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Chevalley Groups of Odd Characteristic as Quadratic Pairs

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Introduction

In the study of Chevalley groups, one naturally studies the relations between two root subgroups corresponding to the long roots. These relations are used by M. Timmesfeld to characterize the characteristic 2 case. Aschbacher and Hall, Jr., studied some cases for the odd characteristic. Besides these relations the root subgroups act on the standard module quadratically, namely, they are generated by a set of elements whose minimal polynomials are $(X - 1)^2$. This observation leads to the study of quadratic pairs which is connected with the p nonstability in the theory of finite groups. John Thompson obtained the complete classification for the case for the prime $p \geqslant 5$ in [7]. For the prime 3, it appears that one has to distinguish between two cases, namely, the case where the root group has order 3 and that whose root group has order greater than 3. By introducing the notion of long r-involution, one can subdivide the first case into subcases. It is known that the first case involves the following groups: the Conway group. O, $G_2(4)$, Suz, HJ.

Main Theorem. *Let (G, M) be a quadratic pair for p such that G is quasisimple. If (G, M) is a quadratic pair for 3 whose root group has order 3, then we also assume that all r-involutions are long and $\theta(X) = \theta_1(X)$ for some $X \in \Sigma$.*

Under these conditions there exist positive integers α and n such that $G/Z(G)$ is isomorphic to one of the following groups where $q = p^\alpha$.

$A_n(q)$ ($n \geqslant 2$ except in the case $q = 3$ where we have $n \geqslant 3$), ${}^2A_n(q)$ ($n \geqslant 2$), $B_n(q)$ ($n \geqslant 3$), $C_n(q)$ ($n \geqslant 2$), $D_n(q)$ ($n \geqslant 3$), ${}^2D_n(q)$ ($n \geqslant 3$), ${}^3D_4(q)$, $G_2(q)$, $F_4(q)$, $E_6(q)$, ${}^2E_6(q)$, $E_7(q)$.

The following theorems in the case of quadratic pair for 3 whose root group has order 3 are also proved in this paper.

Theorem A. *Suppose that (G, M) is a quadratic pair for 3 whose root* *group has order 3 such that G is quasisimple. If there exists X, $Y \in \Sigma$ such that $\langle X, Y\rangle \cong 3^{1+2}$, then (1) $\Sigma = \tilde{\Sigma}$ is a conjugacy class of subgroups, (2) $\theta(E) = \theta_1(E)$ for any $E \in \Sigma$ and (3) all r-involutions are long.*

THEOREM B. *Suppose that (G, M) is a quadratic pair for 3 whose root group has order 3 such that G is quasisimple and $\theta_3(X)$ is the empty set for some $X \in \Sigma$. Then (1) Σ is a conjugacy class of subgroup, (2) $\Sigma = \tilde{\Sigma}$ or $\tilde{\Sigma} = \phi$.*

Furthermore if we let K be the set of abstract groups $\langle E, F\rangle$ where $E, F \in \Sigma$ and $[E, F] \neq 1$, then $K \subset \{SL(2, 3), SL(2, 5)\}$ or $K \subset \{SL(2, 3), 3^{1+2}\}$.

Theorem A, Theorem B and the known Theorem 2.5 give a justification of subdividing the case of quadratic pair for 3 whose root group has order 3 into subcases.

Remark. The original proof of the main theorem in [4, 6, 7] is quite different from the given proof in this paper. On one hand the new proof which we appeal to the tremendous result in [1] gives a uniform treatment of the main theorem. On the other hand the old proof is more elementary and gives a lot of information about the root structures of the groups which we classified. There is some mistake in [4] which the author corrects in the original submitted version. Another remark is the following. Suppose (G, M) is a quadratic pair for 3 such that G is quasisimple. The proof of Lemma 4.1 of [6] shows that if there is $X \in \Sigma$ such that $|X| > 3$, then $|Y| > 3$ for any $Y \in \Sigma$.

1. DEFINITION AND NOTATION

DEFINITION. Let G be a finite group and M a vector space over the finite field $GF(p)$ of p elements where p is an odd prime. We say that (G, M) is a quadratic pair for p if the following is true:

(1.1) M is a faithful irreducible $GF(p)G$ module with $\dim_{GF(p)} M > 1$.

(1.2) G is generated by a set Q of linear transformations of M such that

$$Q = \{g \in G\backslash\{1\} \mid M(g-1)^2 = 0\}.$$

For any subset S of G and any subgroup V of M, let $V_S = C_V(S) = \{v \mid v \in V, vs = v \text{ for all } s \in S\}$ and $V^S = \{v(s-1) \mid v \in V, s \in S\}$. For $\sigma \in G$, set $3^{d(\sigma)} = |M^\sigma|$. If $\sigma \in Q$, then $M^\sigma \subset M_\sigma$. For any integer e, let $Q_e = \{\sigma \in Q \mid d(\sigma) = e\}$.

Let $d = \min_{\sigma \in Q} d(\sigma)$. For each $\sigma \in Q_d$, set $E(\sigma)^* = \{\tau \in Q\backslash\{1\} \mid M^\sigma = M^\tau$ and $M_\sigma = M_\tau\}$ and let $E(\sigma) = E(\sigma) \cup \{1\}$. Then Q_d is partitioned by $E(\sigma)^*$ and $E(\sigma)$ is an elementary abelian p subgroup of G. Let $\Sigma = \{E(\sigma) \mid \sigma \in Q_d\}$.

We call an element of Σ a root subgroup of G. We say that (G, M) is a quadratic pair for 3 whose root group has order 3 if $|X| = 3$ for any $X \in \Sigma$.

For $X \in \Sigma$, let $U(X)$ be the stability subgroup of G of the chain $M \supset M_X \supset M^X \supset 0$. For $\sigma, \tau \in G$, we let $\Delta(\sigma, \tau) = (\sigma - 1)(\tau - 1) + (\tau - 1)(\sigma - 1)$, where σ, τ are regarded as elements of the ring of endomorphisms of M.

For any subgroup H of G, let $\Sigma \cap H = \{X \mid X \in \Sigma \text{ and } X \subset H\}$. Let $Z(G)$ be the center of G.

Let 3^{1+2} be the nonabelian 3-group of order 27, exponent 3, and nilpotent class 2.

All groups considered are of finite orders. Most notations are standard and can be found in [3].

2. Theorem A and Theorem B

Let (G, M) be a quadratic pair for 3 whose root subgroup has order 3 in this section. For the convenience of the reader the following known results are recollected.

THEOREM 2.1. *Let $\sigma, \tau \in Q_d$ and let $H = \langle \sigma, \tau \rangle$. Then H is isomorphic to one of the following groups:*

(a) $SL(2, 3)$, (b) $SL(2, 5)$, (c) $SL(2, 3) \times Z_3$, (d) $Z_3 \times Z_3$, (e) 3^{1+2}.

Furthermore in the case $H \cong 3^{1+2}$ if we let $\lambda = [\sigma, \tau]$, then

(1) $\lambda \in Q_d$,

(2) $M = M_\sigma + M_\tau$, $M^\sigma \cap M^\tau = 0$,

(3) $M_\lambda = (M_\sigma \cap M_\tau) + (M^\sigma + M^\tau)$,
$M^\lambda = (M_\sigma \cap M_\tau) \cap (M^\sigma + M^\tau)$,

(4) $H \subset U(\langle \lambda \rangle)$.

Proof. Lemma 4.2 and Theorem 4.3 of [5].

For each $X \in \Sigma$, let $\theta(X) = \{Y \mid Y \in \Sigma \text{ such that } \langle X, Y \rangle \text{ is not a 3-group}\}$ and $I(X) = \{j \mid j \text{ is the unique involution of } \langle X, Y \rangle \text{ for some } Y \in \theta(X)\}$. We also define $\theta_1(X) = \{Y \mid Y \in \theta(X) \text{ such that } \langle X, Y \rangle \cong SL(2, 3)\}$, $\theta_2(X) = \{Y \mid Y \in \theta(X) \text{ such that } \langle X, Y \rangle \cong SL(2, 5)\}$ and $\theta_3(X) = \{Y \mid Y \in \theta(X) \text{ such that } \langle X, Y \rangle \cong SL(2, 3) \times Z_3\}$. Thus $\theta(X) = \theta_1(X) \cup \theta_2(X) \cup \theta_3(X)$ is a disjoint union of subsets in Σ.

Let i be an involution of G. We say that i is an r-involution of G if there exist $X, Y \in \Sigma$ such that $i \in \langle X, Y \rangle$. Clearly $d(i) \leqslant 2d$. An r-involution i is long if $d(i) = 2d$. An r-involution which is not long is a short r-involution. Let $I_r(G)$ be the set of all r-involutions of G.

Let $\tilde{Q}_d = \{z \in Q_d \mid$ there exists $x, y \in Q_d$ such that $[x, y] = z\}$ and let $\tilde{\Sigma} = \{\langle z \rangle \mid z \in \tilde{Q}_d\}$.

THEOREM 2.2 (Thompson). *If $X \in \tilde{\Sigma}$, $j \in I(X)$, then j is long.*

Proof. Theorem 4.4 of [5].

LEMMA 2.3. *Let $X \in \Sigma$ and $Y \in \theta(X)$. Let $i \in I(X)$ such that $i \in \langle X, Y \rangle$. If $Y \in \theta_2(X)$, then i is long. If $Y \in \theta_3(X)$, then i is short.*

Proof. Lemma 4.6 and Lemma 4.7 of [5].

LEMMA 2.4. *Let $X \in \Sigma$ and $Y \in \theta_1(X)$. Let $\langle x \rangle = X$, $\langle y \rangle = Y$ where x is conjugate to y in $\langle X, Y \rangle$. If the involution i of $\langle X, Y \rangle$ is long, then M has a basis such that, with respect to this basis, the representating matrices of x and y are*

$$\begin{pmatrix} I_d & I_d & 0 \\ 0 & I_d & 0 \\ 0 & 0 & I_{m-2d} \end{pmatrix}, \qquad \begin{pmatrix} I_d & 0 & 0 \\ -I_d & I_d & 0 \\ 0 & 0 & I_{m-2d} \end{pmatrix},$$

respectively. Furthermore, $M = M^X \oplus M^Y \oplus M_X \cap M_Y$.

Proof. Lemma 4.8 of [5].

THEOREM 2.5. *If $X \in \tilde{\Sigma}$, then $\theta(X) = \theta_1(X)$.*

Proof. Theorem 4.9 of [5].

LEMMA 2.6. (a) *If $i \in I(X)$ is a long involution, then i inverts M^X, M/M_X and centralizes M_X/M^X.*

(b) *If $i_1, i_2 \in I(X)$ are long involutions, then $i_1 i_2 \in U(X)$.*

Proof. Lemma 5.1 and Corollary 5.2 of [5].

THEOREM 2.7. *Let $X, Y \in \Sigma$ and $[X, Y] = E \in \Sigma$, then X is conjugate to E.*

Proof. Since G acts irreducibly on M, $O_3(G) = 1$. Hence $P = \langle X, Y \rangle \not\subset O_3(G)$. Since $P \simeq 3^{1+2}$, Y permutes the three one dimensional subspaces of XE, namely, $\langle x \rangle$, $\langle xe \rangle$ and $\langle xe^{-1} \rangle$ where $\langle x \rangle = X$, $E = \langle e \rangle$.

Theorem 2.2 and Lemma 2.3 imply $\theta_3(E) = \phi$. Since $SL(2, 5) \supset SL(2, 3)$, we may choose $Z \in \theta_1(E)$. Let $\langle z \rangle = Z$ and z be conjugated to e in $\langle Z, E \rangle$.

Theorem 2.2 and Lemma 2.4 imply that M has a basis such that with respect to this basis the representing matrix of e is

$$\begin{pmatrix} I_d & I_d & 0 \\ 0 & I_d & 0 \\ 0 & 0 & I_{m-2d} \end{pmatrix}$$

and the representating matrix of z is

$$\begin{pmatrix} I_d & 0 & 0 \\ -I_d & I_d & 0 \\ 0 & 0 & I_{m-2d} \end{pmatrix}.$$

We identify an element of G with its representing matrix with respect to this basis of M. Let

$$i = \begin{pmatrix} -I_d & 0 & 0 \\ 0 & -I_d & 0 \\ 0 & 0 & I_{m-2d} \end{pmatrix}, \qquad \omega = \begin{pmatrix} 0 & I_d & 0 \\ -I_d & 0 & 0 \\ 0 & 0 & I_{m-2d} \end{pmatrix}.$$

Then i and $\omega \in \langle Z, E\rangle$. By Theorem 2.1 we see that $\langle X, Y\rangle \subset U(E)$. Thus

$$x = \begin{pmatrix} I_d & \gamma & \alpha \\ 0 & I_d & 0 \\ 0 & \beta & I_{m-2d} \end{pmatrix}.$$

Since $(x-1)^2 = 0$, $\alpha\beta = 0$. Set $U = U(E)$. Let $U^- = \{u \in U \mid u^i = u^{-1}$ and let $U^+ = C_U(i)$, Thus $U = U^+U^-$ and $U^+ = E$. We label $u \in U$ by $u = (\xi, \eta, \zeta)$, provided that

$$u = \begin{pmatrix} I_d & \zeta & \xi \\ 0 & I_d & 0 \\ 0 & \eta & I_{m-2d} \end{pmatrix}.$$

With this notation we find $(\xi, \eta, \zeta)^i = (-\xi, -\eta, \zeta)$ and $(\xi_1, \eta_1, \zeta_1)(\xi, \eta, \zeta) = (\xi_1 + \xi, \eta_1 + \eta, \zeta_1 + \zeta + \xi_1\eta)$. Thus $(\xi, \eta, \zeta) = (\xi, \eta, \frac{1}{2}\xi\eta)(0, 0, \zeta - \frac{1}{2}\xi\eta)$ where $(\xi, \eta, \frac{1}{2}\xi\eta) \in U^-$ and $(0, 0, \zeta - \frac{1}{2}\xi\eta) \in E$. Let $\sigma = (\alpha, \beta, \frac{1}{2}\alpha\beta) = (\alpha, \beta, 0) = x(0, 0, \gamma - \frac{1}{2}\alpha\beta)$. Thus $\sigma \in XE$. Since $x \notin E$, $\sigma \notin E$. Hence $\langle\sigma\rangle$ is one of the three subgroups of XE permutated by Y. In particular $\langle\sigma\rangle$ is conjugated to X. Let $\tau = \sigma^\omega$ and $\lambda = [\sigma, \tau]$. Then

$$\lambda = \begin{pmatrix} I_d & 0 & 0 \\ 0 & I_d & 0 \\ 0 & 0 & I_{m-2d} - \beta\alpha \end{pmatrix}.$$

Therefore $(\lambda - 1)^2 = 0$ as $\alpha\beta = 0$. If $\beta\alpha \neq 0$, then as $\operatorname{rank}(\beta\alpha) \leqslant \operatorname{rank}(\beta) \leqslant d$, the definition of Q_d implies $\lambda \in Q_d$. Thus $d = \operatorname{rank}(\lambda - 1) \leqslant \operatorname{rank}(\beta\alpha) \leqslant \operatorname{rank}(\beta) \leqslant d$. Therefore $d = \operatorname{rank}(\beta)$. From $x \in Q_d$ one sees $\operatorname{rank}(x - 1) = d$. Since $d = \operatorname{rank}(\beta)$ and

$$x - 1 = \begin{pmatrix} 0 & \gamma & \alpha \\ 0 & 0 & 0 \\ 0 & \beta & 0 \end{pmatrix},$$

$\alpha = 0$. However this implies $\alpha\beta = 0$, a contradiction. Therefore $\beta\alpha = 0$. Let $\delta = e^{\tau}$. Then

$$\delta = \begin{pmatrix} I_d & I_d & \alpha \\ 0 & I_d & 0 \\ 0 & \beta & \beta\alpha + I_{m-2d} \end{pmatrix} = \begin{pmatrix} I_d & I_d & \alpha \\ 0 & I_d & 0 \\ 0 & \beta & I_{m-2d} \end{pmatrix} = e\sigma.$$

Since $\sigma \notin E$, $\delta \notin E$. As $\sigma \in XE$, $\delta \in XE$. Thus $\langle\sigma\rangle$ is one of the three subgroups in XE not equal to E. Hence $\langle\sigma\rangle$ is conjugate to X. Since $\delta = e^{\sigma}$, $\langle\delta\rangle$ is conjugate to E. Therefore X is conjugate to E as required.

We now prove Theorem A. Let $X, Y, Z \in \Sigma$ such that $Z = [X, Y]$. Let Σ_1 be the conjugacy class of subgroups which contains Z. Then $\Sigma_1 \subset \tilde{\Sigma}$. Let $A \in \Sigma$. Suppose there is $B \in \Sigma_1$ such that $A \in \theta(B)$. Theorem 2.2 and Lemma 2.3 imply that $\theta_3(B) = \phi$. Thus A is conjugate to B in this case as $\langle A, B\rangle \cong SL(2, 3)$ or $SL(2, 5)$. Therefore $A \in \Sigma_1$ in this case. Suppose there is $B \in \Sigma_1$ such that $\langle A, B\rangle$ is a nonabelian 3-group. Theorem 2.7 implies $A \in \Sigma_1$ in this case. However G is generated by the elements in Σ_1 and $O_3(G) = 1$. Therefore for any $A \in \Sigma$, there is $B \in \Sigma_1$ such that $[A, B] \neq 1$. The above argument implies $A \in \Sigma_1$. Therefore $\Sigma = \Sigma_1 = \tilde{\Sigma}$ is a conjugacy class of subgroups as required. Theorem 2.2 and Theorem 2.5 imply the rest of the statements of Theorem A.

We conclude this section with the following proof of Theorem B.

Let $E, F \in \Sigma$. By Theorem A we may assume that $[E, F] = 1$ whenever $\langle E, F\rangle$ is a 3-subgroup. Let Σ_1 be the conjugacy class of subgroups containing X. Since G is quasisimple and $O_3(G) = 1$, G is generated by the elements of Σ_1. Thus for any $A \in \Sigma$ there exists $B \in \Sigma_1$ such that $[A, B] \neq 1$. Since $B \in \Sigma_1$, $\theta_3(B) = \phi$. As $A \in \theta(B)$ we see that $\langle A, B\rangle \cong SL(2, 3)$ or $SL(2, 5)$. Hence $A \in \Sigma_1$ and $\Sigma = \Sigma_1$ is a conjugacy class of subgroups. The rest of the proof follows from Theorem 2.5 and Theorem A.

3. Proof of the Main Theorem

THEOREM 3.1. *Let G^* be a simple group. Let i be an involution of G^*. Suppose $C_{G^*}(i)$ contains a subnormal subgroup H such that H contains i and H*

is isomorphic to $SL(2, q)$, q *odd. Then* G^* *is a Chevalley group of odd characteristic or* M_{11}.

Proof. Corollary 3 of [1].

LEMMA 3.2. *There is no vector space* V *over* $GF(p)$ *such that* (M_{11}, V) *is a quadratic pair for* p, p *an odd prime.*

Proof. Case (a). $p = 3$.

There is only one conjugacy class of element of order 3. In this class there are elements a and b such that $\langle a, b\rangle$ is isomorphic to the alternating group on four letters. Since the alternating groups has no quadratic module, the lemma is true in this case.

Case (b). $p = 5$.

There is only one conjugacy class of subgroup of order 5. In this class there are two subgroups A and B such that $\langle A, B\rangle$ is isomorphic to the alternating group on five letters. Since the alternating group has no quadratic module, the lemma is true in this case.

Case (c). $p = 11$.

There is only one conjugacy class of subgroups of order 11. In this class there are two subgroups A and B such that $\langle A, B\rangle$ is isomorphic to $PSL(2, 11)$ which also has no quadratic module. Therefore the lemma holds in this case.

Since 3, 5, 11 are the odd primes which divide $|M_{11}|$, the proof of the lemma is complete.

The proof of the main theorem is broken into three cases.

Case 1. (G, M) is a quadratic pair for 3 whose root group has order 3 and satisfies:

(A.1) G is quasisimple.

(A.2) $I_r(G)$ consists of long r-involutions and $\theta(X) = \theta_1(X)$ for some $X \in \Sigma$.

By Theorem B we see that Σ is a conjugacy class of subgroups. We note that (A.2) is a consequence of Theorem A if $\tilde{Q}_d \neq \varnothing$.

LEMMA 3.3. *If* $X \in \Sigma$, $Y, Z \in \theta(X)$ *such that* $\langle X, Y\rangle \cap \langle X, Z\rangle$ *contains an involution, then* $\langle X, Y\rangle = \langle X, Z\rangle$.

Proof. Let $S = \langle X, Y\rangle$ and let $S^* = \langle X, Y, Z\rangle$. Let i be the involution of S. Since $[S^*, i] = 1$, M^i and M_i are S^*-submodules. Since i is a long involution, S^* induces 1 on M_i. Hence we may identify S^* with a group of automorphisms of $M^i = N$. Let $K = \{O_d, I_d, -I_d\}$. By Lemma 2.4 N has a basis such that with respect to this basis $X = \{X(t) \mid t \in K\}$ and $Y = \{Y(t) \mid t \in K\}$ where

$$X(t) = \begin{pmatrix} I_d & t \\ 0 & I_d \end{pmatrix} \quad \text{and} \quad Y(t) = \begin{pmatrix} I_d & 0 \\ t & I_d \end{pmatrix}.$$

Suppose $\langle Y, Z\rangle \cong 3^{1+2}$. By Theorem 2.1, $N = N_X + N_Z$ and $N^Y \cap N^Z = 0$. Since $\langle Y, Z\rangle$ is a 3-group, $N_X \cap N_Z \neq 0$. From $N^Y \subset N_Y$ and $N^Z \subset N_Z$ we see that N contains $N^Y \oplus N^Z$ properly. Hence $\dim N \ngtr \dim N^Y + \dim N^Z$. However $\dim N = 2d$ and $\dim N^Y = d = \dim N^Z$. The last inequality now reads $2d \ngtr d + d$, a contradiction. Hence $\langle Y, Z\rangle \not\cong 3^{1+2}$.

Next suppose $[Y, Z] = 1$. Let $z \in Z$ such that z is conjugate to $X(1)$ in $\langle X, Z\rangle$. Since $[z, Y] = 1$, $z = \left(\begin{smallmatrix} a & 0 \\ c & a \end{smallmatrix}\right)$, where a, c are $d \times d$ matrices. From $(X(I_d)^{-1}z)^2 = i$ we get $2a^2 = ca$ and $-ca + a^2 = -I$. Hence $a^2 = I_d$. Since $z^3 = 1$, $a^3 = I_d$. Therefore $a = I_d$. By the definition of Σ we see that $z \in Y$ in this case.

Suppose the lemma is false. Let σ be any element in X. Let $W \in \Sigma \cap S$. The above argument shows that $Z^\sigma \in \theta(W)$. Let $\tau \in Z$. Then

$$\tau = \begin{pmatrix} I_d + \alpha & \beta \\ \gamma & I_d + \delta \end{pmatrix},$$

where α, β, γ, δ are $d \times d$ matrices. Since $Z \in \theta(X)$, there is $t \in K$ such that $\Delta(\tau, X(t)) = I_{2d}$. This implies $\gamma = t^{-1}$ and $\alpha + \delta = 0$. Since $Z \in \theta(Y)$, there exists $u \in K$ such that $\Delta(\tau, Y(u)) = I_{2d}$. This implies that $\beta = u^{-1} \in K\backslash\{0\}$. A short calculation shows that

$$\tau^{X(I_d)} = \begin{pmatrix} * & \beta + \alpha - \delta - \gamma \\ * & * \end{pmatrix}.$$

The preceding argument yields $\beta + \alpha - \delta - \gamma \in K\backslash\{0\}$. This implies that $\alpha - \delta \in K$ as $\beta, \gamma \in K$. Since $\alpha + \delta = 0 \in K$, $\alpha \in K$ and $\delta = -\alpha \in K$. Since $\tau \in Q_d$, $(\tau - 1)^2 = 0$. Therefore $\alpha^2 + \beta\gamma = 0$ and $\alpha\beta + \beta\delta = 0$. This shows $\tau \in S$, a contradiction. The proof of the lemma is complete.

LEMMA 3.4. *Let $X \in \Sigma$ and $Y \in \theta(X)$. Let $S = \langle X, Y\rangle$ and let i be the involution of S. Then S is a subnormal subgroup of $C_G(i)$.*

Proof. Let A, B be two members in the conjugacy class of subgroups in $C_G(i)$ which contains X. Then A induces the identity on M_i. Similarly B induces the identity on M_i.

Suppose $\langle A, B\rangle$ is a nonabelian 3-group. Set $T = [A, B]$. Theorem 2.1 implies that $M^i = (M^i)_A + (M^i)_B$ and $(M^i)^A \cap (M^i)^B = 0$. Since $\langle A, B\rangle$ is a 3-group, $(M^i)_{\langle A,B\rangle} \neq 0$. We note that $(M^i)^A = M^A$. Similarly $(M^i)^B = M^B$. Since $(M^i)^A \subset (M^i)_A$ and $(M^i)^B \subset (M^i)_B$, M^i contains $M^A \oplus M^B$ properly. However this implies $2d = \dim M^i \not\geqslant \dim M^A + \dim M^B = 2d$, a contradiction. Hence $[A, B] = 1$ whenever $\langle A, B\rangle$ is a 3-group.

Let $g \in C_G(i)$. It suffices to assume $S \neq S^g$ and show $[S, S^g] = 1$. Let $Z \in \Sigma \cap S_1$. Suppose there exists $W \in S \cap \Sigma$ such that $Z \in \theta(W)$. Let j be the involution of $\langle W, Z\rangle$. Since the restriction of S on M_i is the identity and $g \in C_G(i)$, S^g also induces the identity on M_i. Thus $M_j = M_i$. Since M^i and M_i are $\langle j\rangle$-submodules, $\dim M^j = \dim M^i$ implies that $i = j$. Theorem 3.2 implies that $S^g = \langle Z, W\rangle = S$, a contradiction. Therefore Z commutes with any member of Σ which lies in S. Since Z is arbitrary, $[S^g, S] = 1$ as required. The proof of the lemma is complete.

We now prove the main theorem for Case (1). Since G is quasisimple and $SL(2, 3)$ is solvable, $i \in Z(G)$ by Lemma 3.3. For any subset H of G, let $\bar{H} = HZ(G)/Z(G)$. Then $\bar{S}$ is a subnormal subgroup of $C_{\bar{G}}(\bar{\imath})$ such that $\bar{S}$ contains $\bar{\imath}$ and $\bar{S} \cong SL(2, 3)$. Theorem 3.1 implies that $\bar{G}$ is a Chevalley group of odd characteristic of M_{11}. The argument in Section 26 of [7] shows that a covering group of $E_8(q)$, q odd, does not have a quadratic module. Since the Schur multiplier of M_{11} is 1, Lemma 3.2 implies the main theorem holds in this case.

Case 2. (G, M) is a quadratic pair for 3 whose root group has order greater than 3.

The corresponding lemma of Lemma 3.4 is valid in this case. In the proof of Lemma 3.4 we replace Theorem 2.1 by Theorem 1 and Theorem 2 of [2] and Theorem 3.2 by Theorem 6.1 of [6]. If the involution $i \in Z(G)$, then $G/Z(G) \cong A_1(q)$ where q is a power of 3. Therefore we may assume that $i \notin Z(G)$. The same argument in the proof of case (1) implies the main theorem holds in this case.

Case (3). $p \geqslant 5$.

The argument in this case is the same as in case (2). The only change is the following. We replace Theorem 6.1 of [6] by Lemma 10.2 of [7].

We remark that the results needed in the proof of case (2) and case (3) can be deduced more directly in [6] and [7].

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