The Radiative Transfer Equations: Diffusion Approximation Under Accretiveness and Compactness Assumptions

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Abstract—When the mean free path of photons is small the radiative transfer equations may be approximated by a nonlinear angular diffusion equation. In order to find the diffusion equation, one uses accretiveness assumptions. But these assumptions impose some conditions on the opacity. To overcome this inconvenient, we propose to study the solution of the radiative transfer equations by compactness techniques. © 2001 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

This paper is devoted to a system of two nonlinear PDEs which can be regarded as a perturbation of the well-known transport equation. We consider a flux of photons radiating through a continuous medium, in absence of hydrodynamical motion. The interaction between the medium and the photonic flux can be described by radiative transfer equations. These equations are classical in astrophysics and represent the evolution of a stellar atmosphere in absence of heat conduction. The photons are ruled by a classical equation involving terms describing the absorption and emission. The first mathematical approach of this problem can be found in Mercier [1] (where no proof was given). In the physical problem, the term \( \sigma_\nu(T)B_\nu(T) \) where \( B_\nu(T) \) is known in astrophysics as the Plank's function (as defined below), is the opacity of the matter at temperature given by \( T \) for radiations of frequency \( \nu \). The function \( B_\nu(T) \), after a suitable normalization can be expressed as

\[
B_\nu(T) = \frac{C\nu^3}{e^{\nu/T} - 1}, \text{ where } C \text{ is defined such that } \int_{0}^{\infty} B_\nu(T) \, d\nu = T^4.
\]

The unknowns of our problem are: \( I \equiv I(t, x, \Omega, \nu) \), density probability of photons at position \( x \in X \) (some smooth bounded convex open subset of \( \mathbb{R}^3 \)), with frequency \( \nu(\nu \in \mathbb{R}^+) \), at time
t ≥ 0, with direction Ω ∈ S² (the unit sphere of R³) described by parameter \( T = T(t, x) \) (proportional to the fourth power of the material temperature, at the position \( x \)). The radiative transfer equations (see [1,2]) may be written in the following form:

\[
\frac{1}{c} \partial_t I + \Omega \cdot \partial_x I + \sigma_\nu(T)(I - B_\nu(T)) + \kappa \left( I - \bar{I} \right) = 0, \tag{1.1}
\]

\[
\partial_t \mathcal{E}(T) + \int_0^\infty \int_{S^2} \sigma_\nu(T)(B_\nu(T) - I) \, d\nu \frac{d\Omega}{4\pi} = 0, \tag{1.2}
\]

where

\[
\bar{I} = \int_{S^2} I(\Omega) \frac{d\Omega}{4\pi}.
\]

The boundary condition (specular reflection) is written in the form

\[
I(x, \Omega, \nu, t) = I(x, \mathcal{R}_\nu \Omega, \nu, t), \tag{1.3}
\]

where

\[
\mathcal{R}_\nu \Omega = \Omega - 2(\Omega \cdot n_x)n_x, \quad (n_x \text{ is the outward normal to } X \text{ at the point } x \text{ of } \partial X). \tag{1.4}
\]

Here \( \mathcal{E}(T) \) is internal energy of the matter and \( \kappa \) is Thomson scattering cross section.

We consider Cauchy's problem for these equations, i.e., we impose an initial data

\[
I(x, \nu, \Omega, 0) = I_0(x, \nu); \quad T(x, 0) = T_0(x). \tag{1.5}
\]

To obtain the approximation by angular diffusion (which is an analogous to the approximation by the diffusion of transport equation), we introduce a small parameter \( \epsilon > 0 \) and consider the equations

\[
\frac{1}{\epsilon} \partial_t I_\epsilon + \Omega \cdot \partial_x I_\epsilon + \epsilon \sigma_\nu(T_\epsilon)(I_\epsilon - B_\nu(T_\epsilon)) + \frac{\kappa}{\epsilon} \left( I_\epsilon - \bar{I}_\epsilon \right) = 0, \tag{1.6}
\]

\[
\partial_t \mathcal{E}(T_\epsilon) + \int_0^\infty \int_{S^2} \sigma_\nu(T_\epsilon)(B_\nu(T_\epsilon) - I_\epsilon) \, d\nu \frac{d\Omega}{4\pi} = 0. \tag{1.7}
\]

The boundary condition takes the form

\[
I_\epsilon(x, \Omega, \nu, t) = I_\epsilon(x, \mathcal{R}_\nu \Omega, \nu, t). \tag{1.8}
\]

This scaling is the same as in Bardos-Santos-Sentis [3] for the linear transport equation. It is justified by the fact that in the regime where \( \kappa \) is enough large, on replaces \( \kappa \) by \( \kappa/\epsilon \) and takes \( I_\epsilon \) approximately equivalent to \( \bar{I}_\epsilon \); this allows letting \( \nabla_x \cdot \bar{I}_\epsilon \) go to 0, removing like this the effect with the time scaling. However, in order to observe a diffusion motion at large scale, it is logical to rescale the time variable replacing \( t \) by \( t/\epsilon \) and so \( \partial_t \) by \( \epsilon \partial_t \) adapted to the diffusion. The initial data are written as

\[
I_\epsilon(x, \nu, \Omega, 0) = I_0(x, \nu); \quad T_\epsilon(x, 0) = T_0(x). \tag{1.9}
\]

Observe that \( I|_{\epsilon=0} \) is independent of \( \Omega \), because the system is stable at \( t = 0 \) with the fact that when \( \epsilon \to 0, I_\epsilon - \bar{I}_\epsilon \to 0 \), if not it would appear an initial layer. The parameter \( \epsilon \) is the mean free path of the photons for the scattering Thomson phenomena to the size of the domain \( X \). The angular diffusion approximation consists to study the asymptotic behavior of equations (1.6)-(1.9) when \( \epsilon \to 0 \). Formally, one finds

\[
T_\epsilon \to \theta, \quad I_\epsilon \to J \text{ when } \epsilon \to 0,
\]
where \((\theta, J)\) is the solution of the system of angular diffusion system

\[
\frac{1}{c} \frac{\partial}{\partial t} J - \nabla \cdot \frac{1}{3\kappa} \nabla J + \sigma_\nu(\theta)(J - B_\nu(\theta)) = 0,
\]

\[
\frac{\partial}{\partial t} \mathcal{E}(\theta) + \int_0^\infty \sigma_\nu(\theta)(B_\nu(\theta) - J) \, d\nu = 0.
\]

\[
\frac{\partial J}{\partial n} = 0, \quad \text{(condition de Neumann),}
\]

\[
J(x, \nu, 0) = I_0(x, \nu); \quad T(x, 0) = T_0(x).
\]

We propose to prove two results; the first by monotonicity assumptions which implies the accretiveness of the transfer radiative operator and the second by compactness method: the first result can be stated in the form of the following theorem.

**THEOREM 1.1.** Let \(I_0, E_0\) be of the class \(C^4\) with respect to \(x\) and belong to \(L^1(X \times \mathbb{R}^+)\) and to \(L^1(X)\), respectively. Assume that: for a.e., \(\nu > 0\), \(T \mapsto \sigma_\nu(T)\) is strictly continuous, nonincreasing with value in \(\mathbb{R}^+\); for all \(T > 0, \nu \mapsto \sigma_\nu(T)\) is Borelian; for a.e., \(\nu > 0\), \(T \mapsto \sigma_\nu(T)B_\nu(T)\) is nondecreasing, continuous in \(\mathbb{R}^+\), and \(\sigma_\nu(T)B_\nu(T) \to 0\), when \(T \to 0^+\); \(\mathcal{E}(T)\) is strictly nondecreasing, continuous on \(\mathbb{R}^+\), \(\mathcal{E}(0) = 0\), and \(\mathcal{E}(T) \to +\infty\) when \(T \to +\infty\). Then the solution \((I_\epsilon, \mathcal{E}(T_\epsilon))\) of equations (1.6)–(1.9) converges in \(L^1(X \times S^2 \times \mathbb{R}^+) \times L^1(X)\) to \((J, \mathcal{E}(\theta))\) solution of diffusion equation (1.10)–(1.13).

Physically the monotonicity assumptions made above is not very realistic; this leads up to make those weaker assumptions using compactness techniques and based on the velocity averaging lemma [4]. By this method, it is not very difficult to resolve the evolution problem; so one replaces the last problem by an implicit discretisation in time and denotes \(\lambda = 1/\Delta t\). Hence, one considers the system of equations

\[
\lambda I_\epsilon + \frac{1}{\epsilon} \Omega \cdot \nabla x I_\epsilon + \sigma_\nu(T_\epsilon)(I_\epsilon - B_\nu(T_\epsilon)) + \frac{\kappa}{\epsilon^2} \left( I_\epsilon - \tilde{I}_\epsilon \right) = \lambda I^*,
\]

\[
\lambda \mathcal{E}(T_\epsilon) + \int_0^\infty \sigma_\nu(T_\epsilon)B_\nu(T_\epsilon) \, d\nu - \int_0^\infty \sigma_\nu(T_\epsilon)\tilde{I}_\epsilon \, d\nu = \lambda \mathcal{E}(T^*),
\]

\[
I_\epsilon(x, \Omega, \nu) = I_\epsilon(x, \mathbb{R}^2 \Omega, \nu),
\]

and our result may be reduced to the following theorem.

**THEOREM 1.2.** Assume that \(\sigma_\nu B_\nu\) is bounded for \(k \leq T \leq K\); for p.p. \(\nu \geq 0\), \(T \mapsto \sigma_\nu(T)\) \(T \mapsto \sigma_\nu(T)B_\nu(T)\) are uniformly continuous with respect to \(\nu\); assume that \(0 \leq I_0(x, \nu) \leq B_\nu(T^*)\) on \(X \times \mathbb{R}^+\), and \(0 \leq T_0 \leq T^*\) on \(X\); \(\mathcal{E}(T)\) is strictly nondecreasing, \(\forall \nu \in \mathbb{R}^+, \sigma_\nu(T)\) is strictly nonincreasing and \(\sigma_\nu(T^*)\) is bounded. Moreover, \(T \mapsto \int_0^\infty \sigma_\nu(T)B_\nu(T) \, d\nu\) is nondecreasing. Let \((I_\epsilon, T_\epsilon)\) be the solution of equations (1.14)–(1.16). Then one can extract from \((I_\epsilon, T_\epsilon)\) a subsequence still denote \((I_\epsilon, T_\epsilon)\) such that \(I_\epsilon \to I\) in \(L^1_{\text{loc}}(X \times \mathbb{R})\) and \(T_\epsilon \to T\) in \(L^p_{\text{loc}}(X)\), \(\forall 1 \leq p < +\infty\) and \((I, T)\) is a solution of

\[
\lambda I - \nabla \cdot \frac{1}{3\kappa} \nabla I + \sigma_\nu(T_\epsilon)(B_\nu(T_\epsilon) - I) = \lambda I^*,
\]

\[
\lambda \mathcal{E}(T) + \int_0^\infty \sigma_\nu(T)(B_\nu(T) - I) \, d\nu = \lambda \mathcal{E}(T^*),
\]

\[
I(x, \Omega, \nu) = I(x, \mathbb{R}^2 \Omega, \nu).
\]

**REMARK 1.3.** Observe that the assumptions in the Theorem 1.2 are compatible with Kramer's values of \(\sigma_\nu(T) = (1 - e^{-\nu/T})/\sqrt{T} \nu^3\) since \(\sigma_\nu(T)\) is nondecreasing in \(T\) for all \(\nu\). More \(\int_0^\infty \sigma_\nu(T)B_\nu(T) \, d\nu = (1/\sqrt{T}) \int_0^\infty e^{-\nu/T} \, d\nu = \sqrt{T}\) is nondecreasing in \(T\).

The outline of the paper is organized as follows. The second section deals with the proof of Theorem 1.1. The third section is devoted to the existence of a solution at \(\epsilon\) positive fixed and the maximum principle given by the velocity averaging lemma; finally Theorem 1.2. is proved.
2. PROOF OF THEOREM 1.1

In the system of equations (1.6)–(1.9) one replaces $I$, by

$$J_e = J + \epsilon I_1 + \epsilon^2 I_2$$

(2.1)

and $T_e$ by

$$T_e = \theta,$$

(2.2)

where $(J, \theta)$ is the solution of equations (1.10)–(1.13) and where

$$I_1 = -\frac{1}{\kappa} \Omega \cdot \nabla_x J,$$

(2.3)

$$I_2 = -\frac{1}{\kappa} \Omega \cdot \nabla_x I_1 - \frac{\sigma_\nu(\theta)}{\kappa} (J - B_\nu(\theta)) - \frac{1}{c} \partial_t J$$

$$= -\left(\frac{1}{\kappa}\right)^2 (\Omega \cdot \nabla_x)^2 J - \frac{\sigma_\nu(\theta)}{\kappa} (J - B_\nu(\theta)) - \frac{1}{c\kappa} \partial_t J.$$  

(2.4)

Clearly, $\bar{I}_1 = 0$ and equation (1.3) express that $\bar{I}_2 = 0$.

On the one hand, in $X \times S^2 \times \mathbb{R}^+$

$$\frac{1}{c} \partial_t (I_e - J_e) + \frac{1}{\xi} \Omega \cdot \nabla_x I_1 - \frac{\kappa}{c^2} \left[(I_e - J_e) - \left(\bar{I}_e - \bar{J}_e\right)\right]$$

$$+ \sigma_\nu(T_e)(I_e - B_\nu(T_e)) - \sigma_\nu(\theta)(J_e - B_\nu(\theta))$$

$$= - \left[\frac{\epsilon^2}{c} \partial_t I_2 + \frac{\epsilon}{c} \partial_t I_1 + \epsilon \Omega \cdot \nabla I_2 + \sigma_\nu(\theta) (\epsilon I_1 + \epsilon^2 I_2)\right],$$

(2.5)

and

$$\partial_t [E(T_e) - E(\theta)] + \int_0^\infty \sigma_\nu(T_e) \left(B_\nu(T_e) - \bar{I}_e\right) d\nu - \int_0^\infty \sigma_\nu(\theta) \left(B_\nu(\theta) - \bar{J}_e\right) d\nu = 0. $$

(2.6)

Set

$$R_e = \frac{\epsilon^2}{c} \partial_t I_2 + \frac{\epsilon}{c} \partial_t I_1 + \epsilon \Omega \cdot \nabla I_2 + \sigma_\nu(\theta) (\epsilon I_1 + \epsilon^2 I_2)$$

(2.7)

and observe that the regularity of the solutions of equations (1.10)–(1.13) leads to

$$\sup_{[0,t]} \|R_e\|_{L^1_t, L^\infty} \leq C \epsilon t.$$

(2.8)

Last, for $t = 0$, one has

$$I_e - J_e = \epsilon(I_1 - \epsilon I_2) = C \epsilon.$$

(2.9)

On the other hand,

$$(I_e - J_e)(t, x, \Omega, \nu) - (I_e - J_e)(t, x, R_\Omega, \nu) = J_e(t, x, R_\Omega, \nu) - J_e(t, x, \Omega, \nu)$$

(2.10)

$$= \epsilon^2 I_2(t, x, R_\Omega, \nu) - \epsilon^2 I_2(t, x, \Omega, \nu)$$

since $J$ is independent of $\Omega$, and

$$I_1(t, x, R_\Omega, \nu) - I_1(t, x, \Omega, \nu) = \frac{1}{\kappa} 2(\Omega \cdot n_x)n_x \cdot \nabla_x J$$

$$= \frac{1}{\kappa} 2(\Omega \cdot n_x) \frac{\partial J}{\partial n} = 0.$$  

(2.11)

according to the relation $n_x \cdot \nabla_x I_0 = \partial_\nu I_0 = 0$. After we use the $L^1$-accretiveness properties. To this end, we introduce the Banach spaces

$$E = L^1 \left(X \times S^2 \times \mathbb{R}^+\right) \times L^1(X),$$

(2.12)
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\[ E^+ = \{(I,T); I \in L^1(X \times S^2 \times \mathbb{R}^+) \}, \quad T \in L^1(X), \; T \geq 0, \; I \geq 0 \}, \] (2.13)

and the operators

\[ A(I,T) = (\Omega \cdot \nabla I + \kappa (I - \tilde{I}), 0) \] (2.14)

\[ B(I,T) = (\sigma_v(T)(I - B_v(T)); \int_0^\infty \int_{S^2} \sigma_v(T)(B_v(T) - I) d\Omega \frac{d\Omega}{4\pi}) \] (2.15)

of domain

\[ D = \{(I,T) \in E^+ \; \text{s.t.} \; \sigma_v(T)I \in L^1(X \times S^2 \times \mathbb{R}^+)\}. \] (2.16)

We shall use accretiveness assumptions of Theorem 1.1, by multiplying equation (2.6) by \( \text{sgn}^+(I - J_e) \) and equation (2.7) by \( \text{sgn}^+(T_e - \theta) \) with

\[ \text{sgn}^+ \alpha = 1 \iff \alpha > 0, \quad \forall \alpha \in \mathbb{R}, \]

\[ \text{sgn}^+ \alpha = 0 \iff \alpha \leq 0. \]

One has, in particular for all \( (I,T),(I',T') \in D^2(D) \) with

\[ \varphi = \text{sgn}^+(T - T'); \quad \psi = \text{sgn}^+(I - I'), \]

where the notation \( \langle g \rangle \) stands for

\[ \langle g \rangle = \int \int_{X \times S^2 \times \mathbb{R}^+} g \frac{dx d\Omega}{4\pi} \quad \text{for} \; g \in L^1(X \times S^2 \times \mathbb{R}^+) : \]

\[ \langle \sigma_v(T)(I - B_v(T)) - \sigma_v(T')(I' - B_v(T')) , \psi \rangle \]

\[ + \int_0^\infty \sigma_v(T)(B_v(T) - I) d\nu - \int_0^\infty \sigma_v(T')(B_v(T') - I') d\nu, \varphi \]

\[ = \langle \sigma_v(T)B_v(T) - \sigma_v(T')B_v(T') , \psi \rangle \]

\[ + \langle \sigma_v(T)(I - \sigma_v(T'))B_v(T') - \sigma_v(T')(I' - \sigma_v(T')(B_v(T')) \rangle \psi \]

\[ = \langle \left[ \sigma_v(T)B_v(T) - \sigma_v(T')(B_v(T')) \right] (\varphi - \psi) + \langle \sigma_v(T)I - \sigma_v(T')I' \rangle (\psi - \varphi) \rangle \]

\[ = \left\langle \langle \sigma_v(T)B_v(T) - \sigma_v(T')(B_v(T')) \rangle + \langle \sigma_v(T)B_v(T) - \sigma_v(T')(B_v(T')) \psi \rangle \right\rangle \]

\[ + \langle \sigma_v(T)(I - I') - (I - I') \varphi \rangle \]

\[ + \langle I' \left[ \sigma_v(T) - \sigma_v(T') \right] \psi - (\sigma_v(T) - \sigma_v(T') \psi \rangle \right\rangle, \] (2.17)

and one observes that the assumptions \( \sigma_v(T) \) nonincreasing and \( \sigma_v(T)B_v(T) \) nondecreasing imply that the last term of equation (2.17) is always positive. Thus, multiplying equation (2.7) by \( \text{sgn}^+(T_e - \theta) \) and equation (2.6) by \( \text{sgn}^+(I_e - J_e) \) and combining, thus, obtaining, after integrating from 0 to \( t \)

\[ \|J_e - J_e^+\|_e + \|E(T_e) - E(\theta)^+\| \]

\[ \leq \|I_0 - J_0\|_{L^1_{\Omega,\nu}} + \|I_0 - \Omega_0\|_{L^1_{\Omega}} + \int_0^t \|R_e\|_{L^1_{\Omega,\nu}}(s) ds \]

\[ + \int_0^t \int_{S^2} \frac{1}{\epsilon} (\Omega \cdot n)(I_e - J_e^+)^+ d\sigma d\Omega ds \]

\[ \leq \|I_0 - J_0\|_{L^1_{\Omega,\nu}} + \|I_0 - \Omega_0\|_{L^1_{\Omega}} + \epsilon C + \epsilon \int_0^t \int_{S^2} |I_2| d\sigma d\Omega ds, \] (2.18)

and one concludes. This completes the proof of Theorem 1.1.
3. DIFFUSION BY COMPACTNESS AND PROOF OF THEOREM 1.2

First we recall an existence result of solution at $\varepsilon > 0$ fixed using the maximum principle and compactness assumptions with the velocity averaging lemma.

**Proposition 3.1.** (See [4].) Assume that:
- For all $T$, $u \to \sigma_v(T)$ is a measurable function, a.e., positive; for all $T$, $\nu \to B_v(T)$ is positive and integrable; for all $\nu T \to \sigma_v(T) B_v(T)$ is continuous function from $\mathbb{R}^+$ to $\mathbb{R}^+$ going to 0 when $T \to 0^+$. Last, assume that $E(T)$ is continuous on $\mathbb{R}^+$.
- Then, there exists at $\varepsilon > 0$ fixed a subsequence $(T_\varepsilon, I_\varepsilon)$ solution of the equations (1.14)-(1.16) defined in $L^1(X \times S^2 \times \mathbb{R}^+) \cap L^\infty(X \times S^2 \times \mathbb{R}^+)$. Assume moreover that there exists $k, K > 0$ such that

$$k \leq T^* \leq K \quad \text{and} \quad B_v(k) \leq I^* \leq B_v(K),$$

where $(T^*, I^*)$ is an unique solution of equations (1.17)-(1.19); this imply for the solution

$$k \leq T_\varepsilon \leq K \quad \text{and} \quad B_v(k) \leq I_\varepsilon \leq B_v(K), \quad \forall \varepsilon > 0.$$

Then $I_\varepsilon \to I$ and $T_\varepsilon \to T$ for the weak topology-- in $L^\infty(X \times \Omega \times \mathbb{R}^+)$. We prove in this section the Theorem 1.2.

**Proof of Theorem 1.2.** One cut down to a subsequence still denotes $(I_\varepsilon, T_\varepsilon)$ which converges (cf. Proposition 3.1). We divide our proof in several steps.

1. In the first one, we prove that for any function $\chi \in C^\infty_0(\mathbb{R}^+)$, one has, in the sense of strong convergence in $L^2(X)$, $\lim_{\varepsilon \to 0} \int \chi(\nu) \tilde{I}_\varepsilon \, d\nu = \int \chi(\nu) I \, d\nu$.

2. We show that $\tilde{I}_\varepsilon$ is relatively compact in $L^1_\text{loc}(X \times \mathbb{R}^+)$.  

3. We prove that $T_\varepsilon \to T$ in $L^p(X)$, $\forall 1 \leq p < +\infty$, and pass to the limit in the equation (1.15).

4. Finally, we take the average of equation (1.14) with respect to $\Omega$, letting $\varepsilon$ go to 0 and prove that equation (1.14) converges to $\lambda I - \nabla_x \cdot (1/3\kappa) \nabla_x I + \sigma_v(T)(I - B_v(T)) = \lambda I^*$. 

**First Step.** First we multiply equation (1.14) by $I_\varepsilon$ and integrate with respect to the variables $x, \Omega, \nu$. This leads to

$$\lambda \int I_\varepsilon^2 \, dx \, d\Omega \, d\nu + \int \sigma_v(T_\varepsilon) I_\varepsilon^2 \, dx \, d\Omega \, d\nu + \frac{\kappa}{\varepsilon^2} \left( I_\varepsilon - \tilde{I}_\varepsilon \right)^2 \, dx \, d\Omega \, d\nu$$

$$= \lambda \int I^* I \, dx \, d\Omega \, d\nu + \int \sigma_v(T_\varepsilon) B_v(T_\varepsilon) I_\varepsilon \, dx \, d\Omega \, d\nu. \quad (3.1)$$

In particular, with $I_\varepsilon \leq B_v(K)$

$$\lambda \int I_\varepsilon^2 \, dx \, d\Omega \, d\nu + \int \sigma_v(T_\varepsilon) I_\varepsilon^2 \, dx \, d\Omega \, d\nu + \frac{\kappa}{\varepsilon^2} \left( I_\varepsilon - \tilde{I}_\varepsilon \right)^2 \, dx \, d\Omega \, d\nu$$

$$\leq \frac{\lambda}{2} \int I_\varepsilon^2 \, dx \, d\Omega \, d\nu + \frac{\lambda}{2} \int I_\varepsilon^2 \, dx \, d\Omega \, d\nu + \int \sigma_v(T_\varepsilon) B_v(T_\varepsilon) B_v(K) \, dx \, d\Omega \, d\nu; \quad (3.2)$$

and in particular

$$\|I_\varepsilon - \tilde{I}_\varepsilon\|_{L^2_{x,\Omega,\nu}} = O(\varepsilon). \quad (3.3)$$

Thus, the initial subsequence $(I_\varepsilon, T_\varepsilon)$ in Proposition 3.1 verify

$$\lim_{\varepsilon \to 0} I_\varepsilon = I \equiv I(x, \nu). \quad (3.4)$$

With equation (3.3) one has

$$\Omega \cdot \nabla_x I_\varepsilon = \varepsilon \left[ \lambda (I^* - I_\varepsilon) - \sigma_v(T_\varepsilon)(I_\varepsilon - B_v(T_\varepsilon)) \right] - \frac{\kappa}{\varepsilon} \left( I_\varepsilon - \tilde{I}_\varepsilon \right). \quad (3.5)$$
On other hand, by the maximum principle
\[ \epsilon [\lambda (I^* - I_\epsilon) - \sigma_\nu(T_\epsilon) (I_\epsilon - B_\nu(T_\epsilon))] \to 0, \quad \text{when } \epsilon \to 0 \text{ in } L^\infty(X \times S^2 \times \mathbb{R}^+) \] (3.6)
and
\[ \frac{\kappa}{\epsilon} \left( I_\epsilon(x, \Omega, \nu) - \bar{I}_\epsilon(x, \Omega, \nu) \right) = O(1), \quad \text{in } L^2(X \times S^2 \times \mathbb{R}^+). \] (3.7)
The above estimates show that for all \( \chi \in C_0^\infty(\mathbb{R}^+) \) the functions \( \int \chi(\nu) I_\epsilon(x, \Omega, \nu) \, d\nu \) and \( \Omega \nabla_\nu \int \chi(\nu) I_\epsilon(x, \Omega, \nu) \, d\nu \) still stay in bounded domain of \( L^2(X \times S^2) \). By the velocity averaging theorem [3] it follows that
\[ \int \chi(\nu) I_\epsilon \, d\nu \to \int \chi(\nu) I \, d\nu, \quad \text{strongly in } L^2(X). \] (3.8)

SECOND STEP. We start with the equation
\[ \lambda \Delta_\nu I_\epsilon + \frac{1}{\epsilon} \Omega \cdot \nabla_\nu I_\epsilon + \frac{\kappa}{\epsilon^2} \left( I_\nu^\epsilon - \bar{I}_\nu^\epsilon \right) = \lambda I^* - \sigma_\nu(T_\epsilon) (I_\nu^\epsilon - B_\nu(T_\epsilon)), \] (3.9)
and put
\[ \Delta_h f(\nu) \overset{\text{def}}{=} f(\nu + h) - f(\nu). \] (3.10)
The last equation can be written as
\[ \lambda \Delta_h I_\epsilon + \frac{1}{\epsilon} \Omega \cdot \nabla_\nu \Delta_h I_\epsilon + \frac{\kappa}{\epsilon^2} \left( \Delta_h I_\nu^\epsilon - \Delta_h \bar{I}_\nu^\epsilon \right) + \sigma_\nu(T_\epsilon) \Delta_h I_\epsilon + \sigma_{\nu+h}(T_\epsilon) \Delta_h I_\epsilon \]
\[ = \lambda \Delta_h I^* - \Delta_h \sigma(T_\epsilon) I_\epsilon + \Delta_h (\sigma_\nu B_\nu)(T_\epsilon). \] (3.11)
Taking the absolute value of equation (3.11) and let us appear the function sign leads to
\[ \lambda \int |\Delta_h I_\epsilon| + \frac{1}{\epsilon} \int \Omega \cdot \nabla_\nu |\Delta_h I_\epsilon| + \frac{\kappa}{\epsilon^2} \int \left( \Delta_h I_\nu^\epsilon - \Delta_h \bar{I}_\nu^\epsilon \right) \left( \text{sgn } \Delta_h I_\epsilon - \text{sgn } \Delta_h \bar{I}_\nu^\epsilon \right) \]
\[ + \int \sigma_{\nu+h}(T_\epsilon) |\Delta_h I_\epsilon| \leq \lambda \int \Delta_h I^* \]
\[ + \int B_\nu(T_\epsilon) |\Delta_h \sigma_\nu(T_\epsilon)| + \int |\Delta_h (\sigma_\nu B_\nu)(T_\epsilon)|. \] (3.12)
Since \( (\nu, T) \to \sigma_\nu(T) B_\nu(T) \) is uniformly continuous on any compact
\[ |\Delta_h \sigma_\nu(T_\epsilon) B_\nu(T_\epsilon)| \leq C(h); \] (3.13)
ones has
\[ \int \sup_{k \leq T \leq K} |\Delta_h \sigma_\nu(T_\epsilon) B_\nu(T_\epsilon)| \to 0, \quad \text{when } h \to 0, \] (3.14)
and
\[ \int B_\nu(T_\epsilon) \sup_{k \leq T \leq K} |\Delta_h \sigma_\nu(T_\epsilon)| \to 0, \quad \text{when } h \to 0. \] (3.15)
One deduce that \( \bar{I}_\epsilon \) is relatively compact in \( L^1_{\text{loc}}(X \times \mathbb{R}^+) = L^1(\mathbb{R}^+; L^1(X)). \)

THIRD STEP. We introduce \( T_\epsilon \) solution of equation
\[ \lambda \mathcal{E}(T_\epsilon) + \int_0^\infty \sigma_\nu(T_\epsilon) B_\nu(T_\epsilon) \, d\nu - \int_0^\infty \sigma_\nu(T_\epsilon) \bar{I} \, d\nu = \lambda \mathcal{E}(T^*) , \] (3.16)
unique, since $\sigma_{\nu}(T_*)$ is nonincreasing and $\int_0^\infty \sigma_{\nu}(T_*)B_{\nu}(T_*)\, d\nu$ is nondecreasing. Since $\sigma_{\nu}(T^*)$ is bounded, with equation (3.8) the estimate

$$\int \left| \int_0^\infty \sigma_{\nu}(T_*) \left( I_\epsilon(x, \nu) - \bar{I}(x, \nu) \right) \, d\nu \right| \, dx \leq \int \int_0^A \sigma_{\nu}(T_*) \left( I_\epsilon - \bar{I} \right) \, d\nu \, dx + 2C \int_A^\infty B_{\nu}(K) \, d\nu \leq \varepsilon(A)$$

holds. Consequently,

$$\lambda(\mathcal{E}(T_\epsilon) - \mathcal{E}(T_*)) + \int_0^\infty \sigma_{\nu}(T_\epsilon)B_{\nu}(T_\epsilon)\, d\nu - \int_0^\infty \sigma_{\nu}(T)B_{\nu}(T_*)\, d\nu + \int_0^\infty (\sigma_{\nu}(T_\epsilon) - \sigma_{\nu}(T_*)) \, d\nu = \int \sigma_{\nu}(T_*) \left( \bar{I} - \bar{I} \right) \, d\nu.$$  \hspace{1cm} (3.17)

With the monotonicity assumptions of the Theorem 1.2,

$$\int \sigma_{\nu}(T_*) \left( \bar{I} - \bar{I} \right) \, d\nu \to 0;$$  \hspace{1cm} (3.18)

and

$$\int |T_\epsilon - T_*| \leq C \int \sigma_{\nu}(T_*) \left( \bar{I} - \bar{I} \right) \, d\nu$$  \hspace{1cm} (3.19)

holds; this is enough to assert that

$$T_\epsilon \to T_* \quad \text{in any } L^p_{\text{loc}}(X), \quad 1 \leq p < +\infty,$$  \hspace{1cm} (3.20)

in other words $T_* = T$. Taking the limit in equation (1.15) leads to

$$\lambda \mathcal{E}(T) + \int_0^\infty \sigma_{\nu}(T)B_{\nu}(T)\, d\nu - \int_0^\infty \sigma_{\nu}(T)\bar{I} \, d\nu = \lambda \mathcal{E}(T^*).$$  \hspace{1cm} (3.21)

FOURTH STEP. Now, we use this convergence to pass to the limit in (1.14). Taking the average with respect to $\Omega$ in equation (1.14); it follows:

$$\lambda \bar{I}_\epsilon + \nabla \cdot \frac{1}{\epsilon} \bar{\Omega} \bar{I}_\epsilon + \sigma_{\nu}(T_\epsilon) \left( \bar{I}_\epsilon - B_{\nu}(T_\epsilon) \right) = \lambda \bar{I}^*.$$  \hspace{1cm} (3.22)

With the convergence of $T_\epsilon \to T$, $\bar{I}_\epsilon \to \bar{I} (= I)$ strongly in $L^p$, since $(I_\epsilon, T_\epsilon)$ converges to the limit $(I, T)$, this leads to

$$\sigma_{\nu}(T_\epsilon)(I_\epsilon - B_{\nu}(T_\epsilon)) \to \sigma_{\nu}(T)(I - B_{\nu}(T)), \quad \text{when } \epsilon \to 0.$$

(3.23)

Next, we multiply equation (1.14) by $\epsilon$, and taking the average in angle; it follows:

$$\frac{\kappa}{\epsilon} \left( I_\epsilon - \bar{I}_\epsilon \right) = -\epsilon \lambda(I_\epsilon - I^*_\epsilon) I_\epsilon - \Omega \cdot \nabla I_\epsilon - \epsilon \sigma_{\nu}(T_\epsilon)(I_\epsilon - B_{\nu}(T_\epsilon)),$$

(3.24)

$$\frac{1}{\epsilon} \bar{\Omega} \bar{I}_\epsilon = -\frac{1}{\kappa} \Omega^{\theta^2} \nabla I_\epsilon - \frac{\epsilon}{\kappa} \left[ \lambda \bar{I}_\epsilon + \sigma_{\nu}(T_\epsilon) \left( \bar{I}_\epsilon - B_{\nu}(T_\epsilon) \right) \right].$$  \hspace{1cm} (3.25)

But since

$$-\frac{1}{\kappa} \Omega^{\theta^2} \nabla I_\epsilon \longrightarrow -\frac{1}{3\kappa} \nabla I$$  \hspace{1cm} (3.26)

and

$$\frac{\epsilon}{\kappa} \left[ \lambda \bar{I}_\epsilon + \sigma_{\nu}(T_\epsilon) \left( \bar{I}_\epsilon - B_{\nu}(T_\epsilon) \right) \right] \rightarrow 0, \quad \text{when } \epsilon \to 0 \text{ in } W^{-1,\infty},$$

this implies $I_\epsilon = I$. Finally, we deduce the convergence of equation (1.14) to

$$\lambda I - \nabla x \cdot \frac{1}{3\kappa} \nabla I + \sigma_{\nu}(T)(I - B_{\nu}(T)) = \lambda \bar{I}^*.$$  \hspace{1cm} (3.27)

This completes the proof of Theorem 1.2.
REFERENCES

1. B. Mercier, Influence de la troncature de l'opacité au voisinage de \( \nu = 0 \) en relaxation rayonnement-matière, (France), \textit{Note CEA 2387}, (January 1984).


