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MacLane homology and topological Hochschild homology

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Abstract

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The topological Hochschild homology of a discrete ring is shown to agree with the MacLane homology of that ring.

Introduction

The aim of this paper is to show that the topological Hochschild homology of a discrete ring R in the sense of [2] and the MacLane homology of R (see [10] or [8]) are isomorphic. The method is to show that they are both isomorphic to a certain kind of homology of the category of finitely generated projective R-modules with coefficients in the bifunctor Hom. That the latter agrees with MacLane homology, was shown in [8]; and that it agrees with topological Hochschild homology, is the main result of this paper. In the (appended) last section we describe a related spectral sequence.

1. On MacLane homology

In this section we recall the definition of MacLane homology and the main result of [8].

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Definition 1.1 [10]. The *MacLane homology* of a ring R is defined by

$$H_*^{\mathrm{ML}}(R) = H_*(B(R, Q_*(R), R) \otimes_{R-R} R)\ ,$$

where $Q_*(R)$ is a suitable chain algebra whose homology is isomorphic to the stable homology of Eilenberg–MacLane spaces [6] $H_q(Q_*(R)) = H_{q+n}(K(R, n))$, $n > q$, and $B(R, Q_*(R), R)$ means the two-sided bar construction.

Since $H_0(Q_*(R)) = R$, we have an augmentation map

$$Q_*(R) \to R\ ,$$

which is a map of chain algebras and therefore induces a natural map from MacLane homology to *Hochschild homology*

$$a_* : H_*^{\mathrm{ML}}(R) \to \mathrm{Hoch}_*(R, R)\ .$$

Since $H_1(Q_*(R)) = 0$ it follows that a_0 and a_1 are isomorphisms. If R is an algebra over $\mathbb{Q}$, then $H_q(Q_*(R)) = 0$, $q > 0$. Hence, in this case a_q is an isomorphism for any $q \geq 0$.

Let

$$h_* : K_*^{\mathrm{s}}(R) \to \mathrm{Hoch}_*(R, R)$$

be the natural transformation from *stable K-theory* to Hochschild homology defined in [15] (see also [9]). By [9] the transformation h_i is an isomorphism if $i = 0,1$. If R is an algebra over $\mathbb{Q}$, Goodwillie proved that h_i is an isomorphism for any $i \geq 0$ (see [7]).

It was shown in [12] that h_* has a lifting to MacLane homology: there exists a natural transformation

$$\Theta_* : K_*^{\mathrm{s}}(R) \to H_*^{\mathrm{ML}}(R)$$

such that $h_* = a_* \Theta_*$, and it was also conjectured that Θ_* is an isomorphism. It will be shown elsewhere [13] that stable K-theory and topological Hochschild homology are isomorphic. As a result therefore the map Θ_* is an isomorphism.

The construction of Θ_* is based on Theorem 1.4.

Let $\boldsymbol{C}$ be a small category and

$$D : \boldsymbol{C}^{\mathrm{op}} \times \boldsymbol{C} \to \mathrm{Ab}$$

a bifunctor. Following [1] and [12], we wish to define $H_*(\boldsymbol{C}, D)$, the homology of $\boldsymbol{C}$ with coefficients in D. For any n-simplex

$$\lambda = (A_n \xleftarrow{\lambda_n} A_{n-1} \xleftarrow{\lambda_{n-1}} \cdots \xleftarrow{\lambda_1} A_0)$$

of the nerve of C, we denote $D(A_n, A_0)$ by $D(\lambda)$. Let

$$F_n(C, D) = \bigoplus_{\lambda} D(\lambda)\,,$$

where λ runs through the n-simplices of NC and let in_λ be the inclusion $D(\lambda) \hookrightarrow F_n(C, D)$. We define

$$d_i^n : F_n(C, D) \to F_{n-1}(C, D)\,, \quad 0 \le i \le n\,,$$

by

$$d_i^n \circ \mathrm{in}_\lambda = \begin{cases} \mathrm{in}_{d_0\lambda} \circ D(\mathrm{id}_{A_n}, \lambda_1)\,, & \text{if } i = 0\,, \\ \mathrm{in}_{d_i\lambda}\,, & \text{if } 0 < i < n\,, \\ \mathrm{in}_{d_n\lambda} \circ D(\lambda_n, \mathrm{id}_{A_0})\,, & \text{if } i = n\,, \end{cases}$$

where $d_i\lambda$ is the ith face in NC. Let

$$\delta_n = \sum_{i=0}^{n} (-1)^i d_i^n\,.$$

Then $(F_*(C, D), \delta_*)$ becomes a chain complex.

Definition 1.2. The homology of a category C with coefficients in a bifunctor D is defined by

$$H_*(C, D) = H_*(F_*(C, D), \delta_*)\,.$$

Let $\mathbb{P}(R)$ be the category of finitely generated projective (left) R-modules and

$$\mathcal{F}(R) = (R\text{-mod})^{\mathbb{P}(R)}$$

be the category of all functors from $\mathbb{P}(R)$ to R-mod. Let $P_m : \mathbb{P}(R) \to R$-mod be the functor defined by

$$P_m(X) = R[X^m]\,, \quad m \ge 0\,,$$

where $R[S]$ means the free R-module with base S and X^m means the m-fold product of X with itself. The family P_m, $m \ge 0$, is a family of projective generators in the category $\mathcal{F}(R)$ (see for example [8, 2.5]), therefore the following proposition is a standard fact in homological algebra (see [4]).

Proposition 1.3. *There exists a unique (up to isomorphism) family of functors*

$$H_n(R, -) : \mathcal{F}(R) \to \mathrm{Ab}\,, \quad n \geq 0\,,$$

satisfying the following properties.

(i) *For any short exact sequence of functors*

$$0 \to T_1 \to T \to T_2 \to 0$$

there exists a natural long exact sequence of abelian groups

$$\cdots \to H_{n+1}(R, T_2) \to H_n(R, T_1) \to H_n(R, T) \to H_n(R, T_2) \to \cdots.$$

(ii) *If* $n \geq 1$, *then*

$$H_n(R, P_m) = 0\,, \quad m \geq 0\,.$$

(iii) *One has a natural isomorphism*

$$H_0(R, T) = \mathrm{Hoch}_0(R, L_0^{\mathrm{st}} T(R))\,,$$

where $L_*^{\mathrm{st}} T$ *means the Dold–Puppe* stable derived functors [5]. □

By definition, $L_0^{\mathrm{st}} T(P) = \pi_n T(K(P, n))$ for any $n > 0$ and $P \in \mathbb{P}(R)$. The functor $L_0^{\mathrm{st}} T$ is additive (see [5, 8.3]). Moreover, the rule $T \mapsto L_0^{\mathrm{st}} T$ defines a functor from $\mathcal{F}(R)$ to the category of the additive functors from $\mathbb{P}(R)$ to R-mod, which is a left adjoint to the inclusion (see [11]). The latter category of additive functors is equivalent to the category of *R-bimodules* by $T \mapsto T(R)$. Similar properties hold for stable right derived functors. In particular,

$$\begin{aligned}\mathrm{Hom}_{\mathcal{F}(R)}(I, T) &\cong \mathrm{Hom}_{R\text{-}R}(R, R^0_{\mathrm{st}} T(R)) \\ &\cong \mathrm{Hoch}^0(R, R^0_{\mathrm{st}} T(R))\,,\end{aligned}$$

where $I : \mathbb{P}(R) \to R$-mod is the inclusion. Therefore the following theorem is the dual of the main result of [8].

Theorem 1.4. *Let R be a ring and let*

$$I : \mathbb{P}(R) \to R\text{-mod}$$

be the inclusion. Then there exist natural isomorphisms

$$H_*^{\mathrm{ML}}(R) = H_*(R, I) = H_*(\mathbb{P}(R), \mathrm{Hom})\,. \qquad \square$$

Now we define the transformation Θ_*. Let $F(R)$ be the homotopy fibre of

$$BGL(R) \to (BGL(R))^+ .$$

The group $\pi_1 F(R) = St(R)$ acts on the group of matrices $M(R)$ by conjugation. We denote this local system on $F(R)$ by $M(R)^{con}$. By definition (or by [9]),

$$K^s_*(R) = H_*(F(R), M(R)^{con}) .$$

The inclusion $F(R) \hookrightarrow BGL(R)$ induces a map

$$u_* : K^s_*(R) \to H_*(GL(R), M(R)^{con}) .$$

By a well-known theorem in homological algebra the last groups are isomorphic to the Hochschild homology of $GL(R)$ with coefficients in the bimodule $M(R)$. Let us consider $GL(n, R)$ as a subcategory in $\mathbb{P}(R)$, whose morphisms are the isomorphisms $R^n \to R^n$. Then the restriction of the bifunctor Hom to $GL(n, R)$ is $M_n(R)$. Therefore the inclusion of $GL(n, R)$ in $\mathbb{P}(R)$ induces a homomorphism

$$H_*(GL(n, R), M_n(R)) \to H_*(\mathbb{P}(R), \mathrm{Hom}) .$$

The transformation Θ_* is defined by composition,

$$\begin{aligned} K^s_*(R) &\xrightarrow{u_*} H_*(GL(R), M(R)^{con}) \\ &= \lim_n H_*(GL(n, R), M_n(R)) \longrightarrow H_*(\mathbb{P}(R), \mathrm{Hom}) . \end{aligned}$$

Remark. We have used here that the maps

$$H_*(GL(R, n), M_n(R)) \to H_*(\mathbb{P}(R), \mathrm{Hom})$$

are compatible. This follows from the fact that the endofunctor $- \oplus R : \mathbb{P}(R) \to \mathbb{P}(R)$ induces the identity in homology. Namely let $\mathcal{D}$ be the bifunctor on $\mathbb{P}(R)$ defined by

$$\mathcal{D}(X, Y) = \mathrm{Hom}_R(X \oplus R, Y \oplus R) .$$

Then $\alpha \mapsto \alpha \oplus 0$ defines a transformation $\mathrm{Hom} \to \mathcal{D}$ and therefore a chain map

$$\varphi_* : F_*(\mathbb{P}(R), \mathrm{Hom}) \to F_*(\mathbb{P}(R), \mathcal{D}) .$$

The endofunctor $- \oplus R$ yields a chain map

$$\psi_* : F_*(\mathbb{P}(R), \mathcal{D}) \to F_*(\mathbb{P}(R), \mathrm{Hom}) .$$

Then $\psi_* \varphi_*$ is homotopic to identity, a homotopy is given by $s_n = \sum_{i=0}^{n} (-1)^i \tilde{h}_i^n$, where

$$\tilde{h}_i : F_n(\mathbb{P}(R), \mathrm{Hom}) \to F_{n+1}(\mathbb{P}(R), \mathrm{Hom}) , \quad 0 \le i \le n$$

is defined as follows. Let $\alpha : A_n \to A_0$ be a homomorphism, $\lambda = (A_0 \xrightarrow{\lambda_1} \cdots \xrightarrow{\lambda_n} A_n)$ an n-simplex in the nerve of $\mathbb{P}(R)$, and

$$h_i \lambda = (A_0 \xrightarrow{\lambda_1} \cdots \xrightarrow{\lambda_i} A_i \xrightarrow{\binom{1}{0}} A_i \oplus R \xrightarrow{\lambda_{i+1} \oplus 1} \cdots \xrightarrow{\lambda_n \oplus 1} A_n \oplus R) .$$

Then $\tilde{h}_i$ is given by

$$\tilde{h}_i \circ \mathrm{in}_\lambda(\alpha) = \mathrm{in}_{h_i \lambda}(\alpha, 0) .$$

2. Topological Hochschild homology with coefficients

In this section we define some coefficient systems for topological Hochschild homology and establish some elementary facts. The following definitions are in [2], at least implicitly.

Definition 2.1. A *functor with stabilization* is a functor F from the category of pointed simplicial sets to itself together with a natural transformation

$$\lambda_{X,Y} : X \wedge F(Y) \to F(X \wedge Y)$$

such that:

(i) $\lambda_{X,Y \wedge Z} \circ (\mathrm{id}_X \wedge \lambda_{Y,Z}) = \lambda_{X \wedge Y,Z}$ and

$$\rho_{X \wedge Y,Z} \circ (\lambda_{X,Y} \wedge \mathrm{id}_Z) = \lambda_{X,Y \wedge Z} \circ (\mathrm{id}_X \wedge \rho_{Y,Z}) ,$$

where $\rho_{X,Y} : F(X) \wedge Y \to F(X \wedge Y)$ is defined as the composite

$$F(T_{Y,X}) \circ \lambda_{Y,X} \circ T_{F(X),Y} \quad (T = \text{twist of two factors}) .$$

(ii) If X is n-connected, then $F(X)$ is also n-connected.

(iii) Let $\sigma_X : F(X) \to \Omega F(\Sigma X)$ be the adjoint to $\lambda_{S^1,X}$. Then the limit system

$$\pi_n(FX) \xrightarrow{(\sigma_X)_*} \pi_n \Omega F(\Sigma X) \xrightarrow{(\sigma_{\Sigma X})_*} \pi_n \Omega^2 F(\Sigma^2 X) \to \cdots$$

stabilizes for each n.

Definition 2.2. A *functor with smash product* (FSP) is a functor F with stabilization, together with two natural transformations

$$1_X : X \to F(X)\,, \qquad \mu_{X,Y} : F(X) \wedge F(Y) \to F(X \wedge Y)$$

such that

$$\mu(\mu \wedge \mathrm{id}) = \mu(\mathrm{id} \wedge \mu)\,, \qquad \mu(1_X \wedge 1_Y) = 1_{X \wedge Y}\,,$$
$$\lambda_{X,Y} = \mu_{X,Y} \circ (1_X \wedge \mathrm{id}_{F(Y)})\,, \qquad \rho_{X,Y} = \mu_{X,Y} \circ (\mathrm{id}_{F(X)} \wedge 1_Y)\,.$$

Example 2.3 [2]. Let R be a ring. Then the functor $\tilde{R}$ defined by

$$\tilde{R}(X) = R[X]/R[*]$$

is an FSP.

Definition 2.4. Let F be an FSP and T a functor with stabilization. A *structure of left F-module* on T is a natural transformation

$$\ell_{X,Y} : F(X) \wedge T(Y) \to T(X \wedge Y)$$

such that

$$\ell(\mu \wedge \mathrm{id}) = \ell(\mathrm{id} \wedge \ell)\,, \qquad \lambda_{X,Y} = \ell_{X,Y}(1_X \wedge \mathrm{id}_{T(Y)})\,.$$

The notion of right F-module is defined similarly.

Definition 2.5. A *bimodule* over F is a functor T with stabilization together with a structure of left and right module over F such that

$$\ell_{X,Y \wedge Z}(\mathrm{id}_{F(X)} \wedge r_{Y,Z}) = r_{X \wedge Y,Z}(\ell_{X,Y} \wedge \mathrm{id}_{F(Z)})\,,$$

where r is the structure of right module over T.

The category of F-bimodules is denoted by F-mod-F.

Example 2.6. Let R be a ring and $\tilde{R}$ be the FSP of Example 2.3. Let

$$T : \mathbb{P}(R) \to R\text{-mod}$$

be a functor; by direct limit we may assume T to be extended to the category of projective R-modules which are not necessarily finitely generated. We denote by $T^!$ the composition

$$\text{s.Sets} \xrightarrow{\tilde{R}} \text{s.free } R\text{-mod} \xrightarrow{T} \text{s.}R\text{-mod} \xrightarrow{\text{forgetful}} \text{s.Sets}\,.$$

From the adjoint of the composition

$$\tilde{R}(Y) \to \mathrm{Hom}_R(\tilde{R}(X), \tilde{R}(X \wedge Y)) \xrightarrow{T} \mathrm{Hom}_R(T\tilde{R}(X), T\tilde{R}(X \wedge Y)) ,$$

where the first map is adjoint to the isomorphism $\tilde{R}(X) \otimes \tilde{R}(Y) \to \tilde{R}(X \wedge Y)$, one obtains a pairing

$$T^{!}(X) \wedge \tilde{R}(Y) \to T^{!}(X \wedge Y) . \tag{1}$$

The structure map $R \otimes M \to M$ on any left R-module M determines a pairing

$$\tilde{R}(X) \wedge M \to M \otimes \tilde{\mathbb{Z}}(X)$$

as the composition

$$\tilde{R}(X) \wedge M \to \tilde{R}(X) \otimes M = R \otimes M \otimes \tilde{\mathbb{Z}}(X) \to M \otimes \tilde{\mathbb{Z}}(X) .$$

Therefore we have a map

$$\tilde{R}(X) \wedge T(\tilde{R}(Y)) \to T(\tilde{R}(Y)) \otimes \tilde{\mathbb{Z}}(X) .$$

Composition of this map with the natural embedding

$$T(\tilde{R}(Y)) \otimes \tilde{\mathbb{Z}}(X) = \bigoplus_{X \setminus \{*\}} T(\tilde{R}(Y)) \to T\Big(\bigoplus_{X \setminus \{*\}} \tilde{R}(Y)\Big) = T(\tilde{R}(X \wedge Y))$$

yields the pairing

$$\tilde{R}(X) \wedge T^{!}(Y) \to T^{!}(X \wedge Y) . \tag{2}$$

By 6.9 and 6.12 of [5] the functor $T^{!}$ satisfies the properties (ii) and (iii) of Definition 2.1. The transformations (1) and (2) above determine the structure of $\tilde{R}$-bimodule on $T^{!}$. Therefore we obtain the functor

$$^{!} : \mathcal{F}(R) \to \tilde{R}\text{-mod-}\tilde{R} .$$

Definition 2.7. A *bifunctor with F-action* is a (covariant) bifunctor B which is a functor with stabilization for each variable together with natural transformations

$$\ell_{X,Y,Z} : F(X) \wedge B(Y, Z) \to B(X \wedge Y, Z) ,$$
$$r_{X,Y,Z} : B(X, Y) \wedge F(Z) \to B(X, Y \wedge Z) ,$$

such that $B(-, Y)$ and $B(X, -)$ are left and right F-modules respectively for every X, Y and

$$\ell_{X,Y,Z\wedge W}(\mathrm{id}_{FX} \wedge r_{Y,Z,W}) = r_{X\wedge Y,Z,W}(\ell_{X,Y,Z} \wedge \mathrm{id}_{FW}) .$$

The category of bifunctors with F-action is denoted by F-bif.

Examples 2.8. (i) Let T be an F-bimodule and $T^{\#}$ the bifunctor defined by

$$T^{\#}(X, Y) = T(X \wedge Y) .$$

Then we obtain a functor

$$^{\#} : F\text{-mod-}F \to F\text{-bif} .$$

(ii) Let M be a left F-module and $M \wedge F$ the bimodule defined by

$$(M \wedge F)(X, Y) = M(X) \wedge F(Y) .$$

Then $M \wedge F$ is a bifunctor with F-action and for any bifunctor B with F-action we have

$$\mathrm{Hom}_{F\text{-bif}}(M \wedge F, B) = \mathrm{Hom}_{F\text{-mod}}(M, B(-, S^0)) .$$

Below for any functor

$$E : \boldsymbol{C} \to \text{s.Sets}$$

we denote by $L_C E$ the homotopy colimit of E (see [2]).

Let I be the category whose objects are the natural numbers considered as ordered sets and whose morphisms are injective maps. For any $X \in I$ we denote by $|X|$ the cardinality of X and for any $X = (X_0, \ldots, X_n) \in I^{n+1}$ we let $\bigsqcup X$ denote $X_0 \sqcup X_1 \sqcup \cdots \sqcup X_n$, where $\sqcup$ means *concatenation*.

We are going to define a spectrum $\mathrm{THH}(F, T)$ for each F-bimodule T. To this end, let $\mathrm{THH}(F, T)(m)$ be the simplicial space defined by

$$[n] \mapsto L_{I^{n+1}}(G_n(T)) ,$$

where $G_n(T)$, or more simply G_n, $G_n : I^{n+1} \to \text{s.Sets}$, is the functor

$$G_n(X) = \Omega^{\sqcup X}(S^m \wedge T(S^{X_0}) \wedge F(S^{X_1}) \wedge \cdots \wedge F(S^{X_n})) .$$

The face operators are induced by the natural transformations

$$d_i^n : G_n \to G_{n-1}\partial_i^n , \quad 0 \le i \le n ,$$

where $\partial_i^n : I^{n+1} \to I^n$ is the functor

$$\partial_i^n(X) = \begin{cases} (X_0, \ldots, X_i \sqcup X_{i+1}, \ldots, X_n), & 0 \le i < n, \\ (X_n \sqcup X_0, X_1, \ldots, X_{n-1}), & i = n, \end{cases}$$

and

$$d_i^n(X) = \begin{cases} \Omega^{\sqcup X}(S^m \wedge r \wedge F(S^{X_2}) \wedge \cdots \wedge F(S^{X_n})), & i = 0, \\ \Omega^{\sqcup X}(S^m \wedge T(S^{X_0}) \wedge \cdots \wedge \mu \wedge \cdots \wedge F(S^{X_n})), & 0 < i < n, \\ \Omega^{\sqcup X}(S^m \wedge \ell \wedge F(S^{X_1}) \wedge \cdots \wedge F(S^{X_{n-1}})) \circ \rho, & i = n, \end{cases}$$

here ρ is the map induced by cyclic permutation on I^{n+1}, μ is the multiplication on F, and ℓ and r are the left and right multiplications on T. The degeneracy operators are similar.

Let B be a bifunctor with F-action. Let $\widetilde{\mathrm{THH}}(F, B)(m)$ be the simplicial space

$$[n] \mapsto L_{I^{n+2}}(\tilde{G}_n),$$

where $\tilde{G}_n : I^{n+2} \to$ s.Sets is the functor

$$\tilde{G}_n(Y) = \Omega^{\sqcup Y}(S^m \wedge B(S^{Y_{-1}}, S^{Y_0}) \wedge F(S^{Y_1}) \wedge \cdots \wedge F(S^{Y_n})),$$

here $Y = (Y_{-1}, Y_0, \ldots, Y_n) \in I^{n+2}$. The face operations are induced by the natural transformations

$$\tilde{d}_i^n : \tilde{G}_n \to \tilde{G}_{n-1}\tilde{\partial}_i^n,$$

where $\tilde{\partial}_i^n : I^{n+2} \to I^{n+1}$ is the functor

$$\tilde{\partial}_i^n(Y) = \begin{cases} (Y_{-1}, \ldots, Y_i \sqcup Y_{i+1}, \ldots, Y_n), & 0 \le i \le n, \\ (Y_n \sqcup Y_{-1}, Y_0, \ldots, Y_{n-1}), & i = n, \end{cases}$$

and

$$\tilde{d}_i^n(Y) = \begin{cases} \Omega^{\sqcup Y} B(S^{Y_{-1}}, r) \wedge F(S^{Y_2}) \wedge \cdots \wedge F(S^{Y_n}), \\ \quad i = 0, \\ \Omega^{\sqcup Y} B(S^{Y_{-1}}, S^{Y_0}) \wedge F(S^{Y_1}) \wedge \cdots \wedge \mu \wedge \cdots \wedge F(S^{Y_n}), \\ \quad 0 < i < n, \\ \Omega^{\sqcup Y} B(\ell, S^{Y_0}) \wedge F(S^{Y_1}) \wedge \cdots \wedge F(S^{Y_{n-1}}) \circ \rho, \\ \quad i = n, \end{cases}$$

here ρ is the map induced by cyclic permutation on I^{n+1}. The degeneracy operators are similar.

The rule $m \mapsto \mathrm{THH}(F, T)(m)$ (resp. $\widetilde{\mathrm{THH}}(F, B)(m)$) gives a spectrum with structure maps like those in [2]. The corresponding infinite loopspace is denoted by $\mathrm{THH}(F, T)$ (resp. $\widetilde{\mathrm{THH}}(F, B)$).

By definition $\mathrm{THH}(F, F)$ coincides with $\mathrm{THH}(F)$ from [2].

Proposition 2.9. *Let $p \geq 0$. Then there exists $k \in \mathbb{N}$ such that, for every $X_{-1}, X_0, \ldots, X_n$ with*

$$|X_{-1}|, \ldots, |X_n| \geq k$$

the natural map

$$G_n(X_0, \ldots, X_n) \to L_{I^{n+1}} G_n$$

(resp.

$$\tilde{G}_n(X_{-1}, X_0, \ldots, X_n) \to L_{I^{n+2}} \tilde{G}_n\ )$$

is a p-equivalence.

Proof. In view of property (iii) of Definition 2.1, the proposition follows from [2, Section 1]. $\square$

Proposition 2.10. *Let T be an F-bimodule. Then there exists a natural weak equivalence*

$$\widetilde{\mathrm{THH}}(F, T^{\#}) \to \mathrm{THH}(F, T).$$

Proof. Let $f_n : I^{n+2} \to I^{n+1}$ be the functor

$$f_n(X_{-1}, X_0, \ldots, X_n) = (X_{-1} \sqcup X_0, X_1, \ldots, X_n).$$

Then $\tilde{G}_n = G_n \circ f_n$. Therefore, by Proposition 2.9, f_n yields the weak equivalence

$$L_{I^{n+2}} \tilde{G}_n \to L_{I^{n+1}} G_n.$$

The sequence of maps (f_n), $n \geq 0$, is compatible with the simplicial structure and therefore

$$f_* : \widetilde{\mathrm{THH}}(F, T^{\#}) \to \mathrm{THH}(F, T)$$

is a weak equivalence. $\square$

Definition 2.11. Let $f: T \to T'$ (resp. $f: B \to B'$) be a morphism in F-F-mod (resp. F-bif). We call f a *stable weak equivalence* if for any $n \geq 0$ there exists $m \in \mathbb{N}$ such that for every $k > m$ the map

$$f(S^k): T(S^k) \to T'(S^k)$$

(resp.

$$f(S^k, S^k): B(S^k, S^k) \to B'(S^k, S^k)\ )$$

is an $(n+k)$-equivalence (resp. $(n+2k)$-equivalence).

By Proposition 2.9 any stable equivalence $f: T \to T'$ (resp. $f: B \to B'$) induces a weak equivalence $\mathrm{THH}(F, T) \to \mathrm{THH}(F, T')$ (resp. $\widetilde{\mathrm{THH}}(F, B) \to \widetilde{\mathrm{THH}}(F, T')$).

Proposition 2.12. *Let M be a left F-module. Then there exists a homotopy equivalence* $\widetilde{\mathrm{THH}}(F, M \wedge F) \sim \mathrm{sp}(M)$, *where $M \wedge F$ is the bifunctor with F-action defined in Example* 2.8(ii) *and* $\mathrm{sp}(M)$ *is the infinite loopspace corresponding to the spectrum*

$$\mathrm{sp}(M)(m) = L_l(X \mapsto \Omega^X(S^m \wedge M(S^X))\ .$$

Proof. Product with the unit map $S^0 \to \Omega^n F(S^n)$ induces a contraction in the augmented simplicial space:

$$\widetilde{\mathrm{THH}}(F, M \wedge F)(m) \to \mathrm{sp}(M)(m)\ . \quad \square$$

Proposition 2.13. *Let*

$$T_1 \to T \to T_2$$

be a sequence in the category of F-bimodules such that for every $k \gg 0$ the values of this sequences on S^k is a fibration. Then the natural map of $\mathrm{THH}(F, T_1)$ *to the homotopy fiber of* $\mathrm{THH}(F, T) \to \mathrm{THH}(F, T_2)$ *is a weak equivalence.*

Proof. By well-known properties of bisimplicial sets it is sufficient to show that the map from $L_{I^{n+1}} G_n(T_1)$ to the homotopy fiber of $L_{I^{n+1}} G_n(T) \to L_{I^{n+1}} G_n(T_2)$ is a weak equivalence. But this follows from Proposition 2.9 and Lemma 2.14, which follows easily from the Blakers–Massey theorem. $\square$

Lemma 2.14. *Let*

$$F \to E \to B$$

be a fibration such that F and B are n-connected and let X be an m-connected space. Then

$$\pi_i(E \wedge X, F \wedge X) \to \pi_i(B \wedge X)$$

is an isomorphism if $i \leq 2n + m$. □

3. The main theorem

In this section we prove the following result.

Theorem 3.1. *Let R be a ring. Then there exists a natural isomorphism*

$$H_*^{\mathrm{ML}}(R) \cong \pi_* \mathrm{THH}(\tilde{R}) .$$

By Theorem 1.4, Theorem 3.1 is a particular case of the following theorem.

Theorem 3.2. *Let R be a ring and* $T \in \mathscr{F}(R)$. *Then there exists a natural isomorphism*

$$H_*(R, T) \cong \pi_* \mathrm{THH}(\tilde{R}, T^!) .$$

Proof. Let $H_n'(R, -)$ be the composition

$$\mathscr{F}(R) \xrightarrow{\ !\ } \tilde{R}\text{-}\tilde{R}\text{-mod} \xrightarrow{\mathrm{THH}(\tilde{R},-)} \text{infinite loop spaces} \xrightarrow{\pi_n} \mathrm{Ab} .$$

By Proposition 1.3 it is sufficient to show that the groups $H_n'(R, -)$, $n \geq 0$, satisfy the properties 1.3(i)–1.3(iii).

Let

$$0 \to T_1 \to T \to T_2 \to 0$$

be an exact sequence in $\mathscr{F}(R)$. Then for every $X \in$ s.Sets we obtain an exact sequence of simplicial R-modules

$$0 \to T_1(\tilde{R}(X)) \to T(\tilde{R}(X)) \to T_2(\tilde{R}(X)) \to 0 .$$

Hence $T_1^!(X) \to T^!(X) \to T_2^!(X)$ is a fibration and by Proposition 2.13 the sequence of functors $H_n'(R, -)$ satisfies 1.3(i).

It is clear that $L_{I^{n+1}} G_n(T^!)(m)$ is $(m-1)$-connected, therefore $\mathrm{THH}(\tilde{R}, T^!)(m)$ is also $(m-1)$-connected and from the spectral sequence of bisimplicial sets (see [16]) it follows that

$$\pi_m \mathrm{THH}(\tilde{R}, T^!)(m) = \mathrm{Coker}(\pi_m L_{I^2} G_1(T^!)(m) \rightrightarrows \pi_m L_I G_0(T^!)(m)) .$$

By Proposition 2.9 for any sufficiently large number x we have

$$\begin{aligned}
&\pi_m L_I G_0(T^!)(m) \\
&\quad = \pi_{m+x}(S^m \wedge T(\tilde{R}(S^x))) \\
&\quad = H_{m+x}(S^m \wedge T(\tilde{R}(S^x))) \\
&\quad = H_x(T(\tilde{R}(S^x))) \\
&\quad = \pi_x T(\tilde{R}(S^x)) = L_0^{\mathrm{st}} T(R) ,
\end{aligned}$$

since $\tilde{R}(S^x) = K(R, x)$, $T(\tilde{R}(S^x))$ is $(x-1)$-connected and $S^m \wedge T(R(S^x))$ is $(m+x-1)$-connected. Similarly we obtain

$$\begin{aligned}
&\pi_m L_{I^2} G_0(T^!)(m) \\
&\quad = \pi_{x+y+m}(S^m \wedge T(\tilde{R}(S^x)) \wedge \tilde{R}(S^y)) \\
&\quad = H_{x+y}(T(\tilde{R}(S^x)) \wedge \tilde{R}(S^y)) \\
&\quad = L_0^{\mathrm{st}} T(R) \otimes R ,
\end{aligned}$$

and

$$\begin{aligned}
&\pi_0 \mathrm{THH}(\tilde{R}, T^!) \\
&\quad = \lim_m \pi_m \mathrm{THH}(\tilde{R}, T^!)(m) \\
&\quad = \mathrm{Coker}(L_0^{\mathrm{st}} T(R) \otimes R \rightrightarrows L_0^{\mathrm{st}} T(R)) \\
&\quad = \mathrm{Hoch}_0(R, L_0^{\mathrm{st}} T(R)) .
\end{aligned}$$

For the proof of 1.3(ii) we consider the morphism of left R-modules

$$\tilde{R}^m(X) \xrightarrow{\tilde{R}(1_E)} \tilde{R}(\tilde{R}^m(X)) \to R[\tilde{R}^m(X)]$$

where $E = \tilde{R}^m(X)$. This map yields the morphism in $\tilde{R}$-bif,

$$\tilde{R}^m(X) \wedge \tilde{R}(Y) \to R[\tilde{R}^m(X \wedge Y)] = (P_m^!)^\#(X, Y) .$$

By Lemma 3.3 this map is a stable weak equivalence. Therefore it follows from Proposition 2.10 that

$$\mathrm{THH}(\tilde{R}, P_m^!) = \widetilde{\mathrm{THH}}(\tilde{R}, (P_m^!)^\#) = \widetilde{\mathrm{THH}}(\tilde{R}, \tilde{R}^m \wedge \tilde{R}) .$$

The homotopy groups of the last space are trivial in positive dimensions by Proposition 2.12. □

Lemma 3.3. *Let* $i < 3n - 1$. *Then*

$$\pi_i(K(\pi, n) \wedge K(\tau, n)) \simeq H_i(K(\pi, 2n), \tau) .$$

Proof. We have the following isomorphisms

$$\begin{aligned}
&\pi_i(K(\pi, n) \wedge K(\tau, n)) \\
&= \pi_{i+N}(K(\pi, n) \wedge \Sigma^N K(\tau, n)) && \text{(a)} \\
&= \lim_N \pi_{i+N}(K(\pi, n) \wedge K(\tau, n + N)) && \text{(b)} \\
&= \lim_M \pi_{i-n+M}(K(\pi, n) \wedge K(\tau, M)) && \text{(c)} \\
&= H_{i-n}(K(\pi, n), \tau) && \text{(d)} \\
&= H_i(\Sigma^n K(\pi, n), \tau) && \text{(e)} \\
&= H_i(K(\pi, 2n), \tau) . && \text{(f)}
\end{aligned}$$

The isomorphism (a) follows from the Freudental theorem, (c) and (e) are easy, (d) is the definition of homology in terms of spectra. The validity of (b) and (f) follow from the stable equivalence between $\Sigma^N K(\pi, n)$ and $K(\pi, n + N)$. □

4. Relation with Hochschild homology

The main result of this section is Theorem 4.1, which provides an analog of the Atiyah–Hirzebruch spectral sequence for MacLane homology. The role of the one-point space and of ordinary homology are played by the ring of integers and by Hochschild homology, respectively.

Theorem 4.1. *Let* R *be a ring which is torsion free as abelian group, and* $T \in \mathcal{F}(R)$. *Let* $\tilde{T} \in \mathcal{F}(\mathbb{Z})$ *be the functor defined by*

$$\tilde{T}(X) = T(X \otimes R) ,$$

where $X \in \mathbb{P}(\mathbb{Z})$. *Then there is a natural structure of R-bimodule on* $H_*(\mathbb{Z}, \tilde{T})$ *and there exists a spectral sequence*

$$E^2_{pq} = \mathrm{Hoch}_p(R, H_q(\mathbb{Z}, \tilde{T})) \Rightarrow H_{p+q}(R, T) .$$

Remarks 4.2. (a) When R is an arbitrary ring, there exists a similar spectral sequence

$$E^2_{pq} = \mathrm{Shukla}_p(R, H_q(\mathbb{Z}, \tilde{T})) \Rightarrow H_{p+q}(R, T) \tag{3}$$

where Shukla_* means *Shukla homology* (see [14] for the definition of Shukla (co-)homology). Of course, $\mathrm{Shukla}_*(R, -) = \mathrm{Hoch}_*(R, -)$ if R is torsion free as abelian group. The spectral sequence (3) is obtained from Theorem 4.1 by simplicial approximation of the ring R by using free rings.

(b) When $T = M \otimes_R -$, where M is an R-bimodule, the spectral sequence has the form

$$E^2_{pq} = \mathrm{Shukla}_p(R, H^{\mathrm{ML}}_q(\mathbb{Z}, M)) \Rightarrow H^{\mathrm{ML}}_{p+q}(R, M)\,. \tag{4}$$

By Theorem 3.1 and Bökstedt's calculation (see [3]) we have

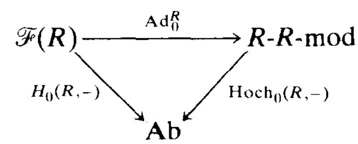
$$\begin{aligned} &H^{\mathrm{ML}}_{2i-1}(\mathbb{Z}, M) = M \otimes \mathbb{Z}/i\,, \\ &H^{\mathrm{ML}}_{2i}(\mathbb{Z}, M) = \mathrm{Tor}(M, \mathbb{Z}/i)\,, \qquad i > 0\,. \end{aligned} \tag{5}$$

The differentials of this spectral sequence are in general nontrivial. For example, when

$$R = M = \mathbb{Z}/p\,,$$

then Bökstedt's calculation [3] shows that $E^2_{**} = E^{2p}_{**}$, d^{2p} is nontrivial and $E^{2p+1}_{**} = E^{\infty}_{**}$.

Proof of Theorem 4.1. For arbitrary $T \in \mathcal{F}(R)$ we denote $L^{\mathrm{st}}_0 T(R)$ by $\mathrm{Ad}^R_0 T$, and the functor P_m of Definition 1.2 is now denoted by P^R_m. By Proposition 1.3(iii), the diagram



(6)

is commutative and $H_*(R, -)$ is the left derived functor of $H_0(R, -)$. Since R is torsion free, the left derived functor of $\mathrm{Hoch}_0(R, -)$ is $\mathrm{Hoch}_*(R, -)$. We denote by Ad^R_* the left derived functor of the functor Ad^R_0. Since

$$\mathrm{Ad}^R_0(P^R_m) = \pi_n(P^R_m(K(R, n)) = (R \otimes R)^m\,, \quad n > 0\,,$$

the functor Ad^R_0 sends projective objects to projective objects, and the spectral sequence for derived functors of the composition of functors for (6) has the form

$$E^2_{pq} = \mathrm{Hoch}_p(R, \mathrm{Ad}^R_q T) \Rightarrow H_{p+q}(R, T) . \tag{7}$$

When $R = \mathbb{Z}$, the spectral sequence (7) is degenerate and we obtain

$$\mathrm{Ad}^{\mathbb{Z}}_* T = H_*(\mathbb{Z}, T) . \tag{8}$$

On the other hand we have

$$(\tilde{P}^R_m)(X) = R[X^m \otimes R] = R \otimes \mathbb{Z}[X^m \otimes R] .$$

Therefore, when R as an abelian group is finitely generated and free, the functor $\tilde{P}^R_m \in \mathcal{F}(\mathbb{Z})$ is a sum of functors of the form $P^{\mathbb{Z}}_K$ and therefore is a projective object in $\mathcal{F}(\mathbb{Z})$. In general, when R is only torsion free as an abelian group, the functor $\tilde{P}^R_m$ is a filtered colimit of such objects and therefore still $\mathrm{Ad}^{\mathbb{Z}}_0$-acyclic. Since $T \mapsto \tilde{T}$ may be regarded as an exact functor from $\mathcal{F}(R)$ to $\mathcal{F}(\mathbb{Z})$, the commutative diagram

$$\begin{array}{ccc} \mathcal{F}(R) & \xrightarrow{T \mapsto \tilde{T}} & \mathcal{F}(\mathbb{Z}) \\ {\scriptstyle \mathrm{Ad}^R_0} \downarrow & & \downarrow {\scriptstyle \mathrm{Ad}^{\mathbb{Z}}_0} \\ R\text{-mod-}R & \xrightarrow{\text{forgetful}} & \mathrm{Ab} \end{array}$$

shows that

$$\text{forgetful} \circ \mathrm{Ad}^R_* T = \mathrm{Ad}^{\mathbb{Z}}_* \tilde{T} .$$

Combining (7) and (8) with this equality we get

$$E^2_{pq} = \mathrm{Hoch}_p(R, H_q(\mathbb{Z}, \tilde{T})) \Rightarrow H_{p+q}(R, T) . \qquad \square$$

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