \neq 0$ on $J \cap C_f$, so $0 \notin f(J \cap C_f)$. Since $f$ is strong Świątkowski, we have either $f \geq 0$ on $J$ or $f \leq 0$ on $J$.

By construction, we have $J \cap [g < 0] \neq \emptyset \neq J \cap [g > 0]$ whenever $J$ is an open interval such that $J \cap F \neq \emptyset$. We obtained a contradiction with (14).

The following problem is open.

**Problem 7.** Characterize the pairs $(A^-, A^+)$ of disjoint subsets of $\mathbb{R}$ for which there exists a strong Świątkowski function which satisfies (6).

References