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Existence of Optimal Controls for a Class of Hereditary Systems with Lagging Control

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In this paper the problem of existence of optimal controls for a class of time lag systems is considered. It is schown that Oğuztöreli's results (Oğuztöreli, the 8.1, p. 184, "Time Lag Control Systems," Academic Press, New York, 1966) can be extended to a class of time lag systems whose "phase velocity" depends also on the past history of control.

1. INTRODUCTION

In this paper we consider the problem of existence of optimal controls for a class of hereditary dynamical systems whose phase velocity at any time depends not only on the past history of states but also on the past history of controls. Oğuztöreli, [Oğuztöreli (1966), Eq. 8.1.1, p. 181] considered hereditary systems whose phase velocity at any moment of time depends on the past values of states and the present value of control.

In this paper it is shown that Oğuztöreli's results can be extended to a class of hereditary systems whose phase velocity depends also on the past history of control.

The system considered is described by a functional differential equation of the form

$$S\begin{cases}\dot{x}(t) = f(t, \pi_t x) + \int_0^t K(t,\tau)\, g(\tau, x(\tau), u(\tau))\, d\tau \qquad \text{a.e.}\\ \text{on } I \,\Delta\, [0, T], \qquad \text{with initial data}\\ x(t) = \hat{x}(t) \text{ for } t \in I_0 \,\Delta\, [-a_0, 0],\ a_0 \geqslant 0,\ \pi_t x \,\Delta\, \{x(\theta),\ -a_0 \leqslant \theta \leqslant t\},\end{cases}$$

where $f: I \otimes C(I_T, E^n) \to E^n$ with $I_T \,\Delta\, [-a_0, T]$;

$$g: I \otimes E^n \otimes E^r \to E^n;$$

and K is a $(n \times n)$ matrix valued kernel defined on the triangle

$$\Delta \triangleq \{(t, \tau) : 0 \leqslant \tau \leqslant t \leqslant T\}.$$

$C(J, E^n)$ denotes the Banach space of continuous functions defined on any compact set $J \subset R$ with values in E^n. $AC(J, E^n) \subset C(J, E^n)$ is the space of absolutely continuous functions on J with values in E^n. Throughout E^n may be assumed to be equipped with the norm $| x | = \sum_{i=1}^{n} | x_i |$ and $C(J, E^n)$ with the norm $\| x \| = \sup_{t \in J} | x(t) |$. The space of $(n \times n)$ matrices, considered as the space of linear operators acting within E^n, is equipped with the usual Euclidean norm $\| K \| = \sup_{1 \leqslant j \leqslant n} (\sum_{i=1}^{n} | K_{ij} |)$.

In optimal control theory the compactness and continuity (in the Hausdorff metric) of attainable sets, and compactness of attainable trajectories are fundamental and in fact proof of existence of optimal controls is based on such properties. Using the results of Proposition 1 and Lemmas 1 and 2 it is shown in Proposition 2, Section 3, that the set of attainable trajectories is conditionally compact. Lemma 3 establishes the equivalence between the original system and a multivalued differential system in a Banach space. Result of Lemma 4 has an independent interest in addition to its usefulness in the text. Results of Lemma 4, Lemma 5 and Corollary 1 are used to prove in Proposition 3 the compactness of the set of trajectories and in Corollary 2 the compactness of the attainable set and its continuity. These results are used in Section 4 for the proof of existence of optimal controls.

Remark. It is of interest to note that the integral term included in the description of the system S arises naturally in economic models that take into account the so-called "absorption lag" [Dobel and Dorfman (1971), p. 24]. In that the rate of growth of capital $\dot{x}(t)$ at time $t \in I$ depends on the past history of both the capital $\{x(\tau), \tau \leqslant t\}$ and investment $\{u(\tau), 0 \leqslant \tau \leqslant t\}$.

2. Basic Assumptions and Definitions

The following assumptions are used throughout the rest of the paper:

A_f : The functional f satisfies the following properties

(i) there exists a bounded measurable function $\alpha \geqslant 0$ so that $| f(t, \pi_t x)| \leqslant \alpha(t)(1 + \| x \|_t)$ for all $x \in C(I_T, E^n)$ and $t \in I$ where $\| x \|_t \triangleq \sup_{s \in I_t} | x(s)|$.

(ii) $| f(t, \pi_t x) - f(t, \pi_t y)| \leqslant \alpha(t) \| x - y \|_t$.

A_K: (i) for each fixed but almost all $\tau \in I$, $K(\cdot, \tau)$ is a bounded measurable (matrix valued) function on $\tau < t \leqslant T$.

(ii) for each $t \in I$, $K(t, \cdot)$ is Lebesgue measurable in $\tau \in [0, t]$ and $\sup_{t \in I} \int_0^t \| K(t, \tau) \| \, d\tau = \hat{K} < \infty$ where $\| K \|$ denotes the usual Euclidean norm.

A_g: The function $g: I \otimes E^n \otimes E^r \to E^n$ is continuous and satisfies the following properties:

(i) $| g(t, \xi, u) | \leqslant \alpha(t)[1 + | \xi |]$ for all $u \in U(t) \subset E^r$ and $t \in I$,

(ii) $| g(t, \xi, u) - g(t, \eta, u) | \leqslant \alpha(t) | \xi - \eta |$ for all $u \in U(t) \subset E^r$.

For admissible controls we choose the set $B = \{u \in L_\infty(I, E^r) : u(t) \in U(t) \text{ a.e. } I\}$ where $U(t)$, $t \in I$ is a function continuous in the Hausdorff metric [Hermes and LaSalle (1969), p. 5] with values in the space of nonempty, compact and convex subsets of the set E^r. For $\hat{x} \in C(J_0, E^n)$ define the set

$$Y = \{x \in C(I_T, E^n) : x(t) = \hat{x}(t), \, t \in I_0 \text{ and } x(\cdot \mid I) \in AC(I, E^n)\},$$

where $x(\cdot \mid I)$ is the vector function x with its domain restrictricted to I.

DEFINITION 1. An element $x \in C(I_T, E^n)$ is said to be a solution of the system S if (i) $x \in Y$ and (ii) $\dot{x}(t) = f(t, \pi_t x) + \int_0^t K(t, \tau) \, g(\tau, x(\tau), u(\tau)) \, d\tau$ a.e. I for some $u \in B$.

For each $t \in I$, $y \in E^n$ define the set valued function $G(t, y) = g(t, y, U(t)) \subseteq E^n$ and for each $x \in Y$ define the sets:

$$H(x) \, \Delta \, \{h: I \to E^n, \text{ measurable} : h(t) \in G(t, x(t)) \text{ a.e.}\}$$

and

$$R(x) \, \Delta \left\{ z: I \to E^n, \text{ measurable: } z(t) = f(t, \pi_t x) + \int_0^t K(t, \tau) \, h(\tau) \, d\tau, \right. \\ \left. \text{a.e. on } I, \, h \in H(x) \right\}.$$

Note that the elements of $H(x)$ and $R(x)$ are measurable n-vector valued functions defined on I.

For an element $x \in Y$, we denote by the symbol x^d the measurable n-vector valued function obtained by differentiating $x(t)$ with respect to t almost everywhere on the interval I and restricting its domain of definition to the interval I. With this preparation we define system S' as

$$S' \begin{cases} x^d \in R(x) \\ \text{with } x(t) = \hat{x}(t) \text{ for } t \in I_0 . \end{cases}$$

An element $x \in Y$ is said to be a solution of the system S' if x satisfies the inclusion property $x^d \in R(x)$. Alternatively if, corresponding to an element $x \in Y$, there exists *no* $y \in R(x) \cap L_1(I, E^n)$ so that $\hat{x}(0) + \int_0^t y(t)\, dt = x(t)$ for each $t \in I$ then x is *not* a response of the system S'. It will be shown in the sequel that the two systems S and S' are equivalent. We need the following definitions in the sequel.

DEFINITION 2. The set $X \subset C(I_T, E^n)$ defined by $X \,\Delta\, \{x \in Y : x^d \in R(x)\}$ is said to be the set of admissible trajectories of the system S.

DEFINITION 3. For each $t \in I$, the set $A(t) \subset E^n$ defined by $A(t) \,\Delta\, \{y \in E^n : y = x(t) \text{ for some } x \in X\}$ is called the attainable set of the system S.

3. Properties of the Attainable Set and Trajectories

In this section we present the properties of the attainable set $A(t)$ $t \in I$ and the set X. We need the following results:

PROPOSITION 1. *Under the hypotheses A_f, A_K and A_g and for each fixed initial data $\hat{x} \in C(I_0, E^n)$ and a control $u \in B$ the system S has one and only one solution $x \in C(I_T, E^n)$.*

Proof. The proof is standard.

The following lemmas are useful and stated without proof.

LEMMA 1. *The set of trajectories X of the system S satisfying the basic assumptions A_f, A_K and A_g is a bounded subset of $C(I_T, E^n)$.*

LEMMA 2. *The set X is an equicontinuous subset of $C(I_T, E^n)$.*

PROPOSITION 2. *Under the assumptions A_f, A_K and A_g the set of trajectories X of the system S is a conditionally compact subset of $C(I_T, E^n)$ (in its topology of uniform convergence).*

Proof. The proof follows from Lemma 1, Lemma 2, and Arzèla–Ascoli theorem [Dunford and Schwartz (1964), th. 7, p. 266].

For the proof of compactness of the set of trajectories X and the attainable set $A(t)$, $t \in I$ we need several lemmas.

LEMMA 3. *The system S is equivalent to the system S' in the sense that every solution of one is the solution of the other (and conversely).*

Proof. An element $x \in Y$ which is a solution of the system S is obviously a solution of the system S'. Conversely if $x \in Y$ and is a solution of the system S' then there exists an $h_x \in H(x)$ so that $x^d = fx + Th_x$ where $(fx)(t) \Delta f(t, \pi_t x)$, $t \in I$ and $(Th_x)(t) \Delta \int_0^t K(t, \tau) h_x(\tau) d\tau$, $t \in I$. By definition of $H(x)$, h_x is a measurable function satisfying the property $h_x(t) \in G(t, x(t))$ a.e. on I. Hence by Filippov–Hermes lemma [Hermes and La Salle (1969), Lemma 8.2, p. 30] there exists a $u_x \in B$ so that $h_x(t) = g(t, x(t), u_x(t))$ a.e. on I. Thus x is also a solution of the system S. This completes the proof.

LEMMA 4. *Let Γ be a closed bounded convex subset of $L_\infty(I, E^n)$ and $E(\Gamma)$ the class of closed convex subsets of the set Γ which is assumed to be equipped with the w^*-topology (L_∞, τ_{L_1}). Then the space $E(\Gamma)$ is metrizable, the metric is a Hausdorff metric ρ_H and $(E(\Gamma), \rho_H)$ is compact.*

Proof. Since Γ is closed, bounded and convex it is w^* closed and hence w^*-compact [Dunford and Schwartz (1964), Cor. 3, p. 424] and since $L_1(I, E^n)$ is separable it follows from (Dunford and Schwartz (1964), Theorem 1, p. 426) that Γ is metrizable with the metric ρ given by

$$\rho(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|(x - y) z_n|}{1 + |(x - y) z_n|},$$

where $x, y \in \Gamma$ and $\{z_n\}$ is a countable dense subset of L_1. The w^*-topology and the metric topologies of Γ are equivalent. Thus it follows from Theorem 4.9.13 [Michael (1951), p. 163] that $E(\Gamma)$ is metrizable. Since (Γ, ρ) is a compact metric space it follows from Propositions 3.5 and 3.6 [Michael (1951), pp. 160–61] that $E(\Gamma)$ can be given the Hausdorff metric ρ_H. The compactness of the metric space $(E(\Gamma), \rho_H)$ follows from Theorem 4.9.12 [Michael (1951), p. 163].

This completes the proof of the lemma.

Note. The duality product appearing in the definition of ρ should cause no confusion.

The metric ρ_H is usually defined as

$$\rho_H(A_1, A_2) \Delta \max \left\{\sup_{x \in A_2} \rho(A_1, x), \sup_{x \in A_1} \rho(x, A_2)\right\},$$

where

$$\rho(A_i, x) = \operatorname*{Inf}_{y \in A_i} \rho(y, x).$$

LEMMA 5. *Suppose for each $t \in I$ and $\xi \in E^n$ the set $G(t, \xi)$ is convex and suppose the assumptions $A_f A_g$ and A_K are satisfied. Then for each $x \in Y$ the set $R(x)$ is a convex and w^*-compact subset of $L_\infty(I, E^n)$.*

Proof. The proof follows from the definition of the set $R(x)$ and the fact that, under the given assumptions, the set $H(x)$ itself is a convex and w^*-compact subset of $L_\infty(I, F^n)$.

As a consequence of Lemmas 1, 4 and 5 we have

COROLLARY 1. *Suppose the hypotheses of Lemma 5 are satisfied. Then there exists a closed bounded and convex subset $\Gamma \subset L_\infty(I, E^n)$ so that*

$$R(x) \in (E(\Gamma), \rho_H) \qquad \textit{for all} \quad x \in X.$$

The following definition is standard and is adopted for use in the present situation.

DEFINITION 4. The set valued function R: $Y \to (E(\Gamma), \rho_H)$ is said to be continuous at the point $x_0 \in Y$ if for every $\delta > 0$ there exists an $\mathscr{E}(\delta, x_0) > 0$ such that $\rho_H(R(x), R(x_0)) \leqslant \delta$ whenever $x \in N_{\mathscr{E}}(x_0) \cap Y$ where $N_{\mathscr{E}}$ is the usual $\mathscr{E}$-neighborhood of the point $x_0 \in C(I_T, E^n)$.

With these preparations we now present our main result of this section.

PROPOSITION 3. *Suppose the hypotheses of Lemma 5 are satisfied and that the set valued function R: $Y \to (E(\Gamma), \rho_H)$ is upper semicontinuous. Then the set of trajectories X of the system S is a compact subset of $C(I_T, E^n)$.*

Proof. By proposition 2, X is sequentially compact and therefore if $x_* \in \bar{X}$ (closure of X) there exists a sequence $\{x_n\} \in X$ such that $x_n \to^u x_*$ (uniformly). It follows from A_f (i), A_g (i) and Lemma 1 that there exists an $\tilde{\alpha} \in L_\infty(I)$, independent of $x \in X$, such that $|\dot{x}(t)| \leqslant \tilde{\alpha}(t)$ a.e. on I. Therefore $x_* \in Y$ and $x_*{}^d$ is a well-defined measurable function on I to E^n. By Lemma 5, $R(x_*)$ along with its closed δ-neighborhood $R^\delta(x_*)(\delta > 0)$ are convex and w^*-compact. Thus the set Γ in Corollary 1 can be chosen large enough so that $E(\Gamma) \ni R(x_*)$, $R^\delta(x_*)$ and $R(x)$ for all $x \in X$. Since R is continuous, for every $\delta > 0$ there exists $n_0(\delta) > 0$ such that $R(x_n) \subset R^\delta(x_*) \in E(\Gamma)$ for all $n \geqslant n_0$. Consequently there exists a subsequence $\{x_{n_k}\}$ such that

$x_{n_k}^d \in R(x_{n_k}) \subset R^\delta(x_*)$ for all integers $k = 1, 2, \ldots$. Since $R^\delta(x_*)$ is w^*-compact or equivalently a compact subset of the metric space (Γ, ρ) there exists a subsequence of the sequence $\{x_{n_k}\}$, again denoted by $\{x_{n_k}\}$, and an element $y \in R^\delta(x_*)$ such that $x_{n_k}^d \rightarrow^{w^*} y$.

The two conditions (i) $x_{n_k} \rightarrow^u x_*$ and (ii) $x_{n_k}^d \rightarrow^{w^*} y$ imply that $x_*^d = y$ [i.e. $\dot{x}_*(t) = y(t)$ a.e. on I and y is measurable] and hence $x_*^d \in R^\delta(x_*)$. But $\delta > 0$, being arbitrary, $x_*^d \in R(x_*)$ and consequently $x_* \in X$. Thus $X = \bar{X}$ and therefore compact.

Remark. The set $R \triangleq \bigcup_{x \in X} R(x)$ is a compact subset of (Γ, ρ).

COROLLARY 2. *The attainable set $A(t)$, $t \in I$ is a compact subset of E^n and it is continuous in the Hausdorff metric.*

Proof. Let z_0 be a limit point of $A(t)$ for $t \in I$, then there is a sequence $z_n \in A(t)$ such that $z_n \rightarrow z_0$. Corresponding to $z_n \in A(t)$ there exists a sequence $x_n \in X$ such that $x_n(t) = z_n$. Since X is compact (Proposition 3) there is a subsequence $\{x_{n_k}\} \subset \{x_n\}$ and $ax^* \in X$ so that $x_{n}. \rightarrow^u x^*$ on I. Clearly $x^*(t) = z_0$. Since $x^* \in X$, $x^*(t) \in A(t)$ and consequently $z_0 \in A(t)$. This proves the closure of the set $A(t)$; its boundedness follows from that of X (Lemma 1). These imply compactness. The continuity follows from the hypotheses A_K, A_g and A_f.

4. EXISTENCE OF OPTIMAL CONTROLS

Using the results of Proposition 3, and Corollary 2 the following results can be obtained.

COROLLARY 3. *Suppose all the hypotheses A_f, A_g and A_K and those of Proposition 3 are satisfied and $N(t)$, $t \in I$ is a continuous set valued function with values in the metric space (Hausdorff metric) of compact subsets of E^n and there exists a $t' \in I$ so that $A(t') \cap N(t')$ is nonempty. Then there exists an optimal control $u^* \in B$ that drives the system S to the target N in minimum time.*

COROLLARY 4. *Suppose all the hypotheses of Proposition 3 are satisfied. Let $M \subset E^n$ be compact and $M \cap A(T)$ is nonempty and φ: $M \rightarrow R$ is a non-negative lower semicontinuous function. Then there exists an optimal control $u^* \in B$ that drives the system S to the target M and yields a minimum to the cost function $\varphi(x(T))$.*

In certain economic problems it is required to maximize the time spent by the trajectory of the system in a desirable region of the phase space. This is solved in the following corollary.

COROLLARY 5. *Suppose all the hypotheses of Proposition* 3 *are satisfied and let* $N(t)$, $t \in I$ *satisfy the hypotheses of Corollary* 3. *Suppose there exists an* $x \in X$ *so that* $\mu\{t \in I : x(t) \in N(t)\} > 0$ *with* μ *the Lebesgue measure on the real line. Then there exists an optimal control* $u^* \in B$ *with the corresponding trajectory* $x^* \in X$ *so that*

$$\mu\{t \in I : x^*(t) \in N(t)\} \geqslant \mu\{t \in I : x(t) \in N(t)\} \qquad \text{for all} \quad x \in X.$$

Remark. It would be interesting to consider the system S containing a functional [Oğuztöreli (1966), p. 171, Eq. 7.5.1] $f : I \otimes C(I_T, E^n) \otimes E^r \to E^n$ depending on the present values of control instead of one independent of controls as done in this paper. This is an open question to the author.

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