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Triangulations and a generalization of Bose's method

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Abstract

We present a nontrivial extension to Bose's method for the construction of Steiner triple systems, generalizing the traditional use of commutative and idempotent quasigroups to employ a new algebraic structure called a 3-tri algebra. Links between Steiner triple systems and 2-$(v,3,3)$ designs via 3-tri algebras are also explored. © 2001 Elsevier Science B.V. All rights reserved.

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1. Background

Let X be a finite set. A *set system* or *configuration* is a pair $(X,\mathscr{A})$, where $\mathscr{A} \subseteq 2^X$. The *order* of the set system is $|X|$. The elements of X are *points* and the elements of $\mathscr{A}$ are *blocks*. A t-(v,k,λ) design is a k-uniform set system $(X,\mathscr{A})$ of order v such that every t-subset of X is contained in precisely λ blocks of $\mathscr{A}$. A 2-$(v,3,1)$ design is a *Steiner triple system* of order v and is denoted by STS(v). A (k,ℓ)-*configuration* in an STS $(X,\mathscr{A})$ is a subset of ℓ blocks in $\mathscr{A}$ whose union is a k-element subset of X. The *Pasch configuration* or *quadrilateral* is the $(6,4)$-configuration on elements (say) a,b,c,d,e,f with blocks $\{a,b,c\},\{a,d,e\},\{f,d,b\}$ and $\{f,c,e\}$. An STS is *anti-Pasch* (or *quadrilateral-free*) if it does not contain the $(6,4)$-configuration.

A 3-*oriented graph* is a graph in which each edge e (with endpoints x and y) has one of three possible orientations: *positive*, *negative*, or *null oriented* from x to y. The edge e is positive oriented from x to y if and only if it is negative oriented from y to x; when e is null oriented the roles of x and y can be freely interchanged. We draw a positive oriented edge from x to y by an arrow from x to y and a null oriented edge without arrows. A 3-oriented graph is *simple* if, for every pair of vertices x and y, the graph contains at most one positive, one negative, and one null oriented

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edge from x to y. In a 3-oriented simple graph we can use without ambiguity $(x,y)^1$, $(x,y)^{-1}$, and $(x,y)^0$ to denote a positive, negative, and null oriented edge from x to y, respectively.

Let G be a 3-oriented simple graph. A path P in G through the vertices $x_0,\ldots,x_n$, $n\geqslant 1$, is denoted by $P=x_0,x_1^{\theta_1},\ldots,x_n^{\theta_n}$, where $\theta_1,\ldots,\theta_n\in\{1,-1,0\}$, if and only if P uses the edges $(x_0,x_1)^{\theta_1},\ldots,(x_{n-1},x_n)^{\theta_n}$. When P is a cycle, we write $P=(x_0^{\theta_0},x_1^{\theta_1}\ldots,x_{n-1}^{\theta_{n-1}})$, with $\theta_0=\theta_n$. If $\theta_0+\theta_1+\cdots+\theta_{n-1}\equiv 0 \bmod \lambda$ for some $\lambda>0$, P is *λ-balanced.* A two-factor of G in which all cycles are λ-balanced is *λ-balanced.* A *triangulation* is a partition of the edges in G into cycles of length 3, and a triangulation is 3-*balanced* if all its paths are 3-balanced. As we soon see, 3-balanced triangulations of a 3-oriented simple graph are closely related to Steiner triple systems.

The graph with v vertices in which each pair of vertices is joined by three parallel edges is denoted by $3K_v$, and $3\bar{K}_v$ denotes the 3-oriented simple graph with v vertices in which each pair x and y of vertices is joined by a positive, a negative, and a null oriented edge from x to y. For both graphs, the vertex sets $V(3K_v)=V(3\bar{K}_v)=\{0,1,\ldots,v-1\}$.

2. A generalization of Bose's method

Bose's method [1] is one of the most important and well-known paradigms in design theory. Our objective is to develop a natural generalization.

Theorem 2.1. *Every* 3*-balanced triangulation of* $3\bar{K}_v$ *yields an* STS$(3v)$.

Proof. Let $\mathscr{T}$ be a 3-balanced triangulation of $3\bar{K}_v$. Let us define:

$$X=\{(a,i)\mid a\in\{0,\ldots,v-1\} \text{ and } i\in\{0,1,2\}\},$$

$$\mathscr{A}_1=\{\{(a,0),(a,1),(a,2)\}\mid a\in\{0,\ldots,v-1\}\}$$

and for each $T=(a^{\theta_a},b^{\theta_b},c^{\theta_c})\in\mathscr{T}$

$$\mathscr{A}_T=\{\{(a,j),(b,(j+\theta_b) \bmod 3),(c,(j+\theta_b+\theta_c) \bmod 3)\}\mid j=0,1,2\}.$$

$\mathscr{A}_T$ is well-defined, since if we use a different representation of T, say $(b^{\theta_b},c^{\theta_c},a^{\theta_a})$, we get

$$A'_T=\{\{(b,k),(c,(k+\theta_c) \bmod 3),(a,(k+\theta_c+\theta_a) \bmod 3)\}\mid k=0,1,2\}.$$

Making the change of variable $k=(j+\theta_b) \bmod 3$, and applying the fact that $\theta_a+\theta_b+\theta_c\equiv 0 \bmod 3$, we find that $A'_T=A_T$. The other representations of T produce the same set.

We claim that $(X,\mathscr{A})$ with $\mathscr{A}=\mathscr{A}_1\cup(\bigcup_{T\in\mathscr{T}}\mathscr{A}_T)$ is an STS$(3v)$. In fact, let $B=\{(a,i),(b,j)\}$ be a two-subset of X; if $a=b$, then $\{(a,0),(a,1),(a,2)\}$ is the unique block in $\mathscr{A}$ containing B; otherwise B is contained in exactly one of the blocks in $\mathscr{A}_T$, where T is the unique triangle in $\mathscr{T}$ containing the edge $(a,b)^{(j-i) \bmod 3}$. $\square$

Bose's method builds Steiner triple systems using a special type of 3-balanced triangulations of $3\bar{K}_v$. A *Bose triangulation* is a 3-balanced triangulation of $3\bar{K}_v$ such that each of its triangles can be expressed as (a^0, b^1, c^{-1}) for appropriate elements $a, b, c \in \{0, \ldots, v-1\}$.

A *latin square* of order n is an $n \times n$ array, each cell of which contains exactly one of the symbols in $\{0, \ldots, n-1\}$, such that each row and each column of the array contains the symbols in $\{0, \ldots, n-1\}$ exactly once. A *quasigroup* of order n is a pair $(Q, \circ)$, where Q is a set of size n and $\circ$ is a binary operation on Q such that for every pair of elements $a, b \in Q$, the equations $a \circ x = b$ and $y \circ a = b$ have unique solutions. The tabular representation of a quasigroup of order n is a latin square of order n.

Proposition 2.2. *Every Bose triangulation produces a commutative and idempotent quasigroup. Conversely every commutative and idempotent quasigroup produces a Bose triangulation.*

Proof. Let $\mathscr{T}$ be a Bose triangulation of $3\bar{K}_v$. If $Q = \{0, \ldots, v-1\}$ and $a, b \in Q$ we define

$$a \circ b = \begin{cases} c & \text{if } (a^0, c^1, b^{-1}) \in \mathscr{T}, \\ a & \text{if } a = b. \end{cases}$$

The binary operation $\circ$ is defined for every pair $a, b \in Q$ because there exists exactly one triangle in $\mathscr{T}$ containing the edge $(a, b)^0$. The operation $\circ$ is commutative and idempotent, as follows. The equation $a \circ x = b$ has only one solution in x because only the triangle (a^0, b^1, x^{-1}) in $\mathscr{T}$ contains the edge $(a, b)^1$ for some x, and the equation $b \circ y = a$ has only one solution in y because only the triangle (b^0, a^1, y^{-1}) in $\mathscr{T}$ contains the edge $(a, b)^{-1}$ for some y. Hence $(Q, \circ)$ is a commutative and idempotent quasigroup.

In the other direction, let $(Q, \circ)$ be a commutative and idempotent quasigroup. Define $\mathscr{T} = \{(a^0, c^1, b^{-1}) \mid a, b \in Q \text{ and } a \circ b = c\}$. Every triangle in this set is well-defined because $(a^0, c^1, b^{-1}) = (b^0, c^1, a^{-1})$. Let a, b be arbitrarily chosen elements in Q, $(a, b)^0$ belongs only to the triangle (a^0, c^1, b^{-1}) for some $c \in Q$ because $\circ$ is a well-defined binary operation. Then $(a, b)^1$ belongs only to the triangle (a^0, b^1, x^{-1}), where x is the unique solution to the equation $a \circ x = b$; and $(a, b)^{-1}$ belongs only to the triangle (b^0, a^1, y^{-1}), where y is the unique solution to the equation $b \circ y = a$. $\mathscr{T}$ is 3-balanced, and it is a Bose triangulation. □

If we take a commutative and idempotent quasigroup $(Q, \circ)$ of order v, build from it the Bose triangulation $\mathscr{T}$ given by Proposition 2.2 and finally build from $\mathscr{T}$ the STS($3v$) given by Theorem 2.1, then the resulting STS is the same as that obtained from $(Q, \circ)$ by using Bose's method directly. Bose triangulations provide only one way to find 3-balanced triangulations of $3\bar{K}_v$, but there are others. There are many possibilities, but we are interested in those 3-balanced triangulations with additional algebraic structure.

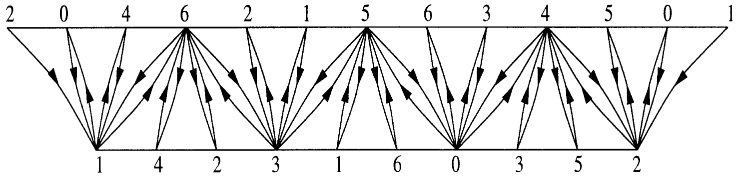

Fig. 1. A uniform triangulation of $3\bar{K}_7$.

A *uniform triangulation* of $3\bar{K}_v$ is a 3-balanced triangulation of $3\bar{K}_v$ such that each of its triangles can be expressed as (a^0, b^1, c^{-1}) or (a^0, b^{-1}, c^1) for appropriate elements $a, b, c \in \{0, \dots, v-1\}$. Triangles of the first type are *positive* and those of the second type, *negative*. A positive triangle cannot be expressed as a negative one, nor vice versa. A Bose triangulation does not permit the mixture of positive and negative triangles, but in a uniform triangulation we admit this possibility. The following uniform triangulation of $3\bar{K}_v$ for $v = 7$ is graphically represented in Fig. 1:

$$\begin{aligned}\mathscr{T}_7 = \{&\{0^0,1^1,2^{-1}\},\{4^0,1^{-1},0^1\},\{6^0,1^1,4^{-1}\},\{1^0,6^1,4^{-1}\},\{4^0,6^{-1},2^1\},\\ &\{2^0,6^1,3^{-1}\},\{2^0,3^1,6^{-1}\},\{1^0,3^{-1},2^1\},\{5^0,3^1,1^{-1}\},\{3^0,5^1,1^{-1}\},\\ &\{1^0,5^{-1},6^1\},\{6^0,5^1,0^{-1}\},\{6^0,0^1,5^{-1}\},\{3^0,0^{-1},6^1\},\{4^0,0^1,3^{-1}\},\\ &\{0^0,4^1,3^{-1}\},\{3^0,4^{-1},5^1\},\{5^0,4^1,2^{-1}\},\{5^0,2^1,4^{-1}\},\{0^0,2^{-1},5^1\},\\ &\{0^0,2^1,1^{-1}\}\}.\end{aligned}$$

When this triangulation is used in the construction of Theorem 2.1 we get an STS(21) isomorphic to the following, reading columns as triples:

```
000000000011111111122222222233333334444444555555666667777888899aabccdd
13579bdfhj3469acfgi345678abe678begi5689abd789abc79beg9aef9abfcgceedhfh
2468acegik578bdehjk9fidcjgkhadcfkhjecgkhjikhdfgjijhfkhbgjekdifiijigkkj
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A direct analysis shows that it is anti-Pasch. It is well known (see [4]) that Bose's method does not produce an anti-Pasch STS(21), so our extension is not trivial.

3. 3-tri algebras

In the same way that Bose's method can be formulated in terms of commutative and idempotent quasigroups, the construction given in Theorem 2.1 can be stated by using 3-tri algebras, algebraic structures that generalize quasigroups.

A 3-*tri algebra* (read as *three triangulation algebra*) of order $v > 0$ is a pair $\Upsilon = (C, \circ)$, where C is a set with cardinality v and $\circ$ is a binary, closed, commutative and idempotent operation over C such that for every pair of distinct elements

$\circ$	0	1	2	3	4	5	6
0	0	2	1	4	1	2	5
1	2	1	3	5	6	3	5
2	1	3	2	6	6	4	3
3	4	5	6	3	0	4	0
4	1	6	6	0	4	2	1
5	2	3	4	4	2	5	0
6	5	5	3	0	1	0	6

Fig. 2. Multiplication table of a 3-tri algebra.

$a, b \in C$ the equations

$$a \circ x = b, \tag{1}$$

$$b \circ y = a, \tag{2}$$

with unknowns x and y, satisfy one and only one of the conditions:

1. There are exactly two solutions for x and none for y.
2. There are exactly two solutions for y and none for x.
3. There is exactly one solution for x and one for y.

Every commutative and idempotent quasigroup is a 3-tri algebra. One example of 3-tri algebra which is not a quasigroup is the pair $(\{0,\ldots,6\},\circ)$, where $\circ$ is the operation shown in Fig. 2. This is the 3-tri algebra used to generate the STS(21) given in Section 2.

The multiplication table of a 3-tri algebra has a structure similar to that of a uniform square. However, an element can appear twice (at most) in a row; an element j does not appear in a row i if and only if i appears twice in the row j. Any idempotent and symmetric matrix with this property corresponds to a 3-tri algebra.

4. 3-tri algebras and 2-(v, 3, 3) designs

Our main interest in 3-tri algebras is their capacity to generalize Bose's method. However, as we show here, they have a strong link with 2-$(v,3,3)$ designs. Let $\Upsilon = (\{0,\ldots,v-1\},\circ)$ be a 3-tri algebra. For every unordered pair $\{i,j\}$ of different elements in $\{0,\ldots,v-1\}$, the set $T_{\Upsilon,\{i,j\}} \stackrel{\text{def}}{=} \{i,j,i\circ j\}$ (or $T_{\{i,j\}}$ when there is no confusion with the 3-tri algebra) is *triple induced by i and j in* Υ. The set $\mathscr{T}_\Upsilon \stackrel{\text{def}}{=} \{T_{\{i,j\}} \mid \{i,j\} \subset \{0,\ldots,v-1\},\ i \neq j\}$ is the *set of triples induced by* Υ.

Let $\Upsilon = (\{0,1,\ldots,7\},\circ)$ be the 3-tri algebra with the operation in Fig. 2, then

$$\begin{aligned}\mathscr{T}_\Upsilon = \{&T_{\{0,1\}} = \{0,1,2\}, T_{\{0,2\}} = \{0,2,1\}, T_{\{0,3\}} = \{0,3,4\}, T_{\{0,4\}} = \{0,4,1\},\\ &T_{\{0,5\}} = \{0,5,2\}, T_{\{0,6\}} = \{0,6,5\}, T_{\{1,2\}} = \{1,2,3\}, T_{\{1,3\}} = \{1,3,5\},\\ &T_{\{1,4\}} = \{1,4,6\}, T_{\{1,5\}} = \{1,5,3\}, T_{\{1,6\}} = \{1,6,5\}, T_{\{2,3\}} = \{2,3,6\},\end{aligned}$$

$$T_{\{2,4\}} = \{2,4,6\}, T_{\{2,5\}} = \{2,5,4\}, T_{\{2,6\}} = \{2,6,3\}, T_{\{3,4\}} = \{3,4,0\},$$
$$T_{\{3,5\}} = \{3,5,4\}, T_{\{3,6\}} = \{3,6,0\}, T_{\{4,5\}} = \{4,5,2\}, T_{\{4,6\}} = \{4,6,1\},$$
$$T_{\{5,6\}} = \{5,6,0\}\}.$$

As we can see it is a 2-$(7,3,3)$ design, and in fact we have the following general result.

Proposition 4.1. *For any* 3-*tri algebra* Υ *of order* v, $\mathscr{T}_\Upsilon$ *is a* 2-$(v,3,3)$ *design.*

Proof. Every pair of distinct elements $a,b \in \{0,\ldots,v-1\}$ belongs to exactly three different triples in $\mathscr{T}_\Upsilon$. One is $T_{\{a,b\}}$, and the other two are:

Case 1: $T_{\{a,x_1\}}$ and $T_{\{a,x_2\}}$, where x_1 and x_2 are the two solutions to Eq. (1), or

Case 2: $T_{\{b,y_1\}}$ and $T_{\{b,y_2\}}$, where y_1 and y_2 are the two solutions to Eq. (2), or

Case 3: $T_{\{a,x_1\}}$ and $T_{\{b,y_1\}}$, where x_1 and y_1 are the solutions to Eqs. (1) and (2). □

$\mathscr{T}_\Upsilon$ is also called the 2-$(v,3,3)$ *design induced by* Υ. Proposition 4.1 is a generalization of the well-known fact (see [2], for example) that an idempotent and commutative quasigroup can be used to produce a 2-$(v,3,3)$ design. A converse is valid for 3-tri algebras:

Proposition 4.2. *Every* 2-$(v,3,3)$ *design generates a family of* 3-*tri algebras.*

Proof. Let $(\{0,\ldots,v-1\},\mathscr{T})$ be a 2-$(v,3,3)$ design. Let $G_{\mathscr{T}}$ be the bipartite graph with bipartition $V_1 = \{\{a,b\} \mid a \neq b,\ a,b \in \{0,\ldots,v-1\}\}$ and $V_2 = \mathscr{T}$, two vertices $\{i,j\} \in V_1$ and $T \in V_2$ being joined by an edge if and only if $\{i,j\} \subset T$. Then $G_{\mathscr{T}}$ is a 3-regular graph. We establish that each of its perfect matchings produces a 3-tri algebra of order v.

Let $M \subset E(G)$ be one such matching. We use the notation $M(i,j) = \{i,j,k\}$ if and only if $(\{i,j\},\{i,j,k\}) \in M$. Define a binary operation $\circ_M$ on $\{0,\ldots,v-1\}$ by

$$i \circ_M j = \begin{cases} k & \text{if } i \neq j \text{ and } M(i,j) = \{i,j,k\}, \\ i & \text{if } i = j. \end{cases}$$

Every set $\{a,b\} \in V_1$ is contained in three and only three triples in $\mathscr{T}$, so there exist two different elements c and d satisfying one of the following:

Case 1: $\{a,b\}$ belongs simultaneously to $M(a,b)$, $M(a,c) = \{a,b,c\}$ and $M(a,d) = \{a,b,d\}$.

Case 2: $\{a,b\}$ belongs simultaneously to $M(a,b)$, $M(b,c) = \{a,b,c\}$ and $M(b,d) = \{a,b,d\}$.

Case 3: $\{a,b\}$ belongs simultaneously to $M(a,b)$, $M(a,c) = \{a,b,c\}$ and $M(b,d) = \{a,b,d\}$.

The solutions for x and y to the equations $a \circ_M x = b$ and $b \circ_M y = a$ are as follows. In Case 1, c and d are solutions in x and y has no solution. In Case 2, c and d

are solutions in y and x has no solution. Finally in Case 3, c is a solution in x and d a solution in y. Then $\circ_M$ is a commutative and idempotent binary operation. We conclude that $(\{0,\dots,v-1\},\circ_M)$ is a 3-tri algebra produced from M. □

5. Uniform triangulations and 3-tri algebras

As we saw in Theorem 2.1, the generalization of Bose's construction rests on our ability to find 3-balanced triangulations of $3\bar{K}_v$. The 3-tri algebras form an intermediate step between 3-balanced triangulations and quasigroups. In fact, 3-tri algebras of order v are 'almost' equivalent to uniform triangulations of $3\bar{K}_v$.

Proposition 5.1. *There exist a one to one correspondence between the set of uniform triangulations of $3\bar{K}_v$ and the set of 3-tri algebras of order v.*

Proof. Let $\mathscr{U}$ be a uniform triangulation of $3\bar{K}_v$. We build the 3-tri algebra $\Upsilon_{\mathscr{U}} = (\{0,\dots,v-1\},\circ_{\mathscr{U}})$, where $i\circ_{\mathscr{U}} j \stackrel{\text{def}}{=} k$ if and only if one of the following three conditions is satisfied:

1. $i=j=k$.
2. $(i^0,k^1,j^{-1})\in\mathscr{U}$.
3. $(i^0,k^{-1},j^1)\in\mathscr{U}$.

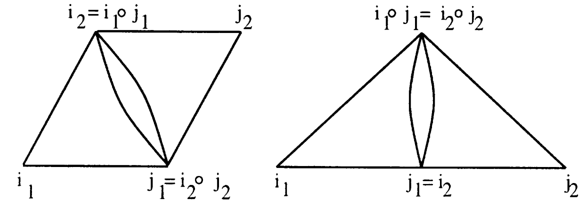
Then $\circ_{\mathscr{U}}$ is a commutative and idempotent binary operation. On the other hand, if a, b are different elements in $\{0,\dots,v-1\}$, then $(a,b)^0\in T_0$, $(a,b)^1\in T_1$ and $(a,b)^{-1}\in T_{-1}$, where T_0,T_1 and T_{-1} are 3-different triangles in $\mathscr{U}$. There exist two different elements $c,d\in\{0,\dots,v-1\}$ such that only one of the following cases is satisfied:

Case 1: $T_1=(a^0,b^1,c^{-1})$ and $T_{-1}=(a^0,b^{-1},d^1)$.
Case 2: $T_1=(b^0,a^{-1},c^1)$ and $T_{-1}=(b^0,a^1,d^{-1})$.
Case 3: $T_1=(a^0,b^1,c^{-1})$ and $T_{-1}=(b^0,a^1,d^{-1})$.

The solutions in x and y to the equations $a\circ_{\mathscr{U}} x=b$ and $b\circ_{\mathscr{U}} y=a$ are as follows. In Case 1, c and d are solutions in x, and y has no solution. In Case 2, c and d are solutions in y, and x has no solution. Finally in Case 3, c is a solution in x and d a solution in y. We conclude that $\Upsilon_{\mathscr{U}}$ is a 3-tri algebra. □

The converse of this proposition does not hold. Only some 3-tri algebras, to be characterized, produce uniform 3-tri algebras of $3\bar{K}_v$. Let $\Upsilon=(\{0,\dots,v-1\},\circ)$ be a 3-tri algebra of order v. The *Bose graph* of Υ, denoted B_Υ, is a graph with the triples in $\mathscr{T}_\Upsilon$ as vertices, two vertices $T_{\{i_1,j_1\}}$ and $T_{\{i_2,j_2\}}$ being joined by an edge if and only if the corresponding triples share a pair $\{i,j\}$ such that $\{i,j\}\neq\{i_1,j_1\}$ and $\{i,j\}\neq\{i_2,j_2\}$. The same idea can be expressed in terms of Υ by saying that $T_{\{i_1,j_1\}}$ and $T_{\{i_2,j_2\}}$ are adjacent if one of the following conditions is true (as shown in Fig. 3):

Condition 1: $j_1=i_2\circ j_2$ and $i_2=i_1\circ j_1$.
Condition 2: $j_1=i_2$ and $i_1\circ j_1=i_2\circ j_2$.



a) Condition 1 (positive edge) b) Condition 2 (negative edge)

Fig. 3. Adjacencies in B_Υ.

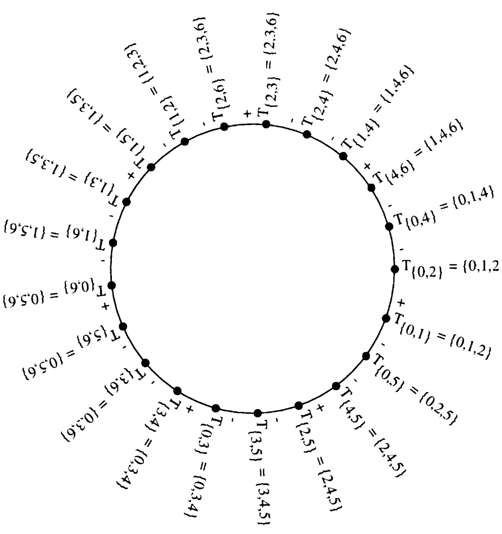


Fig. 4. The Bose graph of the 3-tri algebra in Section 3.

An edge in B_Υ is *positive* if it satisfies Condition 1; otherwise it is *negative*. Fig. 4 depicts the Bose graph of the 3-tri algebra in Fig. 2. In this case the graph is a cycle.

Lemma 5.2. *If* $\Upsilon=(\{0,\dots,v-1\},\circ)$ *is a* 3-*tri algebra, then* B_Υ *is a* 2-*regular simple graph.*

Proof. A triple $T_{\{a,b\}}=\{a,b,c\}$ in $V(B_\Upsilon)=\mathscr{T}_\Upsilon$ is only adjacent to triples containing $\{a,c\}$ and $\{b,c\}$. Since $\mathscr{T}_\Upsilon$ is a 2-$(v,3,3)$ design, other than $T_{\{a,b\}}$ there are only two triples containing $\{a,c\}$. One is $T_{\{a,c\}}$, but it is not adjacent to $T_{\{a,b\}}$. The other is one of the following two possibilities:

Case 1: $T_{\{a,x\}}$, where $a\circ x=c$ and $x\neq b$; or

Case 2: $T_{\{c,y\}}$, where $c\circ y=a$.

The case depends upon the solutions of the equations $a\circ x=c$ and $c\circ y=a$. In either situations such a triple is the only one adjacent to $T_{\{a,b\}}$ which contains $\{a,c\}$.

∘	0	1	2	3	4	5	6
0	0	2	3	1	3	2	1
1	2	1	4	5	5	6	2
2	3	4	2	6	5	3	4
3	1	5	6	3	1	6	4
4	3	5	5	1	4	0	0
5	2	6	3	6	0	5	0
6	1	2	4	4	0	0	6

Fig. 5. Multiplication table of an unsignable 3-tri algebra.

Similarly $T_{\{a,b\}}$ is also adjacent to only one of the following triples containing $\{b,c\}$:

Case 1′: $T_{\{b,x'\}}$ where $b \circ x' = c$ and $x' \neq a$; or

Case 2′: $T_{\{c,y'\}}$ where $c \circ y' = b$.

The triple from Cases 1 and 2, $T_{\{a,b\}}$, and the triple from Cases 1′ and 2′ are different, so $T_{\{a,b\}}$ has degree two and its incident edges are neither loops nor parallel edges in B_Υ. We conclude that this is a 2-regular simple graph. □

Let Υ be a 3-tri algebra of order v. Any function $\sigma : \{\{i,j\} \mid i \neq j \text{ and } i,j \in \{0,\dots,v-1\}\} \to \{+,-\}$ such that for every edge $e = (T_{\{a,b\}}, T_{\{c,d\}})$ in $E(B_\Upsilon)$ $\sigma(a,b) = \sigma(c,d)$ if and only if e is positive is a *signing* of Υ. If Υ has at least one signing it is *signable*; otherwise it is *unsignable*.

Lemma 5.3. *A* 3-*tri algebra* Υ *is signable if and only if every cycle in* B_Υ *has an even number of negative edges.*

Proof. Let σ be a signing of Υ and let $P = T_{\{a_0,b_0\}}, \dots, T_{\{a_k,b_k\}}$ be a path in B_Υ, $\sigma(a_k,b_k) = \sigma(a_0,b_0)(-1)^n$, where n is the number of negative edges in P; so σ is well defined if and only if the number of negative edges in every cycle of B_Υ is even. □

The multiplication table of an unsignable 3-tri algebra is given in Fig. 5. It is unsignable because its Bose graph contains the cycle $(T_{\{4,5\}}, T_{\{5,6\}}, T_{\{4,6\}})$ in which the three edges are negative. Let $\Upsilon = (\{0,\dots,v-1\}, \circ)$ be a signable 3-tri algebra, and let σ be one of its signings. For every pair a,b of different elements in $\{0,\dots,v-1\}$, the 3-oriented cycle $\bar{T}_{\Upsilon,\sigma,a,b} \stackrel{\text{def}}{=} (a^0, (a \circ b)^{\sigma(a,b)}, b^{-\sigma(a,b)})$ (or $\bar{T}_{a,b}$ when there is no confusion with Υ and σ) is the 3-*oriented cycle induced by* Υ, σ, *a and b*. The set $\bar{\mathscr{T}}_{\Upsilon,\sigma} \stackrel{\text{def}}{=} \{\bar{T}_{a,b} \mid a \neq b \text{ and } a,b \in \{0,\dots,v-1\}\}$ is the *set of* 3-*cycles induced by* Υ and σ. The sets $\mathscr{T}_\Upsilon$ and $\bar{\mathscr{T}}_{\Upsilon,\sigma}$ are essentially the same, but in the latter we have chosen orientations.

Proposition 5.4. *If* $\Upsilon = (\{0,\dots,v-1\}, \circ)$ *is a signable* 3-*tri algebra of order v and* σ *is one of its signings, then* $\bar{\mathscr{T}}_{\Upsilon,\sigma}$ *is a uniform triangulation of* $3\bar{K}_v$.

Proof. Let a, b be two different elements in $\{0,\dots,v-1\}$. We establish that each of the edges $(a,b)^0, (a,b)^1$ and $(a,b)^{-1}$ belongs to exactly one 3-cycle in $\bar{\mathscr{T}}_{\Upsilon,\sigma}$. Evidently $(a,b)^0$ belongs only to $\bar{T}_{a,b}$. Now we have three possibilities:

Case 1: The equation $a \circ x = b$ has two solutions in x, say c and d. $(T_{\{c,a\}}, T_{\{a,d\}})$ is a negative edge in B_Υ, so $\sigma(a,d) = -\sigma(a,c)$, and thus $(a,b)^{\sigma(a,c)}$ belongs to $\bar{T}_{a,c} = (a^0, b^{\sigma(a,c)}, c^{-\sigma(a,c)})$ and $(a,b)^{-\sigma(a,c)}$ belongs to $\bar{T}_{a,d} = (a^0, b^{-\sigma(a,c)}, c^{\sigma(a,c)})$. No other 3-cycle in $\bar{\mathscr{T}}_{\Upsilon,\sigma}$ contains $\{a,b\}$.

Case 2: The equation $b \circ y = a$ has two solutions in y. This is similar to Case 1.

Case 3: The equations $a \circ x = b$ and $b \circ y = a$ have one solution in x and one in y, say $x = c$ and $y = d$. $(T_{\{c,a\}}, T_{\{b,d\}})$ is a positive edge in B_Υ, so $\sigma(b,d) = \sigma(a,c)$, and thus $(a,b)^{\sigma(a,c)}$ belongs to $\bar{T}_{a,c} = (a^0, b^{\sigma(a,c)}, c^{-\sigma(a,c)})$ and $(a,b)^{-\sigma(a,c)}$ belongs to $\bar{T}_{b,d} = (b^0, a^{\sigma(b,d)}, c^{-\sigma(b,d)})$. No other 3-cycle in $\bar{\mathscr{T}}_{\Upsilon,\sigma}$ contains $\{a,b\}$.

Since all 3-cycles in $\bar{\mathscr{T}}_{\Upsilon,\sigma}$ have the form of a uniform triangulation we conclude that it is a uniform triangulation of $3\bar{K}_v$. □

6. The Skolem method

We use the idea of Theorem 2.1 to generalize the Skolem method (see [2], for example). Let v be a positive even integer, say $v = 2n$. Denote by $3\bar{K}'_v$ the graph $3\bar{K}_v - \{(a, n+a)^{-1} \mid a \in \{0,\dots,n-1\}\} \cup \{(n+a, n+a)^1 \mid a \in \{0,\dots,n\}\}$. Then $3\bar{K}'_v$ is not simple, since we have replaced a perfect matching of negative edges in $3\bar{K}_v$ by positive loops on the vertices $n, n+1, \dots, 2n-1$.

Theorem 6.1. *Every* 3*-balanced triangulation of* $3\bar{K}'_v$ *yields an* STS$(3v+1)$.

Proof. Let $\mathscr{T}$ be a 3-balanced triangulation of $3\bar{K}'_v$. Let us define:

$$X = \{(a,i) \mid a \in \{0,\dots,n-1\} \text{ and } i \in \{0,1,2\}\} \cup \{\infty\},$$

$$\mathscr{A}_\infty = \{\{(a,(i+1) \bmod 3), (a+n, i), \infty\} \mid a = 0,1,\dots,n-1\},$$

$$\mathscr{A}_1 = \{\{(a,0),(a,1),(a,2)\} \mid a = 0,1,\dots,n-1\}$$

and for each

$$T = (a^{\theta_a}, b^{\theta_b}, c^{\theta_c}) \in \mathscr{T},$$

$$\mathscr{A}_T = \{\{(a,j), (b,(j+\theta_b) \bmod 3), (c,(j+\theta_b+\theta_c) \bmod 3)\} \mid j = 0,1,2\}.$$

In the same manner as in the proof of Theorem 2.1, $(X, \mathscr{A})$ with $\mathscr{A} = \mathscr{A}_\infty \cup \mathscr{A}_1 \cup (\bigcup_{T \in \mathscr{T}} \mathscr{A}_T)$ is an STS$(3v)$. □

It is possible to develop an algebraic structure similar to 3-tri algebras to find 3-balanced triangulations of $3\bar{K}'_v$. However the resulting structure does not share the nice properties of 3-tri algebras and we prefer to omit it.

7. Conclusions

Theorem 2.1 gives us a technique to generalize one of the most important methods to construct Steiner triple systems. The real potential of this construction depends upon our ability to generate 3-balanced triangulations of $3\bar{K}_v$. The 3-tri algebras give some solutions to this problem, but they are not the only possibility. The general problem of determining all 3-balanced triangulations of $3\bar{K}_v$ remains open.

The construction of signable 3-tri algebras is not easy; we have studied some methods which are reported in Ref. [3]. We showed that it is possible to generate 3-tri algebras appropriate for the construction of anti-Pasch Steiner triple systems. These methods are based on an interesting application of the eight queens problem.

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