Determinantal Ideals of Linear Type of a Generic Symmetric Matrix

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INTRODUCTION

Let $I$ be an ideal in a commutative ring, $A$. In this paper we consider the canonical homomorphism $\varphi: S^A(I) \rightarrow R_I(A)$ of the symmetric algebra of $I$ to its Rees algebra. (Recall that $R_I(A) = \bigoplus_{r=0}^{\infty} I^rT^r \subseteq A[T]$, $T$ is an indeterminate over $A$.) Certainly, $\varphi$ is always surjective. When it is an isomorphism, $I$ is called an ideal of linear type.

In [7] C. Huneke determined when the ideal generated by the minors of a given size of a generic matrix is of linear type. In the same article he asked the analogous question for a generic symmetric matrix, in particular whether the ideal generated by the submaximal minors is of linear type. In this paper we give a full answer to this question (Propositions 2.10, 3.1). Our result is

**Theorem A.** Let $X$ be a generic symmetric $n \times n$ matrix over the commutative ring $R$. The ideal $I_t(X) \subseteq R[X_{ij}]$, $1 \leq t \leq n$ is of linear type if and only if $t = 1$, $n-1$, or $n$.

The hard part of this theorem is to establish that the ideal $I_{n-1}(X)$ is of linear type. (The analogous statement when $X$ is a square generic matrix is the main result of [7].) Our approach to this problem is somewhat different from that of Huneke and is closely related to Strickland's article [16]. In order to prove the exactness of the circular complex she constructs an auxiliary algebra and finds a free module-basis for it. It is implicit (but maybe not noted) in [7], that this algebra is isomorphic to the symmetric algebra of the ideal generated by the submaximal minors of a square generic matrix. This remark is the starting point of our approach: In Proposition 2.5 we construct a free basis of the symmetric algebra $S(I_{n-1}(X))$. For this purpose we use a free basis of the ring $R[X_{ij}, Y_{ij}]$.
$I_1(XY)$, where $X = (X_{ij})$ and $Y = (Y_{ij})$ are generic symmetric $n \times n$ matrices. Such a basis was constructed in the author's Master's Thesis (Sofia, 1988). Meanwhile, the necessary results have appeared in the recent article of Ruitenburg [13], and we reproduce them, along with the necessary notation and terminology, in Section 1.

One application of the basis constructed in Proposition 2.5 is a direct proof of the exactness of the "symmetric circular complex," cf. Proposition 2.8. (The corresponding fact in the generic case is due to Strickland [16], cf. also [7].)

**Theorem B.** Let $X$ and $Y$ be generic symmetric $n \times n$ matrices over the commutative ring $R$. Set $B = R[X_{ij}, Y_{ij}]/I_1(XY)$, $1 \leq i, j \leq n$. Let $X$ and $Y$ be the symmetric matrices with entries $X_{ij} = X_{ij} \mod I_1(XY)$, $1 \leq i, j \leq n$, and $Y_{ij} = Y_{ij} \mod I_1(XY)$, $1 \leq i, j \leq n$, respectively. The complex

$$\cdots \to B^n \to B^n \to B^n \to \cdots$$

is then exact.

The method of proof of the linear type of $I_{n-1}(X)$ allows easy proof of the normality of the symmetric algebra (hence of the Rees algebra) when the ground ring is normal. In this direction we prove the following assertion (Proposition 5.1).

**Theorem C.** Let $X$ be a generic symmetric $n \times n$ matrix over a commutative ring $R$. Set $A = R[X_{ij}]$, $1 \leq i, j \leq n$. The symmetric algebra $S = S^t(I_1(X))$, $1 \leq t \leq n$, is a Krull (respectively normal) domain if and only if $R$ is a Krull (respectively, normal) domain, and $t = 1, n - 1$, or $n$. The corresponding divisor class groups are:

1. $\text{Cl}(S) = \text{Cl}(R) \oplus \mathbb{Z}$ when $t = 1$;
2. $\text{Cl}(S) = \text{Cl}(R) \oplus \mathbb{Z}^{n-1}$ when $t = n - 1$;
3. $\text{Cl}(S) = \text{Cl}(R)$ when $t = n$.

The methods used to prove the theorem above also yield the corresponding result in the case of a square generic matrix.

In case $R$ is a field of characteristic zero, W. Bruns [1] has established the normality and computed the divisor class groups of all Rees algebras of determinantal ideals of generic and generic symmetric matrices.

(These results may be compared to the recent general statements [14] on the divisor class group of a normal Rees algebra.)

In the proof of Theorem C we use the fact that the ring $R[X_{ij}, Y_{ij}]/I_1(XY)$ (where $X = (X_{ij})$ and $Y = (Y_{ij})$ are generic symmetric $n \times n$ matrices) is reduced provided $R$ is reduced. This assertion is a part of the following theorem (Proposition 4.1).
THEOREM D. Let $X$ and $Y$ be generic symmetric $n \times n$ matrices over the reduced ring $R$. Set $\delta(k_1, k_2) = I_1(XY) + I_{k_1+1}(X) + I_{k_2+1}(Y) \subseteq R[X_{ij}, Y_{ij}]$ and $B(k_1, k_2) = R[X_{ij}, Y_{ij}]/\delta(k_1, k_2)$. All the rings $B(k_1, k_2)$ are then reduced.

The analogous result in the generic case is proved in [15], cf. also [13]. The results established in this paper were obtained in the author’s Master’s Thesis (Sofia University, 1988).

SOME NOTATIONS

In this paper $X = (X_{ij}), 1 \leq i, j \leq n,$ and $Y = (Y_{ij}), 1 \leq i, j \leq n,$ denote generic symmetric $n \times n$ matrices over a commutative ring, $R$. The polynomial rings $R[X_{ij}], 1 \leq i, j \leq n,$ and $R[X_{ij}, Y_{ij}], 1 \leq i, j \leq n,$ are denoted by $R[X]$ and $R[X, Y]$, respectively. We denote by $\delta(k_1, k_2)$ the ideal $I_1(XY) + I_{k_1+1}(X) + I_{k_2+1}(Y) \subseteq R[X, Y], 0 \leq k_1, k_2 \leq n$. The $R$-algebra $R[X, Y]/\delta(k_1, k_2)$ is denoted by $B(k_1, k_2)$. The group of all invertible $n \times n$ matrices with entries in $R$ is denoted by $GL(n, R)$.

1. THE COORDINATE RING OF THE VARIETY OF PAIRS
SYMMETRIC MATRICES WITH PRODUCT ZERO

In this section we describe an $R$-free basis, constructed in [13], of the ring $B(k_1, k_2)$. For this purpose we need to introduce the language of Young diagrams and Young Tableaux, cf. e.g. [10, 13].

A Young diagram $\sigma$ is a finite subset of $\mathbb{N} \times \mathbb{N}$ such that if $(i, j) \in \sigma$ and $i' \leq i, j' \leq j$ then $(i', j') \in \sigma$. The length $\sigma_i$ of the $i$th row of $\sigma$ is the maximal $j \in \mathbb{N}$ such that $(i, j) \in \sigma$. The Young diagram $\sigma$ is completely determined by the sequence $\sigma_1 \geq \sigma_2 \geq \cdots$. The number of rows $l(\sigma)$ of $\sigma$ is the maximal $i$ such that $\sigma_i > 0$. The degree $|\sigma|$ of $\sigma$ is equal to $\sum_{i=1}^{l(\sigma)} \sigma_i$. The conjugate diagram $\sigma'$ is obtained from $\sigma$ by interchanging rows and columns, namely, $(i, j) \in \sigma$ if and only if $(j, i) \in \sigma$. We think of the Young diagram $\sigma$ as a sequence of $l(\sigma)$ rows of boxes with length $\sigma_1, \sigma_2, \ldots$ respectively.

A Young tableau $A$ on the numbers $[1, ..., n]$ with shape $\sigma$ is a filling of the boxes of $\sigma$ with numbers between 1 and $n$. We think of $A$ as being a (in general nonrectangular) matrix $A = (a_{ij}), a_{ij} \in \{1, ..., n\}, (i, j) \in \sigma$,

$$A = \begin{pmatrix}
a_{11} & \cdots & a_{1\sigma_1} \\
a_{21} & \cdots & a_{2\sigma_2} \\
\vdots & \ddots & \vdots \\
a_{l1} & \cdots & a_{l\sigma_l}
\end{pmatrix},$$

where $l = l(\sigma)$.
A Young tableau $A$ is called *standard* if the numbers in each row of $A$ strictly increase (from left to right) and the numbers in each column of $A$ do not decrease (from top to bottom).

A bitableau $(A \mid B)$ on the numbers $1, \ldots, n$ with *shape* $\sigma$ is a pair of tableaux on the numbers $[1, \ldots, n]$ with the same shape $\sigma$. The bitableaux are used to indicate products of minors of $X$ in the following way. Let $(A \mid B) = (a_{ij}, b_{ij})$; $a_{ij}, b_{ij} \in \{1, \ldots, n\}$; $(i, j) \in \sigma$ be a bitableau with shape $\sigma$. We associate with $(A \mid B)$ the element $(A \mid B)_X \in \mathbb{R}[X]$, $(A \mid B)_X = \left[ a_{i_1, \ldots, i_t}, b_{i_1, \ldots, i_t} \right]_X \cdots \left[ a_{i_1, \ldots, i_t}, b_{i_1, \ldots, i_t} \right]_X$, where $[a_{i_1, \ldots, i_t}, b_{i_1, \ldots, i_t}]_X$ denotes the minor of $X$ involving the rows $a_{i_1, \ldots, a_{i_t}}$ and the columns $b_{i_1, \ldots, b_{i_t}}$. $1 \leq i \leq l = l(\sigma)$. The element $(A \mid B)_Y \in \mathbb{R}[Y]$ is defined in a similar way. Up to sign, $(A \mid B)_X$ does not depend on the order of the numbers in the rows of $A$ and $B$.

Let $\mathcal{P} = [(A \mid B), (C \mid D)]$ be a pair of bitableaux on the numbers $[1, \ldots, n]$ with shapes $\sigma$ and $\tau$, respectively. We associate with $\mathcal{P}$ the polynomial $(A \mid B)_X (C \mid D)_Y \in \mathbb{R}[X, Y]$. By abuse of language we call this polynomial a pair of bitableaux and denote it by the same symbol $[(A \mid B), (C \mid D)]$.

An $R$-free basis of $B(k_1, k_2)$ is formed by a special kind of pairs of bitableaux. Let $[(A \mid B), (C \mid D)]$ be a pair of bitableaux with shape $[\sigma, \tau]$ such that the following hold:

(i) the numbers in each row of $A$, $B$, $C$, and $D$ strictly increase;

(ii) $\sigma_1 + \tau_1 \leq n$.

Then we can associate with $[(A \mid B), (C \mid D)]$ the tableau

$$s[(A \mid B), (C \mid D)] = \begin{pmatrix}
\hat{a}_{i_1} & \ldots & \hat{a}_{i_t}
\hat{c}_{i_1} & \ldots & \hat{c}_{i_t}
\vdots & \ddots & \vdots
\hat{a}_{1w} & \ldots & \hat{a}_{1w}
\hat{c}_{1w} & \ldots & \hat{c}_{1w}
\hat{a}_{1} & \ldots & \hat{a}_{1i}
\hat{b}_{1} & \ldots & \hat{b}_{1i}
\vdots & \ddots & \vdots
\hat{a}_{i} & \ldots & \hat{a}_{i}
\hat{b}_{i} & \ldots & \hat{b}_{i}
\end{pmatrix},$$

where $l = l(\sigma); q = l(\tau); \mu_i = n - \tau_i$ for $1 \leq i \leq q$; the numbers in each row are strictly increasing (from left to right), and such that $\{e_{i_1, \ldots, e_{i_t}} \cup \{\hat{e}_{i_1, \ldots, \hat{e}_{i_t}}\}\} = \{1, \ldots, n\}, \{d_{i_1, \ldots, d_{i_t}} \cup \{\hat{d}_{i_1, \ldots, \hat{d}_{i_t}}\}\} = \{1, \ldots, n\}$ for all $1 \leq i \leq q$. 

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The tableau $s\left[(A \mid B), (C \mid D)\right]$ is a Young tableau since $\sigma_1 + \tau_1 \leq n$. Its shape is

$$(\mu_q, \mu_q, ..., \mu_1, \mu_1, \sigma_1, \sigma_1, ..., \sigma_1, \sigma_1)$$

$$= (n - \tau_q, n - \tau_q, ..., n - \tau_1, n - \tau_1, \sigma_1, \sigma_1, ..., \sigma_1, \sigma_1).$$

A pair of bitableaux $[(A \mid B), (C \mid D)]$ is called standard if it satisfies (i) and (ii) and the tableau $s\left[(A \mid B), (C \mid D)\right]$ is standard.

The following proposition holds [13, 1.2].

**Proposition 1.1.** The standard pairs of bitableaux with shape $[\sigma, \tau]$, such that $\sigma_1 \leq k_1$ and $\tau_1 \leq k_2$, form an $R$-free basis of $B(k_1, k_2)$.

(Of course in Proposition 1.1 we consider each standard pair as an element of $B(k_1, k_2)$ by means of the natural map $R[X, Y] \rightarrow B(k_1, k_2).$)

We shall call a standard pair of bitableaux $[(A \mid B), (C \mid D)]$ with shape $[\sigma, \tau]$ a basic pair for $B(k_1, k_2)$, if $\sigma_1 \leq k_1$ and $\tau_1 \leq k_2$.

The proposition above is proved in [13] only in the case $k_1 = n$, $k_2 = n$, i.e., for the ring $R[X, Y]/I_1(XY)$, but the arguments in the proof cover the general case (cf. also [15, 1.3]).

2. **The Linear Type of the Ideal Generated by the Submaximal Minors of a Generic Symmetric Matrix**

In this section $X$ denotes a generic symmetric $n \times n$ matrix over the commutative ring $R$ and $I = I_{n-1}(X)$ denotes the ideal generated by the $(n-1) \times (n-1)$ minors of $X$. We shall prove that this ideal is of linear type.

First, let us find an explicit form of the symmetric algebra $S(I)$.

Let $M$ be a module over the commutative ring $A$, and let $M$ be generated over $A$ by the elements $m_j \in M$, $j = 1, ..., s$. Let

$$\left\{ \sum_{j=1}^{s} a_j m_j = 0 \mid a_j \in A, i \in A \right\}$$

be the set of all relations between the $m_j$ ($j = 1, ..., s$) over $A$. It is then well known that the symmetric algebra $S^A(M)$ is isomorphic to

$$A[T_1, ..., T_s]/J,$$

where $T_1, ..., T_s$ are indeterminates over $A$ and the ideal $J \subseteq A[T_1, ..., T_s]$ is generated by the linear forms

$$\left\{ \sum_{j=1}^{s} a_j T_j \mid i \in A \right\}.$$
Let \( \tilde{X} = (\tilde{X}_{ij}) \), \( 1 \leq i, j \leq n \), be the matrix of cofactors of \( X \): 
\[
\tilde{X}_{ij} = (-1)^{i+j} \begin{vmatrix} 1 & \cdots & i-1 & i+1 & \cdots & n \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ j-1 & \cdots & 1 & i & \cdots & j+1 \\
\end{vmatrix}_x 
\]
for \( 1 \leq i, j \leq n \). Then \( I \) is generated over \( A = \mathbb{R}[X] \) by the entries of \( \tilde{X} \) and we have the following relations between them:

(i) \( \tilde{X}_{ii} - \tilde{X}_{jj} = 0 \) for all \( 1 \leq i, j \leq n \)

(ii) \( \sum_{k=1}^{n} X_{ik} \tilde{X}_{kj} = 0 \) for \( 1 \leq i \neq j \leq n \)

\[
\sum_{k=1}^{n} X_{ik} \tilde{X}_{ki} - \sum_{k=1}^{n} X_{jk} \tilde{X}_{kj} = 0 \quad \text{for} \quad 1 \leq i, j \leq n.
\]

The first kind of relations arises by the symmetry of \( X \) and the second kind of relations arises by the cofactor expansion \( X\tilde{Y} = (\det X) \text{Id} \) where \( \text{Id} \) denotes the identity \( n \times n \) matrix.

From the resolution of \( I = I_{n-1}(X) \), which can be found in [8], it follows that the relations (i) and (ii) generate all relations between the entries of \( \tilde{X} \) over \( A = \mathbb{R}[X] \).

Thus, we have

\[
S^{4}(I) = A[Y]/J = R[X, Y]/J, \tag{2.1}
\]

where \( X \) and \( Y \) are generic symmetric \( n \times n \) matrices over \( R \) and the ideal \( J \subseteq R[X, Y] \) is generated by the forms

\[
\sum_{k=1}^{n} X_{ik} Y_{kj}, \quad 1 \leq i \neq j \leq n
\]

\[
\sum_{k=1}^{n} X_{ik} Y_{ki} - \sum_{k=1}^{n} X_{jk} Y_{kj}, \quad 1 \leq i, j \leq n.
\]

Set \( b = (XY)_{11} = \sum_{k=1}^{n} X_{1k} Y_{k1} \). Then we have

\[
J = I_{1}(XY - b \text{Id}). \tag{2.2}
\]

Let \( a = (b \mod J) \in S^{4}(I) \). We have

\[
S^{4}(I)/aS^{4}(I) = R[X, Y]/I_{1}(XY) \tag{2.3}
\]

since \( I_{1}(XY - b \text{Id}) + (b) R[X, Y] = I_{1}(XY) \).

Let \( GL(n, R) \) denote the group of all invertible \( n \times n \) matrices with entries in \( R \). Let us consider the left action of \( GL(n, R) \) on \( R[X, Y] \)

\[
^gX = (g'Xg)_{ij}, \quad ^gY = (g^{-1}Y(g^{-1})')_{ij}, \quad g \in GL(n, R), \quad 1 \leq i, j \leq n
\]

or briefly \( ^gX = g'Xg \) and \( ^gY = g^{-1}Y(g^{-1})' \), \( g \in GL(n, R) \).
As is to be expected for geometric reasons, the ideal $J$ is invariant under $GL(n, R)$. We prove this fact below.

First, let us show that

$$\mathcal{S}b \equiv b \mod J \quad \text{for all } g \in GL(n, R). \quad (2.4)$$

We have $b = (XY)_{11}$, hence $\mathcal{S}b = (\mathcal{S}X\mathcal{S}Y)_{11} = (g'XY(g')^{-1})_{11}$. The entries of the matrix

$$g'(XY - b \text{Id})(g')^{-1} = g'XY(g')^{-1} - b \text{Id}$$

belong to $J$ and $\mathcal{S}b - b = (g'XY(g')^{-1} - b \text{Id})_{11}$, which establishes (2.4).

The ideal $\mathcal{S}J$ is generated by the entries of the matrix $\mathcal{S}(XY - b \text{Id})$. We have

$$\mathcal{S}(XY - b \text{Id}) = \mathcal{S}X\mathcal{S}Y - \mathcal{S}b \text{Id} = g'XY(g')^{-1} - \mathcal{S}b \text{Id}$$

$$= (g'(XY - b \text{Id})(g')^{-1}) + ((b - \mathcal{S}b) \text{Id}).$$

Since the entries of the last two matrices belong to $J$ we get that $\mathcal{S}J \subseteq J$ for all $g \in GL(n, R)$, which proves that $J$ is invariant under $GL(n, R)$.

Note that by (2.4) the element $a \in S^d(I)$ is invariant under the induced action of $GL(n, R)$ on $S^d(I)$.

Let us consider $R[X, Y]$ as a $\mathbb{N}$-graded ring in the usual way, namely, $R[X, Y]_0 = R$, and $\deg X_i = 1 = \deg Y_j$ for $1 \leq i, j \leq n$. The ring $S^d(I) = R[X, Y]/I_1(XY - b \text{Id})$ then inherits from $R[X, Y]$ a structure of $\mathbb{N}$-graded ring. Furthermore, its homogenous components are invariant under the induced action of $GL(n, R)$, and when $R = \mathbb{Q}$ (the field of rational numbers), the representation of $GL(n, \mathbb{Q})$ in each homogeneous component is rational.

Now we shall describe an $R$-free basis of $S = R[X, Y]/I_1(XY - b \text{Id})$.

**Proposition 2.5.** The set

$$H = \{a^k \mathcal{P} \mid \mathcal{P} \text{ is a standard pair, } k \geq 0\}$$

is an $R$-free basis of $S$.

**Proof.** By (2.3), $S/aS = R[X, Y]/I_1(XY)$, and by 1.1 the image of $H$ in $S/aS$ generates $S/aS$ over $R$. Since $a$ is a homogeneous element, and $\deg a = 2$, the set $H$ generates $S$ as an $R$-module.

In order to prove the $R$-linear independence of the elements of $H$ it is enough to do it when $R = \mathbb{Q}$. Indeed, then in view of the natural inclusion $\mathbb{Z}[X, Y]/I_1(XY - b \text{Id}) \subseteq \mathbb{Q}[X, Y]/I_1(XY - b \text{Id})$ the claim holds for $R = \mathbb{Z}$, and the general case follows from the natural isomorphism

$$R[X, Y]/I_1(XY - b \text{Id}) = \mathbb{Z}[X, Y]/I_1(XY - b \text{Id}) \otimes_{\mathbb{Z}} R.$$
So from now on let $R = \mathbb{Q}$. We shall use the following facts from the representation theory of $GL(n, \mathbb{Q})$ (see [6]). Let $T$, $U$ and $B \subseteq GL(n, \mathbb{Q})$ be the subgroups of diagonal matrices, upper triangular unipotent matrices, and upper triangular matrices, respectively. Then for each Young diagram $\lambda$ with $l(\lambda) \leq n$, there is a unique irreducible and polynomial representation $F_{\lambda}$ of $GL(n, \mathbb{Q})$ with highest weight vector $\lambda$ (with respect to $B$). It is well known (see [10]) that $\dim F_{\lambda}$ is equal to the number of all standard tableaux on $[1, ..., n]$ with shape $\lambda$. Furthermore, each irreducible rational representation of $GL(n, \mathbb{Q})$ is of the form $F_{\lambda} \otimes \det^h$, $h \in \mathbb{Z}$, where $\det$ denotes the one-dimensional representation $g \mapsto \det g$, $g \in GL(n, \mathbb{Q})$.

We also use the following fact. Let $F$ be rational representation of $GL(n, \mathbb{Q})$, and let $f \in F, f \neq 0$ be a vector such that the one-dimensional vector space $\mathbb{Q}f$ is invariant under $B$. Then $f$ generates an irreducible $GL(n, \mathbb{Q})$-submodule of $F$ with highest weight vector $f$.

Let $[\sigma, \tau]$ be a pair of Young diagrams such that $\sigma_1 + \tau_1 \leq n$. Let $N(\sigma, \tau)$ denote the number of standard pairs with shape $[\sigma, \tau]$. Then it follows from the definition of a standard pair that $N(\sigma, \tau)$ is equal to the number of standard tableaux on $[1, ..., n]$ with shape

$$(n - \tau_q, n - \tau_q, ..., n - \tau_1, n - \tau_1, \sigma_1, \sigma_1, ..., \sigma_l, \sigma_l),$$

where $l = l(\sigma), q = l(\tau)$.

Let $S_d$ denote the homogeneous component of degree $d$ of $S$. Since the set $H$ generates $S$ we have

$$\dim S_d \leq N = \sum N(\sigma, \tau),$$

where the sum is extended over all pairs of Young diagrams $[\sigma, \tau]$ such that $\sigma_1 + \tau_1 \leq n, |\sigma| + |\tau| + 2k = d$ and $k \in \mathbb{N}$.

Now we prove that $\dim S_d \geq N$, which will complete the proof.

Let us fix a pair $[\sigma, \tau]$ such that $\sigma_1 + \tau_1 \leq n, |\sigma| + |\tau| + 2k = d, k \in \mathbb{N}$. Consider the element $a^k s_{[\sigma, \tau]} \in S_d$, where

$$s_{[\sigma, \tau]} = \begin{pmatrix} 1 & \cdots & \sigma_1 & 1 & \cdots & \sigma_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \cdots & \sigma_1 & 1 & \cdots & \sigma_1 \\ n & n-1 & \cdots & n - \tau_1 + 1 & n & n-1 \end{pmatrix},$$

$$l = l(\sigma), q = l(\tau),$$
We have that \( a^k s_{[\sigma, \tau]}(\text{Id}, \text{Id}) = 1 \). On the other hand

\[
(XY - b \text{Id})_0 (\text{Id}, \text{Id}) = 0
\]

for all \( 1 \leq i, j \leq n \), hence \( a^k s_{[\sigma, \tau]} \neq 0 \). The one-dimensional vector space \( \mathbb{Q} a^k s_{[\sigma, \tau]} \) is invariant under \( B \) since \( a^k \) is invariant under \( GL(n, \mathbb{Q}) \) and \( s_{[\sigma, \tau]} \) is a weight vector which is invariant under \( U \). Hence \( a^k s_{[\sigma, \tau]} \) generates an irreducible \( GL(n, \mathbb{Q}) \)-module \( S_{([\sigma, \tau], k)} \supseteq S_d \) with highest weight vector \( a^k s_{[\sigma, \tau]} \).

We have

\[
\text{diag}(t_1, \ldots, t_n) (a^k s_{[\sigma, \tau]}) = (t_1^{2\tau_1} \cdots t_n^{2\tau_n}) (t_1^{2\sigma_1} \cdots t_n^{2\sigma_n}) a^k s_{[\sigma, \tau]}
\]

\[
= (t_1^{2\tau_1 + 2\sigma_1 - 2\tau_1} \cdots t_n^{2\tau_n + 2\sigma_n - 2\tau_n}) (t_1 \cdots t_n)^{-2\tau_1} (a^k s_{[\sigma, \tau]}). \tag{2.6}
\]

Since, on the other hand,

\[
(n - \tau_q, n - \tau_{q-1}, \ldots, n - \tau_1) = (\xi_1 - \xi_n, \xi_1 - \xi_{n-1}, \ldots, \xi_1 - \xi_2, \xi_1 - \xi_1),
\]

we get

\[
(2\xi_1 + 2\xi_1 - 2\xi_n, \ldots, 2\xi_1 - 2\xi_1) = (n - \tau_q, n - \tau_{q-1}, \ldots, n - \tau_1, n - \tau_1, \sigma_1, \sigma_1, \ldots, \sigma_1, \sigma_1).
\]

Hence \( \dim S_{([\sigma, \tau], k)} = N(\sigma, \tau) \).

Furthermore the maximal weight of \( S_{([\sigma, \tau], k)} \) uniquely determines \( \sigma, \tau \), and \( k = \frac{1}{2} (d - |\sigma| - |\tau|) \). Indeed, by (2.6) it is clear that \( 2\tau_1 \) is the minimal \( h \in \mathbb{N} \) such that the representation \( S_{([\sigma, \tau], k)} \otimes \det^h \) is polynomial. Hence the maximal weight of \( S_{([\sigma, \tau], k)} \) determines \( \tau_1 = l(\tau) = q \). Now that \( \tau_1 \) is determined, the Young diagram \( (n - \tau_q, n - \tau_{q-1}, \ldots, n - \tau_1, n - \tau_1, \sigma_1, \sigma_1, \ldots, \sigma_1, \sigma_1) \) is determined as well. Again, since \( q = \tau_1 \) is known, the diagrams \( \sigma \) and \( \tau \) are determined.

The considerations above show that

\[
S_d = \bigoplus S_{([\sigma, \tau], k)},
\]

where the sum is extended over all \( \sigma, \tau \), and \( k \) such that \( \sigma_1 + \tau_1 \leq n \), \( |\sigma| + |\tau| + 2k = d \), and \( k \in \mathbb{N} \). Hence we get

\[
\dim S_d \geq N
\]

which completes the proof of 2.5.

For us, the most important consequence of 2.5 is the following corollary.
COROLLARY 2.7. Let $X$ and $Y$ be generic symmetric $n \times n$ matrices over the commutative ring $R$. Set $b = (XY)_{11} = \sum_{k=1}^{n} X_{1k} Y_{k1}$, $J = I_1(XY - b \text{ Id})$ and $S = R[X, Y]/J$. Then the element $a = (b \mod J) \in S$ is not a zero divisor in $S$.

Proof. Let $s \in S$, $s \neq 0$. Then by 2.5 we can write $s = \sum_{h \in \mathbb{N}, k \geq 0} r_h a^k \mathcal{P}$, $r_h \in R$, $\mathcal{P}$ is a standard pair and at least one $r_h \neq 0$. Then as $\sum_{h \in \mathbb{N}, k \geq 0} r_h a^{k+1} \mathcal{P} \neq 0$, hence $a$ is not a zero divisor in $S$.

This corollary is sufficient to prove the exactness of the "symmetric circular complex." An analogous result for generic matrices is established in [16, 7].

PROPOSITION 2.8. Let $X$ and $Y$ be generic symmetric $n \times n$ matrices over the commutative ring $R$. Set $S = R[X, Y]/I(XY)$ and let $X$ and $Y$ be the $n \times n$-matrices with entries $X_{ij} = X_{ij} \mod I_1(XY)$, $1 \leq i, j \leq n$, and $Y_{ij} = Y_{ij} \mod I_1(XY)$, $1 \leq i, j \leq n$, respectively. The complex

$$\begin{array}{ccccccc}
\mathcal{F} & \rightarrow & B^n & \rightarrow & B^n & \rightarrow & B^n & \rightarrow & \end{array}$$

is then exact.

Proposition 2.8 follows from the next well known lemma [4, Proposition 5.1].

LEMMA 2.9. Let $F$ be a free module over the commutative ring $S$ and let $f, g \in \text{End}_S(F)$ be such that $fg = gf = a \text{ Id}_F$, where $a$ is not a zero divisor in $S$. The complex

$$\begin{array}{ccccccc}
\otimes S/aS & \otimes S/aS & \otimes S/aS & \otimes S/aS & \end{array}$$

is then exact.

Proof of 2.8. It is enough to set in 2.9 $S = R[X, Y]/J$, $a = b \mod J$, $F = S^n$, $f = X \otimes S$, $g = Y \otimes S$.

We are now going to establish the main result of this section.

PROPOSITION 2.10. Let $X$ be a generic symmetric $n \times n$ matrix over the commutative ring $R$. The ideal $I = I_{n-1}(X) \subseteq A = R[X]$ is then of linear type.

Proof. We know from 2.1 that

$$S = S^A(I) = R[X, Y]/I_1(XY - b \text{ Id}),$$
where $X$ and $Y$ are generic symmetric $n \times n$ matrices over $R$ and

$$ b = (XY)_{11} = \sum_{k=1}^{n} X_{1k} Y_{k1}. $$

From now on we shall consider $S$ with the $\mathbb{N}$-grading which corresponds to the natural $\mathbb{N}$-grading of $S^n(I)$, namely, $S_0 = R[X]$ and $\deg \bar{Y}_{ij} = 1$, $1 \leq i, j \leq n$, where $\bar{Y}_{ij} = Y_{ij} \mod J$. In this grading we have $\deg a = 1$.

Let $\bar{Y}$ denote the matrix $(\bar{Y}_{ij})$, $1 \leq i, j \leq n$. The following identity then holds in $S$:

$$ X\bar{Y} = a \text{Id}. \quad (2.11) $$

The ring $S[a^{-1}]$ can be considered as a $\mathbb{Z}$-graded ring in the usual way, $\deg s/a^k = \deg s - k$, where $s \in S$ is a homogenous element. Since $a$ is not a zero divisor in $S$, one can regard $S$ as a subring of $S[a^{-1}]$.

Set $T = a^{1-n} \det \bar{Y} \in S[a^{-1}]$. Using (2.11) we get

$$ T(a^{-1} \det X) = a^{1-n} \det \bar{Y}(a^{-1} \det X) = a^{-n} \det XY = 1 $$

in $S[a^{-1}]$. Thus we have

$$ T^{-1} = a^{-1} \det X \quad (2.12) $$

in $S[a^{-1}]$. Again from 2.11 it follows that $(\det X)Y = aX$, where $X$ denotes the matrix of cofactors of $X$. In $S[a^{-1}]$ we can write $(a^{-1} \det X)Y = X$; hence by (2.12) we get

$$ \bar{Y} = T\bar{X} \quad (2.13) $$

in $S[a^{-1}]$.

We have the inclusion $R[X] \subseteq S[a^{-1}]_0$. Since $\deg T = 1$ and $T$ is not a zero divisor in $S[a^{-1}]$, the element $T$ is transcendental over $R[X]$. Furthermore $\bar{Y} \in R[X][T]$ by (2.13), hence we have

$$ S[a^{-1}] = R[X, \bar{Y}] [a^{-1}] = R[X, T][a^{-1}]. $$

In view of (2.12), $a = T \det X$ and we finally get

$$ S[a^{-1}] = R[X, T][(T \det X)^{-1}], \quad (2.14) $$

where $T$ is indeterminate over $R[X]$.

Now let us consider the Rees algebra $\mathcal{R}_t (A)$ s the subring $R[X][\{ t\bar{X}_{ij} \}]$, $1 \leq i, j \leq n$, of the ring $R[X][t]$, where $t$ is indeterminate over $R[X]$. We write $\mathcal{R}_t (A) = R[X, t\bar{X}] \subseteq R[X, t]$.

The ring $R[X, t\bar{X}][(t \det X)^{-1}]$ is a subring of $R[X, t][(t \det X)^{-1}]$. 
We claim that $R[X, t\tilde{X}][(t\det X)^{-1}] = R[X, t][(t\det X)^{-1}]$. It is enough to show that $t \in R[X, t\tilde{X}][(t\det X)^{-1}]$, but this is clear since $t = (\det t\tilde{X})/(t\det X)^n$.

Let $\phi$ denote the natural map $S^4(I) \to \mathcal{R}_I(A)$. The map $\phi$ is the homomorphism of $R$-algebras $S = R[X, \tilde{Y}] \to R[X, t\tilde{X}]$, such that $\phi(X_{ij}) = X_{ij}, 1 \leq i, j \leq n,$ and $\phi(Y_{ij}) = t\tilde{X}_{ij}, 1 \leq i, j \leq n.$ In particular we have $\phi(a) = t \det X$.

Let us consider the commutative diagram

$$
\begin{array}{ccc}
S^4(I) & \xrightarrow{\phi} & \mathcal{R}_I(A) \\
\downarrow & & \downarrow \\
R[X, T][(T\det X)^{-1}] = S^4(I)[a^{-1}] & \xrightarrow{\phi_a} & \mathcal{R}_I(A)[\phi(a)^{-1}] = R[X, t][(t\det X)^{-1}].
\end{array}
$$

Since $\phi_a(T) = \phi_a(a^{1-n} \det \tilde{Y}) = (t \det X)^{1-n} \det t\tilde{X} = t$,

the map $\phi_a$ is an isomorphism. The left vertical arrow is injective since $a$ is not a zero divisor in $S^4(I)$; hence the map $\phi$ is injective. This proves Proposition 2.10.

Remark. If $R$ is a Noetherian domain then in order to prove the linear type of $I_{n-1}(X)$ we can proceed as in [7]. For this purpose we can use Proposition 4.1, along with the fact that locally $I_{n-1}(X)$ is generated by analytically independent elements. The approach above is chosen because it attains the aim without restrictions on $R$.

3. Determinantal Ideals of Linear Type of a Generic Symmetric Matrix

In addition to Proposition 2.10 we describe below all determinantal ideals of linear type of a generic symmetric matrix.

**Proposition 3.1.** Let $X$ be a generic symmetric $n \times n$ matrix over the commutative ring $R$. The ideal $I_i(X) \subseteq A = R[X]$ is then of linear type if and only if $t = 1, n - 1,$ or $n$.

**Proof.** The ideals $I_i(X)$ and $I_a(X)$ are generated by a regular sequence and so are of linear type [11]. The ideal $I_{n-1}(X)$ is of linear type by 2.10.

In order to prove the necessity of the conditions we use the natural isomorphism $S^4(M) \otimes_A B = S^4(M \otimes_A B)$ where $M$ is $A$-module and $B$ is $A$-algebra.
Assume that the natural homomorphism

\[ S^4(I_i(X)) \to \mathcal{R}_{l_i}(A) \]

is an isomorphism. Tensoring with \( A/I_i(\langle X \rangle) \) we get the natural isomorphism

\[ S^R(I_i(X)/I_1(I_i(X)) I_i(X)) \cong \mathcal{R}_{l_i}(A)/I_1(X) \mathcal{R}_{l_i}(A). \]

The \( R \)-algebra \( \mathcal{R}_{l_i}(A)/I_1(X) \mathcal{R}_{l_i}(A) \) is naturally isomorphic to the \( R \)-subalgebra of \( R[X] \) generated by the \( t \times t \)-minors of \( X \) [2, Proposition 10.16]. In particular, the \( R \)-module \( I_i(X)/I_1(X) I_i(X) \) is isomorphic to the \( R \)-submodule of \( R[X] \) generated by the \( t \times t \)-minors of \( X \). By [3, Theorem 5.1] it follows that this submodule is a free \( R \)-module with basis consisting of all \( t \times t \)-minors \( [a_1, ..., a_i \mid b_1, ..., b_t]_x \) such that \( a_1 < a_2 < \cdots < a_i \), \( b_1 < b_2 < \cdots < b_t \), and \( a_1 \leq b_1, a_2 \leq b_2, ..., a_t \leq b_t \). Thus

\[ S^R(I_i(X)/I_1(X) I_i(X)) \]

is a polynomial ring over \( R \) and the proof will be completed if we show that when \( 1 < t < n - 1 \) the minors of \( X \) described above are not algebraically independent over \( R \).

So, let \( 1 < t < n - 1 \) and in the Plücker relation (see [2, Lemma 4.4])

\[ \sum \epsilon(\pi)[1, ..., t \mid \pi(a_1), ..., \pi(a_t)]_x [\pi(b_1), b_2, ..., b_t \mid 1, ..., t]_x = 0 \]

set \( a_i = i + 2 \) and \( b_i = i \) for \( i = 1, ..., t \). Then we get the identity

\[
\begin{align*}
[1, ..., t \mid 3, ..., \allowbreak t+2]_x & [1, ..., t \mid 1, ..., t]_x \\
- [1, ..., t \mid 1, 3, ..., t + 1]_x [1, ..., t \mid 2, ..., t, t + 2]_x & + [1, ..., t \mid 1, 3, ..., t, t + 2]_x [1, ..., t \mid 2, ..., t + 1]_x = 0.
\end{align*}
\]

The identity above completes the proof of Proposition 3.1.

4. REDUCEDNESS OF THE RINGS \( B(k_1, k_2) \)

In this section we prove the following proposition:

**Proposition 4.1.** All the rings \( B(k_1, k_2) \), \( 1 \leq k_1, k_2 \leq n \), are reduced provided \( R \) is reduced.

In particular, if \( R \) is reduced, then the ring \( B(n, n) = R[X, Y]/I_1(XY) \) is reduced.
Lemma 4.2. If $k_1 \leq 1$ and $k_1 + k_2 \leq n$ then $u = X_{mn} = [n \mid n]_X$ is not a zero divisor in $B(k_1, k_2)$.

Proof. We have $u \neq 0$ since $k_1 \leq 1$. From the definition of a standard pair (see Section 1) and the condition $k_1 + k_2 \leq n$, it follows that the product of $u$ and any basic pair for $B(k_1, k_2)$ is again a basic pair for $B(k_1, k_2)$. Furthermore, if $\mathcal{P}_1 \neq \mathcal{P}_2$ are basic pairs, then $u\mathcal{P}_1 \neq u\mathcal{P}_2$. Since by 1.1 the basic pairs form an $R$-free basis of $B(k_1, k_2)$, Lemma 4.2 is proved.

We shall use 4.2 in an inductive proof. The next two lemmas present the inductive argument.

Lemma 4.3. Let $X$ and $Y$ be generic symmetric $n \times n$ matrices over $R$ and let $W$ be the unipotent $n \times n$ matrix

$$W = \begin{pmatrix}
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
c_1 & c_2 & \cdots & c_{n-1} & 1
\end{pmatrix},$$

where $c_i = -X_{mi}X_{nn}^{-1} = -X_{mi}X_{mn}^{-1}, i = 1 \cdots n - 1.$

Then the symmetric matrices $X'' = W'XW$ and $Y'' = W^{-1}Y(W^{-1})'$ have the form

$$X'' = \begin{pmatrix}
X' & 0 \\
0 & X_{mn}
\end{pmatrix}, \quad Y'' = \begin{pmatrix}
Y' & Y''_{1n} \\
Y''_{n1} & Y_{n,n-1} & \cdots & Y''_{n,n}
\end{pmatrix},$$

where $X'$ and $Y'$ are symmetric $(n - 1) \times (n - 1)$ matrices with entries $X'_{ij} = X_{ij} - X_{in}X_{nj}X_{nn}^{-1}, 1 \leq i, j \leq n - 1$, and $Y'_{ij} = Y_{ij}, 1 \leq i, j \leq n - 1$, respectively, and

$$Y''_{n1} = Y_{n1} = Y_{n1} - \sum_{j=1}^{n-1} c_j Y_{ij}, \quad 1 \leq i \leq n - 1,$$

$$Y''_{nn} = Y_{nn} - 2 \sum_{j=1}^{n-1} c_j Y_{nj} + \sum_{i,j=1}^{n-1} c_i c_j Y_{ij}.$$

Furthermore we have

$$R[X, Y][X_{mn}^{-1}] = R[X_{n1} \cdots X_{nn}][Y_{n1}'' \cdots Y_{nn}''][X_{nn}^{-1}][X', Y'].$$


and the symmetric matrices $X'$ and $Y'$ are generic over the ring $R[X_{n1} \cdots X_{m1}][X_{mn}^{-1}]$.

Proof. By a direct computation.

The next lemma is completely analogous to [15, Lemma 2.12]. In order to avoid confusion we write $\mathcal{R} B(k_1, k_2, n)$ instead of $B(k_1, k_2)$, pointing out the ground ring and the size of $X$ and $Y$.

Lemma 4.4. The rings $\mathcal{R} B(k_1, k_2, n)[u^{-1}]$ and $\mathcal{R} B(k_1 - 1, k_2, n - 1)$ are isomorphic, where $\mathcal{R}$ denotes the ring $R[X_{n1} \cdots X_{m1}][X_{mn}^{-1}]$.

Proof. Let $A$ denote the ring $R[X, Y][X_{mn}^{-1}]$ and set $S = \{X_{n1}, \ldots, X_{mn}\}$, $T = \{Y_{n1}, \ldots, Y_{mn}\}$.

We have

$$\mathcal{R} B(k_1, k_2, n)[u^{-1}] = A/(I_1(XY) + I_{k_1+1}(X) A + I_{k_2+1}(Y) A).$$

The following equations hold in $A$:

$$I_1(XY) A = I_1(X" Y") = I_1(X' Y') + (T) A;$$

$$I_{k_1+1}(X) A = I_{k_1+1}(X") = I_{k_1}(X');$$

$$I_{k_2+1}(Y) A = I_{k_2+1}(Y"),$$

where $X", Y", X'$ and $Y'$ are the matrices described in 4.3.

Since $A = R[S, T][X_{mn}^{-1}][X', Y']$ by 4.3, we have

$$\mathcal{R} B(k_1, k_2, n)[u^{-1}] = R[S][T][X_{mn}^{-1}][X', Y']/(I_1(X' Y') + I_{k_1}(X') + I_{k_2+1}(Y") + (T) A)$$

$$= R[S][X_{mn}^{-1}][X', Y']/(I_1(X' Y') + I_{k_1}(X') + I_{k_2+1}(Y')).$$

The last equation holds since

$$I_{k_2+1}(Y") + (\{T\}) A = I_{k_2+1}(Y") + (T) A.$$

By 4.3 the matrices $X'$ and $Y'$ are generic over $R' = R[S][X_{mn}^{-1}]$, and this completes the proof of Lemma 4.4.

The next proposition contains a part of 4.1.

Proposition 4.5. If $k_1 + k_2 \leq n$ and $R$ is reduced (resp. a domain) then $B(k_1, k_2)$ is reduced (resp. a domain).

Proof. We use induction on $n$. The case $n = 0$ is trivial. Assume that 4.5 holds for $n - 1$. We may assume also that $k_i \geq 1$ without loss of generality,
since $B(0, 0) = R$. Then by 4.2 the element $u$ is not a zero divisor in $R B(k_1, k_2, n)$, hence the natural map
\[ R B(k_1, k_2, n) \to R B(k_1, k_2, n)[u^{-1}] \]
is injective. By 4.4 we have $R B(k_1, k_2, n)[u^{-1}] = R B(k_2 - 1, k_2, n - 1)$. Since the ring $R' = R[X_{n_1} \cdots X_{n_m}][X_{n_1}^{-1}]$ is reduced (resp. a domain) when $R$ has the corresponding property, it follows by the inductive hypothesis that $R B(k_1, k_2, n)$ is reduced (resp. a domain).

Let us fix the pair $(k_1, k_2)$. Then for each pair $(l_1, l_2)$ such that $l_1 \leq k_1$ and $l_2 \leq k_2$, there is a natural homomorphism $B(k_1, k_2) \to B(l_1, l_2)$, which we denote by $\psi_{l_1 l_2}$.

**Lemma 4.6.** Let $(k_1, k_2)$ be a pair such that $k_1 + k_2 \geq n$. The homomorphism
\[
\psi: B(k_1, k_2) \to \prod_{l_1 \leq k_1; l_2 \leq k_2; l_1 + l_2 = n} B(l_1, l_2),
\]
\[
\psi(z) = (\psi_{l_1 l_2}(z)), \quad z \in B(k_1, k_2) (l_1 \leq k_1; l_2 \leq k_2; l_1 + l_2 = n)
\]
is then injective.

Before proving 4.6, let us note that together with 4.5 it implies 4.1.

**Proof of 4.6.** If $\mathcal{P}$ is a basic pair for $B(k_1, k_2)$ then either $\psi_{l_1 l_2}(\mathcal{P})$ is a basic pair for $B(l_1, l_2)$ or $\psi_{l_1 l_2}(\mathcal{P}) = 0$. Furthermore if $\mathcal{P}_1 \neq \mathcal{P}_2$ are basic pairs for $B(k_1, k_2)$ and $\psi_{l_1 l_2}(\mathcal{P}_1) \neq 0$, $\psi_{l_1 l_2}(\mathcal{P}_2) \neq 0$, then $\psi_{l_1 l_2}(\mathcal{P}_1) \neq \psi_{l_1 l_2}(\mathcal{P}_2)$. Thus it is enough to prove that if $\mathcal{P}$ is a basic pair for $B(k_1, k_2)$, then there is a pair $(l_1, l_2)$ such that $l_1 \leq k_1$, $l_2 \leq k_2$, $l_1 + l_2 = n$, and $\psi_{l_1 l_2}(\mathcal{P}) \neq 0$.

Let $[\sigma, \tau]$ be the shape of $\mathcal{P}$. Then we have $\sigma_1 \leq k_1$, $\tau_1 \leq k_2$ and $\sigma_1 + \tau_1 \leq n$ since $\mathcal{P}$ is a basic pair for $B(k_1, k_2)$. Since
\[
\max(\sigma_1, n - k_2) \leq \min(k_1, n - \tau_1),
\]
there is a natural number $l_1$ such that $\sigma_1 \leq l_1 \leq k_1$ and $n - k_2 \leq l_1 \leq n - \tau_1$. Hence $(l_1, l_2) = (l_1, n - l_1)$ is a pair such that $\sigma_1 \leq l_1 \leq k_1$, $\tau_1 \leq l_2 \leq k_2$, and $l_1 + l_2 = n$. The pair $\mathcal{P}$ is basic for $B(l_1, l_2)$, hence $\psi_{l_1 l_2}(\mathcal{P}) \neq 0$. Lemma 4.6 is proved.

In the sequel we shall need the following fact.

**Lemma 4.7.** Let $X$ and $Y$ be generic symmetric $n \times n$ matrices over a domain, $R$. Set $b = (XY)_{ij} = \sum_{k=1}^{n} X_{ik} Y_{kj}$, $J = I_1XY - b \text{Id} \subseteq R[X, Y]$, and $S = R[X, Y]/J$. Let $a$ denote $b \text{ mod } J \in S$. The following then hold:
$aS = \bigcap_{j=0}^{n} P_j$ where $P_j = \mathcal{E}(j, n-j)/J$, $j = 0, \ldots, n$

(ii) the ideals $P_i(j = 0, \ldots, n)$ are prime, and $ht P_i = 1$ ($j = 0, \ldots, n$).

Proof. The homomorphism

$$\psi: R[X, Y]/I_1(XY) \to \prod_{l_1 + l_2 = n} R[X, Y]/\mathcal{E}(l_1, l_2)$$

is injective by 4.6; hence we have

$$I_1(XY) = \bigcap_{j=0}^{n} \mathcal{E}(j, n-j),$$

which yields (i).

The ideals $P_j = \mathcal{E}(j, n-j)/J$ ($j = 0, \ldots, n$) are prime by 4.5. In order to prove that $ht P_j = 1$ ($j = 0, \ldots, n$), note that $P_j \cap \{R - \{0\}\} = \emptyset$ ($j = 0, \ldots, n$). We can invert the elements of $R - \{0\}$ and then the claim that $ht P_j = 1$ follows by Krull's principal ideal theorem [9].

5. THE DIVISOR CLASS GROUP OF THE SYMMETRIC ALGEBRA

This section is devoted to the following question: When is the symmetric algebra $S(I_i(X))$ (where $X$ is a generic symmetric matrix) a normal domain? We consider this question from the point of view of Krull domains. All results about normality follow from the corresponding results about Krull domains, since a Noetherian domain is normal if and only if it is a Krull domain.

As usual, $X$ is a generic symmetric $n \times n$ matrix over the commutative ring $R$. We set $A = R[X]$.

**Proposition 5.1.** The symmetric algebra $S = S^t(I_i(X))$, $1 \leq t \leq n$, is a Krull (resp. normal) domain if and only if $R$ is a Krull (resp. normal) domain and $t = 1, n - 1, or n$. The corresponding divisor class groups are

(i) $Cl(S) = Cl(R) \oplus \mathbb{Z}$ when $t = 1$

(ii) $Cl(S) = Cl(R) \oplus \mathbb{Z}^{n-1}$ when $t = n - 1$

(iii) $Cl(S) = Cl(R)$ when $t = n$.

The proof of the necessity of the conditions of 4.1 is easy. If $S^t(I_i(X))$ is a domain, then $A = R[X]$ is a domain also. If $A$ is any domain and $I$ is an ideal in $A$ then $S^d(I)$ is a domain if and only if $I$ is of linear type [11]. By 3.1 we get that the possible values of $t$ are 1, $n - 1$, or $n$. In each of these
cases $S^d(I_t(X))$ is a free $R$-module (in the case $t=n-1$ by 2.5). Thus if $S^d(I_t(X))$ is a Krull domain, then $R$ also is a Krull domain since the property of a ring being a Krull domain descends in a faithfully flat extension.

The ring $S^d(I_n(X))$ is isomorphic to a polynomial ring over $R$; this proves (iii).

The ring $S^d(I_t(X))$ is isomorphic to $R[Z]/I_2(Z)$, where $Z$ is the matrix of $n(n+1)$ indeterminates over $R$

$$Z = \begin{pmatrix} X_{11} & \cdots & X_{1n} & X_{22} & \cdots & X_{2n} & \cdots & X_{nn} \\ Y_{11} & \cdots & Y_{1n} & Y_{22} & \cdots & Y_{2n} & \cdots & Y_{22} \end{pmatrix}.$$  

It is known that such a ring is a Krull domain when $R$ is one, and that $\text{Cl}(R[Z]/I_2(Z)) = \text{Cl}(R) \oplus \mathbb{Z}$ [2].

It remains to consider the most interesting case when $t = n - 1$. From now on we set $Z = Z + (X)$.

We will prove that $S^d(I)$ is a Krull domain when $R$ is one, by using the following lemma which is an analogue of a lemma of Hironaka, cf., e.g., [12].

**Lemma 5.2.** Let $S$ be a domain and let $a \neq 0$ be an element of $S$ such that the following conditions hold:

(i) $S[a^{-1}]$ is a Krull domain;

(ii) $aS = \bigcap_{j=1}^k P_j$, where $P_j$ ($j = 1, \ldots, k$) are prime ideals, and $htP_j = 1$ ($j = 1, \ldots, k$).

Then $S$ is a Krull domain.

**Proof.** Let $P$ be one of the ideals $P_j$, $j = 1, \ldots, k$. We shall prove that $S_P$ is a discrete valuation ring.

The ring $S_P/aS_P$ is reduced since it is a localization of $S/aS$ which is reduced by (ii). By the reducedness of $S_P/aS_P$ and (ii), it follows that $S_P/aS_P$ is a field, hence $PS_P = aS_P$. Thus the ring $S_P$ has Krull dimension 1 and the maximal ideal of $S_P$ is principal. By a theorem of Cohen [9, Theorem 8], it follows that $S_P$ is Noetherian, therefore $S_P$ is a discrete valuation ring.

The following equation holds in the field of fractions of $S$:

$$S = S[a^{-1}] \cap S_{P_1} \cap \cdots \cap S_{P_k}.$$  

Cf. [9, Theorem 53].

Since $S[a^{-1}]$ and $S_{P_1}, \ldots, S_{P_k}$ are Krull domains, Lemma 5.2 is proved.
PROPOSITION 5.3. Let $X$ be a generic symmetric $n \times n$ matrix over a Krull (resp. normal) domain $R$. Set $A = R[X]$ and $I = I_{n-1}(X) \subseteq A$. Then $S^A(I)$ and $\mathcal{R}_f(A)$ are Krull (resp. normal) domains.

Proof. We apply 5.2. By (2.14) we know that $S^A(I)[a^{-1}] = R[X, T][T \text{det } X]^{-1}$. This ring is a Krull domain since the property is stable by polynomial extensions and localizations. Lemma 4.7 takes care of condition (ii) of 5.2. Hence $S^A(I) = \mathcal{R}_f(A)$ are Krull domains.

PROPOSITION 5.4. With the same assumptions as in 5.3 we have $\text{Cl}(S^A(I)) = \text{Cl}(\mathcal{R}_f(A)) = \text{Cl}(R) \oplus \mathbb{Z}^n$.

Proof. We shall proceed as in [2, Chapt. 8]. Set $S = S^A(I)$. We have the following sequence of homomorphisms:

$$R \xrightarrow{i} S \xrightarrow{j} S[a^{-1}]$$

By 2.5 $S$ is a free $R$-module and we can write

$$\text{Cl}(R) \xrightarrow{\text{Cl}(i)} \text{Cl}(S) \xrightarrow{\text{Cl}(j)} \text{Cl}(S[a^{-1}]),$$

where $\text{Cl}(i)$ and $\text{Cl}(j)$ are defined by extensions of the divisorial ideals [5, Proposition 6.4]. We have $S[a^{-1}] = R[X, T][(T \text{det } X)^{-1}]$, and since $T$ and $\text{det } X$ are prime elements in $R[X, T]$, by virtue of Nagata's Theorem [5, Corollary 7.3] we get $\text{Cl}(S[a^{-1}]) = \text{Cl}(R)$. Hence the sequence (5.5) can be split and $\text{Cl}(S) \cong \text{Cl}(R) \oplus \text{Ker}(\text{Cl}(j))$. Again by Nagata's Theorem [5, Corollary 7.2] it follows that $\text{Ker}(\text{Cl}(j))$ is generated by the classes of the minimal primes of $a$. By 4.7 these are exactly the ideals $P_j = \mathfrak{p}(j, n - j)/J$ ($j = 0, \ldots, n$).

It remains to find all relations between $\text{cl}(P_j)$ ($j = 0, \ldots, n$) in $\text{Cl}(S)$. First, by $aS = \bigcap_{j=0}^n P_j$ it follows that

$$\sum_{j=0}^n \text{cl}(P_j) = \text{cl}(aS) = 0.$$  \hspace{1cm} (5.6)

Now assume that

$$\sum_{j=0}^n t_j \text{cl}(P_j) = 0, \quad t_j \in \mathbb{Z}, \ j = 0, \ldots, n.$$

Then $\sum_{j=0}^n t_j \text{div}(P_j) = \text{div}(fS)$, where $f$ belongs to the field of fractions of $S$. We have $\text{div}(fS[a^{-1}]) = 0$ in $S[a^{-1}]$, hence $f$ and $f^{-1}$ belong to
Since $S[a^{-1}] = R[X, T]/[(T \det X)^{-1}]$ arises from $R[X, T]$ by inversion of the prime elements $T$ and $\det X$, we have

$$f = rT^m(\det X)^q,$$

where $r$ is a unit in $R$, $m \in \mathbb{Z}$, and $q \in \mathbb{Z}$. Set $q' = q - m$. Then we can write

$$f = t(T \det X)^m (\det X)^{q'} = ra^m(\det X)^{q'}.$$

Therefore

$$\sum_{j=0}^{n} t_j \div(P_j) = m \div(aS) + q' \div(\det X). \quad (5.7)$$

By the identity $(\det X)(\det Y) = a^n$ which holds in $S$, it follows that

$$\div(\det X) = \sum_{j=0}^{n} s_j \div(P_j), \quad s_j \in \mathbb{N}, j = 0, ..., n.$$

It follows by (5.7) that all relations between $\cl(P_j)$ in $\cl(S)$ are generated by the relation (5.6) and the relation

$$\sum_{j=0}^{n} s_j \cl(P_j) = 0.$$

Now we shall show that $s_n = 0$ and $s_{n-1} = 1$, thus $\Ker(\cl(j)) \cong \mathbb{Z}^{n-1}$, which will complete the proof of Proposition 5.4.

Let us note that the ring $S = R[X, Y]/I_1(XY - b \Id)$ is naturally bigraded, namely, $S_{i,j} = R$ and $\text{bideg } X_i = (1, 0)$, $\text{bideg } Y_i = (0, 1)$ for $1 \leq i, j \leq n$. We have $P_n = I_1(Y)/J$, thus the ideal $P_n$ is generated by elements of bidegree $(0, 1)$. Since $\text{bideg}(\det X) = (n, 0)$ we get that $\det X \notin P_n$. Hence $s_n = 0$.

By the identity $XY = a \Id$ which holds in $S$, it follows that $(\det X)Y = aX$, where $X$ is the matrix of cofactors of $X$. Hence we have

$$Y_{11}(\det X) = a[2, ..., n | 2, ..., n] X$$

in $S$.

The ideal $P_{n-1} = (I_1(XY) + I_2(Y) + I_n(X))/J$ is generated by elements of bidegree $(1, 1)$, $(0, 2)$, and $(n, 0)$. Hence $Y_{11} \notin P_{n-1}$ and $[2, ..., n | 2, ..., n] X \notin P_{n-1}$ since these elements have bidegree $(0, 1)$ and $(n-1, 0)$, respectively. Let $v$ be the discrete valuation corresponding to $S_{p_{n-1}}$. Then we have

$$s_{n-1} = v(\det X) = v(Y_{11}(\det X)) = v(a[2, ..., n | 2, ..., n] X) = v(a) = 1.$$

Proposition 5.4 is proved.
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REFERENCES

1. W. BRUNS, Algebras defined by powers of determinantal ideals, preprint.