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A new method for obtaining the stress field in plane contacts

J. Vázquez *, C. Navarro, J. Domínguez

Departamento de Ingeniería Mecánica, Universidad de Sevilla, Camino de los Descubrimientos s/n, C.P 41092 Sevilla, Spain

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ABSTRACT

This paper presents two valuable procedures used to calculate the stress field in plane contacts between a punch and a half-plane in partial slip regime. These procedures greatly simplify both the Muskhelishvili potential and the calculation of direct stresses produced on the contact surface, and, therefore, the stress field in the half-plane is simplified as well. The results are applicable when the contacting bodies have isotropic elastic behaviour and identical mechanical properties. It is further assumed that both bodies behave as a semi-infinite medium. To validate the procedures obtained here, they are applied to two cases for which analytical solutions already exist.

1. Introduction

Contact is the principal method of applying loads between deformable solids, and therefore is present in a wide variety of mechanical components. In addition, contacts usually act as stress concentrations, and are thus probable locations for mechanical failure. Some of the most typical mechanical failures involving contact include: fretting, fretting fatigue, wear, fretting wear and false brinelling. These common failures make contact mechanics one of the principal developing areas in solids mechanics and the focus of many researchers.

One essential characteristic of a contact is the presence or absence of friction. Frictionless contacts only exhibit a pressure normal to the contact surface. While in frictional contacts, a shear stress field can also be produced at the contact surface, which, lead to a partial slip condition in which stick and slip zones are developed inside the contact area.

Due to the stress concentration that contacts constitute, the stress and strain fields taking place between two bodies in contact, represent some of the main objectives of contact mechanics, since these fields determine the mechanical behaviour of the material affected by the contact. Despite this, it is only possible to give an analytical solution to these stress and strain fields in a limited number of contact problems. As a result, many numerical methods have been developed. In the case of a two-dimensional contact between a punch and a half-plane, one of the most useful tools for implicitly obtaining the stress and strain fields produced in the interior of both bodies is the Muskhelishvili complex potential (Muskhelishvili, 1954). In the case of partial slip regime this potential is given by the following line integral along the contact zone:

$$\phi(z) = \frac{1}{2\pi i}\int_{contact}\frac{\sigma_{yy}(t,0)+i\sigma_{xy}(t,0)}{t-z}dt, \tag{1}$$

where $\sigma_{yy}(x,0)$ and $\sigma_{xy}(x,0)$ represent the distributions of normal and shear stresses on the surface respectively, and where $t\in\mathbb{R}$ and $z=x+iy\in\mathbb{C}$. It is assumed that the indenter is in the plane xy defined by $y>0$, and the half- plane is in $y\leqslant 0$, as shown in Fig. 1.

Once the complex potential, $\phi(z)$, is obtained, the interior stress field in the half-plane is given by Muskhelishvili (1954):

$$\sigma_{xx}(x,y)+\sigma_{yy}(x,y) = 2\left[\phi(z)+\overline{\phi(z)}\right] \tag{2}$$

$$\sigma_{yy}(x,y)-\sigma_{xx}(x,y)+2i\sigma_{xy}(x,y) = 2\left[(\bar{z}-z)\frac{d}{dz}\phi(z)-\phi(z)-\overline{\phi(\bar{z})}\right], \tag{3}$$

where, if the complex potential is defined by $\phi(z)=u(x,y)+iv(x,y)$, then $\overline{\phi(z)}=u(x,y)-iv(x,y)$ and $\overline{\phi(\bar{z})}=u(x,-y)-iv(x,-y)$.

Another important parameter is the direct stress or normal stress parallel to the surface, $\sigma^t_{xx}(x,0)$, due to the contact shear stress, $\sigma_{xy}(x,0)$, which is given by the Flamant equation (Johnson, 1985):

$$\sigma^t_{xx}(x,0) = \frac{2}{\pi}\int_{contact}\frac{\sigma_{xy}(t,0)}{t-x}dt, \tag{4}$$

where, depending on the location of x, the above integral is understood in the sense of Cauchy principal value – x inside the contact zone –, or in the ordinary (Riemann) sense – x outside the contact zone. The value of this direct stress at the trailing edge of the

* Corresponding author. Tel.: +34 954487387.
E-mail address: jesusvaleo@esi.us.es (J. Vázquez).

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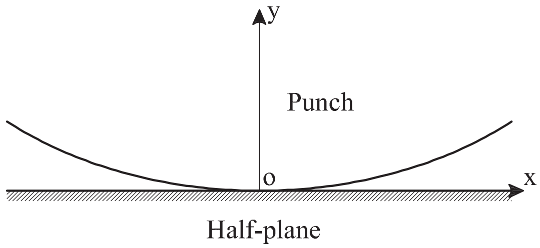

Fig. 1. Position of the Punch and the half-plane in the xy plane.

contact is of great interest in the fretting fatigue phenomenon (Domínguez, 1998; Navarro et al., 2006).

In view of Eqs. (1) and (4), it is clear that in general it is not simple to perform a direct integration of these expressions. In the case of the Muskhelishvili potential, to overcome the integration of the complicated integral shown in Eq. (1), the normal and shear stress fields at the surface, $\sigma_{yy}(x,0)$ and $\sigma_{xy}(x,0)$, can be expressed by means of a Chebyshev or Legendre series (Ciavarella et al., 1998a,b), and then integrated. On the other hand, the integral in Eq. (4) can be evaluated using the Clenshaw–Curtis numerical integration (Nowell and Hills, 1987). To greatly facilitate obtaining analytical expressions for $\phi(z)$ and $\sigma^t_{xx}(x,0)$, this paper presents a novel and simple way to calculate both expressions without the need for performing any type of integration. The procedure used herein is based on the relationship between the Cauchy principal value of a singular integral and the value of such an integral obtained in the ordinary (Riemann) sense. This relationship is given by Muskhelishvili (1954):

$$\frac{1}{2\pi i}C.P.V.\int_a^b \frac{f(t)}{t-t_0}dt = \frac{1}{2}f(t_0) + \frac{1}{2\pi i}\int_a^b \frac{f(t)}{t-t_0}dt, \tag{5}$$

where the abbreviation $C.P.V.$ indicates that the Cauchy principal value of the line integral must be considered. In Eq. (5) again $t \in \mathbb{R}$ and $t_0 \in [a,b]$. The integral on the right side of Eq. (5) cannot be obtained in the ordinary or Riemann sense, but it is possible to evaluate it by means of its indefinite integral

$$\frac{1}{2\pi i}\int \frac{f(t)}{t-t_0}dt, \tag{6}$$

after being evaluated at $t=a$ and $t=b$. With respect to $f(t)$, it must be integrable in an ordinary sense into $[a,b]$.

As an example, considering $f(t)=t$ and $t_0=1/2$ the Cauchy principal value of the next integral

$$C.P.V.\int_{-1}^{1} \frac{t}{t-1/2}dt \tag{7}$$

taking into account that

$$\int \frac{t}{t-1/2}dt = t + \frac{1}{2}\ln(2t-1) \tag{8}$$

can be calculated as follows:

$$\begin{aligned} C.P.V.\int_{-1}^{1} \frac{t}{t-1/2}dt &= \pi i f(1/2) + \left[t + \frac{1}{2}\ln(2t-1)\right]_{t=1} \\ &\quad - \left[t + \frac{1}{2}\ln(2t-1)\right]_{t=-1} \\ &= 2 - \frac{1}{2}\ln(3). \end{aligned} \tag{9}$$

2. Half-Plane surface stress $\sigma^t_{xx}(x,0)$ due to tangential load

In this section it will be assumed that a punch is pressed against the half-plane with a normal force, N, so that a contact zone is generated. Next, a tangential load, $Q < \mu N$, is applied producing a partial slip condition. In this situation, it can be shown that inside the stick zone the following singular integral equation must be fulfilled (Johnson, 1985):

$$C.P.V.\left[-\frac{1}{\pi}\int_{contact} \frac{\sigma_{xy}(t,0)}{t-x}dt\right] = k, \quad \forall\, x \in \text{stick zone}, \tag{10}$$

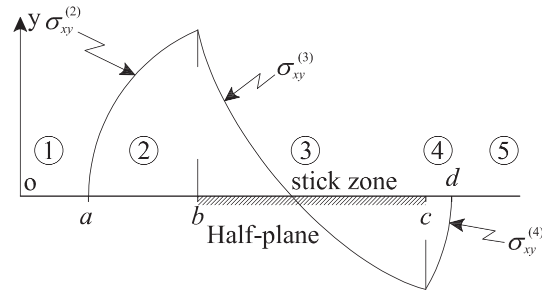

Fig. 2. A general shear stress distribution at the contact surface in partial slip conditions.

where k is a constant that depends on the type of problem under study.

In view of Eqs. (4) and (10), it seems feasible to use the relationship given by Eq. (5) to obtain the stress distribution along the surface $\sigma^t_{xx}(x,0)$. For this, a contact in partial slip conditions with a surface shear stress distribution like the one shown in Fig. 2 can be assumed. In this figure the half-plane surface has been divided into the following 5 areas:

$$\begin{aligned} &\text{Zone 1}: x < a, \quad \sigma_{xy}(x,0) = 0 \\ &\text{Zone 2}: a \leqslant x < b, \quad \sigma_{xy}(x,0) = \sigma^{(2)}_{xy}(x,0) \\ &\text{Zone 3}: b \leqslant x \leqslant c, \quad \sigma_{xy}(x,0) = \sigma^{(3)}_{xy}(x,0) \\ &\text{Zone 4}: c < x \leqslant d, \quad \sigma_{xy}(x,0) = \sigma^{(4)}_{xy}(x,0) \\ &\text{Zone 5}: x > d, \quad \sigma_{xy}(x,0) = 0, \end{aligned}$$

where $\sigma^{(i)}(x,0)$ are the surface stress distributions in the zones $i=1,\ldots,5$. Further developing Eq. (4) and taking into account Eq. (10) the direct stress inside the stick zone ($b \leqslant x \leqslant c$) can be expressed by:

$$\sigma^{t,(3)}_{xx}(x,0) = -2k = \frac{2}{\pi}C.P.V.\left[\int_a^d \frac{\sigma_{xy}(t,0)}{t-x}dt\right], \tag{11}$$

which can also be written as:

$$\begin{aligned} \sigma^{t,(3)}_{xx}(x,0) &= -2k \\ &= \frac{2}{\pi}\left[\int_a^b \frac{\sigma^{(2)}_{xy}(t,0)}{t-x}dt + C.P.V.\int_b^c \frac{\sigma^{(3)}_{xy}(t,0)}{t-x}dt + \int_c^d \frac{\sigma^{(4)}_{xy}(t,0)}{t-x}dt\right]. \end{aligned} \tag{12}$$

Similarly developing Eq. (4) when $x < a$:

$$\sigma^{t,(1)}_{xx}(x,0) = \frac{2}{\pi}\left[\int_a^b \frac{\sigma^{(2)}_{xy}(t,0)}{t-x}dt + \int_b^c \frac{\sigma^{(3)}_{xy}(t,0)}{t-x}dt + \int_c^d \frac{\sigma^{(4)}_{xy}(t,0)}{t-x}dt\right]. \tag{13}$$

Considering Eqs. (12) and (13), it is easy to see that the only difference between the right hand sides of these equations is that the former is calculated as $C.P.V.$ while the latter is calculated in the ordinary sense. Subtracting both equations gives

$$\sigma^{t,(1)}_{xx}(x,0) + 2k = \frac{2}{\pi}\left[\int_b^c \frac{\sigma^{(3)}_{xy}(t,0)}{t-x}dt - C.P.V.\int_b^c \frac{\sigma^{(3)}_{xy}(t,0)}{t-x}dt\right]. \tag{14}$$

Taking into account the relationship given by Eq. (5) and after some algebraic manipulations the following equation is obtained

$$\sigma_{xx}^{t,(1)}(x,0) = -2k - 2i\sigma_{xy}^{(3)}(x,0). \tag{15}$$

The simplicity of the above procedure cannot be denied, producing as it does, quickly and directly the direct stress on the contact surface $\sigma_{xx}^{t}(x,0)$ given only the shear stresses developed on the surface and, regardless of the indenter geometry.

Similarly to the previous procedure, it is possible to obtain the direct stress field, $\sigma_{xx}^{t,(i)}(x,0)$, on the rest of the surface.

To obtain $\sigma_{xx}^{t,(2)}(x,0)$, first Eq. (4) is developed when $a \leqslant x < b$, which leads to:

$$\sigma_{xx}^{t,(2)}(x,0) = \frac{2}{\pi}\left[C.P.V.\int_a^b \frac{\sigma_{xy}^{(2)}(t,0)}{t-x}dt + \int_b^c \frac{\sigma_{xy}^{(3)}(t,0)}{t-x}dt + \int_c^d \frac{\sigma_{xy}^{(4)}(t,0)}{t-x}dt\right]. \tag{16}$$

Therefore, recalling Eq. (13) the stress $\sigma_{xx}^{t,(2)}(x,0) - \sigma_{xx}^{t,(1)}(x,0)$ must be equal to

$$\sigma_{xx}^{t,(2)}(x,0) + 2k + 2i\sigma_{xy}^{(3)}(x,0) = \frac{2}{\pi}\left[C.P.V.\int_a^b \frac{\sigma_{xy}^{(2)}(t,0)}{t-x}dt - \int_a^b \frac{\sigma_{xy}^{(2)}(t,0)}{t-x}dt\right]. \tag{17}$$

Re-calculating and considering Eq. (5) leads to

$$\sigma_{xx}^{t,(2)}(x,0) = -2k - 2i\sigma_{xy}^{(3)}(x,0) + 2i\sigma_{xy}^{(2)}(x,0). \tag{18}$$

The distribution $\sigma_{xx}^{t,(5)}(x,0)$ can be found in a similar manner to that of $\sigma_{xx}^{t,(1)}(x,0)$, producing

$$\sigma_{xx}^{t,(5)}(x,0) = -2k - 2i\sigma_{xy}^{(3)}(x,0). \tag{19}$$

Similarly to the way in which $\sigma_{xx}^{t,(2)}(x,0)$ is calculated, the distribution $\sigma_{xx}^{t,(4)}(x,0)$ is now achieved thusly:

$$\sigma_{xx}^{t,(4)}(x,0) = -2k - 2i\sigma_{xy}^{(3)}(x,0) + 2i\sigma_{xy}^{(4)}(x,0). \tag{20}$$

Although the procedure developed here has been applied to a contact area with two sliding areas and one adhesion area, it is adaptable for other contact configurations, such as a single slip zone commonly occurring in rolling contacts (Carter, 1926).

3. The Muskhelishvili potential

The Muskhelishvili potential, given above by Eq. (1), can be divided into two parts, one due to the normal stress distribution $\sigma_{yy}(x,0)$, designated by $\phi^n(z)$, and the other due to the shear stress $\sigma_{xy}(x,0)$ and designated as $\phi^t(z)$, so that

$$\phi^n(z) = \frac{1}{2\pi i}\int_a^d \frac{\sigma_{yy}(t,0)}{t-z}dt \tag{21}$$

$$\phi^t(z) = \frac{1}{2\pi}\int_a^d \frac{\sigma_{xy}(t,0)}{t-z}dt. \tag{22}$$

Eq. (21) is quite similar to the integral equation that relates the profile of the indenter, $\upsilon(x)$, with the normal stress distribution, $\sigma_{yy}(x,0)$, which is given by

$$\frac{d}{dx}\upsilon(x) = C.P.V.\left[-\frac{A}{\pi}\int_a^d \frac{\sigma_{yy}(t,0)}{t-x}dt\right] \tag{23}$$

In Eq. (23), A is a constant that in plane strain conditions is equal to $4(1-\nu^2)/E$ and where, again, except for the constants involved, the only difference between Eqs. (21) and (23) lies in the way in which both integrals are obtained.

After using algebraic operations and evaluating the relationship given by Eq. (5), Eq. (23) can be expressed as

$$\frac{1}{2\pi i}\int_a^d \frac{\sigma_{yy}(t,0)}{t-x}dt = \frac{i}{2A}\upsilon'(x) - \frac{1}{2}\sigma_{yy}(x,0) \tag{24}$$

A comparison of Eq. (24) with the definition of $\phi^n(z)$ previously given by Eq. (21) shows that, after exchanging x for z

$$\phi^n(z) = \frac{i}{2A}\upsilon'(z) - \frac{1}{2}\sigma_{yy}(z). \tag{25}$$

This result was partially achieved by Adibnazari and Sharafbafi (2008), but, because the term, $i\upsilon'(z)/2A$, was not included directly in that work, it is necessary to obtain such a term by means of the complete form, both real and imaginary, of the pressure distribution which is obtained by inverting the integral equation Eq. (23). This fact illustrates that the method presented here is one of direct application; extra integration is made unnecessary.

The term of the potential $\phi(z)$, corresponding to $\sigma_{xy}(x,0)$, $\phi^t(z)$, can be obtained using the expressions given by Eqs. (12) and (22). After some algebraic operations and using the relationship given by Eq. (5), Eq. (12) can be expressed as follows:

$$\frac{1}{2\pi}\left[\int_a^b \frac{\sigma_{xy}^{(2)}(t,0)}{t-x}dt + \int_b^c \frac{\sigma_{xy}^{(3)}(t,0)}{t-x}dt + \int_c^d \frac{\sigma_{xy}^{(4)}(t,0)}{t-x}dt\right] = -\frac{k}{2} - \frac{1}{2}i\sigma_{xy}^{(3)}(x,0). \tag{26}$$

Comparing Eq. (26) with Eq. (22), and exchanging x for z yields

$$\phi^t(z) = -\frac{k}{2} - \frac{1}{2}i\sigma_{xy}^{(3)}(z), \tag{27}$$

Again, this result shows a similarity with the result previously obtained by Adibnazari and Sharafbafi (2008), but with two important differences: in the previous work, a full sliding condition was considered and the term, $-k/2$, was not included.

With these clarifications in mind the Muskhelishvili potential, $\phi(z)$, is obtained directly as

$$\phi(z) = \phi^n(z) + \phi^t(z) = \frac{1}{2}\left[-\sigma_{yy}(z) - i\sigma_{xy}^{(3)}(z) + \frac{i}{A}\upsilon'(z) - k\right]. \tag{28}$$

This result shows that the stress field at any point in the interior of the half-plane can be easily calculate by utilising the analytical expressions of the normal and shear stresses on the area of contact. The parameter, k, which is defined by the problem under study, and the profile of the indenter, $\upsilon(x)$, are parameters that must be known. Thus, this procedure avoids the integration of Eq. (1) to obtain the Muskhelishvili complex potential.

4. Application examples

To verify the above results, they are applied in the following sections to two cases for which analytical solutions already exist. The first case analysed is one in which a cylindrical punch with radius, R, is pressed against the half-plane with a normal force, N. Then a tangential force, Q, and a bulk stress parallel to the surface, σ, are applied simultaneously. In the second case, a wedge-shaped punch is initially pressed against a half-plane with a normal load,N, and later, a tangential force, Q, is applied to the punch.

4.1. Contact between a half-plane and cylinder in the presence of bulk and tangential load

This situation is shown schematically in Fig. 3, which qualitatively shows the stress distributions developed on the contact surface.

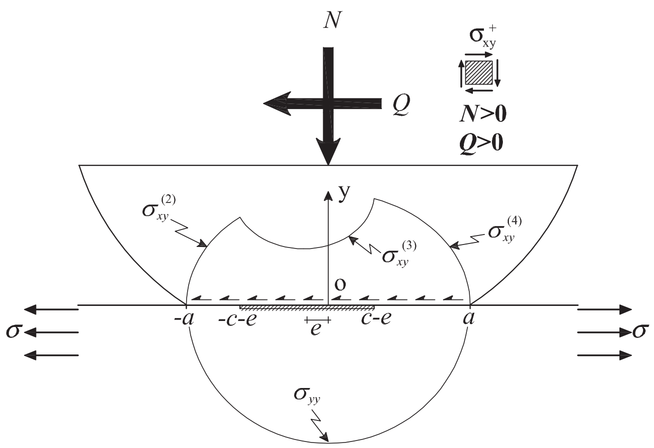

Fig. 3. Schematic view for a cylindrical contact subjected to normal and tangential loads.

In this situation the distribution of normal stresses at the contact surface is defined by Johnson (1985) and Hills and Nowell (1994):

$$\sigma_{yy}(x,0) = -p_0\sqrt{1-(x/a)^2}, \quad |x| \leqslant a \tag{29}$$

where $a=\sqrt{\frac{8NR(1-\nu^2)}{\pi E}}$, $p_0=\frac{2N}{\pi a}$ and ν and E are the Poisson and Young's modulus, respectively. Furthermore, due to the simultaneous application of the tangential load Q and the bulk load, σ, an eccentric stick zone along the contact surface is developed in $|x+e| \leqslant c$, where $c=a\sqrt{1-Q/(\mu N)}$ and $e=\frac{R\sigma(1-\nu^2)}{\mu E}$. The shear stress distribution in this zone is given by

$$\sigma_{xy}^{(3)}(x,0) = -\mu p_0\sqrt{1-\left(\frac{x}{a}\right)^2} + \mu p_0\frac{c}{a}\sqrt{1-\left(\frac{x+e}{c}\right)^2}, \quad |x+e| \leqslant c. \tag{30}$$

In the slip zone, the shear stresses are defined by:

$$\sigma_{xy}^{(2)}(x,0) = \sigma_{xy}^{(4)}(x,0) = -\mu p_0\sqrt{1-(x/a)^2}, \quad |x+e|>c,\ |x| \leqslant a \tag{31}$$

Before applying the results of the previous sections, it is important to note that the function $\sqrt{z}$ has two possible branches, i.e., $\pm\sqrt{z}$, and therefore it is necessary to choose the branch that makes physical sense for the problem at hand. For example, to obtain σ_{xx}^{t} when $x<-a$, and considering the sign of the shear stress distribution at the surface, it is expected that $\sigma_{xx}^{t,(1)}(x,0)<0$ and $\sigma_{xx}^{t,(1)}(-\infty,0)\to 0$. Applying Eq. (15), taking the appropriate branches for $\sqrt{z}$ and remembering that in this situation $k=\sigma/4$ (Hills and Nowell, 1994), the following expression is obtained

$$\sigma_{xx}^{t,(1)}(x,0) = -\sigma/2 - 2i\left[\mu p_0\sqrt{1-(x/a)^2} - \mu p_0\frac{c}{a}\sqrt{1-\left(\frac{x+e}{c}\right)^2}\right], \quad x<-a. \tag{32}$$

Similarly after applying Eq. (18) $\sigma_{xx}^{t,(2)}(x,0)$ the following equation is obtained

$$\begin{aligned}\sigma_{xx}^{t,(2)}(x,0) &= -\sigma/2 - 2i\left[\mu p_0\sqrt{1-(x/a)^2} - \mu p_0\frac{c}{a}\sqrt{1-\left(\frac{x+e}{c}\right)^2}\right] \\ &\quad + 2i\mu p_0\sqrt{1-(x/a)^2} \\ &= -\sigma/2 + 2i\mu p_0\frac{c}{a}\sqrt{1-\left(\frac{x+e}{c}\right)^2}, \quad -a \leqslant x < -c-e.\end{aligned} \tag{33}$$

The calculation of $\sigma_{xx}^{t,(3)}(x,0)$ by means of Eq. (12) is trivial, and therefore $\sigma_{xx}^{t,(3)}(x,0)=-\sigma/2$. While obtaining the appropriate branches for $\sigma_{xx}^{t,(5)}(x,0)$, it must be taken into account that $\sigma_{xx}^{t,(5)}(x,0)>0$ and $\sigma_{xx}^{t,(5)}(\infty,0)\to 0$, leading to

$$\sigma_{xx}^{t,(5)}(x,0) = -\sigma/2 + 2i\left[\mu p_0\sqrt{1-(x/a)^2} - \mu p_0\frac{c}{a}\sqrt{1-\left(\frac{x+e}{c}\right)^2}\right], \quad x>a. \tag{34}$$

Finally, to obtain $\sigma_{xx}^{t,(4)}$ similar considerations as those taken to calculate Eq. (33) must be utilised. They lead to

$$\sigma_{xx}^{t,(4)} = -\sigma/2 - 2i\mu p_0\frac{c}{a}\sqrt{1-\left(\frac{x+e}{c}\right)^2}, \quad c-e<x\leqslant a. \tag{35}$$

The method used here can be validated observing that the results shown in Eqs. (32)–(35), are mathematically identical to those given in Hills and Nowell (1994).

As mentioned later, the direct stress acting at the trailing edge of the contact zone is an important parameter. In this case it peaks at $x=a$ with the value

$$\sigma_{xx}^{t}(a,0) = -\sigma/2 + 2\mu p_0\frac{c}{a}\sqrt{\left(\frac{a+e}{c}\right)^2-1}. \tag{36}$$

The maximum value of $\sigma_{xx}^{t}(a,0)$ is reached when a full sliding condition is set, i.e., $c=0$. This maximum value is obtained by taking the limit of Eq. (36) when $c\to 0^+$. Therefore, the value of $\sigma_{xx}^{t}(a,0)$ in a full sliding condition is, as expected, the following:

$$\lim_{c\to 0^+}\left(-\sigma/2 + 2\mu p_0\frac{c}{a}\sqrt{\left(\frac{a+e}{c}\right)^2-1}\right) = -\sigma/2 + 2\mu p_0\left(\frac{a+e}{a}\right) = 2\mu p_0. \tag{37}$$

To obtain the Muskhelishvili potential, in addition to the distributions of surface stresses, it was necessary to know the profile of the indenter, in this case $v(x)=x^2/(2R)$. Again, as similarly performed with direct stresses $\sigma_{xx}^{t,(i)}$, the proper branch of the function $\sqrt{z}$ must be taken. With this in mind and applying Eqs. (25) and (27), the Muskhelishvili potential for $y<0$ (within the half-plane) is given by

$$\phi^{n}(z) = \begin{cases}\frac{1}{2}i\left(\frac{z}{AR}+\frac{p_0}{a}\sqrt{z^2-a^2}\right) = i\frac{p_0}{2a}\left(z+\sqrt{z^2-a^2}\right), & x<0,\\ \frac{1}{2}i\left(\frac{z}{AR}-\frac{p_0}{a}\sqrt{z^2-a^2}\right) = i\frac{p_0}{2a}\left(z-\sqrt{z^2-a^2}\right), & x\geqslant 0,\end{cases} \tag{38}$$

$$\phi^{t}(z) = \begin{cases}-\frac{\sigma}{8}+\frac{\mu p_0}{2a}\left[\sqrt{z^2-a^2}-\sqrt{(z+e)^2-c^2}\right], & x<-e,\\ -\frac{\sigma}{8}+\frac{\mu p_0}{2a}\left[\sqrt{z^2-a^2}+\sqrt{(z+e)^2-c^2}\right], & -e\leqslant x<0,\\ -\frac{\sigma}{8}-\frac{\mu p_0}{2a}\left[\sqrt{z^2-a^2}-\sqrt{(z+e)^2-c^2}\right], & x\geqslant 0.\end{cases} \tag{39}$$

The complex potential offered by Eqs. (38) and (39) is identical to those obtained in Hills and Nowell (1994). Finally, when $c\to 0$ and no bulk stress is present, i.e., $\sigma=0$, the classical solution of a sliding cylindrical punch is recovered (Hills and Nowell, 1994)

$$\phi(z) = \frac{p_0}{2a}(i+\mu)\left(z+\sqrt{z^2-a^2}\right), \quad x<0, \tag{40}$$

$$\phi(z) = \frac{p_0}{2a}(i+\mu)\left(z-\sqrt{z^2-a^2}\right), \quad x\geqslant 0, \tag{41}$$

which indicates that the method presented here, is also adaptable for full sliding conditions.

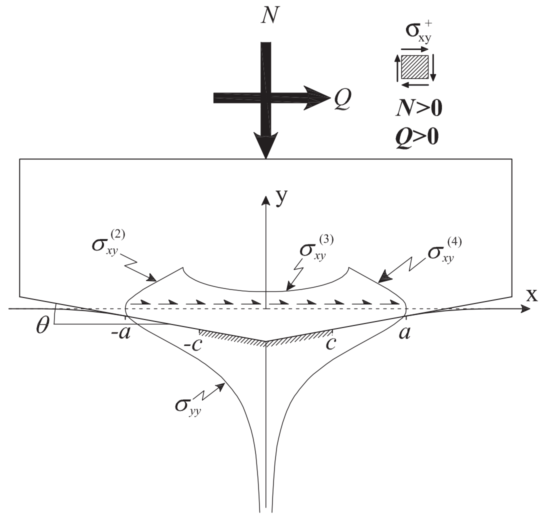

Fig. 4. Schematic view for a contact between a shallow wedge and a half-plane.

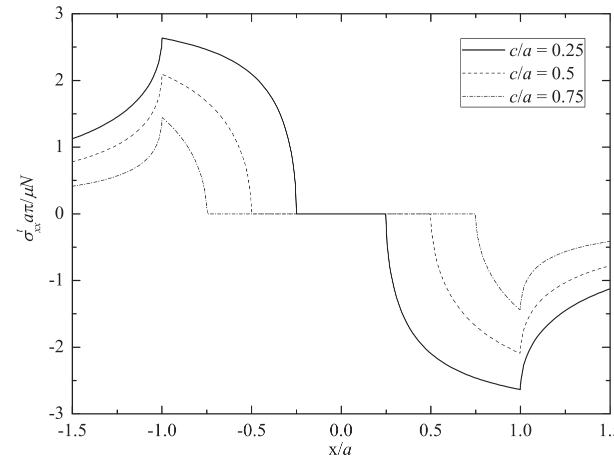

Fig. 5. Direct stress $\sigma_{xx}^{t}(x,0)$for a wedge indenter for different c/a values.

4.2. Half-plane and shallow wedge with normal and tangential load

Fig. 4 shows the type of contact to analyse, which consists of a wedge indenter first subjected to a normal load N and subsequently to a tangential load Q, that lead to a contact in partial slip conditions. This configuration has been studied previously by other authors (Ciavarella, 1998; Truman et al., 1995), wherein they show the shear stress field at the surface developed in partial slip conditions. Also in Truman et al. (1995), in the case of global sliding, the stress field in the half-plane is obtained by means of Chebyshev polynomials.

The surface contact stress due to the normal load can be written as (Johnson, 1985)

$$\sigma_{yy}(x,0) = -\frac{2\tan\theta}{\pi A}\cosh^{-1}\left(\frac{a}{|x|}\right), \tag{42}$$

where $a = AN/(2\tan\theta)$. The surface shear stress distribution developed by Q is given by Ciavarella (1998):

$$\sigma_{xy}^{(2,4)}(x,0) = \frac{2\mu\tan\theta}{\pi A}\cosh^{-1}\left(\frac{a}{|x|}\right), \quad c < |x| \leqslant a, \tag{43}$$

$$\sigma_{xy}^{(3)}(x,0) = \frac{2\mu\tan\theta}{\pi A}\left[\cosh^{-1}\left(\frac{a}{|x|}\right) - \cosh^{-1}\left(\frac{c}{|x|}\right)\right], \quad |x| \leqslant c, \tag{44}$$

where $c = a(1 - |Q/(\mu N)|)$ is the half-width of the stick zone. Once again, the function $\cosh^{-1}\left(\frac{1}{z}\right)$ has two branches, i.e., $\pm\cosh^{-1}\left(\frac{1}{z}\right)$, so it is necessary to select the branch that is physical relevant to the problem at hand. With these considerations, given that $k = 0$ and applying Eqs. (15), (18)–(20) the following expressions for $\sigma_{xx}^{t,(i)}(x,0)$ are obtained

$$\sigma_{xx}^{t,(1)}(x,0) = \frac{4i\mu\tan\theta}{\pi A}\left[\cosh^{-1}\left(\frac{a}{|x|}\right) - \cosh^{-1}\left(\frac{c}{|x|}\right)\right], \quad x < -a, \tag{45}$$

$$\sigma_{xx}^{t,(2)}(x,0) = -\frac{4i\mu\tan\theta}{\pi A}\cosh^{-1}\left(\frac{c}{|x|}\right), \quad -a \leqslant x < -c, \tag{46}$$

$$\sigma_{xx}^{t,(3)}(x,0) = 0, \tag{47}$$

$$\sigma_{xx}^{t,(4)}(x,0) = \frac{4i\mu\tan\theta}{\pi A}\cosh^{-1}\left(\frac{c}{|x|}\right), \quad c < x \leqslant a, \tag{48}$$

$$\sigma_{xx}^{t,(5)}(x,0) = -\frac{4i\mu\tan\theta}{\pi A}\left[\cosh^{-1}\left(\frac{a}{|x|}\right) - \cosh^{-1}\left(\frac{c}{|x|}\right)\right], \quad x > a. \tag{49}$$

These expressions can be transformed considering that for $|x| < 1$, $\cosh^{-1}(x) = i\cos^{-1}(x)$ and when $x < 0$, $\cos^{-1}(|x|) = \pi - \cos^{-1}(x)$. Therefore

$$\sigma_{xx}^{t,(1)}(x,0) = \frac{4\mu\tan\theta}{\pi A}\left[\cos^{-1}\left(\frac{a}{x}\right) - \cos^{-1}\left(\frac{c}{x}\right)\right], \quad x < -a, \tag{50}$$

$$\sigma_{xx}^{t,(2)}(x,0) = \frac{4\mu\tan\theta}{\pi A}\left[\pi - \cos^{-1}\left(\frac{c}{x}\right)\right], \quad -a \leqslant x < -c, \tag{51}$$

$$\sigma_{xx}^{t,(3)}(x,0) = 0, \tag{52}$$

$$\sigma_{xx}^{t,(4)}(x,0) = -\frac{4\mu\tan\theta}{\pi A}\cos^{-1}\left(\frac{c}{x}\right), \quad c < x \leqslant a, \tag{53}$$

$$\sigma_{xx}^{t,(5)}(x,0) = \frac{4\mu\tan\theta}{\pi A}\left[\cos^{-1}\left(\frac{a}{x}\right) - \cos^{-1}\left(\frac{c}{x}\right)\right], \quad x > a. \tag{54}$$

Fig. 5 shows the stress $\sigma_{xx}^{t}(x,0)$ obtained for different values of the ratio c/a.

Again, the direct stress, $\sigma_{xx}^{t}(x,0)$, peaks at the trailing edge of the contact zone. In this situation, the maximum value is obtained at $x = -a$:

$$\sigma_{xx}^{t}(-a,0) = -\frac{4i\mu\tan\theta}{\pi A}\cosh^{-1}\left(\frac{c}{a}\right) = \frac{4\mu\tan\theta}{\pi A}\cos^{-1}\left(\frac{c}{a}\right) \tag{55}$$

As previously stated, the maximum value of $\sigma_{xx}^{t}(-a,0)$ is reached when a full sliding condition is met, and again $c \to 0^{+}$, being this maximum value equal to $\sigma_{xx}^{t}(-a,0) = 2\mu\tan\theta/A$.

To obtain the Muskhelishvili potential, it is necessary to define the profile of the indenter, which in this case is given by

$$v(x) = \begin{cases} -x\tan\theta, & x < 0, \\ x\tan\theta, & x \geqslant 0. \end{cases} \tag{56}$$

Remembering Eqs. (56) and (25), it is easy to see that $\phi^{n}(z) = \pm i\tan\theta/(2A) - \sigma_{yy}(z)/2$. However, Eqs. (2) and (3) show that the interior stress state is not affected when the potential $\phi^{n}(z)$ is replaced by $\phi^{n}(z) + i\beta$ (where β is a real constant), and therefore $\phi^{n}(z) = -\sigma_{yy}(z)/2$.

It is important to note that to implement Eqs. (2) and (3) in a numerical code, it is more effective to write the function $\cosh^{-1}(C/z)$ as

$$\cosh^{-1}\left(\frac{C}{z}\right) = \pm i\cos^{-1}\left(\frac{C}{z}\right), \tag{57}$$

where C is a real constant. This allows for easier selection of the appropriate branch.

Hence, taking the appropriate branches, the Muskhelishvili potential for the case under study is given by Eqs. (25) and (27), and therefore

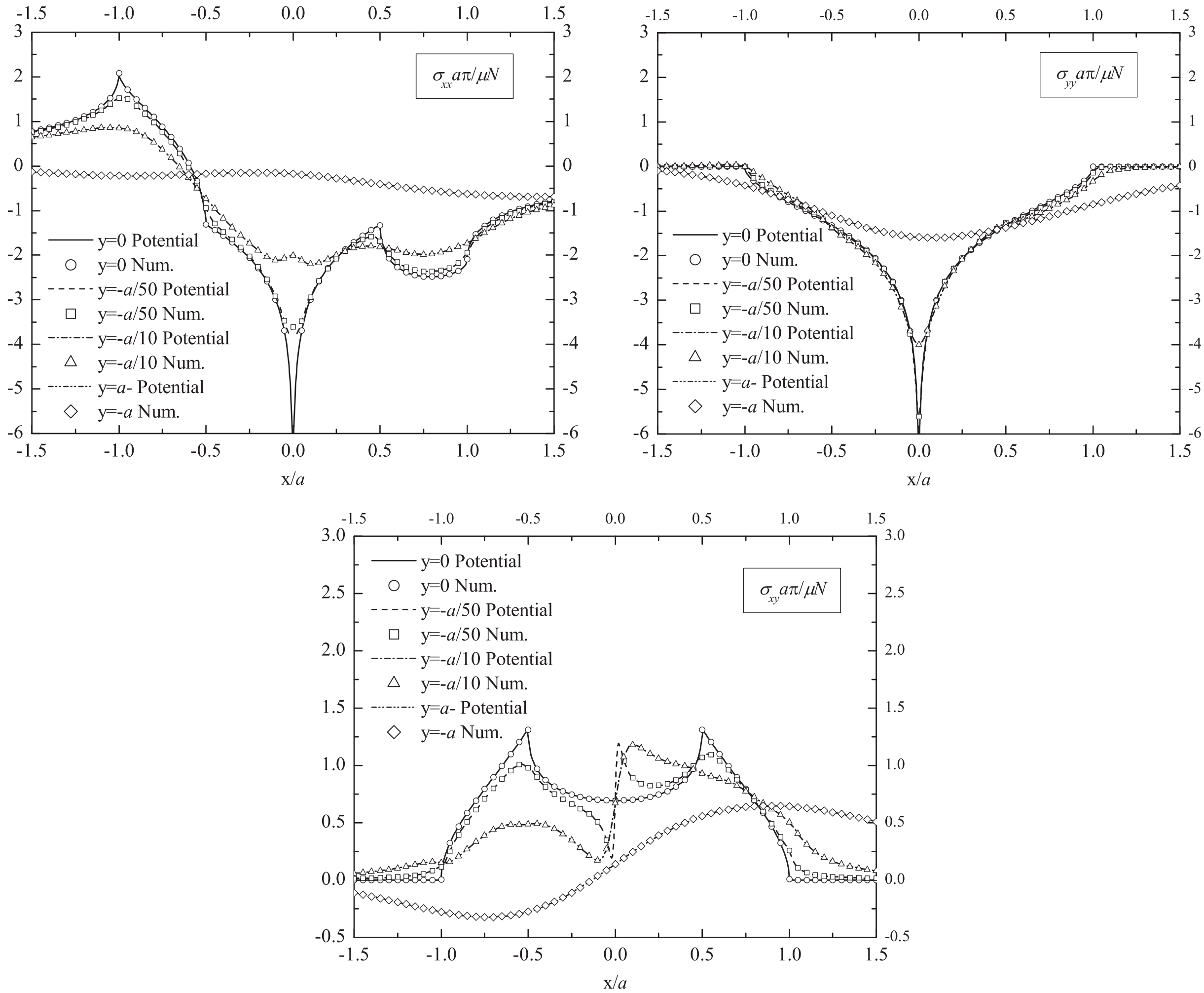

Fig. 6. Half-plane stress field for a wedge indenter in partial slip condition $c/a = 0.5$ at differents depths.

$$\phi^n(z) = -\frac{i\tan\theta}{\pi A}\cos^{-1}\left(\frac{a}{z}\right), \tag{58}$$

$$\phi^t(z) = \frac{\mu\tan\theta}{\pi A}\left[\cos^{-1}\left(\frac{a}{z}\right) - \cos^{-1}\left(\frac{c}{z}\right)\right]. \tag{59}$$

With the exception of a pure complex constant, the expression given by Eq. (58) is mathematically identical to that obtained by Sackfield et al. (2005) for an inclined punch when the tilt angle, α, is equal to zero.

Fig. 6 shows, with a solid line, the stress field produced in the half-plane and obtained by the potential defined by Eqs. (58) and (59) for different depths and with a ratio of $c/a = 0.5$. In this case, the comparison has to be made with the values obtained by the numerical integration of the surface stresses by means of the following equations (Johnson, 1985):

$$\sigma_{xx}(x,y) = -\frac{2y}{\pi}\int_{-a}^{a}\frac{\sigma_{yy}(s,0)(x-s)^2}{\left((x-s)^2+y^2\right)^2}ds - \frac{2}{\pi}\int_{-a}^{a}\frac{\sigma_{xy}(s,0)(x-s)^3}{\left((x-s)^2+y^2\right)^2}ds \tag{60}$$

$$\sigma_{yy}(x,y) = -\frac{2y^3}{\pi}\int_{-a}^{a}\frac{\sigma_{yy}(s,0)}{\left((x-s)^2+y^2\right)^2}ds - \frac{2y^2}{\pi}\int_{-a}^{a}\frac{\sigma_{xy}(s,0)(x-s)}{\left((x-s)^2+y^2\right)^2}ds \tag{61}$$

$$\sigma_{xy}(x,y) = -\frac{2y^2}{\pi}\int_{-a}^{a}\frac{\sigma_{yy}(s,0)(x-s)}{\left((x-s)^2+y^2\right)^2}ds - \frac{2y}{\pi}\int_{-a}^{a}\frac{\sigma_{xy}(s,0)(x-s)^2}{\left((x-s)^2+y^2\right)^2}ds, \tag{62}$$

As shown in Fig. 6, the numerical results, those plotted with the circular symbol, agree perfectly with those obtained using the complex potential $\phi(z)$, thus reaffirming the method described in this paper.

5. Conclusions

Two new relationships for two-dimensional contacts in partial slip conditions have been found. These relationships enable a quick and easy method for obtaining two important parameters in the field of contact mechanics. The procedure developed herein shows that if the normal and shear stress distribution at the contact surface is known, it is possible to easily obtain the complete stress field in the entire half-plane. The first relationship allows one to obtain, analytically and explicitly, the direct stress field at the surface, $\sigma^t_{xx}(x,0)$, developed by surface shear stress $\sigma_{xy}(x,0)$. The second of these relationships, and perhaps the most important, is applicable to the Muskhelishvili complex potential, from which the complete stress field in the interior of the half-plane is implicitly obtained.

Although the methods described herein are applicable to contacts with a half-plane as defined in a Cartesian coordinate system, it is possible to make some modifications in the formulation to take into account a curvilinear coordinate system. Finally, despite the simple procedures, the resultant expressions for any other case would not be completely defined, because the application would involve taking the correct branch of a multivalued function.

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