Similarity between preferential models

Zhaohui Zhu\textsuperscript{a,b,c}

\textsuperscript{a}Department of Computer Science, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, PR China
\textsuperscript{b}State Key Lab of Novel Software Technology, Nanjing University, Nanjing 210093, PR China
\textsuperscript{c}Shanghai Key Lab of Intelligent Information Processing, Fudan University, Shanghai 200433, PR China

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Abstract

The notion of bisimulation is an important concept in process algebra and modern modal logic. This paper explores the notion of B-similarity, which is a kind of bisimulation between preferential models. We characterize the equivalence of preferential models in terms of B-similarity. However, this result is applicable only for preferential models of finite depth. To overcome this defect, we introduce a weak notion of similarity called M-similarity, and obtain a result corresponding to Hennessy–Milner Theorem and Keisler–Shelah’s Isomorphism Theorem in modal logic and first-order logic, respectively. As its application, we investigate the expressive power of Boolean combinations of conditional assertions (BCA, for short), and prove that BCAs are the fragments of first-order language preserved under M-similarity. Moreover, we obtain a characterization for elementary classes defined by BCAs. A notion of first-order translation originating from modal logic plays an important role in this paper. In order to illustrate that first-order translation is a powerful tool in the study of nonmonotonic logic, some model-theoretic results about preferential models are proved based on this translation.

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1. Introduction

The notion of bisimulation is a familiar concept in modal logic and process algebra, which supplies us with powerful tools for investigating properties of Labeled Transition System (LTS, for short) \cite{2,14,13,16}. Roughly speaking, a bisimulation is a relation between two LTSs in which related states have identical atomic information and matching transition possibilities. It is well known that bisimulation provides a method to characterize modal equivalence, more formally, the following theorem is fundamental in modal logic \cite{2}:

\textit{Let }\tau\textit{ be a modal similarity type, and let }M_1\textit{ and }M_2\textit{ be }\tau\textit{-models, and }w_1\textit{ and }w_2\textit{ two states in }M_1\textit{ and }M_2, \textit{respectively. Then, }w_1\textit{ and }w_2\textit{ satisfy the same modal formulas (i.e., }w_1\textit{ and }w_2\textit{ are modally equivalent) if and only if there exists a bisimulation between ultrafilter extensions of }M_1\textit{ and }M_2\textit{ which links principal ultrafilters generated by }w_1\textit{ and }w_2, \textit{respectively.}

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When we consider only image-finite Kripke models [2], there is a more succinct result so-called Hennessy–Milner Theorem which states that modal equivalence is coincident with bisimulation for image-finite models. These theorems play a similar role as Keisler–Shelah’s Isomorphism Theorem in first-order logic [7], which asserts that two first-order models are elementarily equivalent if and only if they have isomorphic ultrapowers.

Recently, in the field of belief change and nonmonotonic logic, a notion of similarity has been introduced by Bochman [5], which is a kind of bisimulation for structures that consist of some theories endowed with a preferential relation. These structures are called epistemic states and regarded as a generalization of common representations suggested for belief change [3–5]. The notion of similarity plays an important role in exploring properties of epistemic states that determine their behavior in nonmonotonic reasoning and belief change, technical detail may be found in [5].

This paper will concern with the notion of similarity for preferential models. Two preferential models are said to be equivalent if they generate the same preferential relation. Then, an interesting theoretical problem is raised: is it possible to establish a result about equivalence and similarity for preferential models, which corresponds to the results of modal and first-order logic mentioned in the above? In other words, is it possible to provide a characterization for the equivalence of two preferential models in terms of similarity? This paper will give a positive answer.

The rest of the paper is organized as follows: Section 2 recalls some related definitions and results from nonmonotonic logic and first-order model theory. In Section 3, a notion of first-order translation is introduced and explored, which is a bridge between preferential relations and first-order logic. Based on this notion, we can import results and techniques from first-order logic to nonmonotonic logic. Sections 4 and 5 concern with the link between the equivalence and similarity of preferential models. Section 6 shows some model-theoretic properties of preferential models through first-order translation.

2. Preliminaries

In this section, some related definitions and results that have appeared in the literature will be recalled.

2.1. Preferential inference relations and models

We consider formulae of classical propositional calculus built over a set of atomic formulae denoted ℓ plus two constants ⊤ and ⊥ (the formulae true and false, respectively). The set of all well formed formulae in ℓ will be denoted by Form(ℓ). A valuation is a function v : ℓ ∪ {⊤, ⊥} → {0, 1} such that v(⊤) = 1 and v(⊥) = 0. We use lower case letters of the Greek alphabet to denote formulae, the letters v, n, m, v1, v2, etc. to denote valuations, and Val(ℓ) to denote the set of all valuations for ℓ. For any valuation v, we denote the set {x : v |= x and x ∈ Form(ℓ)} by Th(v).

The notation ⊢ x means that the formula x is a tautology and v |= x means that the valuation v satisfies x where compound formulae are evaluated as usual. If Σ is a set of valuations, then Σ |= x means that v |= x for any valuation v ∈ Σ. For any Γ ⊆ Form(ℓ), we denote the set {v ∈ Val(ℓ) : v |= x for any x ∈ Γ} by mod(Γ).

A nonmonotonic inference relation is a binary relation over formulae which satisfies some Horn or non-Horn conditions defined in the style of Gentzen. Following Gabbay [9], this paper uses the relation symbol |∼ to denote nonmonotonic consequence to distinguish it from monotonic logical consequence. If both x and β are formulae, then the sequence x|∼β is called a conditional assertion.

Let S be a set and ≤ a strict partial order over S. For any V ⊆ S, V is said to be smooth if for any t ∈ V, either t is itself minimal in V, i.e., there is no w ∈ V such that w < t, or there exists s ∈ V such that s < t and s is minimal in V. The set of all minimal elements of V with respect to ≤ will be denoted by min(V).

Following the definition in [10], a preferential model W for a language ℓ is a triple (S, l, <), where S is a nonempty set, the elements of which are called states, the interpretation function l : S → Val(ℓ) assigns a valuation to each state, and ≤ is a strict partial order on S satisfying the following smoothness condition: for any formula x, the set ||x||W =def {s : s ∈ S and l(s) = x} is smooth. If there is no ambiguity, we shall write ||x|| for ||x||W. A model W is said to be injective if the function l is injective. For any X ⊆ S, we denote the set {v ∈ Val(ℓ) : ∃s ∈ X and l(s) = v}) by l(X).

Let W be a preferential model for ℓ. The inference relation |∼W generated by W is defined as follows: |∼W is a binary relation over Form(ℓ) such that, for any formulae x and β, x|∼W β iff for any s minimal in ||x||, l(s) |= β. We denote the set {β : x|∼W β} by CW(x).
For any preferential models $W_1$ and $W_2$ for the same language, if $|\sim_{W_1}| = |\sim_{W_2}|$ then $W_1$ will be said to be equivalent to $W_2$ and denoted by $W_1 \equiv W_2$.

2.2. First-order models and ultraproducts

A first-order language $\mathcal{L}$ consists of three kinds of symbols: relation symbols, function symbols and constant symbols. Since this paper concerns only first-order languages without function symbols, we assume that $\mathcal{L}$ does not contain function symbols when we recall some related concepts and results from first-order model theory. More general definitions may be found in [7]. This paper denotes relation symbols and constant symbols by capital Latin letters and lower case Latin letters $c$ with subscripts, respectively. Each relation symbol $P$ of $\mathcal{L}$ is associated with a natural number $n \geq 1$, which means $P$ is an $n$-placed relation. A language $\mathcal{L}'$ is said to be a simple expansion of $\mathcal{L}$ if $\mathcal{L} \subseteq \mathcal{L}'$ and all symbols in $\mathcal{L}'$ but not in $\mathcal{L}$ are constant symbols. In this case, $\mathcal{L}'$ may be written as $\mathcal{L}' = \mathcal{L} \cup X$, where $X$ is the set of new constant symbols.

A model $M$ for $\mathcal{L}$ is a pair $(A, \varepsilon)$, where $A$ is the domain of $\mu$ which is a nonempty set, and $\varepsilon$ is an interpretation function such that, for any $n$-placed relation symbol $P$ in $\mathcal{L}$, $\varepsilon(P)$ is an $n$-placed relation over $A$, and for any constant symbol $c$, $\varepsilon(c)$ is a $\mu$-placed relation over $A$. As usual, we denote $\varepsilon(P)$ (resp. $\varepsilon(c)$) by $P^M$ (resp. $c^M$). A model $M_1 = (A_1, \varepsilon_1)$ is said to be a submodel of $M_2 = (A_2, \varepsilon_2)$ if $A_1 \subseteq A_2$, $\varepsilon_1(c) = \varepsilon_2(c)$ for each constant symbol $c$ in $\mathcal{L}$ and, for each relation symbol $P$ in $\mathcal{L}$, $\varepsilon_1(P)$ is the restriction of $\varepsilon_2(P)$ to $A_1$.

Let $a_1, a_2, \ldots, a_n$ be a sequence in $A$ and $x(x_1 \ldots x_n)$ a first-order formula whose free variables are among $x_1, x_2, \ldots, x_n$. The notation $\mu \vDash x[a_1 \ldots a_n]$ means that formula $x(x_1 \ldots x_n)$ is satisfied in the model $\mu$ under the assignment $[a_1 \ldots a_n]$, whose formal definition may be found in [7]. The theory of $\mu$, in symbols $Th(\mu)$, is the set of all sentences (i.e., formulas without free variables) which are true in $\mu$. Two first-order models are said to be elementarily equivalent if they have the same theory. Let $\Sigma(x_1 \ldots x_n)$ be a set of first-order formulas and every formula $\sigma$ in $\Sigma(x_1 \ldots x_n)$ contains at most the variables $x_1, x_2, \ldots, x_n$ free, $\mu \vDash \Sigma[a_1 \ldots a_n]$ means that $\mu \vDash \sigma[a_1 \ldots a_n]$ for each $x(x_1 \ldots x_n)$ in $\Sigma(x_1 \ldots x_n)$. The set $\Sigma$ is said to be realized in $\mu$ if, for some sequence $a_1, a_2, \ldots, a_n \in A$, we have $\mu \vDash \Sigma[a_1 \ldots a_n]$. Otherwise, we say that $\mu$ omits $\Sigma(x_1 \ldots x_n)$.

Let $I$ be a nonempty set and $D$ an ultrafilter over $I$. Suppose $\mu_i$ is a model for each $i \in I$. The Cartesian product of $A_i$ ($i \in I$) (notation: $\Pi_{i \in I} A_i$) is the set of all functions $f$ with domain $I$ such that $f(i) \in A_i$ for each $i \in I$. For any two functions $f, g \in \Pi_{i \in I} A_i$, $f$ and $g$ are said to be $D$-equivalent, in symbols $f =^D g$, if and only if $i \in I : f(i) = g(i) \in D$. We use $\Pi_D f$ to denote the equivalence class $\{g \in \Pi_{i \in I} A_i : f =^D g\}$. As usual, we use $\Pi_D \mu$ to denote the ultraproduct of $\{\mu_i\}_{i \in I}$ modulo $D$. The formal definition of the ultraproduct may be found in [7]. In the case when all the models $\mu_i$ are the same, say $\mu_i = \mu$, the ultraproduct may be written $\Pi_D \mu$, and is called the ultrapower of $\mu$ modulo $D$.

The following result is the ‘fundamental theorem’ of ultraproducts, which is an important and basic theorem in first-order model theory.

Theorem 2.1. Let $I$ be a nonempty set, and let $D$ be an ultrafilter over $I$ and $\mu_i$ be a model for first-order language $\mathcal{L}$ for each $i \in I$. Then

1. Given any formula $\psi(x_1, x_2, \ldots, x_n)$ of $\mathcal{L}$ and $f^1_D \ldots f^n_D \in \Pi_D A_i$, we have

$$\Pi_D [\mu_i \vDash \psi[f^1_D] \ldots f^n_D] \text{ if and only if } \{i \in I : \mu_i \vDash \psi[f^1] \ldots f^n(i)\} \in D.$$

2. For any sentence $\beta$ of $\mathcal{L}$,

$$\Pi_D [\mu_i \vDash \beta] \text{ if and only if } \{i \in I : \mu_i \vDash \beta\} \in D.$$

Proof. See Theorem 4.1.9 in [7]. ∎

For any two models $\mu_1$ and $\mu_2$ for the same language, $\mu_1$ is an elementary submodel of $\mu_2$ iff $\mu_1$ is a submodel of $\mu_2$ and for all formulas $\beta(x_1 \ldots x_n)$ and all elements $a_1 \ldots a_n$ in $\mu_1$, we have $\mu_1 \vDash \beta[a_1 \ldots a_n]$ iff $\mu_2 \vDash \beta[a_1 \ldots a_n]$. An elementary embedding of $\mu_1$ into $\mu_2$ is an isomorphism $f$ of $\mu_1$ onto an elementary submodel of $\mu_2$, denoted by $f : \mu_1 \leq \mu_2$. Let $I$ be a nonempty set, $D$ an ultrafilter over $I$ and $\mu$ a model. The natural embedding $d$ of $\mu$ into $\Pi_D \mu$...
Lemma 3.2. For any formula \( \phi \), the function \( d(\phi) \) is the equivalence class of the constant function with value \( a \), i.e., \( d(a) = f_D \), where \( f(i) = a \) for each \( i \in I \). It is well known that \( d : \mu \leq H_D \mu \).

3. First-order translation

In order to explore the relationship between modal logic and first-order logic, modal logicians introduced a technology called standard translation [2], which provides an avenue to use results and techniques from first-order model theory and plays an important role in establishing the correspondence theory in modal logic [2]. This paper borrows this technology and introduces a similar translation for preferential inferences.

We first define our correspondence languages—that is, the languages we will translate conditional assertions into. For \( \ell \) a proposition language, let \( \mathcal{L}_\ell \) be the first-order language with equality which consists of unary relation symbols \( P_0, P_1, \ldots \) corresponding to the proposition letters \( p_0, p_1, \ldots \) in \( \ell \), and an binary relation symbol \( R \).

**Definition 3.1.** Let \( x \) be a first-order variable. The translation function \( \text{Tr}_x(\cdot) \) taking propositional formulas in \( \mathcal{L}_\ell \) to first-order formulas in \( \mathcal{L}_\ell \) is defined as follows:

1. \( \text{Tr}_x(p) = \text{def} \ P(x) \) for any \( p \in \ell \),
2. \( \text{Tr}_x(\neg \beta) = \text{def} \ \neg \text{Tr}_x(\beta) \),
3. \( \text{Tr}_x(\beta \lor \theta) = \text{def} \ \text{Tr}_x(\beta) \lor \text{Tr}_x(\theta) \), and
4. \( \text{Tr}_x(\top) = \text{def} \ x = x \).

Clearly, for any preferential model \( M \) for \( \ell \), since \( M \) provides an interpretation for each symbol in \( \mathcal{L}_\ell \), it can act as a first-order model for the language \( \mathcal{L}_\ell \). Formally, we have the following definition:

**Definition 3.2.** Given a preferential model \( M = \langle S, l, < \rangle \) for a language \( \ell \), the model \( \mu_M = \langle S, R^M, P^M \rangle \) for the first-order language \( \mathcal{L}_\ell \) is defined as follows:

1. \( R^M = \text{def} < \).
2. \( P^M = \text{def} \ \| p \|_M = \{ s \in S : l(s) \models p \} \) for any \( p \in \ell \).

**Lemma 3.1.** Let \( M = \langle S, l, < \rangle \) be a preferential model for a language \( \ell \). For any formula \( z \) and \( s \in S \), we have

1. \( l(s) \models z \iff \mu_M \models \text{Tr}_x(z)[s] \).
2. \( \| z \| = \| \beta \| \iff \mu_M \models \forall x(\text{Tr}_x(z) \leftrightarrow \text{Tr}_x(\beta)) \).
3. \( s \in \text{min}(\| z \|) \iff \mu_M \models \text{min}(z, x)[s] \), where

\[
\text{min}(z, x) = \text{def} \ \text{Tr}_x(z) \land \forall z(\text{Tr}_x(z) \rightarrow \neg R(z, x)).
\]

**Proof.** (1) By induction on the complexity of \( z \); (2) Immediately follows from (1); (3) Straightforward. \( \square \)

Thus, for any formula \( z \) in \( \ell \) and preferential model \( M \), the first-order formula \( \text{Tr}_x(z) \) simulates \( z \) in first-order model \( \mu_M \). Moreover, the smoothness of the set \( \| z \| \) can be depicted as \( \mu_M \models \text{smooth}(z) \), where

\[
\text{smooth}(z) = \text{def} \ \forall x(\text{Tr}_x(z) \land \exists y(\text{Tr}_y(z) \land R(y, x)) \rightarrow \exists y(\text{min}(x, y) \land R(y, x))).
\]

More formally, we have

**Lemma 3.2.** For any formula \( z \) of propositional language \( \ell \) and first-order model \( \mu \) for \( \mathcal{L}_\ell \), the following are equivalent:

1. The set \( \{ s : \mu \models \text{Tr}_x(z)[s] \} \) is smooth with respect to \( R^\mu \).
2. \( \mu \models \text{smooth}(z) \).

**Proof.** Straightforward. \( \square \)

We now turn to another translation function \( (\cdot)^\circ \), which translates conditional assertions into first-order sentences. In fact, translating conditional assertions into other languages has appeared in the literature, for instance, Boutilier presents a method for translating them into modal formulas [6].
**Theorem 3.5.** For any preferential model \( W \) and ultrafilter \( D \)

**Corollary 3.4.** For any preferential relation \( \sim \) in \( \ell \), we get

**Proof.** (1)⇒(2) Assume that \( s \) is any state such that

\[ \mu_M \models Tr_x(\ell) \land \neg \exists y (Tr_y(\ell) \land R(y, x)) [s]. \]

To complete the proof, it is enough to show \( \mu_M \models Tr_x(\ell)[s] \). By the assumption and (3) from Lemma 3.1, we have \( s \in \text{min}(\|\ell\|) \). So, \( l(s) \models \beta \) comes from \( \ell \models \sim_M \beta \). Further, by (1) from Lemma 3.1, we obtain \( \mu_M \models Tr_x(\ell)[s] \), as desired.

(2)⇒(1) Let \( s \in \text{min}(\|\ell\|) \). It is enough to show \( l(s) \models \beta \). Clearly, by (3) from Lemma 3.1, \( \mu_M \models Tr_x(\ell) \land \neg \exists y (Tr_y(\ell) \land R(y, x))[s] \) immediately follows from \( s \in \text{min}(\|\ell\|) \). Since \( \mu_M \models (\ell \models \sim \beta)^\circ \), i.e.,

\[ \mu_M \models \forall x ((Tr_x(\ell) \land \neg \exists y (Tr_y(\ell) \land R(y, x))) \rightarrow Tr_x(\beta)), \]

we get \( \mu_M \models Tr_x(\beta)[s] \). So, \( l(s) \models \beta \).  

As an immediate consequence of the above lemma, we obtain

**Corollary 3.4.** For any preferential relation \( \sim \) and preferential model \( M \), \( \sim \) can be generated by \( M \) (i.e., \( \sim = \sim_M \)) if and only if \( \mu_M \models |\sim| \).

We know that, for any preferential model \( M \) for \( \ell \), \( \mu_M \) is a first-order model for \( \mathbb{S}_\ell \). In reverse, given a model \( \mu \) for \( \mathbb{S}_\ell \) such that:

1. \( \mu \models \forall x \sim R(x, x) \),
2. \( \mu \models \forall x \forall y \forall z (R(x, y) \land R(y, z) \rightarrow R(x, z)) \), and
3. \( \mu \models \text{smooth}(\ell) \), for any \( \ell \in \text{Form}(\ell) \),

then \( \mu \) induces a preferential model \( M_\mu = \langle S, l, \prec \rangle \) for \( \ell \) described as follows:

4. \( S \) is the domain of \( \mu \).
5. For any \( s \in S, l(s) = \{ p \in \ell : s \in P^p \}. \)
6. \( \prec = R^\mu \).

Thus, given a preferential model \( W \), since \( \mu_W \) satisfies the conditions (1), (2) and (3), by Theorem 2.1, so does the ultrapower \( \Pi_D \mu_W \) for any ultrafilter \( D \). Hence, \( \Pi_D \mu_W \) can induce a preferential model \( M_{\Pi_D \mu_W} \) in the above manner.

**Convention:** In the following, for convenience, we denote the preferential model \( M_{\Pi_D \mu_W} \) by \( \Pi_D W \). On the other hand, we also denote the ultrapower \( \Pi_D \mu_W \) by \( \Pi_D W \) when we can understand the meaning of \( \Pi_D W \) from its context.

**Theorem 3.5.** For any preferential model \( W \) and ultrafilter \( D \), we have

1. \( \Pi_D W \) is a preferential model.
2. \( \sim_W = \sim_{\Pi_D W} \).
3. \( d : \mu_W \preceq \Pi_D W \), where \( d \) is natural embedding.

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1. In this paper, we give the valuation as for a Herbrand model, that is identifying the subset of variables with its characteristic function.
Theorem 4.2. Let \( M \) will be said to be

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Example 4.1. Let $\ell$ be an infinite language, and let $m$ and $n$ be valuations such that $m \neq n$. Consider the following two injective models: $W_1 = (U - \{n\}, id_1, \emptyset)$ and $W_2 = (U - \{m\}, id_2, \emptyset)$, where $id_1$ and $id_2$ are the identity functions over $U - \{n\}$ and $U - \{m\}$, respectively, and $U$ is the set of all valuations. By the infiniteness of the language, it is easy to see that

$$\models W_1 \beta \iff \models W_2 \beta,$$

for any pair of formulas $\alpha$ and $\beta$.

So, $W_1 \equiv W_2$. On the other hand, since $n \notin U - \{n\}$ and $n \in U - \{m\}$, we have $W_1 \not\models_B W_2$, so, $W_1 \not\models_B W_2$. Similarly, $W_2 \not\models_B W_1$. □

Example 4.2. Let $\ell$ be an infinite language. Consider the following two injective ranked models $W_1$ and $W_2$ in Fig. 1, where $n$ is a valuation and $U$ is the set of all valuations.

By the infiniteness of the language, we get $W_1 \equiv W_2$ trivially, however, we have $W_1 \models_B W_2$ and $W_1 \not\models_B W_2$. But, $W_2 \models_B W_1$ holds, in fact, $W_2 \models_B W$ for any preferential model $W$ for $\ell$. □

Nonetheless, it is possible to show a restricted converse to Theorem 4.2. To this end, let’s firstly recall a basic concept from first-order model theory which will play an important role in the following work.

Let $\mu$ be a model for first-order language $\Im$ and $X \subseteq A$, where $A$ is the domain of $\mu$. The expansion $(\mu, a)_{a \in X}$ is a model for $\Im \cup \{c_a : a \in X\}$ which has the same interpretations for old symbols as $\mu$, and interprets $c_a$ by $a$ itself for each $a \in X$. Let $\Sigma(x)$ be a set of formulas of $\Im$. $\Sigma(x)$ is said to be consistent with the theory of $\mu$ if the set $\Sigma(x) \cup Th(\mu)$ can be realized. Consequently, by the compactness, $\Sigma(x)$ is consistent with $Th(\mu)$ if and only if every finite subset of $\Sigma(x)$ is realized in some model of $Th(\mu)$. \(^2\)

**Definition 4.4.** Let $\eta$ be a cardinal. A model $\mu$ for first-order language $\Im$ is said to be $\eta$-saturated iff for every subset $X \subseteq A$ (the domain of $\mu$) of power $|X| < \eta$, the expansion $(\mu, a)_{a \in X}$ realizes every set $\Sigma(x)$ of the language $\Im \cup \{c_a : a \in X\}$ which is consistent with the theory of $(\mu, a)_{a \in X}$.

It is easy to see that, if $\mu$ is $\eta$-saturated then it is $\xi$-saturated for any cardinal $\xi < \eta$. For convenience, we recall a notion introduced in [18] as follows.

**Definition 4.5.** Let $M = \langle S, l, \prec \rangle$ be a preferential model for a language $\ell$. $M$ is said to be $P$-saturated if, for any formula $\alpha$,

$$l(min(\|\alpha\|)) = \{m : m \in Val(\ell) \text{ such that } m \models C_M(\alpha)\}.$$  

**Proposition 4.3.** For any finite preferential model $M$ for a language $\ell$, $\mu_M$ is $\omega_0$-saturated.

**Proof.** Let $X$ be a finite subset of $S_M$ and $\Sigma(x)$ a set of formulas of $\Im \cup \{c_a : a \in X\}$. Suppose that $\Sigma(x)$ is consistent with the theory of $(\mu_M, a)_{a \in X}$. Hence, $\Sigma(x) \cup Th((\mu_M, a)_{a \in X})$ can be realized in some model $\mu_0$. Thus, $(\mu_M, a)_{a \in X}$ is

\(^2\) Refer to Proposition 2.2.7 in [7].
elementarily equivalent to $\mu_0$. Further, since $(\mu_M, a)_{a \in X}$ is finite, $(\mu_M, a)_{a \in X}$ is isomorphism to $\mu_0$. Consequently, the expansion $(\mu_M, a)_{a \in X}$ realizes $\Sigma(x)$. So, $\mu_M$ is $\omega_0$-saturated.

**Definition 4.6.** Let $m$ be a valuation for $\ell$. The following set will be called the *diagram* of $m$ and denoted by $A_m(x)$.

$$A_m(x) = \{ P(x) : p \in \ell \And m \models p \} \cup \{ \neg P(x) : p \in \ell \And m \not\models p \}.$$ 

Clearly, for any preferential model $M$ and $s \in S_M$, we have

$$\mu_M \models A_m[s] \text{ if and only if } l_M(s) = m.$$ 

**Lemma 4.4.** Let $M$ be a preferential model for a language $\ell$ and $\mu_M$ be $\omega_0$-saturated. Then $M$ is $P$-saturated.

**Proof.** It is enough to show that, for any formula $z$, we have

$$l(\text{min}(\|z\|)) = \{ m : m \in \text{Val}(\ell) \And m \models \Sigma_M(z) \}.$$ 

Since the claim is trivial when $\|z\|$ is empty, we assume it is nonempty. Let $m \in \text{Val}(\ell)$ such that $m \models \Sigma_M(z)$. We set

$$\Sigma_m(x) = A_m(x) \cup \{ Tr_x(z) \And \neg \exists y(Tr_1(z) \And R(y, x)) \}.$$ 

Now we demonstrate the following claims.

**Claim 1.** Let $\Sigma_0(x)$ be any nonempty finite subset of $\Sigma_m(x)$. Then, there exists an element $v \in S_M$ such that:

1. $\mu_M \models \Sigma_0[v]$.
2. $\mu_M \models Tr_x(z) \And \neg \exists y(Tr_y(z) \And R(y, x))[v]$.

If $\Sigma_0(x) \cap A_m(x) = \emptyset$ then the sentence $Tr_x(z) \And \neg \exists y(Tr_y(z) \And R(y, x))$ is the only element in $\Sigma_0(x)$. Since $\|z\| \neq \emptyset$, by the smoothness, $\text{min}(\|z\|) \neq \emptyset$. Clearly, for any $v \in \text{min}(\|z\|)$, we have

$$\mu_M \models Tr_x(z) \And \neg \exists y(Tr_y(z) \And R(y, x))[v].$$ 

Next, we consider another case in which $\Sigma_0(x) \cap A_m(x) \neq \emptyset$. We put

$$\Sigma_0^-(x) = \Sigma_0(x) - \{ Tr_x(z) \And \neg \exists y(Tr_y(z) \And R(y, x)) \}.$$ 

Clearly, $\Sigma_0^-(x) \subseteq A_m(x)$. Suppose that $\mu_M \not\models \Sigma_0^-(x)[v]$ for any $v$ such that

$$\mu_M \models Tr_x(z) \And \neg \exists y(Tr_y(z) \And R(y, x))[v].$$ 

Thus, we have

$$\mu_M \models Tr_x(z) \And \neg \exists y(Tr_y(z) \And R(y, x)) \rightarrow \neg Tr_x(\beta), \text{ where}$$

$$\beta = \left( \bigwedge_{p(x) \in \Sigma_0^-(x)} p \right) \bigwedge_{\neg p(x) \in \Sigma_0^-(x)} \neg p.$$ 

So, $\mu_M \models (z \neg \beta)^\circ$. Further, by Lemma 3.3, we get $\neg \beta \in C_M(x)$, which contradicts $m \models C_M(x)$ and $m \models \beta$. Hence, there exist some states in $S_M$ satisfying the conditions (1) and (2).

**Claim 2.** $m \in l(\text{min}(\|z\|))$.

By Claim 1, since $\mu_M$ is $\omega_0$-saturated, there exists an element $v \in S_M$ such that $\mu_M \models \Sigma_m[v]$. So, $\mu_M \models Tr_x(z) \And \neg \exists y(Tr_y(z) \And R(y, x))[v]$, further, by (3) from Lemma 3.1, we have $v \in \text{min}(\|z\|)$. Moreover, since $\mu_M \models A_m[v]$, $l(v) = m$. Hence, $m \in l(\text{min}(\|z\|))$, as desired.

---

3 Refer to Proposition 1.3.19 in [7].
Lemma 4.5. Let $M = (S, l, \prec)$ be a preferential model for a language $\ell$ and $\mu_M$ be $\omega_0$-saturated, and let $m \in \text{Val}(\ell)$ and $m \models C_M(z)$. On the other hand, the opposite inclusion can be proved trivially. So, the proof is complete. □

Consequently, by Proposition 4.3 and Lemma 4.4, all finite preferential models are $P$-saturated.

**Lemma 4.5.** Let $M = (S, l, \prec)$ be a preferential model for a language $\ell$ and $\mu_M$ be $\omega_0$-saturated, and let $m \in \text{Val}(\ell)$ and $\Sigma$ be a nonempty set of formulas of $\ell$. Then $M$ satisfies the following condition (4.1):

If $m \in l(\min(||\Sigma_0||))$ for any nonempty finite subset $\Sigma_0$ of $\Sigma$ then there exists $s \in S$ such that $l(s) = m$ and $s \in \bigcap_{x \in \Sigma} \min(||z||)$.

(4.1)

**Proof.** Let $m$ be any valuation such that $m \in l(\min(||\bigvee \Sigma_0||))$ for any nonempty finite subset $\Sigma_0$ of $\Sigma$. We put

$$\Gamma(x) = A_m(x) \cup \{Tr_\ell(z) \land \forall y(Tr_y(z) \rightarrow \neg R(y, x)) : \varphi \in \Sigma\}.$$ 

Clearly, it suffices to prove that $\Gamma(x)$ is realized in $\mu_M$. Further, since $\mu_M$ is $\omega_0$-saturated, we only need to show that any finite subset of $\Gamma(x)$ is realized in $\mu_M$.

Let $\Gamma_0(x)$ be any finite subset of $\Gamma(x)$. If $\Gamma_0(x) \subseteq A_{m(x)}$, then $\Gamma_0(x)$ is realized by any $t \in S$ such that $l(t) = m$. Since $\Sigma \neq \emptyset$ and $m \in l(\min(||\bigvee \Sigma_0||))$ for any nonempty finite subset $\Sigma_0$ of $\Sigma$, such states must exist. Next, we consider another case in which

$$\Gamma_0(x) \cap \{Tr_\ell(z) \land \forall y(Tr_y(z) \rightarrow \neg R(y, x)) : \varphi \in \Sigma\} \neq \emptyset.$$ 

We set

$$\Sigma_0 = \{\varphi \in \Sigma : Tr_\ell(z) \land \forall y(Tr_y(z) \rightarrow \neg R(y, x)) \in \Gamma_0(x)\}.$$ 

Clearly, $\Sigma_0$ is a finite subset of $\Sigma$. So, $m \in l(\min(||\bigvee \Sigma_0||))$, hence, there exists a state $s \in S$ such that $l(s) = m$ and $s \in \min(||\bigvee \Sigma_0||)$. Consequently, by (3) from Lemma 3.1, we have

$$\mu_M \models Tr_\ell(\bigvee \Sigma_0) \land \forall y(Tr_y(\bigvee \Sigma_0) \rightarrow \neg R(y, x))[s]$$

and

$$\mu_M \models A_m[s].$$

Since $Tr_\ell(\bigvee \Sigma_0) = \bigvee_{x \in \Sigma_0} Tr_\ell(z)$, we get

$$\forall y(Tr_y(\bigvee \Sigma_0) \rightarrow \neg R(y, x)) \models \bigwedge_{x \in \Sigma_0} \forall y(Tr_y(z) \rightarrow \neg R(y, x)).$$

Hence, we obtain

$$\mu_M \models \{\forall y(Tr_y(z) \rightarrow \neg R(y, x)) : \varphi \in \Sigma_0\}[s].$$

On the other hand, for any $\varphi \in \Sigma_0 \subseteq \Sigma$, since $m \in l(\min(||\varphi||))$, we have $l(s) \models \varphi$, i.e., $\mu_M \models Tr_\ell(z)[s]$. So,

$$\mu_M \models \{Tr_\ell(z) \land \forall y(Tr_y(z) \rightarrow \neg R(y, x)) : \varphi \in \Sigma_0\}[s]$$

Further, $\mu_M \models \Gamma_0[s]$ comes from $\mu_M \models A_m[s]$. □

By the way, it is obvious that condition (4.1) in the above lemma is implied by the following condition (4.2):

If $\Sigma(m) \neq \emptyset$ then there exists $s \in S$ such that $s \in \bigcap_{x \in \Sigma(m)} \min(||z||)$ and $l(s) = m$, where $\Sigma(m) = \{\varphi \in \text{Form}(\ell) : m \in l(\min(||\varphi||))\}$. (4.2)
We now take up the questions: Does condition (4.1) imply (4.2)? For any preferential model $M$, does $\omega_0$-saturation of $\mu_M$ imply (4.2)? Both answers are $no$ because of the proposition as follows:

**Proposition 4.6.** Let $M = \langle S, l, < \rangle$ be a preferential model for a language $\ell$. If $\mu_M$ is $\omega_0$-saturated then the following are equivalent:

(i) $M$ satisfies condition (4.2).

(ii) $|\sim_M$ satisfies the following condition WDR:

$$C_M(\alpha \lor \beta) \subseteq Cn(C_M(\alpha) \cup C_M(\beta)), \text{ for any formulas } \alpha \text{ and } \beta.$$ 

**Proof.** (i $\Rightarrow$ ii) Let both $\alpha$ and $\beta$ be any formula. Assume that, for some formula $\gamma$, we have $\gamma \in C_M(\alpha \lor \beta)$ and $\gamma \notin Cn(C_M(\alpha) \cup C_M(\beta))$. Hence, there exists a valuation $m \in Val(\ell)$ such that

$$m \models C_M(\alpha) \cup C_M(\beta) \cup \{\neg \gamma\}.$$ 

So, by Lemma 4.4, $\alpha, \beta \in \Sigma(m)$. Further, by the condition (i), there exists a state $s \in S$ such that $l(s) = m$ and $s \in min(\|\alpha\|) \cap min(\|\beta\|)$. Since

$$min(\|\alpha\|) \cap min(\|\beta\|) \subseteq min(\|\alpha \lor \beta\|),$$

we get $s \in min(\|\alpha \lor \beta\|)$, which contradicts $l(s) \not\models \neg \gamma$ and $\gamma \in C_M(\alpha \lor \beta)$.

(ii $\Rightarrow$ i) Let $\Sigma_0 = \{z_1, \ldots, z_n\}$ be any nonempty finite subset of $\Sigma(m)$. Since $m \models C_M(z_i)$ for each $i \in \{1 \ldots n\}$, by WDR, we get $m \models C_M(\bigvee_{1 \leq i \leq n} z_i)$. Consequently, by Lemma 4.4, there exists a state $t \in S$ such that $l(t) = m$ and $t \in min(\{\|\bigvee_{1 \leq i \leq n} z_i\|\})$. Further, by Lemma 4.5, there exists a state $s \in S$ such that $l(s) = m$ and $s \in \bigcap_{z \in \Sigma(m)} min(\|z\|)$. □

The condition WDR is introduced by Freund in order to characterize injective inference relations [8]. In the finite framework, Freund obtains a representation theorem for preferential relations satisfying the condition WDR in terms of injective preferential models as follows:

Let $\ell$ be a logical finite language and $|\sim$ a preferential inference relation in $\ell$. Then, the relation $|\sim$ satisfies WDR if and only if there exists an injective preferential model $W$ such that $|\sim = |\sim_W$.

Unfortunately, the above result is false if the language is infinite [15]. In the literature [8], a notion of standard model is defined, which is a special kind of injective model. An injective preferential model $W = (S, l, <)$ is said to be a standard model if $mod(C_W(\alpha)) = \{l(s) : s \in min(\|\alpha\|)\}$ for any formula $\alpha$. Thus, an injective model is standard if and only if it is P-saturated. In [18], a notion of a valuation structure is defined, which consists of worlds ordered by a binary relation introduced in [17]. A canonical approach is also presented, through which we can obtain an injective preferential model for any preferential relation satisfying WDR. In particular, the following representation result is established, which provides the semantical characterization for the family of all preferential inference relations satisfying WDR:

A preferential inference relation $|\sim$ satisfies WDR if and only if there exists a standard model $W$ such that $|\sim_W = |\sim$.

For any preferential model $M$, there exists a $\omega_0$-saturated model $M^*$ such that $M \equiv M^*$ (see Theorem 4.11 in this paper). However, it is false that any preferential model satisfies the condition WDR. In fact, WDR does not always hold even for injective models [15]. Thus, by Proposition 4.6, the conditions (4.1) and (4.2) are not equivalent.

**Remark.** From the proofs of Lemmas 4.4, 4.5 and Proposition 4.6, it is easy to see that we only need 1-saturation for these proofs to go through.

Now, let’s return to our subject matter. In order to show a restricted converse to Theorem 4.2, we need to recall a notion introduced in [1]. A preferential model $W$ is said to be parsimonious iff for every state $s \in S$ there is a formula $\alpha$ such that $s \in min(\|\alpha\|)$ [1]. Clearly, given a preferential $M = \langle S, l, < \rangle$, the restriction model of $M$ to the set $\{s : s \in S$
Proposition 4.7. Let $M = \langle S, I, \prec \rangle$ be a preferential model. Then the preferential model $M^\ast$ is parsimonious, where $M^\ast = \langle S^\ast, l^\ast, \prec^\ast \rangle$ is defined as follows:

1. $S^\ast = \{ s : s \in S \text{ and } s \in \text{min}([\|\varnothing\|]) \text{ for some formula } \varnothing \}$.
2. $l^\ast(s) = l(s) \text{ for any } s \in S^\ast$.
3. $\prec^\ast = \prec \cap (S^\ast)^2$.

Proof. Straightforward. □

The above construction induces a mapping $par : M \mapsto M^\ast$. Obviously, $M \equiv par(M)$ and, for any formula $\varnothing$, $l(\text{min}([\|\varnothing\|])) = l^\ast(\text{min}([\|\varnothing\|]_{par(M)}))$. Moreover, if $M$ is $P$-saturated then so is $par(M)$.

Lemma 4.8. Let $W$ be a preferential models for a language $\ell$ and $\mu_W$ be $\omega_0$-saturated, and let $s \in S_W$. Suppose that $s \notin \text{min}([\|\varnothing\|])$ for any formula $\varnothing$. Then, there exists a state $t \in S_W$ such that $t \prec s$ and $l(t) = l(s)$.

Proof. We put

$$\Gamma(x) = \Delta_m(x) \cup \{R(x, c_s)\},$$

where $l(s) = m$ and $c_s$ is a new constant symbol interpreted by $s$ in the model $(\mu_W, s)$. To prove this lemma, we need the following auxiliary result:

Claim. Let $\Gamma_0(x)$ be any finite subset of $\Gamma(x)$. Then, $\Gamma_0(x) \cup \{R(x, c_s)\}$ is realized in $(\mu_W, s)$.

Suppose not. Since $s \notin \text{min}([\|\varnothing\|])$ for any formula $\varnothing$, we get $s \notin \text{min}([\|\text{true}\|])$. So, for some $v \in S_W$, we have $v \prec s$. Consequently, $(\mu_W, s) \models R(x, c_s)[v]$. Hence, $\Gamma_0(x) \cap \Delta_m(x) \neq \emptyset$. Let

$$\beta = \left( \bigwedge_{p(x) \in \Gamma_0(x)} p \right) \wedge \left( \bigwedge_{\neg p(x) \in \Gamma_0(x)} \neg p \right).$$

Since $(\mu_W, s)$ omits $\Gamma_0(x) \cup \{R(x, c_s)\}$, we have

$$(\mu_W, s) \models \forall x (Tr_x(\beta) \rightarrow \neg R(x, c_s)).$$

Hence, we obtain

$$(\mu_W, s) \models \forall x (Tr_x(\beta) \rightarrow \neg R(x, c_s))[s].$$

On the other hand, $\mu_W \models Tr_x(\beta)[s]$ comes from $l(s) \models \beta$. Therefore, by (3) from Lemma 3.1, we get $s \in \text{min}([\|\beta\|])$, which contradicts the assumption that $s \notin \text{min}([\|\varnothing\|])$ for any formula $\varnothing$.

Now we can arrive the conclusion as desired. Since $\mu_W$ is $\omega_0$-saturated, by the above claim, $\Gamma(x)$ is realized in $(\mu_W, s)$. So, there exists a state $t \in S_W$ such that $t \prec s$ and $l(t) = l(s)$. □

In the following, a preferential model $\langle S, I, \prec \rangle$ is said to be well founded if there are no infinite sequences decreasing with respect to $\prec$, that is, no infinite sequences $s_0, s_1, \ldots$ such that $s_1 < s_0, s_2 < s_1, \ldots, s_{i+1} < s_i, \ldots$, for $i < \omega_0$.

Lemma 4.9. Let $W_i$ be a preferential model for a language $\ell$ and $\mu_{W_i}$ be $\omega_0$-saturated, for $i = 1, 2$. If $W_1$ is well founded then

$$\models \neg W_2 \text{ implies } par(W_2) \models_B par(W_1).$$

Proof. For $i = 1, 2$, we denote $par(W_i) = \langle S_i, I_i, \prec_i \rangle$ by $M_i$. Since $\mu_{W_i}$ is $\omega_0$-saturated, by Lemma 4.4, $M_i$ is $P$-saturated. Let $s_1 \in S_1$ and $l_1(s_1) = m$. We now prove that there exists a state $s_2 \in S_2$ satisfying the following conditions.

1. $l_2(s_2) = m$.
2. $\forall t_2 \in S_2 (t_1 < t_2 \rightarrow \exists t_1 \in S_1 (t_1 \prec t_1 \wedge l_1(t_1) = l_2(t_2)))$. 
Since $M_1$ is parsimonious, there exists a formula $z$ such that $s_1 \in \min(\|z\|_{M_1})$. We set
\[ \Omega(m) = \{ s \in S_2 : s \in \min(\|z\|_{M_2}) \text{ such that } l_2(s) = m \}. \]

From $C_{M_1}(z) \subseteq C_{M_1}(z)$ and $m \models C_{M_1}(z)$, we have $m \models C_{M_1}(z)$. Further, since $M_2$ is $P$-saturated, it is easy to see that $\Omega(m) \neq \emptyset$. We shall show that there exists a state $s_2 \in \Omega(m)$ satisfying the condition (2). Suppose not. Then, for any $s \in \Omega(m)$, there exists a state $t \in S_2$ such that
\[
(3)\ t \prec_1 s, \text{ and }
(4)\ \forall t_1 \in S_1 (t_1 \prec_1 s_1 \Rightarrow l_1(t_1) \neq l_2(t)).
\]

Now, for each $s \in \Omega(m)$, we choose a state $t_s \in S_2$ satisfying the above conditions (3) and (4). To induce a contradiction, we need to show the following claims.

**Claim 1.** There exists a state $t_1 \in S_1$ such that $t_1 \prec_1 s_1$.

Suppose not. Then, $m \in l_1(\min(\|\text{true}\|_{M_1}))$. Since $M_2$ is $P$-saturated and $\sim_{W_2} \subseteq \sim_{W_1}$, there exists a state $s \in S_2$ such that
\[ l_2(s) = m \text{ and } s \in \min(\|\text{true}\|_{M_2}). \]

Further, from $m \models z$, we get $s \in \min(\|z\|_{M_2})$, so, $s$ belongs to $\Omega(m)$ and there is no $t \in S_2$ such that $t \prec_2 s$, which contradicts the assumption.

**Claim 2.** For any $s \in \Omega(m)$, there exists a formula $z_s$ such that $l_2(t_s) \models \neg z_s$ and, for any $t_1 \prec_1 s_1$, $l_1(t_1) \models z_s$.

Let $s$ be any state in $\Omega(m)$, and let $c_{s_1}$ be a new constant symbol interpreted by $s_1$ itself in ($\mu_{M_1}, s_1$) and ($\mu_{W_1}, s_1$). We put
\[ \Gamma_{t_s}(x) = A_{l_2(t_1)}(x) \cup \{ R(x, c_{s_1}) \}. \]

Clearly, $\Gamma_{t_s}(x)$ is a set of formulas of the first-order language $\exists \ell \cup \{ c_{s_1} \}$. Since $l_1(t) \neq l_2(t)$ for any $t \prec_1 s_1$, the model ($\mu_{M_1}, s_1$) omits the set $\Gamma_{t_s}(x)$. We now verify that the model ($\mu_{W_1}, s_1$) omits the set $\Gamma_{t_s}(x)$ too.

Assume not. Thus, for some state $t_1 \in S_{W_1}$, we have ($\mu_{W_1}, s_1) \models \Gamma_{t_s}(t_1)$. So, $t_1 \prec_{W_1} s_1$ and $l_{W_1}(t_1) = l_2(t_1)$. Since ($\mu_{M_1}, s_1$) omits the set $\Gamma_{t_s}(x)$, $t_1 \not\in S_1$. So, $t_1 \notin \min(\|\beta\|_{W_1})$ for any formula $\beta$. Consequently, by Lemma 4.8, there exists a state $t_2 \in S_{W_1}$ such that:
\[(1)\ t_2 \prec_{W_1} t_1,
(2)\ l_{W_1}(t_2) = l_{W_1}(t_1).\]

Since the relation $\prec_{W_1}$ is transitive, we obtain $t_2 \prec_{W_1} s_1$, further, ($\mu_{W_1}, s_1) \models \Gamma_{t_s}(t_2)$ comes from $l_{W_1}(t_2) = l_2(t_1)$. Similarly, $t_2 \notin \min(\|\beta\|_{W_1})$ for any formula $\beta$, and there exists a state $t_3 \in S_{W_1}$ such that $t_3 \prec_{W_1} t_2$ and $l_{W_1}(t_2) = l_{W_1}(t_3)$. Iterating this process, we obtain an infinite decreasing chain:
\[ \ldots \prec_{W_1} t_{i+1} \prec_{W_1} t_i \prec_{W_1} t_3 \prec_{W_1} t_2 \prec_{W_1} t_1 \ (i < \omega_0). \]

Hence, a contradiction follows from the well-foundedness of $W_1$. Therefore, ($\mu_{W_1}, s_1$) omits $\Gamma_{t_s}(x)$, as desired.

Further, since $\mu_{W_1}$ is $\omega_0$-saturated, for some finite subset $\Gamma_0 \subseteq \Gamma_{t_s}(x)$, ($\mu_{W_1}, s_1$) omits $\Gamma_0$. From the above claim, $\Gamma_0 \cap A_{l_2(t_s)}(x) \neq \emptyset$. We set
\[ z_s = \left( \bigvee_{P(x) \in \Gamma_0} \neg p \right) \lor \left( \bigvee_{P(x) \in \Gamma_0} p \right).
\]

Clearly, $l_2(t_s) \models \neg z_s$ and, for any $t_1 \prec_1 s_1$, we have $l_1(t_1) \models z_s$, otherwise, by Claim 1, $\Gamma_0$ can be realized by some state $t \prec_1 s_1$ in ($\mu_{W_1}, s_1$).

**Claim 3.** Let $\Phi = \{ z \lor \neg z_s : s \in \Omega(m) \} \cup \{ z \}$. For any nonempty finite subset $\Phi_0 \subseteq \Phi$, we have $m \in l_{W_2}$($\min(\| \lor \Phi_0 \|_{W_2})$).
Clearly, since \( m \models \alpha \), we have
\[
m \models \sqrt{\Phi_0}.
\]

For any \( t_1 \prec_1 s_1 \), from \( s_1 \in \min(\|\alpha\|_{M_1}) \), we have \( l_1(t_1) \models \neg \alpha \). Further, since \( l_1(t_1) \models \alpha_s \) for each \( s \in \Omega(m) \), we have \( l_1(t_1) \models \neg (\sqrt{\Phi_0}) \). Consequently, \( s_1 \in \min(\|\sqrt{\Phi_0}\|_{M_1}) \). Hence, \( m \in l_1(\min(\|\sqrt{\Phi_0}\|_{M_1})) \). Since
\[
C_{W_2} (\sqrt{\Phi_0}) \subseteq C_{W_1} (\sqrt{\Phi_0}) \quad \text{and} \quad m \models C_{W_2} (\sqrt{\Phi_0}),
\]
we get \( m \models C_{W_2} (\sqrt{\Phi_0}) \). Moreover, since \( \mu_{W_2} \) is \( \omega_0 \)-saturated, by Lemma 4.4, we obtain \( m \in l_{W_2} (\min(\|\sqrt{\Phi_0}\|_{W_2})) \).

**Claim 4.** There exists a state \( s^* \in S_2 \) such that \( s^* \in \bigcap_{\beta \in \Phi} \min(\|\beta\|_{M_2}) \) and \( l_2(s^*) = m \).

From the above claim and Lemma 4.5, since \( W_2 \) is \( \omega_0 \)-saturated, there exists a state \( s^* \in W_2 \) such that \( l_{W_2}(s^*) = m \) and \( s^* \in \bigcap_{\beta \in \Phi} \min(\|\beta\|_{W_2}) \). Further, since \( M_2 = \text{par}(W_2) \), we have \( s^* \in \bigcap_{\beta \in \Phi} \min(\|\beta\|_{M_2}) \) and \( l_2(s^*) = m \).

Now we can get a contradiction as desired. Since \( \alpha \in \Phi \), we have \( s^* \in \Omega(m) \). However, by the assumption, there exists a state \( t_{s^*} \in S_2 \) such that \( t_{s^*} \prec_2 s^* \), moreover, by Claim 2, we obtain \( l_2(t_{s^*}) \models \alpha \lor \neg \alpha_{s^*} \). Consequently, a contradiction follows from \( s^* \in \min(\|\alpha \lor \neg \alpha_{s^*}\|_{M_2}) \).

**Corollary 4.10.** Let \( W_i \) be a preferential model and \( \mu_{W_i} \) be \( \omega_0 \)-saturated, for \( i = 1, 2 \). If both \( W_1 \) and \( W_2 \) are well founded, then
\[
W_1 \equiv W_2 \implies \text{par}(W_1) \preceq_{B} \text{par}(W_2).
\]

**Proof.** Immediately follows from Lemma 4.9. \( \square \)

**Remark.** From the proofs of Lemmas 4.8 and 4.9, it is easy to see that we only need 2-saturation for these proofs to go through.

We now turn our attention to the existence of \( \omega_0 \)-saturated preferential models. Before this issue is addressed, we recall some basic concepts and results from first-order model theory [7].

Let \( I \) be a nonempty set and \( \beta \) a cardinal number, and let \( f, g \) be functions on the set \( S_\beta(\beta) \) of all finite subsets of \( \beta \) into the set \( S(I) \) of all subsets of \( I \). We say that \( g \preceq f \) iff for all \( s \in S_\beta(\beta) \), \( g(s) \subseteq f(s) \). The function \( f \) is said to be monotonic iff \( s, v \in S_\beta(\beta) \) and \( s \subseteq v \) implies \( f(v) \subseteq f(s) \), and \( f \) is said to be additive iff \( f(s \cup v) = f(s) \cap f(v) \) for any \( s, v \in S_\beta(\beta) \).

Let \( \alpha \) be an infinite cardinal. An ultrafilter \( D \) over \( I \) is said to be \( \alpha \)-good iff it satisfies the following condition: for every cardinal \( \beta < \alpha \) and every monotonic function \( f \) on \( S_\beta(\beta) \) into \( D \), there exists an additive function \( g \) on \( S_\beta(\beta) \) into \( D \) such that \( g \preceq f \). An ultrafilter \( D \) is said to be countably incomplete if there exists a countable set \( E \subseteq D \) such that \( \cap E \notin D \).

**Theorem 4.11.** For any preferential model \( M \) for any language \( \ell \), there exists a preferential model \( M^* \) such that
\begin{enumerate}
\item \( M \equiv M^* \), and
\item The model \( \mu_{M^*} \) is \( \omega_0 \)-saturated.
\end{enumerate}

**Proof.** Let \( I \) be any set of power \( \alpha \) such that \( \omega_0 + |\mathfrak{I}_\ell| < \alpha \). Then, there exists an \( \alpha^+ \)-good countably incomplete ultrafilter \( D \) over \( I \), where \( \alpha^+ \) is the least cardinal greater than \( \alpha \). From \( \alpha < \alpha^+ \), \( D \) is \( \alpha \)-good. Hence, the ultrapower \( \Pi_D M \) is \( \alpha \)-saturated. \(^5\) Since
\[
\omega_0 \leq \omega_0 + |\mathfrak{I}_\ell| < \alpha,
\]
\( \Pi_D M \) is also \( \omega_0 \)-saturated. On the other hand, by Theorem 3.5, \( \Pi_D M \) is a preferential model such that \( \Pi_D M \equiv M \). \( \square \)

\(^4\) Refer to Theorem 6.1.4 in [7].
\(^5\) Refer to Theorem 6.1.8 in [7].
However, in order to show a restricted converse to Theorem 4.2, we expect that $M^*$ is well founded. But, an ultraproduct $II_D M$ can not insure this even if $M$ is well founded.\footnote{In fact, the well-foundedness of $M$ does not always imply the well-foundedness of $II_D M$, however, this implication holds if $D$ is $\omega_1$-complete, i.e., for any $E \subseteq D$ of power $|E| < \omega_1$, we have $\cap E \in D$.} Thus, we introduce the following notion.

**Definition 4.7.** A preferential model $M$ is said to be of *finite* depth if and only if there exists a natural number $n$ such that the length of any chain contained in $M$ is smaller than $n$.

**Lemma 4.12.** Let $M$ be a preferential model of finite depth. Then, $II_D M$ is well founded for any ultrafilter $D$.

**Proof.** Suppose that the length of any chain contained in $M$ is smaller than $n$ and $II_D M$ contains the following infinite decreasing chain:

$$\ldots < D f_D^{i+1} < D f_D^i < D \ldots < D f_D^1 (i < \omega_0).$$

So, for any $i < \omega_0$, $\{ j : f(i+1)(j) <_M f(i)(j) \} \in D$.\footnote{Refer to the definition of the reduced product (see, for example, 4.1.6 in [7]).} So, we have

$$\emptyset \neq \bigcap_{i < n+1} \{ j : f(i+1)(j) <_M f(i)(j) \} \in D.$$

Let $k \in \bigcap_{i < n+1} \{ j : f(i+1)(j) <_M f(i)(j) \}$. Then, the model $M$ contains a chain with the length $n + 1$ as follows:

$$f^{n+1}(k) <_M f^n(k) <_M \ldots <_M f^2(k) <_M f^1(k).$$

Thus, a contradiction raises, as desired. \(\square\)

**Theorem 4.13.** Let $M_1$ and $M_2$ be two preferential models, and let $M_1$ be of finite depth. Then

$$|\sim_{M_2} \subseteq |\sim_{M_1} \text{ if and only if } \par(II_D M_2) \Rightarrow_B \par(II_D M_1) \text{ for some ultrafilter } D.$$

**Proof.** ($\Leftarrow$) By Lemma 4.1, we have $|\sim_{par(II_D M_2)} \subseteq |\sim_{par(II_D M_1)}$, further, $|\sim_{M_2} \subseteq |\sim_{M_1}$ comes from $par(II_D M_1) \equiv M_1$ and $par(II_D M_2) \equiv M_2$.

($\Rightarrow$) Similar to Theorem 4.11, for some ultrafilter $D$, both $II_D M_1$ and $II_D M_2$ are $\omega_0$-saturated. Further, by Theorem 3.5, $|\sim_{II_D M_2} \subseteq |\sim_{II_D M_1}$ comes from $|\sim_{M_2} \subseteq |\sim_{M_1}$. Hence, by Lemma 4.9 and 4.12, $par(II_D M_2) \Rightarrow_B par(II_D M_1)$. \(\square\)

**Corollary 4.14.** Let $M_1$ and $M_2$ be two preferential models of finite depth. Then

$$M_1 \equiv M_2 \text{ if and only if } \par(II_D M_1) \Leftrightarrow_B \par(II_D M_2) \text{ for some ultrafilter } D.$$

**Proof.** Follows from Theorem 4.13. \(\square\)

Obviously, for any preferential models $M_1$ and $M_2$ such that both $par(M_1)$ and $par(M_2)$ have finite depth, $M_1 \equiv M_2$ if and only if $par(II_D par(M_1)) \Leftrightarrow_B par(II_D par(M_2))$ for some ultrafilter $D$. By the way, $II_D par(M)$ is isomorphism to $par(II_D M)$ for any finite model $M$. In the rest of this section, we will show that the above theorem may be expressed in a more succinct form.
**Lemma 4.15.** Let $M = (S, l, \prec)$ be a well-founded preferential model and $\mu_M$ be $\omega_0$-saturated, and let $s \in S$. Suppose that $s \notin \min(||\beta||)$ for any formula $\beta$. Then there exists a state $t \in S$ such that:

1. $t \prec s$.
2. $l(t) = l(s)$.
3. $t \in \min(||\beta||)$ for some formula $\beta$.

**Proof.** Otherwise, by Lemma 4.8, there exists an infinite decreasing chain $s = s_0 \succ s_1 \succ s_2 \succ \ldots$ such that $l(s_i) = l(s)$ for any $i < \omega_0$, which contradicts the well-foundedness of $M$. □

Hence, for any preferential model $M$, if $M$ is well-founded and $\mu_M$ is $\omega_0$-saturated, then $M$ is valuation parsimonious [18], that is, for any $m \in l_M(S_M)$, there exists a formula $\beta$ such that $m \in l_M(\min(||\beta||))$.

**Lemma 4.16.** Let $M$ be a well-founded preferential model and $\mu_M$ be $\omega_0$-saturated. Then, $M \leftrightarrow_B \par(M)$.

**Proof.** It is enough to show the following claims.

**Claim 1.** $\par(M) \rightarrow_B M$.

Let $s \in S_M$. If $s \in \min(||\beta||_M)$ for some formula $\beta$, then $s \in \par(M)$ and $l_M(s) = l_{\par(M)}(s)$. Further, since $\forall \beta \in \par(M)$, we have $s \leq M \par(M)\leq s_M$. Now, we consider another case in which $s \notin \min(||\beta||_M)$ for any formula $\beta$. By Lemma 4.15, there exists a state $t \in S_M$ such that $t \prec_M s$, $l(t) = l(s)$ and $t \in \min(||\beta||_M)$ for some formula $\beta$. Clearly, $t \in S_{\par(M)}$ and $l_M(t) = l_{\par(M)}(t)$, and $s \leq M \par(M)$ comes from $t \leq M \par(M) \in M_M$.

**Claim 2.** $M \rightarrow_B \par(M)$.

Let $t \in S_{\par(M)}$. Clearly, $t \in S_M$ and $l_M(t) = l_{\par(M)}(t)$. Thus, it is enough to show $t \leq M \par(M) \leq t_M$. Let $s \in S_M$ such that $s \prec_M t$. If $s \in \min(||\beta||_M)$ for some formula $\beta$, then $s \in \par(M)$, $l_M(s) = l_{\par(M)}(s)$ and $s \prec \par(M) = t$. If $s \notin \min(||\beta||_M)$ for any formula $\beta$, then, by Lemma 4.15, there exists a state $k \in S_M$ such that $k \prec_M s$, $l(k) = l(s)$ and $k \in \min(||\beta||_M)$ for some formula $\beta$. Hence, $k \in \par(M)$, $l_M(k) = l_{\par(M)}(k)$ and $k \prec \par(M) = t$. □

Since $\leftrightarrow_B$ is transitive, we have the following result:

**Theorem 4.17.** Let $M_1$ and $M_2$ be two preferential models of finite depth. Then the following are equivalent:

1. $M_1 \equiv M_2$.
2. $\par(\Pi_D M_1) \leftrightarrow_B \par(\Pi_D M_2)$ for some ultrafilter $D$.
3. $\Pi_D M_1 \leftrightarrow_B \Pi_D M_2$ for some ultrafilter $D$.

**Proof.** (1)$\iff$(2) Follows from Corollary 4.14.

(3)$\implies$(1) Comes from Theorems 3.5 and 4.2.

(1)$\implies$(3) By Corollary 4.14, for some ultrafilter $D$, $\par(\Pi_D M_1) \leftrightarrow_B \par(\Pi_D M_2)$ and both $\Pi_D M_1$ and $\Pi_D M_2$ are $\omega_0$-saturated. Further, by Lemma 4.16 and 4.12, $\Pi_D M_i \leftrightarrow_B \par(\Pi_D M_i)$ for $i = 1, 2$. So, by the transitivity of $\leftrightarrow_B$, we have $\Pi_D M_1 \leftrightarrow_B \Pi_D M_2$. □

Similarly, due to the transitivity of $\leftrightarrow_B$, we have

**Theorem 4.18.** Let $M_1$ and $M_2$ be two preferential models, and let $M_1$ be of finite depth. Then the following are equivalent:

1. $\ll M_2 \subseteq \ll M_1$.
2. $\par(\Pi_D M_2) \rightarrow_B \par(\Pi_D M_1)$ for some ultrafilter $D$.
3. $\Pi_D M_2 \rightarrow_B \Pi_D M_1$ for some ultrafilter $D$.

**Corollary 4.19.** Let both $M_1$ and $M_2$ be two finite preferential models. Then

1. $M_1 \equiv M_2$ if and only if $M_1 \leftrightarrow_B M_2$.
2. $\ll M_2 \subseteq \ll M_1$ if and only if $M_2 \rightarrow_B M_1$. 


Proof. For i = 1, 2, since $M_i$ is finite, $M_i$ is isomorphism to $\Pi_D M_2$ for any ultrafilter $D$. □

5. M-Similarity between preferential models

In the last section, we explore the notion of B-similarity. A limitation of Theorem 4.17 lies in that it is applicable only for preferential models of finite depth. To supply this gap, this section will introduce a weak notion of similarity called M-similarity and explore the relationship between M-similarity and equivalence of preferential models. For any preferential models, we will establish a similar result as Theorem 4.17 in terms of M-similarity. Moreover, as its application, the expressive power of Boolean combinations of conditional assertions will be investigated.

5.1. M-Similarity and equivalence

Given a preferential model $M = (S, l, <)$, we will denote the set \{s $\in S : s \in \text{min}(\|x\|)$ for some formula $x$\} by $MF(M)$.

Definition 5.1. For any two preferential models $M_1 = (S_1, l_1, <_1)$ and $M_2 = (S_2, l_2, <_2)$ for the same language, $M_1$ will be said to be semi-M-similar to $M_2$ (notation: $M_1 \rightsquigarrow M_2$) if

\[\forall s_2 \in MF(M_2) \exists s_1 \in S_1 (l_1(s_1) = l_2(s_2) \land s_2 \upharpoonright M_2 \sqsubseteq s_1 \upharpoonright M_1).\]

Proposition 5.1. Let $M_1$ and $M_2$ be two preferential models. Then

1. $M_1 \rightarrow_B M_2$ implies $M_1 \rightarrow M_2$.
2. $\text{par}(M_1) \rightarrow_B \text{par}(M_2)$ iff $\text{par}(M_1) \rightarrow \text{par}(M_2)$.

Proof. Straightforward. □

Proposition 5.2. Let $M_1$ and $M_2$ be preferential models, and let $s_2 \in MF(M_2)$ and $s_1 \in S_1$. Then

\[l_1(s_1) = l_2(s_2) \land s_2 \upharpoonright M_2 \sqsubseteq s_1 \upharpoonright M_1 \text{ implies } s_1 \in MF(M_1).\]

Proof. Since $s_2 \in MF(M_2)$, we may suppose that $s_2 \in \text{min}(\|x\|)_{M_2}$ for some formula $x$. So, $l_1(s_1) \vdash x$ comes from $l_1(s_1) = l_2(s_2)$. Similar to Lemma 4.1, we can show that $s_1 \in \text{min}(\|x\|_{M_1})$. Hence, $s_1 \in MF(M_1)$, as desired. □

Thus, $M_1 \rightarrow M_2$ may be equivalently defined as

\[\forall s_2 \in MF(M_2) \exists s_1 \in MF(M_1) (l_1(s_1) = l_2(s_2) \land s_2 \upharpoonright M_2 \sqsubseteq s_1 \upharpoonright M_1).\]

Applying the definition of $\rightarrow$ in the above fashion, it is easy to show that the relation $\rightarrow$ is transitive.

By the way, since the set $MF(M)$ is the domain of $\text{par}(M)$, it is possible for someone to raise the following conjectures:

Conjecture 5.1. $M_1 \rightarrow M_2$ implies $\text{par}(M_1) \rightarrow \text{par}(M_2)$.

Conjecture 5.2. $\text{par}(M_1) \rightarrow \text{par}(M_2)$ implies $M_1 \rightarrow M_2$.

Conjecture 5.3. $l_1 (\text{min}(\|x\|_{M_1})) \sqsupseteq l_2 (\text{min}(\|x\|_{M_2}))$ for any $x$ implies $M_1 \rightarrow M_2$.

The following example provides negative answers for the above conjectures.

Example 5.1. Let $\ell$ be an infinite language. Consider injective ranked models $W_1$, $W_2$ and $W_3$ with the graphical shape in Fig. 2, where $m$ is a valuation such that $m \models p$, $U = \{v \in \text{Val}(\ell) : v \models \neg p\}$ and $n \in U$. It is easy to see that $W_1 \rightarrow W_3$. However, due to the infiniteness of the language $\ell$, $n \notin \text{par}(W_3)$, thus, $\text{par}(W_1) \rightarrow \text{par}(W_3)$ comes from

\[m \in \text{par}(W_3) \text{ and } n \not<_{\text{par}(W_3)} m.\]
On the other hand, $\text{par}(W_3) \rightarrow \text{par}(W_2)$ and, for any $z$, we have

$$l_2(\min(||z||_{W_2})) = l_3(\min(||z||_{W_3})).$$

But, since $n \prec_3 m$ and $n \not\prec_2 m$, we obtain $W_3 \nrightarrow W_2$. □

However, for finite preferential models, Conjectures 5.1, 5.2 and 5.3 hold. More generally, we have

**Proposition 5.3.** Let $M_i$ be a well-founded preferential model and $\mu_{M_i}$ be $\omega_0$-saturated for $i = 1, 2$. Then,

$$M_1 \rightarrow M_2 \text{ if and only if } \text{par}(M_1) \rightarrow \text{par}(M_2).$$

In particular, for any finite preferential models $M_1$ and $M_2$, $M_1 \rightarrow M_2$ if and only if $\text{par}(M_1) \rightarrow \text{par}(M_2)$.

**Proof.** ($\Rightarrow$) Suppose that $M_1 \rightarrow M_2$. By Claim 1 from Lemma 4.16 and (1) from Proposition 5.1, we get $\text{par}(M_1) \rightarrow M_1$, further, due to the transitivity of $\rightarrow$, we have

$$\text{par}(M_1) \rightarrow M_2.$$

On the other hand, by Claim 2 from Lemma 4.16 and (1) from Proposition 5.1, we obtain

$$M_2 \rightarrow \text{par}(M_2).$$

Hence, $\text{par}(M_1) \rightarrow \text{par}(M_2)$ comes from the transitivity of $\rightarrow$.

($\Leftarrow$) Suppose that $\text{par}(M_1) \rightarrow \text{par}(M_2)$. First, by (2) from Proposition 5.1, $\text{par}(M_1) \rightarrow_B \text{par}(M_2)$. Next, by Claims 1 and 2 from Lemma 4.16, we have

$$M_1 \rightarrow_B \text{par}(M_1) \text{ and } \text{par}(M_2) \rightarrow_B M_2.$$

So, $M_1 \rightarrow_B M_2$ comes from the transitivity of $\rightarrow_B$. Thus, by (1) from Proposition 5.1, we obtain $M_1 \rightarrow M_2$, as desired. □

**Proposition 5.4.** Let $M_i$ be a well-founded preferential model and $\mu_{M_i}$ be $\omega_0$-saturated for $i = 1, 2$. Then,

$$M_1 \rightarrow M_2 \text{ if and only if } M_1 \rightarrow_B M_2.$$

In particular, for any finite preferential models $M_1$ and $M_2$, $M_1 \rightarrow M_2$ if and only if $M_1 \rightarrow_B M_2$.

**Proof.** By Proposition 5.3 and (2) from Proposition 5.1, it is easy to see that

$$M_1 \rightarrow M_2 \text{ if and only if } \text{par}(M_1) \rightarrow_B \text{par}(M_2).$$

Further, by Claims 1 and 2 from Lemma 4.16 and the transitivity of $\rightarrow_B$, we have

$$M_1 \rightarrow_B M_2 \text{ if and only if } \text{par}(M_1) \rightarrow_B \text{par}(M_2).$$

Consequently, $M_1 \rightarrow M_2$ if and only if $M_1 \rightarrow_B M_2$, as desired. □
Furthermore, from the above proposition and Corollary 4.19, it is easy to see that Conjecture 5.3 holds for finite preferential models. Now, we turn to issue concerning the relationship between the equivalence and M-similarity.

**Lemma 5.5.** $M_2 \rightarrow M_1$ implies $\sim_{M_2} \subseteq \sim_{M_1}$.  

**Proof.** Similar to Lemma 4.1. $\square$

**Definition 5.2.** Let $M_1$ and $M_2$ be two preferential models for the same language. $M_1$ and $M_2$ will be said to be *M-similar* (notation: $M_1 \leftrightarrow M_2$) if $M_1 \rightarrow M_2$ and $M_2 \rightarrow M_1$.

Obviously, the relation $\leftrightarrow$ is an equivalence relation.

**Theorem 5.6.** $M_1 \leftrightarrow M_2$ implies $M_1 \equiv M_2$.  

**Proof.** Immediately follows from Lemma 5.5 and Definition 5.2. $\square$

Similar to B-similarity, two equivalent preferential models do not need to be M-similar. For instance, consider $W_2$ and $W_3$ in Example 5.1. Since $W_2 = \text{par}(W_3)$, we have $W_2 \equiv W_3$, but $W_2 \not\leftrightarrow W_3$. The rest of this subsection will concern itself with showing a restricted converse to the above theorem.

**Lemma 5.7.** Let $M_1 = (S_1, l_1, \prec_1)$ and $M_2 = (S_2, l_2, \prec_2)$ be two preferential models for a language $\ell$, and let both $\mu_{M_1}$ and $\mu_{M_2}$ be $\omega_0$-saturated. Then, $\sim_{M_2} \subseteq \sim_{M_1}$ implies $M_2 \rightarrow M_1$.  

**Proof.** Similar to Lemma 4.9. But, for the integrality, we give the proof here. Let $s_1 \in MF(M_1)$ and $l_1(s_1) = m$. It is enough to show that there exists a state $s_2 \in S_2$ such that

1. $l_2(s_2) = m$.
2. $\forall t_2 \in S_2(t \prec_2 s_2 \Rightarrow \exists t_1 \in S_1(t_1 \prec_1 s_1 \land l_1(t_1) = l_2(t_2))$. Since $s_1 \in MF(M_1)$, we have $s_1 \in \text{min}(\|x\|_{M_1})$ for some formula $x$. We put

$$\Omega(m) = \{s \in S_2 : s = \text{min}(\|x\|_{M_2}) \text{ and } l_2(s) = m\}.$$  

Similar to Lemma 4.9, $\Omega(m) \neq \emptyset$. We will prove that there is a state $s_2 \in \Omega(m)$ satisfying condition (2). Suppose not. Hence, for every $s \in \Omega(m)$, there exists a state $t \in S_2$ satisfying the following conditions:

3. $t \prec_2 s$, and
4. $\forall t_1 \in S_1(t_1 \prec_1 s_1 \Rightarrow l_1(t_1) \neq l_2(t))$.

So, for every $s \in \Omega(m)$, we can choose a state $t_s \in S_2$ satisfying the above conditions (3) and (4). To complete the proof, we need to demonstrate the following claims.

**Claim 1.** There exists a state $t_1 \in S_1$ such that $t_1 \prec_1 s_1$.  

Similar to Claim 1 in Lemma 4.9.

**Claim 2.** For any $s \in \Omega(m)$, there exists a formula $\varphi_s$ such that $l_2(t_s) \models \varphi_s$ and, for any $t_1 \prec_1 s_1$, $l_1(t_1) \models \varphi_s$.

Let $c_s$ be a new constant symbol interpreted by $s_1$ in $(\mu_{M_1}, s_1)$, and let

$$\Gamma_s(x) = A_{l_2(t_s)}(x) \cup \{R(x, c_s)\}.$$  

Since $l_1(t) \neq l_2(t_s)$ for any $t \prec_1 s_1$, the model $(\mu_{M_1}, s_1)$ omits the set $\Gamma_s(x)$. Further, since $\mu_{M_1}$ is $\omega_0$-saturated, there exists a finite subset $\Gamma_0 \subseteq \Gamma_s(x)$ omitted by $(\mu_{M_1}, s_1)$. By Claim 1, it is easy to see that $\Gamma_0 \cap A_{l_2(t_s)}(x) \neq \emptyset$. We set

$$\varphi_s = \left( \bigvee_{p(x) \in \Gamma_0} \neg p \right) \lor \left( \bigvee_{p(x) \in \Gamma_0} p \right).$$  

Clearly, $l_2(t_s) \models \varphi_s$. On the other hand, for any $t_1 \prec_1 s_1$, we have $l_1(t_1) \models \varphi_s$, otherwise, by Claim 1, $\Gamma_0$ can be realized by some state $t \in S_1$ such that $t \prec_1 s_1$.  

Claim 3. Let $\Phi = \{x \lor \neg x_s : s \in \Omega(m) \} \cup \{x\}$. For any nonempty finite subset $\Phi_0 \subseteq \Phi$, we have $m \in l_2(\min(\|\Phi_0\|_{M_1}))$.

Since $m \models x$, we have $m \models \bigvee \Phi_0$. For any $t_1 \prec_1 s_1$ and $s \in \Omega(m)$, from $l_1(t_1) \models x$ and $s_1 \in \min(\|x\|_{M_1})$, we have $l_1(t_1) \models \neg(\bigvee \Phi_0)$. Consequently, $s_1 \in \min(\|\bigvee \Phi_0\|_{M_1})$. Hence, $m \in l_1(\min(\|\bigvee \Phi_0\|_{M_1}))$ and $m \models C_{M_1}(\bigvee \Phi_0)$. Since $C_{M_1}(\bigvee \Phi_0) \subseteq C_{M_2}(\bigvee \Phi_0)$, we have $m \models C_{M_2}(\bigvee \Phi_0)$. Further, since $\mu_{M_2}$ is $\omega_0$-saturated, by Lemma 4.4, $m \in l_2(\min(\|\bigvee \Phi_0\|_{M_2}))$.

Now, a contradiction is raised. By Claim 3 and Lemma 4.5, since $\mu_{M_2}$ is $\omega_0$-saturated, we have $s^* \in \bigcap_{s \in \Phi} \min(\|s\|_{M_2})$ and $l_2(s^*) = m$ for some state $s^* \in S_2$.

Further, $s^* \in \Omega(m)$ follows from $x \in \Phi$. However, by the assumption and Claim 2, we get $t_{s^*} \prec_2 s^*$ and $l_2(t_{s^*}) \models \neg x_{s^*}$ for some state $t_{s^*} \in S_2$. Thus, it follows that $s^* \notin \min(\|x \lor \neg x_{s^*}\|_{M_2})$, a contradiction, as desired. □

Corollary 5.8. Let $M_1$ and $M_2$ be two preferential models, and let both $\mu_{M_1}$ and $\mu_{M_2}$ be $\omega_0$-saturated. Then

$$M_1 \equiv M_2 \text{ implies } M_1 \equiv M_2.$$ 

Proof. Immediately follows from Lemma 5.7. □

Now, we arrive at the main result of this section which plays a similar role as Keisler–Shelah’s Isomorphism Theorem [7] and Hennessy–Milner Theorem [2] in first-order logic and modal logic, respectively.

Theorem 5.9. For any preferential models $M_1$ and $M_2$, we have

1. $|\sim_{M_2}| \subseteq |\sim_{M_1}|$ iff $II_D M_2 \equiv II_D M_1$ for some ultrafilter $D$.
2. $M_1 \equiv M_2$ iff $II_D M_1 \equiv II_D M_2$ for some ultrafilter $D$.

In particular, if both $M_1$ and $M_2$ are finite, then

3. $|\sim_{M_2}| \subseteq |\sim_{M_1}|$ iff $M_2 \equiv M_1$.
4. $M_1 \equiv M_2$ iff $M_1 \equiv M_2$.

Proof. Similar to Theorem 4.11, there exists an ultrafilter $D$ such that $II_D M_2$ and $II_D M_1$ are $\omega_0$-saturated. Then, (1) and (2) come from Theorem 3.5, 5.6, Lemma 5.5, 5.7 and Corollary 5.8. Conclusions (3) and (4) follow from Corollary 4.19 and Proposition 5.4. □

5.2. Expressive power of Boolean combinations of conditional assertions

From the first-order translation defined in Section 3, we know that conditional assertions may be regarded as fragments of first-order languages. This subsection will explore the characterization for first-order sentences which are equivalent to some translations of conditional assertions in terms of M-similarity. More formally, we will show that a first-order sentence is equivalent to a Boolean combination of the translations of conditional assertions if and only if it is preserved under $\leftrightarrow$.

Definition 5.3. Given a propositional language $\ell$, $BCA(\ell)$ is the least set satisfying the following conditions:

1. $BCA(\ell)$ is a set of sentences of first-order language $\Sigma_\ell$.
2. For any $x, \beta \in \text{Form}(\ell)$, $(x \lor \beta) \circ \in BCA(\ell)$.
3. If $\theta \in BCA(\ell)$ then $\neg \theta \in BCA(\ell)$.
4. If $\theta_1, \theta_2 \in BCA(\ell)$ then $\theta_1 \land \theta_2, \theta_1 \lor \theta_2 \in BCA(\ell)$.

Any sentence in $BCA(\ell)$ is said to be a boolean combination of conditional assertions. On the other hand, we use $PBCA(\ell)$ to denote the least set satisfying the above conditions (1), (2) and (4), and sentences in $PBCA(\ell)$ are called positive boolean combinations of conditional assertions.
**Lemma 5.10.** Boolean combinations of conditional assertions are preserved under $\iff$. That is, for any preferential models $M_1$ and $M_2$, if $M_1 \iff M_2$ then

$$\mu_{M_1} \models \theta \iff \mu_{M_2} \models \theta$$

for any $\theta \in BCA(\ell)$.

**Proof.** Proceeding by induction on the complexity of $\theta$. It is easy to carry out for $\theta$ with the format $\neg \theta_1, \theta_1 \lor \theta_2$ and $\theta_1 \land \theta_2$, where $\theta_1, \theta_2 \in BCA(\ell)$. In the case where $\theta = (\phi|\sim \beta)\circ$ for some formula $\phi, \beta \in Form(\ell)$, this is done by observing that the following are equivalent:

$$\mu_{M_1} \models (\phi|\sim \beta)\circ$$

if $\phi|\sim \mu_{M_1} \beta$ (By Lemma 3.3)

if $\phi|\sim \mu_{M_2} \beta$ (By $M_1 \iff M_2$ and Theorem 5.6)

if $\mu_{M_2} \models (\phi|\sim \beta)\circ$ (By Lemma 3.3). □

**Lemma 5.11.** Positive boolean combinations of conditional assertions are preserved under $\rightarrow$. That is, for any preferential models $M_1$ and $M_2$, if $M_1 \rightarrow M_2$ then

$$\mu_{M_1} \models \theta$$

implies $\mu_{M_2} \models \theta$ for any $\theta \in PBCA(\ell)$.

**Proof.** By Lemma 5.5 and proceeding by induction on $\theta$, omitted. □

Given a propositional language $\ell$, let $\phi$ be a first-order sentence of the language $\mathfrak{A}_\ell$. The formula $\phi$ will be said to be $P$-equivalent to a (positive) boolean combination of conditional assertions if and only if, for some formula $\beta \in BCA(\ell)$ (resp., $\beta \in PBCA(\ell)$), we have

$$\mu_M \models \phi \iff \mu_M \models \beta$$

for any preferential model $M$ for $\ell$.

It is easy to see that the above condition (5.1) is equivalent to

$$\{\phi_{pos}\} \cup \{\text{smooth}(\delta) : \delta \in Form(\ell)\} \models \phi \iff \beta,$$

where

$$\phi_{pos} = \def \forall x \neg R(x, x) \land \forall x \forall y \forall z (R(x, y) \land R(y, z) \rightarrow R(x, z)).$$

As a matter of convenience, we put

$$\Sigma_P(\ell) = \{\phi_{pos}\} \cup \{\text{smooth}(\delta) : \delta \in Form(\ell)\}.$$

If there is no ambiguity, we shall write $\Sigma_P$ for $\Sigma_P(\ell)$. Clearly, the $P$-equivalence is exactly the $A$-equivalence$^8$ in first-order logic if we set $A = \Sigma_P$.

**Theorem 5.12.** Let $\ell$ be a propositional language and $\phi$ be a first-order sentence of the language $\mathfrak{A}_\ell$. Then, $\phi$ is $P$-equivalent to a boolean combination of conditional assertions iff $\phi$ is preserved under $\iff$.

**Proof.** ($\Rightarrow$) Immediately follows from Lemma 5.10.

($\Leftarrow$) Suppose that $\phi$ is preserved under $\iff$. We put

$$BCA(\phi) = \{\phi : \phi \in BCA(\ell) \text{ such that } \Sigma_P \models \phi \rightarrow \phi\}.$$

Let $M$ be any preferential model for $\ell$ such that $\mu_M \models BCA(\phi)$. We will show $\mu_M \models \phi$. This leads to two claims being demonstrated as follows.

**Claim 1.** There exists a preferential model $W$ such that $\mu_W \models (|\sim \phi)\circ \cup \{\phi\}$.

---

$^8$ In first-order logic, two formulas $\phi$ and $\psi$ are said to be $A$-equivalent iff $A \models \phi \iff \psi$. 
Suppose not. So, we have
\((\sim M) \cup \Sigma_p \vdash \neg \phi\).

By the compactness, there exists a finite subset \(\Sigma \subseteq (\sim M)\) such that
\[\Sigma_p \vdash \phi \rightarrow \neg \wedge \Sigma.\]

Clearly, \(\neg \wedge \Sigma \in BCA(\ell)\), hence, \(\neg \wedge \Sigma \in BCA(\phi)\), which contradicts \(\mu_M \models BCA(\phi)\) and \(\mu_M \models \Sigma\).

Claim 2. \(\mu_M \models \phi\).

By the above claim, we may suppose \(\mu_\ell \models (\sim M) \cup \{\phi\}\) for some preferential model \(\mu_\ell\). So, by Corollary 3.4, \(\mu_\ell \equiv M\). By Theorem 5.9, \(PBCA(\ell) \equiv PBCA(\phi)\) for some ultrafilter \(D\). From \(\mu_\ell \models \phi\) and \(\mu_\ell \leq \Pi_D W\), we have \(\Pi_D W \models \phi\). Further, since \(\phi\) is preserved under \(\equiv\), we have \(\Pi_D M \models \phi\). Consequently, \(\mu_M \models \phi\) comes from \(\mu_M \leq \Pi_D M\).

Returning now to the proof of the theorem. By Claim 2, for any preferential model \(\mu_\ell\) for \(\ell\), \(\mu_M \models BCA(\phi)\) implies \(\mu_M \models \phi\). So, by the compactness, there exists a finite subset \(\Gamma \subseteq BCA(\phi)\) such that
\[\Sigma_p \cup \Gamma \models \phi.\]

On the other hand, we have
\[\Sigma_p \cup \{\phi\} \models \Gamma.\]

Hence, \(\Sigma_p \models \phi \iff \Gamma \models BCA(\ell)\), as desired. \(\square\)

Theorem 5.13. Let \(\ell\) be a propositional language and \(\phi\) be a first-order sentence of the language \(\exists_\ell\). Then, \(\phi\) is \(P\)-equivalent to a positive boolean combination of conditional assertions iff \(\phi\) is preserved under \(\equiv\).

Proof. \((\Rightarrow)\) Comes from Lemma 5.11.

\((\Leftarrow)\) Assume that \(\phi\) is preserved under \(\equiv\). We set
\[PBCA(\phi) = \{\phi : \phi \in PBCA(\ell)\} \text{ such that } \Sigma_p \vdash \phi \rightarrow \phi\}.

Firstly, we will show that \(\mu_M \models \phi\) for any preferential model \(M\) such that \(\mu_M \models PBCA(\phi)\). Let \(M\) be any model satisfying \(\mu_M \models PBCA(\phi)\). The following two claims are proved in turn.

Claim 1. Let \(A = \{-(\phi | \sim \theta)^{\phi} : \beta | \phi \models \theta \text{ and } \beta, \theta \in Form(\ell)\}\). Then, \(\mu_\ell \models A \cup \{\phi\}\) for some preferential model \(W\).

Suppose not. Hence, \(A \cup \Sigma_p \models \neg \phi\). It then follows from the compactness that, for some finite subset \(\Sigma \subseteq A\), we have
\[\Sigma_p \cup \{\phi\} \models \neg \wedge \Sigma.\]

Clearly, there exists a formula \(\delta \in PBCA(\ell)\) such that \(\models \neg \wedge \Sigma \iff \delta\). Thus, we obtain
\[\Sigma_p \models \phi \rightarrow \delta.\]

So, \(\delta \in PBCA(\phi)\). Further, a contradiction comes from \(\mu_M \models PBCA(\phi)\), \(\mu_M \models \Sigma\) and \(\models \wedge \Sigma \iff \neg \delta\).

Claim 2. \(\mu_M \models \phi\).

By Claim 1, there exists a preferential model \(W\) such that \(\mu_W \models A \cup \{\phi\}\). Thus, \(\Pi_D W \models \phi\) comes from \(\mu_W \models \phi\) and \(\mu_W \leq \Pi_D W\). For any \(\beta, \theta \in Form(\ell)\), since \(\mu_W \models \phi\) \(\beta | \phi \models \theta\) implies \(\beta | \phi \models \theta\). So, \(\models \sim W \subseteq \sim M\). Hence, by Theorem 5.9, we get \(\Pi_D W \models \Pi_D M\) for some ultrafilter \(D\). Because \(\Pi_D W \models \phi\) and \(\phi\) is preserved under \(\equiv\), we obtain \(\Pi_D M \models \phi\). So, \(\mu_M \models \phi\) follows from \(\mu_M \leq \Pi_D M\).
Consequently, for each preferential model $M$ such that $\mu_M \models PBCA(\ell)$, we have $\mu_M \models \forall$. So,
\[
\Sigma_p \cup PBCA(\ell) \models \forall.
\]

It follows from the compactness that $\Sigma_p \cup \Gamma \models \exists$ for some finite subset $\Gamma \subseteq PBCA(\ell)$. Moreover, from $\Gamma \subseteq PBCA(\ell)$, we have
\[
\Sigma_p \cup \{\exists\} \models \Gamma.
\]

Thus, $\Sigma_p \models \forall \iff \bigwedge \Gamma$, as desired. $\square$

In model theory, a class $K$ of models for $\mathcal{L}$ is said to be an elementary class if, for some set $\Sigma$ of sentences of $\mathcal{L}$, $K$ is defined by $\Sigma$, that is, $K$ is exactly the class of all models of $\Sigma$. There exists a well known theorem so-called Elementary Class Theorem\(^9\) which provides a characterization for elementary classes. Clearly, given a preferential relation $\models \exists$ in $\mathcal{L}$, the class of all preferential models for $\models \exists$ is an elementary class because it can be defined by the set $\Sigma_p \cup \{\exists\}$. Thus, a natural problem raises at this point, namely how to characterize this kind of elementary classes? More generally, how to characterize elementary classes of preferential models defined by sentences in $BCA(\ell)$ or $PBCA(\ell)$? Since conditional assertions are just fragments of first-order languages, Elementary Class Theorem does not provide answers to these questions directly. In the following, we address this issue. We firstly recall a basic theorem from model theory, which is an ultraproduct version of the compactness theorem.

**Theorem 5.14.** Let $\Sigma$ be a set of first-order sentences. Let $I_o(\Sigma)$ be the set of all finite subsets of $\Sigma$, and for each $i \in I_o(\Sigma)$, let $\mu_i$ be a model of $i$. Then there exists an ultrafilter $D$ over $I_o(\Sigma)$ such that the ultraproduct $\Pi_D \mu_i$ is a model of $\Sigma$.

**Proof.** See Corollary 4.1.11 in [7]. $\square$

Let $K$ be a nonempty class of preferential models for a language $\ell$. $K$ is said to be closed under $\leftrightarrow$ if, for any models $M_1$ and $M_2$, $M_1 \leftrightarrow M_2$ and $M_1 \in K$ implies $M_2 \in K$. Similarly, $K$ is said to be closed under $\rightarrow$ if, for any models $M_1$ and $M_2$, $M_1 \rightarrow M_2$ and $M_1 \in K$ implies $M_2 \in K$. $K$ is said to be closed under ultraproducts if every ultraproduct $\Pi_D M_i$ of a family of models $M_i \in K$ belongs to $K$. $K$ is said to be closed under ultrapowers if $M \in K$ implies the ultrapower $\Pi_D M_i \in K$ for any ultrafilter $D$. In the following, we use $\overline{K}$ to denote the complement of $K$ within the class of all preferential models for $\mathcal{L}$.

$K$ is said to be defined by (positive) boolean combinations of conditional assertions if there exists a set $\Sigma \subseteq BCA(\ell)$ ($\Sigma \subseteq PBCA(\ell)$, respectively) such that, for any preferential model $W$ for $\ell$, $\mu_W \models \Sigma$ if and only if $W \in K$, that is, the class $\{\mu_M : M \in K\}$ is an elementary class defined by $\Sigma_p \cup \Sigma$. Moreover, if such set $\Sigma$ is finite, then, $K$ is said to be finitely defined by boolean combinations of conditional assertions.

**Theorem 5.15.** Let $K$ be a nonempty class of preferential models for a language $\ell$. $K$ is defined by boolean combinations of conditional assertions if and only if

1. $K$ is closed under $\leftrightarrow$.
2. $K$ is closed under ultraproducts.
3. $\overline{K}$ is closed under ultrapowers.

**Proof.** ($\Rightarrow$) Straightforward.

($\Leftarrow$) We put
\[
\Sigma = \{\theta \in BCA(\ell) : \theta \text{ holds in all } \mu_M \text{ such that } M \in K\}.
\]

Let $M$ be a preferential model for $\ell$ such that $\mu_M \models \Sigma$. It is enough to show that $M \in K$. Let $i$ be any finite nonempty subset of $(\models \neg M)^\circ$. We will show that there exists a model $W_i \in K$ such that $\mu_{W_i} \models i$. Suppose not. Then, for any $W \in K$, $\mu_W \models \neg \bigwedge_{\theta \in i} \theta$. Clearly, $\neg \bigwedge_{\theta \in i} \theta \in BCA(\ell)$, so, $\neg \bigwedge_{\theta \in i} \theta \in \Sigma$, which contradicts $\mu_M \models \Sigma$ and $\mu_M \models i$. Consequently, by Theorem 5.14, there exists an ultrafilter $D$ such that $\Pi_D W_i \models (\models \neg M)^\circ$. So, $\Pi_D W_i \equiv M$, moreover,

\(^9\) See Theorem 4.1.12 and Corollary 6.1.16 in [7].
by (2), $\Pi_D W_i \in K$. On the other hand, by Theorem 5.9, $\Pi_{D^*}(\Pi_D W_i) \leftrightarrow \Pi_{D^*} M$ for some ultrafilter $D^*$. Since $K$ is closed under ultraproducts, we have $\Pi_{D^*}(\Pi_D W_i) \in K$. Further, $\Pi_{D^*} M \in K$ follows from (1). Since $\overline{K}$ is closed under ultrapowers, we obtain $M \in K$, as desired. □

**Corollary 5.16.** Let $K$ be a nonempty class of preferential models for a language $\ell$. $K$ is finitely defined by boolean combinations of conditional assertions if and only if

1. $K$ is closed under $\leftrightarrow$.
2. $K$ is closed under ultraproducts.
3. $\overline{K}$ is closed under ultraproducts.

**Proof.** ($\Leftarrow$) Since $K$ is closed under $\leftrightarrow$, so is $\overline{K}$. Hence, by Theorem 5.15, both $K$ and $\overline{K}$ are defined by boolean combinations of conditional assertions. Let $K$ and $\overline{K}$ be defined by $\Sigma_1$ and $\Sigma_2$ respectively. So, $\Sigma_1 \cup \Sigma_2 \cup \Sigma_P$ is inconsistent. By the compactness, for some finite set $\Gamma_1 \subseteq \Sigma_1$ and $\Gamma_2 \subseteq \Sigma_2$, $\Gamma_1 \cup \Gamma_2 \cup \Sigma_P$ is inconsistent. It is easy to see that $K$ and $\overline{K}$ is defined by $\Gamma_1$ and $\Gamma_2$, respectively.

($\Rightarrow$) Let $K$ be defined by finite set $\Sigma \subseteq BCA(\ell)$. Thus, (1) and (2) follows from Theorem 5.15. Since $\Sigma$ is a finite subset of $BCA(\ell)$, $\neg(\bigwedge \Sigma) \in BCA(\ell)$. Clearly, $\overline{K}$ is defined by $\neg(\bigwedge \Sigma)$. So, by Theorem 5.15, $\overline{K}$ is closed under ultraproducts. □

**Theorem 5.17.** Let $K$ be a nonempty class of preferential models for a language $\ell$. $K$ is defined by positive boolean combinations of conditional assertions if and only if

1. $K$ is closed under $\rightarrow$.
2. $K$ is closed under ultraproducts.
3. $\overline{K}$ is closed under ultrapowers.

**Proof.** ($\Rightarrow$) Straightforward.

($\Leftarrow$) Let

$$\Sigma = \{ \theta \in PBCA(\ell) : \theta \text{ holds in all } \mu_M \text{ such that } M \in K \}.$$ 

Suppose that $M$ is a preferential model for $\ell$ such that $\mu_M = \Sigma$. It is sufficient to show that $M \in K$. We put

$$\Gamma = \{ \neg(\alpha|\neg\beta)^o : \alpha \vdash_M \beta \text{ and } \alpha, \beta \in \text{Form}(\ell) \}.$$ 

Let $i$ be any finite nonempty subset of $\Gamma$. We will prove that there exists a model $W_i \in K$ such that $\mu_{W_i} \models i$. Suppose not. Then, for any $W \in K$, $\mu_W \models \neg(\bigwedge_{\theta \in i} \theta)$. Clearly,

$$\models \neg(\bigwedge_{\theta \in i} \theta) \leftrightarrow \delta \text{ for some formula } \delta \in PBCA(\ell).$$

So, $\delta \in \Sigma$, which contradicts $\mu_M \models \Sigma$ and $\mu_M \models i$. Consequently, by Theorem 5.14, there exists an ultrafilter $D$ such that $\Pi_D W_i \models \Gamma$. Hence, $\models \neg(\Pi_D W_i) \subseteq \models M$. Further, by Theorem 5.9, $\Pi_{D^*}(\Pi_D W_i) \rightarrow \Pi_{D^*} M$ for some ultrafilter $D^*$. Due to (2), we have $\Pi_{D^*}(\Pi_D W_i) \in K$. Thus, $\Pi_{D^*} M \in K$ comes from the closeness of $K$ under $\rightarrow$. By (3), we have $M \in K$, as desired. □

**Corollary 5.18.** Let $K$ be a nonempty class of preferential models for a language $\ell$. $K$ is finitely defined by positive boolean combinations of conditional assertions if and only if

1. $K$ is closed under $\rightarrow$.
2. $K$ is closed under ultraproducts.
3. $\overline{K}$ is closed under $\leftrightarrow$.
4. $\overline{K}$ is closed under ultraproducts.

**Proof.** ($\Rightarrow$) Let $K$ be defined by finite set $\Sigma \subseteq PBCA(\ell)$. Thus, (1) and (2) follows from Theorem 5.17. Since $\Sigma$ is a finite subset of $PBCA(\ell)$, $\neg(\bigwedge \Sigma) \in BCA(\ell)$. It is easy to see that $\overline{K}$ is defined by $\neg(\bigwedge \Sigma)$. So, by Theorem 5.15, $\overline{K}$ is closed under ultraproducts and $\leftrightarrow$.

($\Leftarrow$) Similar to Corollary 5.16. □
Given a class $K$ of preferential models for a language $\ell$, $K$ is said to be defined by a preferential relation $\simeq$ if, for any preferential model $M$, $M \in K$ if and only if $\simeq_M = \simeq$. Then, we have the following result:

**Theorem 5.19.** Let $K$ be a nonempty class of preferential models for a language $\ell$. $K$ is defined by a preferential relation if and only if

1. For any $M_1, M_2 \in K$, there exists an ultrafilter $D$ such that $\Pi_D M_1 \iff \Pi_D M_2$.
2. $K$ is closed under $\iff$.
3. $K$ is closed under ultrapowers.
4. $\overline{K}$ is closed under ultrapowers.

**Proof.** ($\Rightarrow$) Easy.

($\Leftarrow$) Since $M \equiv \Pi_D M$ for any preferential model $M$ and ultrafilter $D$, by (1) and Theorem 5.6, we have $M_1 \equiv M_2$ for any $M_1, M_2 \in K$. Hence, we may suppose that $\simeq_M = \simeq$ for any $M \in K$. Let $W$ be a preferential model such that $\simeq = \simeq_W$. Thus, $W \equiv M$ for some (equivalently, every) model $M \in K$. So, by Theorem 5.6, $\Pi_D W \iff \Pi_D M$ for some ultrafilter $D$. By (2) and (3), we have $\Pi_D W \in K$, further, $W \in K$ follows from (4).

### 6. Some model-theoretic results about preferential models

Since first-order translation provides an approach of using results and methods from first-order model theory, it is a powerful tool of exploring properties of preferential models and preferential inferences. To illustrate this, we will give some model-theoretic results about preferential models. These results immediately follow from first-order translation and first-order model theory, however, it seems to me that it is nontrivial to show them without the help of first-order model theory.

In the following, for convenience, the pair $\langle S, < \rangle$ is called a poset if the binary relation $<$ over $S$ is transitive and irreflexive. Let $\Omega$ be a class of posets, a preferential model $\langle S, l, < \rangle$ is said to be from $\Omega$ if the poset $\langle S, < \rangle$ belongs to $\Omega$. We use $\Omega(\ell)$ to denote the class of all preferential models for $\ell$ coming from $\Omega$.

**Definition 6.1 (Zhu et al. [19]).** Let $\simeq$ be an inference relation in $\ell$ and $\ell_0$ a sublanguage of $\ell$, the reduct of $\simeq$ with respect to $\ell_0$ is $\simeq \cap (\text{Form}(\ell_0))^2$ and denoted by $\simeq_{\ell_0}$.

**Definition 6.2 (Zhu et al. [19]).** Let $W = \langle S, l, < \rangle$ be a preferential model for $\ell$ and $\ell_0$ a sublanguage of $\ell$. The reduct of $W$ with respect to $\ell_0$ is a triple $\langle S_0, l_0, <_0 \rangle$ such that $S_0 = S$, $<_0 = <$, and for any $s \in S$, $l_0(s)$ is the reduction of $l(s)$ with respect to $\ell_0$ (i.e., $l_0(s) = l(s) \cap \ell_0$). In the following, the triple $\langle S_0, l_0, <_0 \rangle$ will be denoted by $W_{\ell_0}$.

**Proposition 6.1 (Compactness).** Let $\Omega$ be a class of posets which is closed under ultraproducts, and let $\simeq$ be a preferential relation in a language $\ell$. If for any finite sublanguage $\ell_0 \subseteq \ell$, the reduction $\simeq_{\ell_0}$ has models from $\Omega$, then so does the relation $\simeq$ itself.

**Proof.** Let $\Sigma = \simeq_{\ell_0}$. We demonstrate the following claims.

**Claim 1.** For any $i \in I_\ell(\Sigma)$, we have $\mu_{M_i} \models i$ for some preferential model $M_i$ from $\Omega(\ell)$.

Clearly, $i$ is a set of sentences of $\mathcal{L}_\ell$. We put

$$\ell_0 = \{ p : p \in \ell \text{ such that corresponding relation symbol } P \text{ occurs in } i \}.$$ 

Since $i$ is finite, so is $\ell_0$. So, there exists a preferential model $M_i^* = \langle S_i^*, l_i^*, <_i^* \rangle$ from $\Omega$ such that $\simeq_{M_i^*} = \simeq_{\ell_0}$. Obviously, $i \subseteq (\simeq_{M_i^*})^\ell$. So, by Lemma 3.3, $\mu_{M_i^*} \models i$. Since $\mu_{M_i^*}$ is a model for the language $\mathcal{L}_{\ell_0}$, in order to complete the proof of this claim, we need an expansion of $\mu_{M_i^*}$ to the language $\mathcal{L}_\ell$. Thus, we define $M_i = \langle S_i, l_i, <_i \rangle$ as follows:

1. $S_i = S_i^*$.
2. $<_i = <_{i^*}$.
3. For any $p \in \ell$ and $s \in S_i$, $p \in l_i(s)$ iff $p \in \ell_0$ and $p \in l_i^*(s)$.
We will prove that $M_i$ is a preferential model for $\ell$. It is enough to show $M_i$ is smooth. Let $\pi(p_1, p_2, \ldots, p_n)$ be any formula of $\ell$ and $p_1, p_2, \ldots, p_n$ be all propositional symbols occurring in $\pi$ but not in $\ell_0$. It is easy to see that
\[ \|\pi(p_1, p_2, \ldots, p_n)\|_{M_i} = \|\pi(\bot, \bot, \ldots, \bot)\|_{M_i}, \]
where $\pi(\bot, \bot, \ldots, \bot)$ is the formula obtained from $\pi(p_1, p_2, \ldots, p_n)$ by substituting $\bot$ for $p_i (i = 1, 2, \ldots, n)$. Since the set $\|\pi(\bot, \bot, \ldots, \bot)\|_{M_i}$ is smooth, so is $\|\pi(p_1, p_2, \ldots, p_n)\|_{M_i}$. Hence, $M_i$ is a preferential model for $\ell$ and $(S_i, <_i) \in \Omega$. Obviously, $\mu_{M_i}$ is an expansion of $\mu_{M_i}$ to the language $\cal{L}_\ell$ and $\mu_{M_i} \vDash i$.

Claim 2. There exists a preferential model $W$ from $\Omega$ such that $|\sim_W = |\sim|$.

By Claim 1 and Theorem 5.14, there exists an ultrafilter $D$ over $I_\ell(\Sigma)$ such that $\Pi D M_i \vDash \Sigma$. Similar to Theorem 3.5, it is easy to see that $\Pi D M_i$ is a preferential model such that $|\sim_{\Pi D M_i} = |\sim$. Further, since $\Omega$ is closed under ultraproducts, $\Pi D M_i$ is from $\Omega$.

Let $\zeta$ be a property of posets definable by first-order language, in other words, the class $\{ (S, <): (S, <) \text{ is a poset satisfying } \zeta \}$ be an elementary class. Then, the above proposition implies that: for any inference relation $|\sim$, if any reduct of $|\sim$ with respect to finite sublanguage may be generated by models with the property $\zeta$, then so does $|\sim$ itself.

Proposition 6.2 (Existence of $P$-saturated model). Let $\Omega$ be a class of posets which is closed under ultraproducts. For any language $\ell$ and preferential model $M$ from $\Omega$, there exists a preferential model $W$ such that:

1. $W$ comes from $\Omega$.
2. $M \equiv W$.
3. $W$ is $P$-saturated.

Proof. Similar to Theorem 4.11. □

Thus, any preferential model $M$ has an equivalent model which is $P$-saturated and has the same first-order properties as $M$.

Proposition 6.3. Let $|\sim$ be a preferential relation in $\ell$. If $|\sim$ may be generated by some infinite preferential models, then it may be generated by infinite models of any given power $\pi \geq \|\ell\|$.

Proof. Let $\Sigma = |\sim^{\circ} \cup \Sigma_P$. Since $|\sim$ can be generated by an infinite preferential model, $\Sigma$ has infinite model. So, by Löwenheim–Skolem–Tarski Theorem\(^\text{10}\) from first-order model theory, for any cardinal number $\pi \geq \|\ell\| = \|\Sigma\|$, $\Sigma$ has model with size $\pi$. Thus, by Corollary 3.4, $|\sim$ may be generated by infinite models of any given power $\pi \geq \|\ell\|$. □

We denote the class of all injective models for $\ell$ by $IM(\ell)$. For any finite language $\ell$, since there are only finitely many injective models up to isomorphism and those injective models are finite, the class $\{ \mu_M: \mu \in IM(\ell) \}$ can be characterized by a first-order sentence of the language $\cal{L}_\ell$. However, it is false when the language is infinite.

Proposition 6.4. For any infinite language $\ell$, the class $\{ \mu_M: \mu \in IM(\ell) \}$ can not be characterized by first-order sentences. In other words, it is not an elementary class of $\cal{L}_\ell$.

Proof. Suppose that there exists a set $\Pi$ of first-order sentences of $\cal{L}_\ell$ such that $\mu_M = \Pi$ if $M \in IM(\ell)$. Since $\ell$ is infinite, there exists an infinite model in $IM(\ell)$ (for instance, considering an infinite antichain). Hence, $\Pi$ has infinite models. So, by Löwenheim–Skolem–Tarski Theorem, $\Pi$ has an infinite model $\mu_M$ with size $\pi > 2|\ell|$. Clearly, $M \not\in IM(\ell)$, a contradiction. □

Remark. A preferential model may be regarded as a Kripke model of basic modal language which contains only diamond $\diamond$, thus it makes perfect sense to consider whether the class $IM(\ell)$ is definable by basic modal formulas.

\(^\text{10}\)See Corollary 2.1.6 in [7].
Since any modal formula is equivalent to some first-order sentences when considering global truth on models [2], by the above proposition, the class $IM(\ell)$ is indefinable by basic modal formulas for any infinite language $\ell$. Moreover, even for finite language $\ell$, since $IM(\ell)$ is not closed under disjoint unions, the class $IM(\ell)$ is also indefinable by basic modal formulas.

One of the important topics in the study of nonmonotonic inference relations is establishing representation theorems for them. A number of representation theorems have been established in the literature [1,8,10,11,15,17,18]. Suppose that $\Delta$ is a set of properties of inference relations (e.g., Horn or non-Horn conditions defined in the style of Gentzen) and $\Omega(\ell)$ is a class of preferential models for a language $\ell$. A representation theorem $RTH(\Omega(\ell), \Delta)$ usually consists of the following two statements:

(i) If an inference relation $|\sim$ satisfies all properties in $\Delta$ then there exists a preferential model $W \in \Omega(\ell)$ such that $|\sim = |\sim_W$.

(ii) For any preferential model $W \in \Omega(\ell)$, the relation $|\sim_W$ satisfies all properties in $\Delta$.

A general rule [5] is a rule of the form

$$\Gamma \models \Sigma(p_1, p_2, \ldots, p_n),$$

where both $\Gamma$ and $\Sigma$ are finite sets of conditional assertions and $p_1, p_2, \ldots, p_n$ are all propositional symbols occurring in $\Gamma \models \Sigma$. Following [5], we say that a relation $|\sim$ in $\ell$ satisfies the general rule $\Gamma \models \Sigma(p_1, p_2, \ldots, p_n)$ if and only if, for any formulas $x_1, x_2, \ldots, x_n$ of $\ell$,

$$\Gamma \left\{ \frac{x_i}{p_i} \right\} \subseteq |\sim \text{ implies } \Sigma \left\{ \frac{x_i}{p_i} \right\} \cap |\sim \neq \emptyset,$$

where $\Sigma \left\{ \frac{x_i}{p_i} \right\}$ (resp., $\Gamma \left\{ \frac{x_i}{p_i} \right\}$) is a set of conditional assertions obtained from $\Sigma$ (resp., $\Gamma$) by substituting $x_i$ for $p_i(i = 1, 2, \ldots, n)$. Obviously, both Horn and non-Horn rules introduced in the literature [1,8,10–12] are general rules.

**Theorem 6.5.** Let $\Omega$ be a class of posets which is closed under ultraproducts, and let $\Delta$ be a set of general rules. Then the following are equivalent:

1. $RTH(\Omega(\ell), \Delta)$ holds for any finite language $\ell$.
2. $RTH(\Omega(\ell), \Delta)$ holds for any language $\ell$.

**Proof.** (1$\Rightarrow$2) It is enough to demonstrate the following claims.

**Claim 1.** For any language $\ell$ and any preferential model $M \in \Omega(\ell)$, $|\sim_M$ satisfies $\Delta$.

Suppose that, for some general rule $\Gamma \models \Sigma(p_1, p_2, \ldots, p_n) \in \Delta$, $|\sim_M$ does not satisfy it. So, there are formulas $x_i (1 \leq i \leq n)$ such that

$$\Gamma \left\{ \frac{x_i}{p_i} \right\} \subseteq |\sim_M \text{ and } \Sigma \left\{ \frac{x_i}{p_i} \right\} \cap |\sim_M = \emptyset.$$

Let $\ell_0 = \bigcup_{1 \leq i \leq n} \text{atm}(x_i)$, where $\text{atm}(x_i)$ is the set of all propositional symbols occurring in $x_i$. So, $M_{\not\models \ell_0}$ does not satisfy the rule $\Gamma \models \Sigma$. Further, since $\ell_0$ is finite, a contradiction comes from (1) and $M_{\not\models \ell_0} \in \Omega(\ell_0)$.

**Claim 2.** For any language $\ell$ and preferential relation $|\sim$ in $\ell$, if $|\sim$ satisfies $\Delta$ then there exists a model $M \in \Omega(\ell)$ such that $|\sim_M = |\sim$.

Since $|\sim$ satisfies $\Delta$, so does the reduction $|\sim_{\not\models \ell_0}$ for any finite sublanguage $\ell_0 \subseteq \ell$. So, by (1), for any finite sublanguage $\ell_0 \subseteq \ell$, we have $|\sim_{\not\models \ell_0} = |\sim_{M_{\not\models \ell_0}}$ for some model $M_{\not\models \ell_0} \in \Omega(\ell_0)$. Thus, by Proposition 6.1, we obtain $|\sim_M = |\sim_M$ for some model $M \in \Omega(\ell)$, as desired.

(2$\Rightarrow$1) Trivially. \(\square\)

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11 The definition of the disjoint unions may be found in [2].
Consequently, given an elementary class $\Omega$, in order to establish the representation theorem $RTH(\Omega(\ell), \Lambda)$ for any language $\ell$, it is enough to consider only the finite language case.

7. Conclusion

In this paper we explored the notion of similarity for preferential models and characterized the equivalence of models in terms of similarity. As application of the main theorem obtained in this paper, we investigated the expressive power of conditional assertions and provided the characterization for the class of preferential models defined by Boolean combinations of conditional assertions.

First-order translation originating from modal logic is of basic importance in this paper, through which we can apply results and techniques from first-order model theory to nonmonotonic logic. Thus, we believe that first-order translation is a powerful tool in the study of nonmonotonic inference relations, moreover, such idea is useful for any nonclassical logic if its semantic may be expressed in first-order logic.

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