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Topology and its Applications 141 (2004) 1–20
Topology and its Applications
www.elsevier.com/locate/topol

Connected sums of knots and weakly reducible Heegaard splittings ☆

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Received 28 January 2003; received in revised form 18 August 2003

Abstract

This paper studies the question of whether minimal genus Heegaard splittings of exterior spaces of knots which are connected sums are weakly reducible or not. Furthermore it is shown that the Heegaard splittings of the knots used by Morimoto to show that tunnel number can be sub-additive are all strongly irreducible. These are the first examples of strongly irreducible minimal genus Heegaard splittings of composite knots. We also give a characterization of when is a set of primitive annuli on a handlebody simultaneously primitive. This characterization is different from that given in [Gordon, Topology Appl. 27 (1997) 285].

MSC: 57N25

Keywords: Heegaard splittings; Weakly reducible; Strongly irreducible; Connected sum

1. Introduction

For some time it is known that there is a connection between the existence of closed incompressible surfaces in a 3-manifold and the nature of its Heegaard splittings. See for example [1,3,5,7,11,12,14,16]. In this paper we begin to explore this connection with respect to essential surfaces with boundary and as a first step study spaces containing essential annuli. A special case of manifolds which contain essential (i.e., incompressible non-boundary parallel) annuli are exterior spaces of connected sums of knots in S^3. These manifolds are obtained from two knot exterior spaces by gluing them together along a meridional annulus A.

☆ Supported by The Fund for Promoting Research at the Technion grant 100-053.
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0166-8641/$ – see front matter
doi:10.1016/j.topol.2003.08.003

Given a Heegaard splitting for a manifold M (i.e., a decomposition $M = V_1 \cup V_2$, $V_1 \cap V_2 = \Sigma$, where V_i, $i = 1, 2$, are compression bodies and $\Sigma = \partial V_1 = \partial V_2$ is the Heegaard surface) let $\mathcal{C}_i$ denote the set of all essential simple curves on Σ which bound disks in V_i. Define: $d(V_1, V_2) = \min\{d(C_1, C_2) \mid C_i \in \mathcal{C}_i(\Sigma)\}$, where $d(C_1, C_2)$ is measured in the curve complex $\mathcal{C}$ of Σ. In particular a Heegaard splitting will be reducible if $d(V_1, V_2) = 0$, weakly reducible if $d(V_1, V_2) \leqslant 1$ and strongly irreducible if $d(V_1, V_2) \geqslant 2$. Note that any knot exterior of a knot which is a connected sum, contains at least two essential tori. It is a result of Hempel [4] and Thompson [17] that if a manifold contains an essential torus then any Heegaard splitting (V_1, V_2) has $d(V_1, V_2) \leqslant 2$. In general we have a result by Hartshorn [Ha] that if an irreducible 3-manifold M contains a closed incompressible surface of genus g the distance of any Heegaard splitting (V_1, V_2) of M is less than or equal to $2g$.

As the Euler characteristic of an annulus is 0, just like that of a torus, and it is also a twice punctured 2-sphere one might "hope" that the theorem of Hartshorn might be extended to say that an essential annulus in a 3-manifold with torus boundary will imply that $d(V_1, V_2) \leqslant 1$. In other words: Any Heegaard splitting of such a manifold will be weakly reducible. Evidence in this direction is in [5] where the authors describe a very large class of knots in S^3 for which the connected sum yields manifolds with a minimal genus Heegaard splitting which are weakly reducible. In this direction we prove the following theorems:

Theorem 4.1. *Given knots K_1, K_2 and $K = K_1 \# K_2$ in S^3 for which the tunnel number satisfies $t(K) = t(K_1) + t(K_2) + 1$, i.e., $t(K)$ is super additive, then there is a minimal genus Heegaard splitting of $E(K)$ which is weakly reducible.*

Theorem 4.2. *Let K_1, K_2 and $K = K_1 \# K_2$ be knots in S^3 and (V_1^i, V_2^i), $i = 1, 2$, be Heegaard splittings for $E(K_i)$. If (V_1^1, V_2^1) and (V_1^2, V_2^2) induce a Heegaard splitting (V_1, V_2) of $E(K)$ then (V_1, V_2) is a weakly reducible Heegaard splitting.*

In particular this theorem says that if one of $E(K_1)$ or $E(K_2)$ has a μ-primitive minimal genus Heegaard splitting then $E(K_1 \# K_2)$ will have a weakly reducible Heegaard splitting of genus $g_1 + g_2 - 1$, where $g_i = \text{genus}(E(K_i))$.

Theorem 5.3. *Let $K = K_1 \# K_2 \subset S^3$ be a knot. Any Heegaard surface Σ for $E(K)$ which does not contain any Σ horizontal surfaces is weakly reducible.*

Finally:

Theorem 5.6. *Let K_1, K_2 be prime knots in S^3 and $K = K_1 \# K_2$. Assume that $t(K) = t(K_1) + t(K_2)$ and $t(K_i) \leqslant 2$. Furthermore, assume that a minimal tunnel system for K minimaly intersects a decomposing annulus A in a single point, then there is a Heegaard splitting of $E(K)$ of minimal genus which is weakly reducible.*

However the connection between the distance of Heegaard splittings and the existence of an essential annulus is more complicated as shown by the following theorem. Let K_n denote the knots as in [10]:

Theorem 6.1. *Let $K_n \subset S^3$ be the knot as in Fig.* 6 *and $K(\frac{\alpha}{\beta}) \subset S^3$ a 2-bridge knot determined by $\frac{\alpha}{\beta} \subset \mathbb{Q}$. Let K denote the connected sum $K_n \# K(\frac{\alpha}{\beta})$, then the Heegaard splitting of $E(K)$ determined by the minimal tunnel system for K* (*as in Fig.* 6) *is strongly irreducible.*

These are the first examples of strongly irreducible Heegaard splittings of exteriors of connected sums. These knots have the property that $g(E(K_1 \# K_2)) = g(E(K_1)) + g(E(K_2)) - 2$, where $g(\ )$ denotes the genus of the manifold in brackets. Hence a minimal genus Heegaard splitting of $E(K_1 \# K_2)$ cannot possibly be induced by Heegaard splittings of the two knot spaces.

In light of the above I would like to propose the following conjecture:

Conjecture 1.1. *Given two knots K_1, K_2 in S^3 for which the tunnel number $t(K)$ satisfies $t(K_1 \# K_2) = t(K_1) + t(K_2)$, then there is a minimal genus Heegaard splitting of $E(K)$ which is weakly reducible.*

The situation is further complicated by the possibility of a positive answer to the following open question:

Question 1.2. *Can a* 3-*manifold M have both weakly reducible and strongly irreducible minimal genus Heegaard splittings*?

For definitions of the above terminology see Sections 2 and 4.

2. Preliminaries

Throughout the paper K_1 and K_2 will be knots in S^3 and $K = K_1 \# K_2$ will denote the connected sum of K_1 and K_2. The knots K_i will be called the *summands* of the *composite knot* K. Let $N()$ denote an open regular neighborhood in S^3. An incompressible surface in a knot complement $E(K)$, $K \subset S^3$ is called *meridional* if it has boundary components which are meridian curves of $\partial E(K)$.

Recall that (S^3, K) is obtained by removing from each space (S^3, K_i), $i = 1, 2$, a small 3-ball intersecting K_i in a short unknotted arc and gluing the two remaining 3-balls along the 2-sphere boundary so that the pair of points of K_1 on the 2-sphere are identified with the pair of points of K_2. If we denote $S^3 - N(K)$ by $E(K)$ then $E(K)$ is obtained from $E(K_i)$, $i = 1, 2$, by identifying a meridional annulus A_1 on $\partial E(K_1)$ with a meridional annulus A_2 on $\partial E(K_2)$. A knot $K \subset S^3$ is *prime* if it is not a connected sum of two non-trivial knots. The annulus $A_1 = A_2$ will be denoted by A and called the *decomposing annulus*. If both knots K_1, K_2 are prime then the decomposing annulus is unique up to isotopy

A *tunnel system* for an arbitrary knot $K \subset S^3$ is a collection of properly embedded arcs $\{t_1, \ldots, t_n\}$ in $S^3 - N(K)$ so that $S^3 - N(K \cup t_1 \cup \cdots \cup t_n)$ is a handlebody.

Given a tunnel system for a knot $K \subset S^3$ note that the closure of $N(K \cup t_1 \cup \cdots \cup t_n)$ is always a handlebody denoted by V_1 and the handlebody $S^3 - N(K \cup t_1 \cup \cdots \cup t_n)$ will be denoted by V_2. For a given knot $K \subset S^3$ the smallest cardinality of any tunnel system is called the *tunnel number* of K and is denoted by $t(K)$.

A compression body V is a compact orientable and connected 3-manifold with a preferred boundary component $\partial_+ V$ and is obtained from a collar of $\partial_+ V$ by attaching 2-handles and 3-handles, so that the connected components of $\partial_- V = \partial V - \partial_+ V$ are all distinct from S^2. The extreme cases, where V is a handlebody, i.e., $\partial_- V = \emptyset$, or where $V = \partial_+ V \times I$, are allowed. Alternatively we can think of V as obtained from $(\partial_- V) \times I$ by attaching 1-handles to $(\partial_- V) \times \{1\}$. An annulus in a compression body will be called a *spanning* (*or vertical*) *annulus* if it has one boundary component on $\partial_+ V$ and the other on $\partial_- V$.

Given a knot $K \subset S^3$ a *Heegaard splitting* for $E(K)$ is a decomposition of $E(K)$ into a compression body V_1 and a handlebody $V_2 = S^3 - \mathrm{int}(V_1)$. Hence, a tunnel system $\{t_1, \ldots, t_n\}$ in $S^3 - N(K)$ for K determines a Heegaard splitting of genus $n + 1$ for $E(K)$.

When considering knot complements the operation of connected sum is well defined and not dependent on the choice of the removed trivial ball pair (B, t) as any two such ball pairs are isotopic in $E(K)$. However when we are studying the additional structure of Heegaard splittings of composite knot complements we must be careful as it is not clear that an isotopy of the ball pairs can induce an isotopy of the meridional annulus preserving the Heegaard surface.

Given a Heegaard splitting (V_1, V_2) for $S^3 - N(K_1 \# K_2)$ we will choose a decomposing annulus A which intersects the compression body V_1 in two spanning annuli A_1^*, A_2^* and a *minimal* collection of disks $\mathcal{D} = \{D_1, \ldots, D_l\}$. Note also that A intersects V_2 in a connected incompressible planar surface.

Let $\mathcal{E} = \{E_1, \ldots, E_{t(K)+1}\}$ be a complete meridian disk system for V_2, chosen to minimize the intersection $\mathcal{E} \cap A$. Since V_2 is a handlebody it is irreducible and we can assume that no component of $\mathcal{E} \cap A$ is a simple closed curve.

When we cut $E(K)$ along a decomposing annulus A any Heegaard splitting (V_1, V_2) of $E(K)$ induces Heegaard splittings on both of $E(K_1)$ and $E(K_2)$, as follows: Set $V_1^i = (V_1 \cap E(K_i)) \cup_{\mathcal{D} \cup A_1^* \cup A_2^*} N(A)$, it is a compression body as it is a union of an *annulus* $\times I$ and some 1-handles along the two vertical annuli and a collection of disks. Now set $V_2^i = V_2 - N(A)$, it is a handlebody since the annulus A meets V_2 in an incompressible connected planar surface P which separates V_2 into two components each of which is a handlebody. Hence the pair (V_1^i, V_2^i) is a Heegaard splitting for $E(K_i)$ and will be referred to as the *induced Heegaard splitting* of $E(K_i)$.

We say that a curve on a handlebody is *primitive* if there is an essential disk in the handlebody intersecting the curve in a single point. An annulus A on H is primitive if its core curve is primitive. A Heegaard splitting (V_1, V_2) for $S^3 - N(K)$ will be called *μ-primitive* if there is a spanning annulus $A \subset V_1$ such that $\partial A = \mu \cup \alpha$ where μ is a meridian and α is a primitive curve on ∂V_2. Note that a curve on a handlebody H is *primitive* if it represents a primitive element in the free group $\pi_1(H)$.

Two Heegaard splittings (V_1^i, V_2^i) for $E(K_i)$, respectively, induce a decomposition of $E(K)$ into (V_1, V_2). We can think of V_1^i as a union of $(\partial E(K_i) \times I) \cup 1$-*handles*, hence if we consider the ball pair $(B_i, N(t_i))$ and remove it from $E(K_i)$ we can think of the decomposing annulus $A_i = \partial B_i - N(\partial t_i)$ as the union of two vertical annuli A_1^{*i}, A_2^{*i} and a meridional annulus $A_i \subset \partial E(K_i) \times \{1\} \subset \partial V_1^i = \partial V_2^i$. We obtain V_1 by gluing the compression bodies V_1^1 and V_1^2 along the two vertical annuli and V_2 by gluing V_2^1 and V_2^2 along a meridional annulus. Hence V_1 is always a compression body but V_2 is a handlebody if and only if the meridional annulus is a primitive annulus in V_2^i for one of $i = 1$ or $i = 2$. In this case we will say that (V_1, V_2) is the *induced Heegaard splitting of* $E(K)$ induced by (V_1^i, V_2^i), $i = 1, 2$.

3. Interior tunnels

Consider now a Heegaard splitting (V_1, V_2) for $E(K)$ the exterior of $K = K_1 \# K_2$, where $\partial E(K) \subset V_1$ and in which the decomposing annulus A meets V_1 in disks and two vertical annuli. Since the annulus A meets V_2 in a connected planar surface P it separates V_2 into two components each of which is a handlebody. We will denote the handlebodies $\mathrm{cl}(V_2 - A) \cap E(K_i)$ by V_2^i, respectively. However $V_1 - A$ might have many components.

Definition 3.1. A component of $\mathrm{cl}(V_1 - A)$ which is disjoint from $\partial E(K_i)$ and intersects A in n disks will be called an *n-float* (see Fig. 2).

Remark. Note that a n-float is either a 3-ball or a handlebody if its spine is not a tree. Furthermore there are always exactly two components of $\mathrm{cl}(V_1 - A)$ not disjoint from $\partial E(K_i)$ (one in each of $E(K_1)$ and $E(K_2)$) and each one is a handlebody of genus at least one as V_1 is a compression body with a T^2 boundary. We denote these special components by N_1 and N_2 depending on whether they are contained in $E(K_1)$ or $E(K_2)$, respectively.

Consider now any one of the meridian disks $E_i \subset \mathcal{E}$ of V_2. On E_i we have a collection of arcs corresponding to the intersection with the decomposing annulus. These arcs, as indicated in Fig. 1, separate E_i into sub-disks where disks on opposite sides of arcs are contained in opposite sides of A, i.e., in $E(K_1)$ or $E(K_2)$, respectively. So each sub-disk is contained in either $E(K_1)$ or $E(K_2)$. The boundary of these sub-disks is a collection of alternating arcs $\bigcup(\alpha_i \cup \beta_i)$ where α_i are arcs on A and β_i are arcs on some component of $\mathrm{cl}(V_1 - A)$.

Proposition 3.2. *Let K_1 and K_2 be knots in S^3 and let $K, A, \mathcal{E}$ be the connected sum, a minimal intersection decomposing annulus and a meridional system for some Heegaard splitting of $E(K)$ as above. Then*

(a) *the β arc part of the boundary of an outermost sub-disk in E cannot be contained in a n-float of genus* 0.

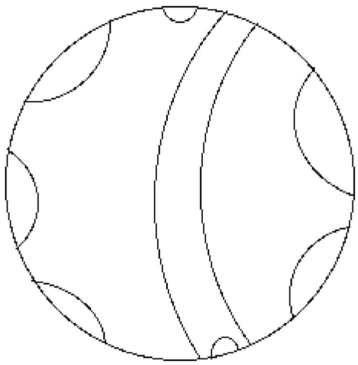


Fig. 1.

(b) *if the β arc part of the boundary of an outermost sub-disk in E is contained in an N_i component, $i = 1$ or 2, and if K_i, $i = 1, 2$, are prime the genus of N_i is greater than one.*

Proof. Denote an outermost sub-disk of some E_j by Δ and suppose it is cut off by an arc α on A. By the "Facts" proved in [9, pp. 41–42], any such outermost arc α must have both end points on a single disk $D_i \subset A$ which belongs to some n-float of genus 0. Furthermore $\alpha \cup D_i \subset A$ must separate the boundary components of A. Assume further that $\partial\Delta = \alpha \cup \beta$ where β is an arc on the n-float meeting D_i in exactly two points $\partial\beta = \partial\alpha$. On ∂D_i there is a small arc γ so that $\gamma \cup \beta$ is a simple closed curve on the n-float bounding a disk D there, since the n-float has no genus (see Fig. 2 below). Furthermore $\gamma \cup \alpha$ is a simple closed curve on A which together with a boundary component of A bounds a sub-annulus of A. Hence $\gamma \cup \alpha$ bounds a disk D' on the decomposing 2-sphere of K intersecting K in a single point. Thus we obtain a 2-sphere $D \cup \Delta \cup D'$ which intersects the knot K in a single point. This is a contradiction finishing case (a).

For case (b), assume that the outermost disk Δ is contained in N_1, say, and that genus of N_1 is one (coming from the fact that it is pierced by the knot). As before we have $\partial\Delta = \alpha \cup \beta$ where β is an arc on N_1 and a small arc γ so that $\gamma \cup \beta$ is a simple closed curve on N_1. If $\gamma \cup \beta$ bounds a disk in N_1 we have the same proof as in case (a). If $\gamma \cup \beta$ does not bound a disk on N_1 we consider small sub-arcs β_1 and β_2 of β which are respective closed neighborhoods of $\partial\beta$. These arcs together with a small arc δ on $\partial N_1 - \partial E(K_1)$ and γ bound a small band b on ∂N_1. Notice that $b \cup_{\beta_1,\beta_2} \Delta$ is an annulus A'. The annulus A' together with the sub-annulus A'' of A cut off by $\alpha \cup \gamma$ defines an annulus $A' \cup_{\alpha\cup\gamma} A''$ which determines an isotopy of a meridian curve in ∂A to a simple closed curve λ on ∂N_1. Note that N_1 is a solid torus and $\pi_1(N_1) = \mathbb{Z}$ which is generated by a meridian μ of $E(K_1)$. Adding a meridian disk to N_1 to obtain N_1' and using the loop theorem on N_1' we can conclude that $[\lambda] = \mu^n \in \pi_1(E(K_1))$. However $[\lambda]$ and μ cobound an annulus in $E(K_1)$. Hence $[\lambda] = \mu \in \pi_1(N_1)$ (see Fig. 3).

Now we can consider the annulus $(A - A'') \cup A'$. If it is non-boundary parallel then since both knots K_1, K_2 are prime it must be a decomposing annulus which has at least one less disk component intersection than A in contradiction to the choice of A. If it is boundary

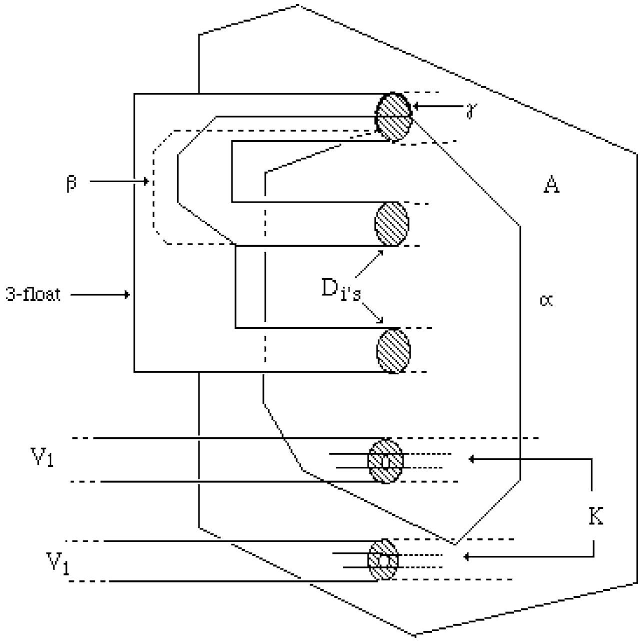


Fig. 2.

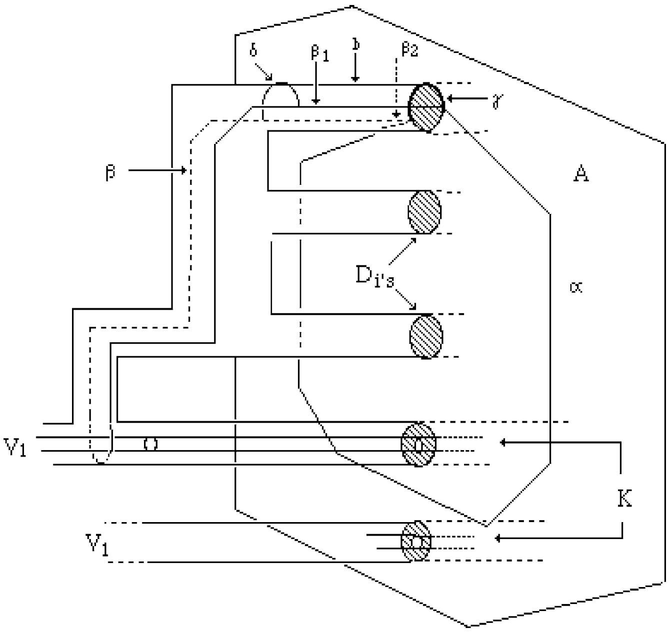


Fig. 3.

parallel, then as above, we have $A'' \cup A'$ as a decomposing annulus with a smaller number of disks. Again in contradiction to the choice of A. So genus N_1 cannot be one and this finishes case (b). $\square$

Corollary 3.3. *Let $K_1, K_2 \subset S^3$ be prime knots. Then every minimal genus Heegaard splitting (V_1, V_2) for $E(K)$, $K = K_1 \# K_2$ has a spine which contains a circle disjoint from a minimal intersection decomposing annulus A for K.*

Proof. Consider a meridional system and a decomposing annulus as in Proposition 3.2. Since the β part of an outer-most disk must be contained in a float of genus greater or equal to one if it is not on N_i or greater or equal to two if it is N_i we must have a 1-*handle* on the float to create the genus. The core arc of this 1-handle genetares the circle in the spine disjoint from the decomposing annulus A. $\square$

4. Super additive and additive knots I

Given knots $K_1, K_2 \subset S^3$ then the knot $K = K_1 \# K_2$ falls into one of three possibilities.

(i) $t(K) = t(K_1) + t(K_2) + 1$,
(ii) $t(K) = t(K_1) + t(K_2)$,
(iii) $t(K) \leqslant t(K_1) + t(K_2) - 1$.

Recall that Heegaard splittings $(V_1^i, V_2^i),\ i = 1, 2$, of $E(K_i)$ induce a Heegaard splitting (V_1, V_2) of $E(K_1 \# K_2)$ if and only if one of $(V_1^i, V_2^i),\ i = 1, 2$, has a primitive meridian. If(V_1, V_2) is induced then we have $t(K) \leqslant t(K_1) + t(K_2)$. Therefore Case (ii) splits into two subcases: (a) (V_1, V_2) is induced by $(V_1^i, V_2^i),\ i = 1, 2$, and (b) (V_1, V_2) is not induced by $(V_1^i, V_2^i),\ i = 1, 2$. In this section we will deal with Case (i) and Case (ii)(a). Case (ii)(b) will be discussed in the next section. In Case (i) we have:

Theorem 4.1. *Given knots K_1, K_2 and $K = K_1 \# K_2$ in S^3 for which the tunnel number satisfies $t(K) = t(K_1) + t(K_2) + 1$, i.e., $t(K)$ is super additive, then there is a minimal genus Heegaard splitting of $E(K)$ which is weakly reducible.*

Proof. No one of the two knots has a Heegaard splitting where the meridian is a primitive element since $t(K) = t(K_1) + t(K_2) + 1$. A primitive meridian would mean that the Heegaard splittings of the knots will induce a Heegaard splitting of the connected sum which would make the tunnel number additive or less. Now drill a tunnel in V_2^i with end points on opposite sides of the meridian curve on ∂V_2^i for one of the knots K_i and add it as a 1-handle to V_1^i thus making the meridian primitive at the expense of increasing the genus by 1. The two Heegaard splittings will now induce a Heegaard splitting on the connected sum which is of genus $t(K) + 1$. It is minimal since $t(K) = g - 1$ and weakly reducible by Proposition 4.2. $\square$

For Case (ii)(a) we have the following theorem:

Theorem 4.2. *Let K_1, K_2 and $K = K_1 \# K_2$ be knots in S^3 and (V_1^i, V_2^i), $i = 1, 2$, be Heegaard splittings for $E(K_i)$. If (V_1^1, V_2^1) and (V_1^2, V_2^2) induce a Heegaard splitting (V_1, V_2) of $E(K)$ then (V_1, V_2) is a weakly reducible Heegaard splitting.*

Proof. We can assume that the decomposing annulus A intersects the Heegaard splitting V_1, V_2 as follows: It intersects V_1 in two vertical annuli and V_2 in one meridional annulus. (This is a consequence of the fact that (V_1, V_2) is induced by the respective Heegaard splittings). Choose two essential disks D_1^1 and D_1^2 for V_1 on both sides of A, for example cocore disks for tunnels. Note that $D_1^1 \subset V_1^1$ and $D_1^2 \subset V_1^2$. The handlebody V_2 is obtained from V_2^1 and V_2^2 by gluing them along the meridional annulus A. Since V_2 turns out to be a handlebody A must be a primitive annulus in at least one of V_2^1 or V_2^2, say V_2^1. So there is at least one essential disk D_2 in V_2^1 which is disjoint from A and hence is also an essential disk in V_2. But D_2 is disjoint from D_1^2 as D_1^2 is also disjoint from A and is on the opposite side. Hence the Heegaard splitting (V_1, V_2) is weakly reducible. □

Remark 4.3. Case (i)(a) is very common indeed, e.g., any two knots which realize a minimal Heegaard splitting in a $2n$-plat projection with the canonical tunnel systems will have a weakly reducible Heegaard splitting when composed (see [5]).

5. Additive knots II

In this section we consider Case (ii)(b): In this case both knots cannot have minimal genus Heegaard splittings with primitive meridians. Knots with this property, called also *fiendish knots*, are very elusive and their existence was first proved in [13] and first examples were given in [15]. The knots considered in both [13] and [15] satisfy $t(K) = t(K_1) + t(K_2) + 1$ so they fall into Case (i). For fiendish knots we have the following conjecture (see also [12, Conjecture 1.5]):

Conjecture 5.1. *Knots $K_1, K_2 \subset S^3$ will satisfy $t(K) = t(K_1) + t(K_2) + 1$ if and only if both $E(K_1)$ and $E(K_2)$ do not have minimal genus Heegaard splittings with primitive meridians.*

Note that Conjecture 5.1 implies Conjecture 1.1. As if Conjecture 5.1 is true then Case (ii)(b) cannot arise as all such knots will be in Case (i) and we are done. Conjecture 5.1 is known for knots which do not contain essential surfaces with meridian boundary components [12, Theorem 1.6]. We have the following:

Definition 5.2. An incompressible meridional surface S in a knot complement $E(K)$ will be called *Σ horizontal* if it is not an annulus and it is contained in a Heegaard surface Σ of $E(K)$ as a sub-surface, except for annuli collar neighborhoods of the meridian boundary components of S. These annuli will have one boundary component on the surface Σ and the other on $\partial E(K)$.

Theorem 5.3. *Let $K = K_1 \# K_2 \subset S^3$ be a knot. Any Heegaard surface Σ for $E(K)$ which does not contain any Σ horizontal surfaces is weakly reducible.*

Proof. Assume in contradiction that (V_1, V_2) is a strongly irreducible Heegaard splitting for $E(K_1 \# K_2)$. Let $\Sigma = \partial V_1 = \partial V_2$ be the Heegaard surface and let A be the decomposing annulus for the connected sum minimizing the intersection with Σ. We can assume (see [11, Lemma 2.3]) that after an isotopy of the annulus $A \cap \Sigma$ is a collection of essential curves on both A and Σ. Hence, as we assumed that V_1 is the compression body containing $\partial E(K_1 \# K_2)$ then $V_1 \cap A$ is composed of two vertical annuli A_1^*, A_2^* and a minimal collection of essential annuli $A_1, \ldots, A_d$ and $V_2 \cap A$ is composed of a minimal collection of essential annuli $B_1, \ldots, B_{d+1}$. By Lemma 2.1 of [11] we can find essential disks D_1, D_2 in V_1, V_2, respectively, which are disjoint from $A_1, \ldots, A_d$ and $B_1, \ldots, B_{d+1}$. Since A_1^*, A_2^* share a boundary component with B_1 and B_{d+1} we can conclude that the disks D_1, D_2 are disjoint from A. The annulus A splits each of V_1 and V_2 into two unions of handlebodies $\bigcup_r V_{1,r}^i$ and $\bigcup_s V_{2,s}^i$, respectively, where $i = 1, 2$ depending if the component is in $E(K_1)$ or $E(K_2)$. If the disks D_1, D_2 are contained in $V_{1,r}^i$ and $V_{2,s}^j$, respectively, $i, j \in \{1, 2\}$, for different values of i and j then $\partial D_1 \cap \partial D_2 = \emptyset$ as both of D_1 and D_2 are disjoint from A. Hence the Heegaard splitting (V_1, V_2) is weakly reducible in contradiction. So we can assume that both of D_1 and D_2 are contained in $V_{1,r}^i$ and $V_{2,s}^i$ for the same i, say $i = 1$, i.e., on the same side of A. Consider now the components of $\Sigma - A$ contained in V_1^2 and V_2^2. An innermost disk argument shows that each of these components must be incompressible in V_1^2 and V_2^2 as otherwise we obtain a compressing disk D_3 disjoint from A which is disjoint from both D_1 and D_2 and hence the Heegaard splitting (V_1, V_2) is weakly reducible in contradiction. The boundary curves of any component of $\Sigma - A$ contained in V_1^2 and V_2^2 are essential curves on the meridional decomposing annulus A and hence are isotopic to meridian curves in $E(K_2)$. Therefore they are isotopic to meridian curves in $E(K)$. Thus these components of $\Sigma - A$ are horizontal surfaces. Since we assumed that such surfaces do not exist in $E(K)$ we obtain a contradiction to our assumption that (V_1, V_2) is a strongly irreducible Heegaard splitting of$E(K)$. □

Remark 5.4. A result of similar nature is mentioned by Morimoto (see [11, Remark 4.3]): If $K_i \subset M_i$ are knots then $E(K_1 \# K_2)$ always has a weakly reducible Heegaard splitting of minimal genus if none of M_1 and M_2 have Lens space summands and none of $E(K_1)$ and $E(K_2)$ contains meridional essential surfaces. It seems that the conditions in Theorem 5.3 are weaker.

We will now specialized to the situation where there is a tunnel system for K with a single tunnel minimally intersecting the decomposing annulus in a single point. More precisely: $E(K)$ has a minimal genus Heegaard splitting so that $t(K) = t(K_1) + t(K_2)$ and $V_1 \cap A$ consists of two spanning annuli and a single disk. This is clearly a subset of Case (ii)(b). However to the best of my knowledge all examples of minimal tunnels systems of composite knots which have tunnels intersecting the decomposing annulus essentially do so exactly once.

Before we specialize we need the theorem below which is true in a more general setting. It is of independent interest as it gives a new characterization for when a set of primitive curves on a handlebody is simultaneously primitive (compare [2]).

Given a collection of annuli $A_1, \ldots, A_n$ on the boundary of a handlebody H we say that they are *simultaneously primitive* if there exists a collection $D_1, \ldots, D_n$ of disjoint essential disks so that $D_i \cap A_i$ is a single essential arc in A_i and if $i \neq j$ then $D_i \cap A_j = \emptyset$.

Theorem 5.5. *Let H_1 and H_2 be two handlebodies and let $B_1, \ldots, B_n$ be a set of disjoint mutually non-parallel incompressible primitive annuli in ∂H_1. Let $C_1, \ldots, C_n$ be any collection of incompressible non-primitive disjoint annuli in ∂H_2. Then $B_1, \ldots, B_n$ are simultaneously primitive in H_1 if and only if $H_1 \cup_{\{B_1=C_1,\ldots,B_n=C_n\}} H_2$ is a handlebody.*

Proof. Assume first that the annuli $B_1, \ldots, B_n$ are simultaneously primitive in H_1. The proof will be by induction on n. For $n = 1$ we can glue H_1 to H_2 along B_1 and C_1 to obtain a manifold N_1. Since the annuli B_1 and C_1 are incompressible we have that $\pi_1(N_1) = \pi_1(H_1) *_{\mathbb{Z}} \pi_1(H_2)$. The generator of the $\mathbb{Z}$ is a primitive element in the free group $\pi_1(H_1)$ so $\pi_1(N_1)$ is a free group. It now follows from the Loop Theorem that N_1 is a handlebody. Assume by induction that $N_{n-1} = H_1 \cup_{\{B_1=C_1,\ldots,B_{n-1}=C_{n-1}\}} H_2$ is a handlebody. The annulus B_n is disjoint from the annuli $B_1, \ldots, B_{n-1}$ and $C_1, \ldots, C_n$ and is still primitive in N_{n-1} as the annuli $B_1, \ldots, B_n$ are simultaneously primitive and non-parallel and hence there is an essential disk D in N_{n-1} which is disjoint from $B_1, \ldots, B_{n-1}$ and $C_1, \ldots, C_n$ and which intersects B_n in a single arc. Now N_n is obtained from N_{n-1} by gluing the primitive annulus B_n to the annulus C_n. Hence $\pi_1(N_n) = \pi_1(H_{n-1}) *_{\mathbb{Z}}$ is an HNN extension of the free group $\pi_1(H_{n-1})$ where two $\mathbb{Z}$-subgroups are identified and the generator of one of them is a primitive element. It follows that $\pi_1(N_n)$ is a free group and again by the Loop Theorem $N_n = H_1 \cup_{\{B_1=C_1,\ldots,B_n=C_n\}} H_2$ is a handlebody.

For the proof in the other direction: Assume that $H_1 \cup_{\{B_1=C_1,\ldots,B_n=C_n\}} H_2$ is a handlebody H and let $B = \{B_1, \ldots, B_n\}$ and $C = \{C_1, \ldots, C_n\}$ be as in the theorem. Let H_1' be the result of cutting H_1 along a maximal set of compression disks of $\partial H_1 - \bigcup B_i$. Note that gluing H_1' to H_2 along B and C yields a handlebody. As it is obtained from the handlebody H by cutting it along disks which are disjoint from both of B and C. Up to relabeling we may assume that $B' = \{B_1, \ldots, B_k\}$ is the set of annuli in B which are a longitudinal annulus of some solid torus component V_i, $i = 1, \ldots, k$, of H_1' containing no other B_j. Denote by $H_1'' = H_1' - \bigcup V_i$, and let $B'' = B - B'$. There is no compressing disk in H_1'' intersecting B'' in a single essential arc. As any such disk would define another torus components V_j containing some annulus B_j, $j \notin \{1, \ldots, k\}$, and no other annulus.

Let C' and C'' be the corresponding subsets of C. Then $H_1' \cup_{B=C} H_2$ can be obtained by gluing $V_1, \ldots, V_k$ to H_2 along B' and C' to obtain a manifold H_2', and then gluing H_1'' to H_2' along B'' and C''. The manifold H_2' is a handlebody and is homeomorphic to H_2 by the definition of B' and the first part of the theorem. Hence the annuli C'' are still non-primitive annuli on $\partial H_2'$. If B is not simultaneously primitive then B'' is non-empty, hence after gluing the remaining components of H_1' to H_2', the surface $B'' = C''$ is an essential surface in the handlebody $H_1' \cup H_2 = H_1'' \cup H_2'$ because there is no compressing or boundary compressing disk for this surface, which contradicts the fact that there are no essential non-disk surfaces in a handlebody. □

Further evidence in the direction of Conjecture 1.1 is the following:

Theorem 5.6. *Let K_1, K_2 be prime knots in S^3 and $K = K_1 \# K_2$. Assume that $t(K) = t(K_1) + t(K_2)$ and $t(K_i) \leqslant 2$. Furthermore, assume that a minimal tunnel system for K minimaly intersects a decomposing annulus A in a single point, then there is a Heegaard splitting of $E(K)$ of minimal genus which is weakly reducible.*

Proof. Let (V_1, V_2) be the Heegaard splitting of $E(K)$ determined by the minimal tunnel system which intersects the decomposing annulus A in a single point. We can therefore assume that $V_1 \cap A = A_1^* \cup A_2^* \cup D_1$. The once punctured annulus $A \cap V_2$ has two boundary components coming from the vertical annuli A_1^*, A_2^* and denoted by C_1^*, C_2^*, respectively, and one boundary component ∂D_1 coming from the tunnel. As A intersects V_1 minimally $A - D_1$ is an incompressible planar surface in a handlebody and hence is boundary compressible. A boundary compression cannot be on an arc connecting C_i^*, $i = 1, 2$, to ∂D_1 as then we could use the compressing disk to isotope the tunnel off A. Such an arc will be called of *type I*. Furthermore a boundary compression cannot be on an arc connecting C_1^* to C_2^* as then A will be boundary parallel in contradiction. Hence the boundary compressing arc will connect ∂D_1 to itself and since it is non-trivial it must separate C_1^* and C_2^*. Such an arc will be called of *type II* (Compare also [9]).

Choose a meridional system of disks $\mathcal{E} = E_1, \ldots, E_{t(K)+1}$ for V_2. Each disk in $\mathcal{E}$ must intersect D_1 as otherwise the Heegaard splitting will be weakly reducible and we are done. An outermost arc of intersection α on some E_i separates a boundary compressing sub-disk $\Delta \subset E_i$ and from the previous paragraph α is an arc of type II on A.

We can boundary compress A along Δ or alternatively isotope $\partial V_1 = \partial V_2$ along Δ. Doing the second operation does not change A or the isotopy class of the Heegaard splitting (V_1, V_2), but does change the intersection of the “new” Heegaard surface, also denoted by $\partial V_1 = \partial V_2$, with A. The result is that now $A \cap V_1 = A_1^* \cup A_2^* \cup A_1$, where A_1 is an essential sub-annulus of A which contains the disk D_1. The intersection $A \cap V_2 = B_1 \cup B_2$, where B_1, B_2 are also essential sub-annuli of A (as in Fig. 4).

Let V_1^i denote the components of $V_1 - A$ and V_2^i denote the components of $V_2 - A$. Assume that the disk Δ is contained in $E(K_k)$, $k = 1$ or $k = 2$. Note that isotoping the Heegaard surface ∂V_1 along Δ changes the *induced* Heegaard splitting only on the knot complement containing Δ (i.e., on $E(K_k)$ only!). On the induced Heegaard splitting of $E(K_k)$ this isotopy is equivalent to cutting V_2^k along Δ to obtain W_2^k and adding the 2-handle $N(\Delta)$ to V_1^k to obtain W_1^k. It is possible that in this case W_1^k might not be a handlebody. It is also possible that Δ is a separating disk in V_2^k and in this case, W_2^k might have two components $W_2^{k,1}$ and $W_2^{k,2}$.

The annuli B_1, B_2 are essential annuli contained in V_2 which together separate V_2. Hence, when we cut V_2 along them we obtain a handlebody W_2^j, $j \neq k$, and if neither of B_1 or B_2 is separating a handlebody W_2^k. If one of B_1 or B_2 is separating then $V_2 \cap E(K_k)$ splits into two handlebodies $W_2^{k,1}$ and $W_2^{k,2}$ This is the situation corresponding to the disk Δ being a separating disk in V_2^k. Denote the “traces” of B_1 and B_2 on W_2^i by B_1^i, B_2^i, $i = 1, 2$.

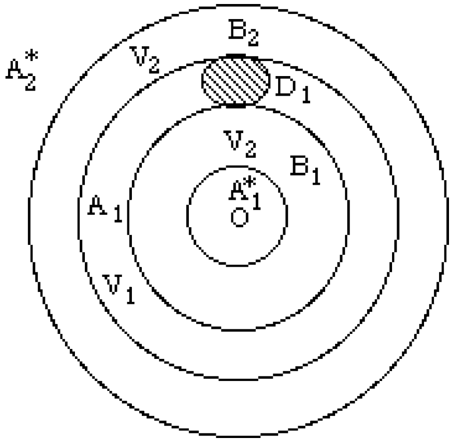


Fig. 4.

Since $t(K) = t(K_1) + t(K_2)$ and only one tunnel gets split into two arcs by cutting along A it follows that after cutting (V_1, V_2) along A there are two possibilities: The induced Heegaard splitting (V_1^1, V_2^1) of $E(K_1)$ is of minimal genus and (V_1^2, V_2^2) of $E(K_2)$ is of minimal genus plus one or vice versa. Up to relabeling the knots we assume that $E(K_1)$ is of minimal genus.

Claim 1. *If one of B_1^1 or B_2^1 is primitive in $W_2^1 \subset E(K_1)$ then $E(K)$ has a weakly reducible Heegaard splitting of minimal genus.*

Proof of Claim 1. If the disk Δ is contained in $E(K_2)$ then (W_1^1, W_2^1) is a Heegaard splitting of minimal genus for $E(K_1)$. So, if either B_1^1 or B_2^1 is a primitive annulus on W_2^1 (which, in this case, is equal to V_2^1 less a collar) we will treat an isotopic image of the primitive annulus B_1^1 or B_2^1 respectively on $\partial E(K_1)$ as a decomposing annulus. Now glue $E(K_1)$ to $E(K_2)$ along this annulus to obtain a Heegaard splitting of $E(K)$ which is of minimal genus (as that of (V_1, V_2)) and is weakly reducible by Theorem 4.2.

If, on the other hand, the disk Δ is contained in $E(K_1)$ then recall that we obtain V_2^1 from W_2^1 by identifying together the two "traces" (copies) of the disk Δ on W_2^1, i.e., adding a 1-handle to these traces. These traces intersect both of ∂B_1^1 and ∂B_2^1 in a single arc each. Hence if one of B_1^1 or B_2^1 is primitive in W_2^1 it would also be primitive in V_2^1, regardless of whether B_1^1 and B_2^1 are separating or not. We now use the same argument as above to obtain a weakly reducible Heegaard splitting of the same genus as that of (V_1, V_2) of $E(K)$. □

Thus we can assume that both of the annuli B_1^1 and B_2^1 are not primitive in $W_2^1 \subset E(K_1)$. Since $V_2 = W_2^2 \cup_{B_1^2 = B_1^1, B_2^2 = B_2^1} W_2^1$ is a handlebody it follows that B_1^2 and B_2^2 must be primitive in W_2^2: By setting $B_1 = B_1^2$, $B_2 = B_2^2$ and $C_1 = B_1^1$, $C_2 = B_2^1$ we satisfy the conditions of Theorem 5.5 and can conclude that B_1^2 and B_2^2 are simultaneously primitive in W_2^2. If it happens that W_2^2 has more than one component we certainly have disjoint annuli intersecting disjoint disks in a single arc. We will refer to this situation as the annuli being *extended simultaneously primitive*.

Claim 2. *If B_1^2, B_2^2 are simultaneously primitive or extended simultaneously primitive on $W_2^2 \subset E(K_2)$ the complement with the non-minimal genus Heegaard splitting, then $E(K)$ is a weakly reducible Heegaard splitting of minimal genus.*

Proof of Claim 2. The induced Heegaard splitting of $E(K_2)$ is not of minimal genus, thus it is of genus at least three (It is induced by a tunnel system containing at least two tunnels, i.e., one "interior tunnel" (by Corollary 3.3) and the "half" tunnel coming from the split tunnel crossing A). Assume that the disk Δ is contained in $E(K_2)$ so after cutting V_2^2 along Δ we obtain either a handlebody of genus at least two with two simultaneously primitive annuli on it or a disjoint union of two handlebodies one of which has at least genus two with two extended simultaneously primitive annuli on them.

Thus in both cases there is at least one essential disk D_2 in W_2^2 (a separating disk in the first case), which is disjoint from B_1^2 and B_2^2 and hence from A. Since $V_2 = W_2^2 \cup_{B_1^2 = B_1^1, B_2^2 = B_2^1} W_2^1$, as before, the disk D_2 is an essential disk in V_2 which is disjoint from A and hence from the essential disk $D_1^* \subset V_1$ which is the image of the disk D_1 pushed slightly into $E(K_1)$. Thus the Heegaard splitting (V_1, V_2) of $E(K)$ is weakly reducible and we are done (see Fig. 5).

Assume therefore that the disk Δ is contained in $E(K_1)$. If $t(K_1) = 1$ then V_2^1 is a genus two handlebody and after cutting V_2^1 along Δ we obtain either one or two solid tori (depending if Δ is separating or not) embedded in S^3 with non-primitive annuli on their boundary. Extend these annuli into V_1^1 so that one boundary component of each annulus is a meridian curve on $\partial E(K_1)$. When attaching disks to these meridional curves one obtains a Lens space contained in S^3, which is a contradiction.

If $t(K_1) = 2$ then V_2^1 is a genus three handlebody and after cutting V_2^1 along Δ we obtain either one solid torus component with one or two non-primitive annuli on its

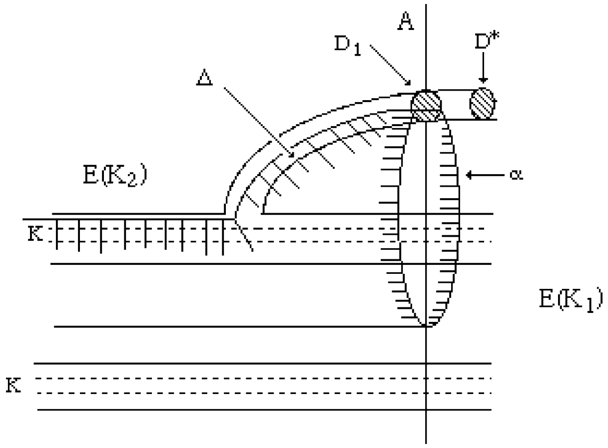


Fig. 5.

boundary (if Δ is separating) or a genus two handlebody with two non-primitive annuli on its boundary (if Δ is non-separating). In both cases the components are embedded in S^3. The first case is dealt with as in the previous paragraph. In the second case, first note that the genus three Heegaard splitting (V_1^1, V_2^1) for $E(K_1)$ induces, by filling meridional disks, a Heegaard splitting for S^3. Now after adding to V_2 two meridional disks along the annuli and cutting along Δ we obtain a 2-sphere $S \subset (S^3, K_1)$ which intersects K_1 in four points. In particular S bounds a 3-ball on both sides. If we change the order of cutting along Δ and adding disks by first adding the two meridional disks to the meridional annuli on V_2^1 we obtain a solid torus W_2 with Δ as its unique meridional disk.

The complement $W_1 = S^3 - W_2$ can be obtained from the genus three compression body V_1^1 as follows: Fill $\partial_- V_1^1$ with $N(K_1)$ to get a pair (V, K_1). Now cut the pair (V, K_1) along meridional disks corresponding to the meridional annuli on ∂V_1^1. These annuli are not parallel on ∂V_1^1 so we get a solid torus W_1 whose unique meridian disk Δ' is a cocore disk of one of the 1-handles of V_1^1 and is therefore disjoint from K_1.

Now since we obtained S from W_1 and W_2 the disks Δ and Δ' are a canceling pair. But this implies that the minimal genus Heegaard splitting (V_1^1, V_2^1) is reducible in contradiction. Hence this case cannot happen and the proof of Claim 2 is complete. □

This completes the proof of the theorem. □

6. Sub-additive knots

In this section we consider connected sums of knots $K_n \subset S^3$ as in Fig. 6 and 2-bridge knots $K(\frac{\alpha}{\beta}) \subset S^3$ determined by $\frac{\alpha}{\beta} \subset \mathbb{Q}$. These are the only examples so far of prime knots $K_1, K_2 \subset S^3$ so that $t(K) = t(K_1) + t(K_2) - 1$. For these examples we have:

Theorem 6.1. *Let $K_n \subset S^3$ be the knot as in Fig.* 6 *and $K(\frac{\alpha}{\beta}) \subset S^3$ a* 2*-bridge knot determined by $\frac{\alpha}{\beta} \subset \mathbb{Q}$. Let K denote the connected sum $K_n \# K(\frac{\alpha}{\beta})$, then the Heegaard splitting of $E(K)$ determined by the minimal tunnel system for K* (*as in Fig.* 6) *is strongly irreducible.*

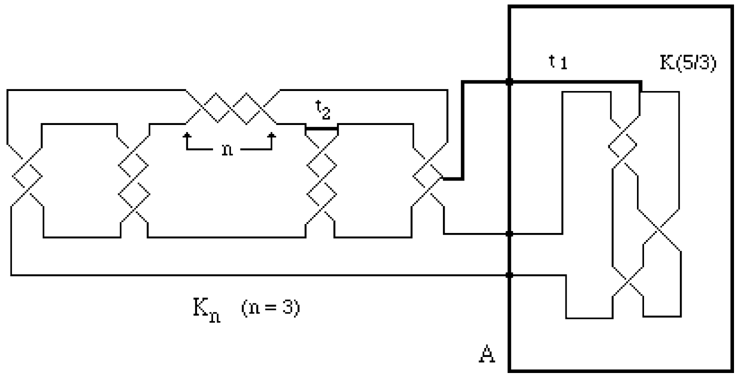


Fig. 6.

In Fig. 6, A denotes the decomposing annulus and t_1, t_2 denote the unknotting tunnels.

Proof. Since $E(K(\frac{\alpha}{\beta}))$ has a genus two Heegaard splitting (as $K(\frac{\alpha}{\beta})$ is a tunnel number one knot) and is irreducible, the Heegaard splitting is strongly irreducible. Otherwise we could compress the Heegaard surface to both sides and obtain an essential 2-sphere in contradiction. Similarly any Heegaard splitting of minimal genus three of a hyperbolic knot is strongly irreducible: As the knot complement is irreducible we can compress at most twice (once to each side). But then, by compressing the Heegaard surface we obtain an incompressible non-boundary parallel torus in contradiction to the fact that the knot is hyperbolic (see [8]). The knots K_n are alternating knots and not torus knots so by Corollary 1 of [6] they do no contain incompressible non-boundary parallel tori and hence are hyperbolic.

Note that $E(K)$ induces minimal genus Heegaard splittings, of genus two and three respectively, on both of $E(K(\frac{\alpha}{\beta}))$ and $E(K_n)$. By slightly abusing notation we will denote the components of $E(K) - A$ by $E(K(\frac{\alpha}{\beta}))$ and $E(K_n)$.

As in Fig. 7 let D denote the cocore disk of the tunnel t_1 which intersects the decomposing annulus A and let D' denote the cocore disk of t_2 the tunnel interior to $E(K_n)$. We can choose the disks $\mathcal{E} = \{D, D'\}$ as a meridional system of disks for the compression body V_1. Note also that A minimizes the intersection with V_1 as if $A \cap V_1 = \emptyset$ the Heegaard genus of $E(K)$ would be additive and equal to three.

Let F be the Heegaard splitting surface $\partial V_1 = \partial V_2$, and let $F_1 = F \cap E(K(\frac{\alpha}{\beta}))$, and $F_2 = F \cap E(K_n)$. For each essential disk $\mathcal{D}_1$, $\mathcal{D}_2$ in V_1 and V_2 respectively, we choose a representative in their isotopy class so that $\mathcal{D}_i \cap A$ is minimal; in particular, each component of $\partial\mathcal{D}_i \cap F_j$, $i, j \in \{1, 2\}$, is an essential circle or essential arc on F_j, and each component of $\mathcal{D}_i \cap A$ is an arc.

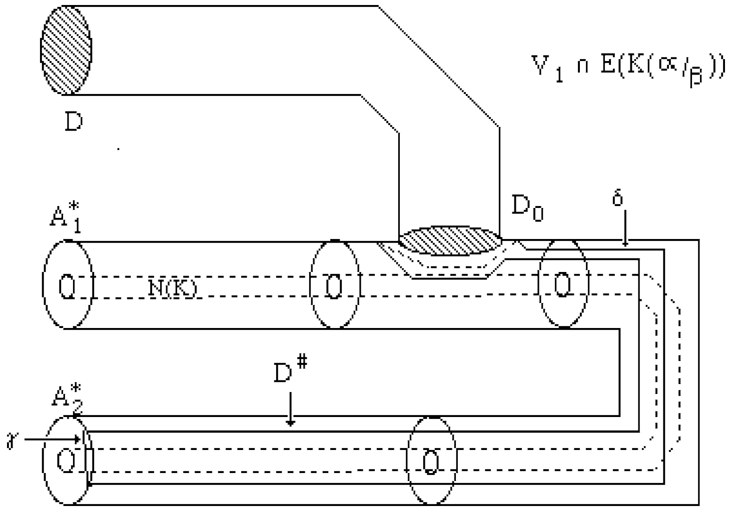


Fig. 7.

Claim. *Let E be an essential disk in V_1 minimizing the intersection with A in its isotopy class. Then*:

(a) *If $E \cap A \neq \emptyset$ then the outermost sub-disk $E^\#$ of $E - A$ is an essential disk in the components $V_1^2 \subset E(K_n)$ or $V_1^1 \subset E(K(\frac{\alpha}{\beta}))$ of $V_1 - A$, depending on which side of A contains $E^\#$. If it is in $E(K(\frac{\alpha}{\beta}))$ then $\partial E^\# = \gamma \cup \delta$ where γ is an inessential arc on one of the vertical annuli A_i^* and δ is an arc on $\partial V_1 - A$ as indicated in Fig. 7.*

(b) *If $E \cap A = \emptyset$ and E is contained in the $E(K(\frac{\alpha}{\beta}))$ component then E is parallel to D.*

Proof. (a) Note that $\partial E^\#$ is the union of two arcs $\gamma \subset A$ and δ. If $E^\#$ is inessential we could isotope $E^\#$ off A. This is a contradiction to the choice of E.

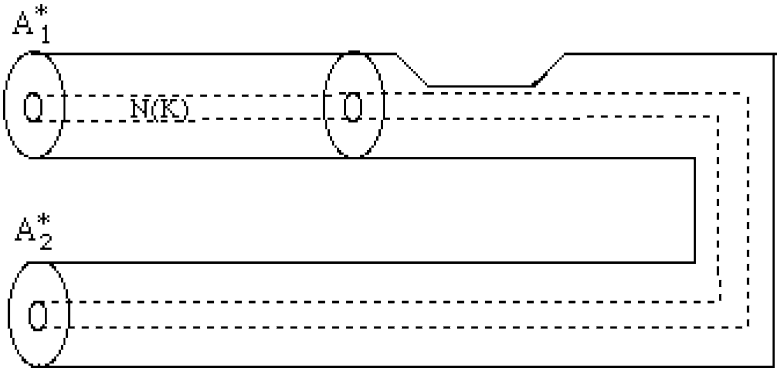
Assume now that $E^\#$ is contained in $E(K(\frac{\alpha}{\beta}))$. Note that $V_1 \cap E(K(\frac{\alpha}{\beta}))$ is a solid torus whose fundamental group is generated by a meridian of $E(K)$. If the curve γ is also contained in the disk D then $\partial E^\#$ is isotopic to a curve which represents a power of the meridian in $\pi_1(E(K))$ which is a contradiction as the meridian has infinite order in $\pi_1(E(K))$. So $E^\# \cap D = \emptyset$. Consider now the disk D_0 which is the intersection of $N(t_1)$ with the component of $N(\partial E(K)) - A$ contained in $E(K(\frac{\alpha}{\beta}))$. If $E^\# \cap D_0 = \emptyset$ then since $E^\# \cap \partial E(K) = \emptyset$ the disk $E^\#$ is an inessential disk in this component of $V_1 - A$ which is a solid torus. If $E^\# \cap D_0 \neq \emptyset$ then since this solid torus is irreducible we can reduce the intersection by isotoping $E^\#$ off the neighborhood of the half tunnel until $E^\#$ is isotopic to D_0.

(b) If E is contained in the component $E(K(\frac{\alpha}{\beta})) \cap V_1$ then, as above, since it is in the component of $V_1 - A$ which is a solid torus and cannot intersect A it is isotopic to D_0 which is parallel to D. $\square$

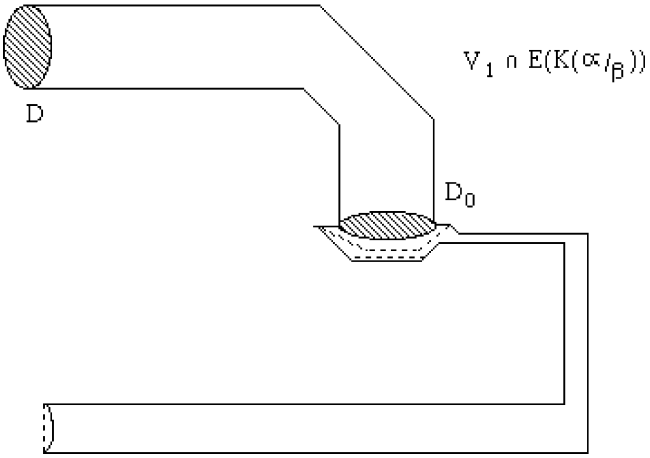
Assume in contradiction that the Heegaard splitting (V_1, V_2) is weakly reducible and let $\mathcal{D}_1, \mathcal{D}_2$ be a pair of essential disks in V_1 and V_2 respectively, so that $\mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset$. As the single component of $E(K(\frac{\alpha}{\beta}) \cap V_1)$ is of type $N_{K(\frac{\alpha}{\beta})}$, in the terminology of Proposition 3.2(b), of genus one it follows from Proposition 3.2 that all outermost disks of $\mathcal{D}_2 \cap A$ are in $E(K_n)$. Note, further, that (V_1, V_2) induces the original Heegaard splitting on $E(K_n)$ which is strongly irreducible by the first paragraph of the proof.

If the disk $\mathcal{D}_1 \cap A = \emptyset$ then it is either contained in $E(K_n)$ or parallel to the disk D: As if it is not in $E(K_n)$ it must be a non-essential disk in the solid torus V_1^1 and these are parallel to D. In the first case it is essential in the strongly irreducible induced Heegaard splitting on $E(K_n)$ and so must intersect the outermost sub-disks of any essential disk $\mathcal{D}_2 \subset V_2$: Note that all outermost sub-disks of V_2 which are contained in $E(K_n)$ are essential disks in the strongly irreducible Heegaard splitting induced on $E(K_n)$. In the second, case as all outermost sub-disks of V_2 intersect the parallel copy of $D \subset E(K_n)$ it follows that the corresponding disks of V_2 must run through the annulus A and intersect $D = \mathcal{D}_1$.

If the disk $\mathcal{D}_1 \cap A \neq \emptyset$ then assume first, that the outermost sub-disk $\mathcal{D}^\# \subset \mathcal{D}_1$ is in the $E(K_n)$ component of $E(K) - A$. By the above claim $\mathcal{D}^\#$ is an essential disk there. Since the induced Heegaard splitting on $E(K_n)$ is strongly irreducible any two outermost sub-disks of $\mathcal{D}_1$ and $\mathcal{D}_2$ in $E(K_n)$ must intersect.



(a)



(b)

Fig. 8.

If the outermost sub-disks of $\mathcal{D}_1$ are in $E(K(\frac{\alpha}{\beta}))$ then by the claim above if we cut this component of V_1 along $\mathcal{D}^\#$ we obtain two components one of which is a solid torus and the other is a 3-ball $\mathcal{B}$ (see Fig. 8(a) and (b)).

Consider now an essential disk $\mathcal{D}_2$ in V_2. If $\mathcal{D}_2 \cap A = \emptyset$ then $\mathcal{D}_2$ is an essential disk in V_2^1 or V_2^2, the two components of $V_2 - A$, depending on which side of A the disk $\mathcal{D}_2$ is. Hence $\mathcal{D}_2$ is an essential disk in the handlebody part of the induced Heegaard splitting on either $E(K(\frac{\alpha}{\beta}))$ or $E(K_n)$. However these Heegaard splittings are strongly irreducible so $\mathcal{D}_2$ must intersect D the cocore disk of t_1 as it is an essential disk in the corresponding V_1^1 or V_1^2. This implies that $\mathcal{D}_2$ must intersect the decomposing annulus which is a contradiction. Hence $\mathcal{D}_2 \cap A$ is non-empty.

Let $\mathcal{D}^* \subset \mathcal{D}_2$ be a sub-disk, which is outermost among all sub-disks of $\mathcal{D}_2 - A$ which are contained in the $E(K(\frac{\alpha}{\beta}))$ component of $E(K) - A$. Let $\alpha_1, \ldots, \alpha_n$ be the components of $\mathcal{D}^* \cap A$, then for all but one, say α_1, the arcs α_i are ourtermost arcs of D_2 and hence are

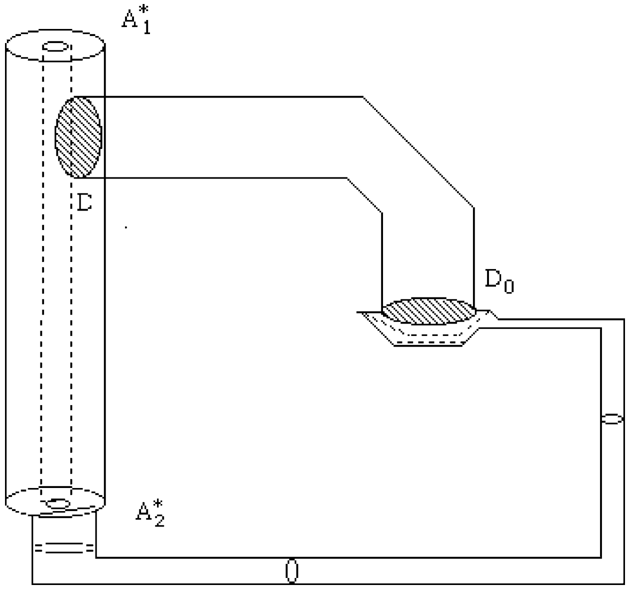


Fig. 9.

of type II (as in the proof of Theorem 5.6). Hence $\alpha_2, \ldots, \alpha_n$ have both end points on D, the cocore disk of the tunnel t_1. The arc α_1 may be of type II or type I in which case it has one end point on one of ∂A_1^* or ∂A_2^*, and one on ∂D.

Since we are assuming that $\mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset$, in both cases $\partial \mathcal{D}^* \cap F_1$ is a set of arcs contained in the annular sub-surface of F_1 depicted in Fig. 8(b) and Fig. 9 with all but at most one endpoint on ∂D. Since by assumption all these arcs must be essential in F_1, it follows that $n = 1$ and α_1 is of type I. But this contradicts the fact that an outermost arc of intersection cannot be of type I as then we can reduce the intersection of A with V_1 in contradiction to the choice of A. Thus we have showed that any two essential disks in V_1 and V_2 must intersect and hence the Heegaard splitting (V_1, V_2) is strongly irreducible. $\square$

Remark 6.2. The induced Heegaard splitting of genus three on $E(K_n \# K(\frac{\alpha}{\beta}))$ is a stabilization of the minimal Heegaard splitting (V_1, V_2) discussed above. This can be seen as follows: Remove a regular neighborhood of a short arc τ on A connecting ∂D to one of the vertical annuli, say A_1^* from V_2 and add it as a 1-handle to V_1. The arc τ is of type I on some meridional disk E of V_2 and since there is only one tunnel crossing A it bounds a sub-disk Δ on E. Hence the cocore disk of $N(\tau)$ intersects Δ in a single point and therefore the pair $(V_1 \cup N(\tau), V_2 - N(\tau))$ is a stabilized Heegaard splitting for $E(K_n \# K(\frac{\alpha}{\beta}))$. However we can slide the tunnel off A by splitting it and sliding along $N(\tau)$. We obtain a isotopic Heegaard splitting with no tunnels crossing A which is isotopic to the Heegaard splitting of $E(K_n \# K(\frac{\alpha}{\beta}))$ which is induced by the two "standard" Heegaard splittings of $E(K_n)$ and $E(K(\frac{\alpha}{\beta}))$.

Acknowledgement

I would like to thank Ying-Qing Wu for suggesting the current proof of Theorem 5.5. and other helpful remarks.

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