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On the Distributional Fourier Duality and Its Applications

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Sampling theorems for bandlimited functions or distributions are obtained by exploiting the topological isomorphism between the space $\mathscr{E}'(\mathbb{R})$ of distributions of compact support on $\mathbb{R}$ and the Paley–Wiener space PW of entire functions satisfying an estimate of the form $|f(z)| \le A(1+|z|)^N e^{B|\operatorname{Im} z|}$ for some constants $A, B, N \ge 0$. We obtain sampling theorems for f in PW by expanding its Fourier transform T in a series converging in the topology of $\mathscr{E}'(\mathbb{R})$ and whose coefficients are samples taken from f. By Fourier duality, we obtain a sampling theorem for f in the space PW. These sampling expansions converge, in fact, uniformly on compact sets of $\mathbb{C}$, since convergence in the topology of PW implies uniform convergence on compact sets of $\mathbb{C}$. This procedure allows us to recover previous sampling theorems in a unified way. We also present further expansions of Paley–Wiener functions obtained by expanding their Fourier transform as a series involving Legendre or Hermite polynomials.

Key Words: uniform and nonuniform sampling; distributions of compact support; Fourier duality, Paley–Wiener spaces.

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1. INTRODUCTION

The Whittaker–Shannon–Kotel'nikov sampling theorem, hereafter the WSK theorem, states that any function $f \in L^2(\mathbb{R})$, bandlimited to $[-\pi, \pi]$, i.e., such that the support of its Fourier transform is contained in $[-\pi, \pi]$, may be reconstructed from its samples $\{f(n)\}_{n \in \mathbb{Z}}$ on the integers as

$$f(z) = \sum_{n=-\infty}^{\infty} f(n)\operatorname{sinc}(z-n), \tag{1}$$

where sinc denotes the cardinal sine $\operatorname{sinc}(z) = \sin \pi z / \pi z$ (see [15]). The choice of the interval $[-\pi, \pi]$ is arbitrary. The same result applies to any compact interval $[-\pi\sigma, \pi\sigma]$ taking the samples in $\{n/\sigma\}$ and replacing π with π/σ in the cardinal sines.

This theorem and its numerous offsprings have been proved in many different ways, e.g., using Fourier expansions, the Poisson summation formula, contour integrals, etc. (see, for instance, [15] and [6]). But the most elegant proof is probably the one due to Hardy [5], using that the Fourier transform $\mathscr{F}$ is an isometry between $L^2[-\pi, \pi]$ and the Paley–Wiener space $PW_\pi = \{f \in L^2(\mathbb{R}) \cap \mathscr{C}(\mathbb{R}), \operatorname{supp} \hat{f} \subseteq [-\pi, \pi]\}$, where $\hat{f}$ denotes the Fourier transform $\mathscr{F}(f)$ of f. For any $f \in PW_\pi$ one has

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{f}(\omega) e^{i\omega z}\, d\omega,$$

so any value $f(t_n)$ of f is the inner product in $L^2[-\pi, \pi]$ and $\hat{f}$ and the complex exponential $e^{-it_n\omega}$. Furthermore, the classical Paley–Wiener theorem shows that PW_π coincides with the space of entire functions of exponential type at most π whose restriction to the real axis is square integrable, i.e.,

$$PW_\pi = \left\{f \in \mathscr{H}(\mathbb{C}) : |f(z)| \leq Ae^{\pi|z|},\ f|_{\mathbb{R}} \in L^2(\mathbb{R})\right\}. \tag{2}$$

The key point in Hardy's proof is that an expansion converging in $L^2[-\pi, \pi]$ is transformed by $\mathscr{F}^{-1}$ into another expansion which converges in the topology of PW_π. This implies, in particular, that it converges uniformly on compact sets of the complex plane (to be precise, it converges on horizontal strips of $\mathbb{C}$). Choosing the first expansion in such a way that the coefficients are samples of f, or of some function related to f (its derivatives, its Hilbert transform, etc.) provides different sampling theorems for functions in PW_π. This Fourier duality technique can also be applied to the multidimensional case, or to the so-called multiband case of functions whose Fourier transform has support on the union of a finite number of disjoint sets of finite Lebesgue measure (see [6] for more details).

The set of bandlimited functions can be enlarged tremendously if, instead of taking the usual Fourier transform, we consider the Fourier transform in the sense of Schwartz distributions and look for functions whose Fourier transform is a distribution with compact support. This enables us, for instance, to consider complex exponentials multiplied by polynomials, or functions in $L^p(\mathbb{R})$ for $p > 2$. In this case, the functions of the enlarged space, hereafter denoted by PW, are characterized by the Paley−Wiener−Schwartz theorem, which asserts that the distributional Fourier transform

$$\begin{aligned} \mathscr{F}\colon PW &\to \mathscr{E}'(\mathbb{R}), \\ f &\mapsto T = \mathscr{F}(f), \end{aligned} \tag{3}$$

such that $f(z) = (1/2\pi)\langle T, e^{i\omega z}\rangle$, $z \in \mathbb{C}$, is a linear isomorphism between the space $\mathscr{E}'(\mathbb{R})$ of distributions with compact support and the functional space PW whose elements are entire functions satisfying an estimate of the form $|f(z)| \le A(1 + |z|)^N e^{B\,|\mathrm{Im}\, z|}$ for some constants $A, B, N \ge 0$ (see [1, Theorem 1.4.15]).

Many different authors (among them Campbell, Pfaffelhuber, and Lee) have obtained in this more general setting sampling expansions converging either in a functional or in a distributional sense (see [15] for a detailed account of the results and convergences). However, bear in mind that not all functions in PW admit a Shannon expansion (1) converging in a functional sense: the derivative δ_a' of the Dirac delta in a point $a \in (-\pi, \pi)$, for instance, has an inverse Fourier transform $f(z) = ize^{-iaz}$, whose samples $f(n)$ are $O(|n|)$ as $|n| \to \infty$. Thus, the corresponding expansion (1) diverges.

The aim of the present paper is to describe the most general framework in which Fourier duality allows us to recover, in a unified way, some of the aforementioned sampling theorems on PW. Given a function $f \in PW$, we expand its Fourier transform in $\mathscr{E}'(\mathbb{R})$ in such a way that the coefficients are either samples of f or other quantities related to f. Applying the inverse Fourier transform on that expansion leads to another expansion converging to f in PW. The key point is that, according to the Paley−Wiener−Schwartz−Ehrenpreis theorem, one can endow PW with an appropriate topology so that the isomorphism (3) becomes topological. Thus, although we do not have an isometry as in Hardy's setting, we can still transfer convergence back and forth from one space to the other through $\mathscr{F}$ or $\mathscr{F}^{-1}$. Furthermore, we do not need to check for uniform convergence on compact sets, since, as we will see, this is a direct consequence of convergence in the topology of PW. This suggests that this methodology may be useful in further developing sampling theory for entire functions whose Fourier transform is a distribution with compact support.

The paper is organized as follows: Section 2 is devoted to reviewing some basic facts concerning the inductive limit topology on PW associated with certain subspaces PW_n. In particular, we show that convergence in this topology implies uniform convergence on compact sets of $\mathbb{C}$. In Section 3 we make use of Fourier duality to prove four well-known sampling theorems, three in the functional case and one in the distributional one. Finally, we present in Section 4 some further expansions in Paley–Wiener space using inverse Fourier transforms of classical orthogonal polynomials.

2. PRELIMINARIES

As stated in the Introduction, we deal with the space

$$PW = \Big\{ f \in \mathscr{H}(\mathbb{C}) : \exists A, B \geq 0,\ N \in \mathbb{N} \cup \{0\},$$
$$|f(z)| \leq A(1 + |z|)^N e^{B|\operatorname{Im} z|}\ \forall z \in \mathbb{C} \Big\},$$

which, according to the Paley–Wiener–Schwartz theorem, is the image of $\mathscr{E}'(\mathbb{R})$ by the distributional inverse Fourier transform $\mathscr{F}^{-1}$. We describe the topology on PW which makes $\mathscr{F}$ a topological isomorphism. For this, we first consider, for each $n \in \mathbb{N} \cup \{0\}$, the space PW_n of entire functions such that

$$\|f\|_n = \sup_{z \in \mathbb{C}} \left(|f(z)| e^{-n|\operatorname{Im} z|} (1 + |z|)^{-n} \right) < \infty.$$

Each PW_n is a Banach space, endowed with the norm $\|\cdot\|_n$. Furthermore, we have $PW_n \subset PW_{n+1}$ for each n with continuous inclusion and $PW = \bigcup_{n=0}^{\infty} PW_n$. The *inductive limit topology* on PW is defined as the finest locally convex topology under which all the inclusions i_n: $PW_n \to PW$ are continuous (see [10, Chap. V], [7]). Considering this topology in PW, the Paley–Wiener–Schwartz–Ehrenpreis theorem asserts that the Fourier transform is a topological isomorphism between PW and $\mathscr{E}'(\mathbb{R})$ equipped with the strong topology $\beta(\mathscr{E}'(\mathbb{R}), \mathscr{E}(\mathbb{R}))$, i.e., the topology of uniform convergence on weakly bounded sets of $\mathscr{E}(\mathbb{R}) = C^{\infty}(\mathbb{R})$ (see [1, Theorem 1.4.16]).

One can easily see that convergence in a *particular* PW_n implies uniform convergence on compact sets of $\mathbb{C}$. This implies the continuity of every inclusion i_n: $PW_n \to PW$, where PW is equipped with the topology of uniform convergence on compact sets. Now, since the inductive limit topology is by definition the finest one making the i_n continuous, we obtain the following result.

LEMMA 1. *Any sequence in PW converging in the inductive limit topology is also uniformly convergent on compact subsets of* $\mathbb{C}$.

As a consequence, any expansion converging in $\mathscr{E}'(\mathbb{R})$ is automatically transformed by $\mathscr{F}^{-1}$ in an expansion converging uniformly on compact sets.

We conclude with a remark on convergence in $\mathscr{E}'(\mathbb{R})$. Since $C^\infty(\mathbb{R})$ is a Montel space [12, Sect. 34.4], every weakly converging sequence in its strong dual space $\mathscr{E}'(\mathbb{R})$ is also strongly converging. Hence, we have that $T_n \to T$ in $\mathscr{E}'(\mathbb{R})$ if and only if $\langle T_n, \varphi\rangle \to \langle T, \varphi\rangle$ for every $\varphi \in \mathscr{E}(\mathbb{R})$ (see [12, Corollary 2 in Section 34.4]).

3. SAMPLING THEOREMS

We begin this section by recovering classical sampling theorems in the functional case. For this, we use the fact that convergence in $L^2[-\pi, \pi]$ implies convergence in $\mathscr{E}'(\mathbb{R})$, since if $f_n \to 0$ in $L^2[-\pi, \pi]$ and $\varphi \in \mathscr{E}(\mathbb{R})$, one can use the Cauchy–Schwarz inequality to bound

$$\left|\int_{-\pi}^{\pi} f_n \varphi\right| \leq \|f_n\|_2 \|\varphi \chi_{[-\pi, \pi]}\|_2, \tag{4}$$

which goes to zero as $n \to \infty$ ($\chi_{[-\pi, \pi]}$ denotes the characteristic function of the interval $[-\pi, \pi]$).

We now apply the duality argument to prove the WSK theorem and its nonuniform version, the Paley–Wiener–Levinson (hereafter PWL) theorem.

THEOREM 3.1 (WSK theorem). *Let* $f \in PW_\pi$, *then*

$$f(z) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi(z-n)}{\pi(z-n)},$$

where the convergence is uniform on compact sets of $\mathbb{C}$.

Proof. We expand the Fourier transform $\hat{f}$ of f as

$$\hat{f}(\omega) = \sum_{n=-\infty}^{\infty} f(n) e^{-in\omega}$$

in the orthonormal basis $\{e^{-in\omega}\}_{n\in\mathbb{Z}}$ of $L^2[-\pi, \pi]$. This series converges in $\mathscr{E}'(\mathbb{R})$, since it converges in $L^2[-\pi, \pi]$. Taking inverse Fourier transforms, we obtain

$$f(z) = \mathscr{F}^{-1}(\hat{f})(z) = \sum_{n=-\infty}^{\infty} f(n) \mathscr{F}^{-1}(e^{-in\omega})(z)$$

in PW. ∎

Instead of an orthonormal basis, we may use a Riesz basis of complex exponentials $\{e^{-it_n\omega}\}_{n\in\mathbb{Z}}$, where $\{t_n\}_{n\in\mathbb{Z}}\subset\mathbb{R}$. A sufficient condition for this to be a Riesz basis is the so-called Kadec condition

$$\sup_{n\in\mathbb{Z}}|t_n-n|<\tfrac{1}{4} \tag{5}$$

(see [14]). Now, if we denote by $\{g_n(\omega)\}_{n\in\mathbb{Z}}$ the biorthogonal basis of $\{e^{-it_n\omega}\}_{n\in\mathbb{Z}}$, it is a well-known fact that the inverse Fourier transform of g_n is given by

$$\mathscr{F}^{-1}(g_n)(z)=\frac{G(z)}{(z-t_n)G'(t_n)}, \tag{6}$$

where $G(z)$ denotes the infinite product

$$(z-t_0)\prod_{n=1}^{\infty}\left(1-\frac{z}{t_n}\right)\left(1-\frac{z}{t_{-n}}\right) \tag{7}$$

(see [9] and [8]; see also [4] for an alternative proof, using results from [11]). Now we prove the PWL theorem.

THEOREM 3.2 (PWL theorem). *Let $f\in PW_\pi$, let $\{t_n\}_{n\in\mathbb{Z}}\subset\mathbb{R}$ satisfy* (5), *and let $G(z)$ be given by* (7). *Then*

$$f(z)=\sum_{n=-\infty}^{\infty}f(t_n)\frac{G(z)}{(z-t_n)G'(t_n)}$$

uniformly on compact sets of the complex plane.

Proof. We expand the Fourier transform $\hat{f}(\omega)$ of f with respect to $\{g_n(\omega)\}$, the biorthogonal basis of $\{e^{-it_n\omega}\}$. The coefficients of this expansion are the inner products $\langle\hat{f},e^{-it_n\omega}\rangle_{L^2[-\pi,\pi]}$, i.e., the samples $f(t_n)$. Now, since

$$\hat{f}(\omega)=\sum_{n=-\infty}^{\infty}f(t_n)g_n(\omega)$$

in $L^2[-\pi,\pi]$ (and, hence, also in $\mathscr{E}'(\mathbb{R})$), taking inverse Fourier transforms we obtain

$$f(z)=\sum_{n=-\infty}^{\infty}f(t_n)\mathscr{F}^{-1}(g_n)(z)$$

in PW. Thus, the result follows from (6). ∎

We may also use Fourier duality to obtain the following sampling theorem [6, Sect. 6.4] for the Paley–Wiener spaces PW_{π}^{p} of functions whose Fourier transform is in $L^p[-\pi, \pi]$ for $1 < p < 2$ (the case $p \geq 2$ is covered by the WSK theorem, since in that case $L^p[-\pi, \pi] \subseteq L^2[-\pi, \pi]$).

THEOREM 3.3. *Let $f \in PW$ be such that its Fourier transform $\hat{f}$ is in $L^p[-\pi, \pi]$ for $1 < p < 2$. Then*

$$f(z) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi(z-n)}{\pi(z-n)}$$

uniformly on compact sets of the complex plane.

Proof. We may expand $\hat{f}$ in $L^p[-\pi, \pi]$ as

$$\hat{f}(\omega) = \sum_{n \in \mathbb{Z}} c_n e^{-in\omega} \chi_{[-\pi, \pi]},$$

since $\{e^{-in\omega}\}$ is a Schauder basis of $L^p[-\pi, \pi]$ for $1 < p < \infty$ (see [14]).

Now, recall that convergence in $L^p[-\pi, \pi]$ implies convergence in $\mathscr{E}'(\mathbb{R})$ (the argument is the same used to obtain (4) with Hölder's inequality replacing Cauchy–Schwarz's). Thus, making use of standard distributional calculus one readily identifies the coefficients $\{c_n\}$ in the expansion of $\hat{f}$, since

$$\begin{aligned} f(m) &= \frac{1}{2\pi} \langle \hat{f}, e^{im\omega} \rangle = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} c_n \langle e^{-in\omega} \chi_{[-\pi, \pi]}, e^{im\omega} \rangle \\ &= \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} c_n 2\pi \delta_{nm} = c_m, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ stands for the integral from $-\pi$ to π. In this way, we arrive at

$$\hat{f}(\omega) = \sum_{n \in \mathbb{Z}} f(n) e^{-in\omega} \chi_{[-\pi, \pi]}$$

in $\mathscr{E}'(\mathbb{R})$. Applying $\mathscr{F}^{-1}$ leads to the result. ∎

Note that a function f in the conditions of Theorem 3.3 is also in the Bernstein space B_{π}^{q} of entire functions of exponential type at most π whose restriction to the real axis belongs to $L^q(\mathbb{R})$ for $(1/p) + (1/q) = 1$ (see [6] for more details).

In the remainder of this section we focus on sampling theorems for functions of PW which are the inverse Fourier transform of some distribution with compact support on the real line. Let $T \in \mathscr{E}'(\mathbb{R})$ be a distribution with support supp T contained, say, in the open interval $(-\pi, \pi)$ and consider its 2π-periodic extension

$$T_{\mathrm{per}} = T * \Delta_{2\pi},$$

where $\Delta_{2\pi} = \sum_{n=-\infty}^{\infty} \delta_{2n\pi}$ is the Dirac comb with period 2π. Then it is well known [13] that T_{per} is in $\mathscr{S}'(\mathbb{R})$, the space of tempered distributions, and that it admits a Fourier expansion

$$T_{\mathrm{per}} = \sum_{n=-\infty}^{\infty} c_n e^{-in\omega}$$

converging in the topology of $\mathscr{S}'(\mathbb{R})$. Furthermore, the coefficients above are $c_n = (1/2\pi)\langle T, e^{in\omega}\rangle = f(n)$ for $f = \mathscr{F}^{-1}(T)$, and $c_n = O(|n|^p)$ for some $p \in \mathbb{Z}$. To apply our duality argument, however, we need convergence to T in the space $\mathscr{E}'(\mathbb{R})$. To do this we introduce a convergence factor, i.e., a smooth function which is identically 1 on a neighborhood of supp T (note that we may always choose this function to be even, so that its Fourier transform is a real function). Making use of this device, one can prove

LEMMA 2. *Let* $T \in \mathscr{E}'(\mathbb{R})$ *with support contained in the open interval* $(-\pi, \pi)$, $f = \mathscr{F}^{-1}(T) \in PW$, *and* $\hat{\theta} \in \mathscr{D}(\mathbb{R})$ *with* supp $\hat{\theta} \subset (-\pi, \pi)$ *and* $\hat{\theta}(x) \equiv 1$ *on an open set containing* supp T. *Then*

$$T = \sum_{n=-\infty}^{\infty} f(n) e^{-in\omega}\hat{\theta}$$

with convergence in $\mathscr{E}'(\mathbb{R})$.

Proof. First, since $\hat{\theta}(x) \equiv 1$ on an open set containing supp T, one has $\hat{\theta}T_{\mathrm{per}} = T$. Multiplication by $\hat{\theta}$ is a continuous operation in $\mathscr{S}'(\mathbb{R})$, so

$$T = \sum_{n=-\infty}^{\infty} c_n e^{-in\omega}\hat{\theta}$$

in $\mathscr{S}'(\mathbb{R})$. Furthermore, the convergence is also in $\mathscr{E}'(\mathbb{R})$, since for any $\varphi \in \mathscr{E}(\mathbb{R})$ we have $\langle T, \varphi\rangle = \langle \hat{\theta}T_{\mathrm{per}}, \varphi\rangle = \langle T, \hat{\theta}\varphi\rangle$. This implies

$$\langle T, \varphi\rangle = \lim_{N\to\infty} \left\langle \sum_{-N}^{N} c_n e^{-in\omega}, \varphi\hat{\theta} \right\rangle = \lim_{N\to\infty} \left\langle \sum_{-N}^{N} c_n e^{-in\omega}\hat{\theta}, \varphi \right\rangle,$$

which concludes the proof. ∎

We are now in the position of proving Campbell's theorem [2]:

THEOREM 3.4 (Campbell's theorem). *Let $f \in PW$, $T = \mathscr{F}(f)$, and $\hat{\theta} \in \mathscr{D}(\mathbb{R})$ as in the statement of Lemma* 2. *Then*

$$f(z) = \sum_{n=-\infty}^{\infty} f(n)\theta(z-n) \tag{8}$$

with uniform convergence on compact sets of $\mathbb{C}$.

Proof. Applying the inverse Fourier transform to the equality in Lemma 2 we obtain

$$f(z) = \sum_{n=-\infty}^{\infty} f(n)\mathscr{F}^{-1}\big(e^{-in\omega}\hat{\theta}(\omega)\big)(z) = \sum_{n=-\infty}^{\infty} f(n)\theta(z-n) \tag{9}$$

in PW. ∎

We end with two final comments related to expansion (8). The first concerns the convergence factor $\hat{\theta}$ introduced in Lemma 2 to obtain convergence in $\mathscr{E}'(\mathbb{R})$. The price we have to pay for this convergence is that it forces us to oversample, i.e., to sample $f(z)$ at frequency 1, which is larger than the Nyquist frequency $m(\operatorname{supp} T)/2\pi$, where $m(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}$. Note, finally, that (8) lacks the appearance of a cardinal series. However, we may introduce the characteristic function $\chi_{[-\pi,\pi]}$ in (9), obtaining

$$\begin{aligned} f(z) &= \sum_{n=-\infty}^{\infty} f(n)\mathscr{F}^{-1}\Big(e^{-in\omega}\chi_{[-\pi,\pi]}\hat{\theta}(\omega)\Big)(z) \\ &= \sum_{n=-\infty}^{\infty} f(n)\Big[\mathscr{F}^{-1}\Big(e^{-in\omega}\chi_{[-\pi,\pi]}\Big) * \theta\Big](z) \\ &= \sum_{n=-\infty}^{\infty} f(n)(\tau_n \operatorname{sinc} * \theta)(z), \end{aligned}$$

where $\tau_n \operatorname{sinc}(t) = \operatorname{sinc}(t-n)$. Thus, the sampling functions of (8) are given through a convolution involving the inverse Fourier transform of the convergence factor. This convolution structure in the sampling expansion is not unusual and, in fact, has previously appeared in the work of Feichtinger and Gröchenig [3].

4. ORTHOGONAL POLYNOMIAL EXPANSIONS

In the previous section we have made use of complex exponential expansions in $\mathscr{E}'(\mathbb{R})$ as a starting point to apply Fourier duality. However, one may also consider different expansions in $\mathscr{E}'(\mathbb{R})$, involving, for instance, classical families of orthogonal polynomials. We devote this section to present a couple of examples of what kind of expansions may be obtained in this way. It is important to note that these are not sampling theorems, since the coefficients of the series are no longer samples of the Paley–Wiener function.

We may take, for instance, the Legendre polynomials $P_n(x)$, leading to so-called Bessel–Neumann expansions in

$$PW_1 = \left\{ f \in L^2(\mathbb{R}) \cap \mathscr{C}(\mathbb{R}), \operatorname{supp} \hat{f} \subseteq [-1, 1] \right\}.$$

It is well known that $\{\sqrt{n + (1/2)}\, P_n(x)\}$ is an orthonormal basis of $L^2[-1, 1]$ and that

$$\mathscr{F}(\Gamma_n)(x) = P_n(x)\, \chi_{[-1,1]}(x)$$

for any $n \in \mathbb{N} \cup \{0\}$, where

$$\Gamma_n(t) = \frac{i^n}{\sqrt{2\pi}} t^{-1/2} J_{n+1/2}(t) \tag{10}$$

and $J_{n+1/2}(t)$ is the Bessel function of half odd integer order. We first deal with the functional case.

THEOREM 4.1. *Let $f \in PW_1$. Then*

$$f(z) = \sum_{n=0}^{\infty} a_n \frac{i^n}{\sqrt{2\pi}} z^{-1/2} J_{n+1/2}(z) \tag{11}$$

uniformly on compact subsets of $\mathbb{C}$, where

$$a_n = \left(n + \tfrac{1}{2}\right) \langle \hat{f}, P_n \rangle_{L^2[-1,1]}.$$

Proof. We expand the Fourier transform $\hat{f}$ of f as

$$\hat{f}(x) = \sum_{n=0}^{\infty} a_n P_n(x)$$

in the orthonormalized Legendre basis of $L^2[-1, 1]$. The result is obtained applying the inverse Fourier transform. ∎

As in the previous section, the same kind of result may be extended to the distributional case. To obtain an analogue of the Bessel−Neumann expansion (11) for functions whose Fourier transform is a distribution T of $\mathscr{E}'(\mathbb{R})$ with $\operatorname{supp} T \subset (-1,1)$ we need to expand T with respect to the Legendre polynomials with convergence in $\mathscr{E}'(\mathbb{R})$. Again, this forces us to introduce a convergence factor $\hat{\theta}$ as in Section 3.

THEOREM 4.2. *Let* $T \in \mathscr{E}'(\mathbb{R})$ *with* $\operatorname{supp} T \subset (-1,1)$ *and* $\hat{\theta} \in \mathscr{D}(\mathbb{R})$ *with* $\operatorname{supp} \hat{\theta} \subset (-1,1)$ *and* $\hat{\theta}(x) \equiv 1$ *on an open set containing* $\operatorname{supp} T$. *If* $f = \mathscr{F}^{-1}(T)$, *then*

$$f(z) = \sum_{n=0}^{\infty} a_n (\Gamma_n * \theta)(z), \tag{12}$$

with uniform convergence on compact sets of $\mathbb{C}$, *where* $a_n = \langle T, P_n \rangle$ *and* $\Gamma_n(t)$ *is given by* (10).

Proof. We may mirror the proof of [13], Proposition 6.3, to ensure that

$$T = \sum_{n=0}^{\infty} a_n P_n \chi_{[-1,1]} \hat{\theta},$$

with convergence in $\mathscr{E}'(\mathbb{R})$. Again using Fourier duality we obtain (12). ∎

We may also use Hermite functions instead of exponentials. The Fourier transform of the normalized Hermite functions

$$h_n(x) = \frac{H_n(x) e^{-x^2/2}}{\pi^{1/4} 2^{n/2} (n!)^{1/2}}, \qquad n \in \mathbb{N} \cup \{0\},$$

is $\widehat{h_n}(\omega) = (-i)^n \sqrt{2\pi}\, h_n(\omega)$, $n \in \mathbb{N} \cup \{0\}$, where H_n denotes the nth Hermite orthogonal polynomial. Now, the Fourier transform $\hat{f}$ of any $f \in PW_\pi$ may be expanded as

$$\hat{f}(\omega) = \sum_{n=0}^{\infty} a_n h_n \chi_{[-\pi,\pi]}(\omega)$$

converging in $L^2[-\pi,\pi]$ (hence, also in $\mathscr{E}'(\mathbb{R})$), where

$$a_n = \langle \hat{f}, h_n \rangle_{L^2[-\pi,\pi]}. \tag{13}$$

Thus, applying the inverse Fourier transform we obtain

THEOREM 4.3. *Let* $f \in PW_\pi$. *Then*

$$f(z) = \sum_{n=0}^{\infty} \frac{i^n}{\sqrt{2\pi}} a_n (h_n * \operatorname{sinc})(z),$$

with uniform convergence on compact subsets of $\mathbb{C}$, *where* a_n *is given by* (13).

In the distributional case we need, as usual, to introduce a suitable convergence factor $\hat{\theta}$ to achieve convergence in $\mathscr{E}'(\mathbb{R})$ of the Hermite series, which is known to converge in $\mathscr{S}'(\mathbb{R})$ (see [13, Theorem 6.7]).

THEOREM 4.4. *Let* $f \in PW$, $T = \mathscr{F}(f)$ *with* $\operatorname{supp} T \subset (-A, A)$, *and* $\hat{\theta} \in \mathscr{D}(\mathbb{R})$ *with* $\operatorname{supp} \hat{\theta} \subset (-A, A)$ *and* $\hat{\theta}(x) \equiv 1$ *on an open set containing* $\operatorname{supp} T$. *Then*

$$T = \sum_{n=0}^{\infty} a_n h_n \hat{\theta}$$

with convergence in $\mathscr{E}'(\mathbb{R})$, *where* $a_n = \langle T, h_n \rangle$. *Furthermore*,

$$f(z) = \sum_{n=0}^{\infty} \frac{a_n i^n}{\sqrt{2\pi}} (h_n * \theta)(z),$$

with uniform convergence on compact sets of $\mathbb{C}$.

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