J. Math. Anal. Appl. 373 (2011) 376–383

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Stability of multi-additive mappings in non-Archimedean normed spaces

Krzysztof Ciepliński

Institute of Mathematics, Pedagogical University, Podchorążych 2, 30-084 Kraków, Poland

ARTICLE INFO

Article history:
Received 27 April 2010
Available online 3 August 2010
Submitted by R. Curto

Keywords:
Stability
Multi-additive mapping
Non-Archimedean space

ABSTRACT

A function $f : V^n \longrightarrow W$, where V is a commutative semigroup, W is a linear space and $n \geqslant 1$ is an integer, is called multi-additive if it is additive in each variable. In this paper we prove the generalized Hyers–Ulam stability of multi-additive mappings in non-Archimedean normed spaces, using the so-called direct method.

1. Introduction

Throughout this paper $\mathbb{N}$ stands for the set of all positive integers, while $\mathbb{Q}$ denotes the set of all rationals.

Let us recall that a function $f : V^n \longrightarrow W$, where V is a commutative semigroup, W is a linear space and $n \in \mathbb{N}$, is called *multi-additive* or *n-additive* if it is additive (satisfies Cauchy's functional equation) in each variable, that is

$$f\left(x_1, \ldots, x_{i-1}, x_i + x_i', x_{i+1}, \ldots, x_n\right) = f(x_1, \ldots, x_n) + f\left(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n\right),$$
$$i \in \{1, \ldots, n\},\ x_1, \ldots, x_{i-1}, x_i, x_i', x_{i+1}, \ldots, x_n \in V.$$

Some basic facts on such mappings can be found for instance in [25], where their application to the representation of polynomial functions is also presented (see also [27]).

In this paper we prove the generalized Hyers–Ulam stability both of the above system and an equation which characterizes it in non-Archimedean normed spaces, using the so-called direct (Hyers) method.

Let us recall that by a *non-Archimedean field* we mean a field $\mathbb{K}$ (throughout the paper we assume that all considered fields are of characteristics different from 2) equipped with a function (*valuation*) $|\cdot| : \mathbb{K} \longrightarrow [0, \infty)$ such that

$$|r| = 0 \quad \text{if and only if} \quad r = 0,$$
$$|rs| = |r||s|, \quad r, s \in \mathbb{K},$$

and

$$|r + s| \leqslant \max\left\{|r|, |s|\right\}, \quad r, s \in \mathbb{K}.$$

In any non-Archimedean field we have $|1| = |-1| = 1$ and $|n| \leqslant 1$ for all $n \in \mathbb{N}$.

Let X be a linear space over a field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$. A function $\|\cdot\| : X \longrightarrow [0, \infty)$ is said to be a *non-Archimedean norm* if it satisfies the following conditions:

E-mail address: kc@up.krakow.pl.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2010.07.048

$$\|x\| = 0 \quad \text{if and only if} \quad x = 0,$$

$$\|rx\| = |r|\|x\|, \quad r \in \mathbb{K},\ x \in X,$$

and

$$\|x + y\| \leqslant \max\{\|x\|, \|y\|\}, \quad x, y \in X.$$

Then $(X, \|\cdot\|)$ is called a *non-Archimedean space*. In any such a space a sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy if and only if $(x_{n+1} - x_n)_{n\in\mathbb{N}}$ converges to zero. By a *complete non-Archimedean space* we mean one in which every Cauchy sequence is convergent.

In 1899, K. Hensel (see [20]) discovered the p-adic numbers as a number theoretical analogue of power series in complex analysis. Fix a prime number p. For any nonzero rational number x, there exists a unique integer n_x such that $x = \frac{a}{b}p^{n_x}$, where a and b are integers not divisible by p. Then $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x, y) = |x - y|_p$ is denoted by $\mathbb{Q}_p$, and it is called the *p-adic number field*. In fact, $\mathbb{Q}_p$ is the set of all formal series $x = \sum_{k\geqslant n_x}^{\infty} a^k p_k$, where $|a_k| \leqslant p - 1$ are integers. The addition and multiplication between any two elements of $\mathbb{Q}_p$ are defined naturally. The norm $|\sum_{k\geqslant n_x}^{\infty} a^k p_k|_p = p^{-n_x}$ is a non-Archimedean norm on $\mathbb{Q}_p$ and it makes $\mathbb{Q}_p$ a locally compact field (see for instance [18] and [37]).

During the last three decades p-adic numbers have gained the interest of physicists for their research in particular in problems coming from quantum physics, p-adic strings and superstrings (see for instance [24]).

Speaking of the stability of a functional equation we follow the question raised in 1940 by S.M. Ulam: "when is it true that the solution of an equation differing slightly from a given one, must of necessity be close to the solution of the given equation?".

The first answer (in the case of Cauchy's functional equation in Banach spaces) to Ulam's question gave D.H. Hyers (see [21]). After his result a great number of papers (see for instance [6,12,13,22,23,26,35] and the references given there) on the subject have been published, generalizing Ulam's problem and Hyers's theorem in various directions and to other functional equations (as the words "differing slightly" and "be close" may have various meanings, different kinds of stability can be dealt with).

The classical works on the stability of additive mappings are [21,2,4,34,15,36,5,16], whereas some recent results on this topic can be found in [17,7,31,28,29]. On the other hand, for some outcomes on stability of multi-additive functions we refer the reader to [1,3,14,11] (see also [25]).

In 2007, M.S. Moslehian and Th.M. Rassias (see [31]) proved the generalized Hyers–Ulam stability of the Cauchy and quadratic functional equations in non-Archimedean normed spaces. After their results some papers (see for instance [19,30, 32,10]) on the stability of other equations in such spaces have been published.

In this paper we deal with the generalized Hyers–Ulam stability, in the spirit of D.G. Bourgin (see [4]) and P. Găvruţă (see [16]), of multi-additive mappings in non-Archimedean normed spaces.

To finish this introductory section let us finally mention some studies of W. Prager and J. Schwaiger (see [33]) and the author (see [8–10]) concerning different kinds of stability of multi-Jensen functions (introduced by W. Prager and J. Schwaiger with the connection with generalized polynomials), that is functions satisfying (under some additional assumptions on V) Jensen's functional equation in each variable.

2. Results

First, we prove the stability of the system of equations defining the multi-additive mapping.

Theorem 1. *Let V be a commutative semigroup and W be a complete non-Archimedean space. Assume also that $n \in \mathbb{N}$ and for every $i \in \{1, \ldots, n\}$, $\varphi_i : V^{n+1} \longrightarrow [0, \infty)$ is a mapping such that for each $(x_1, \ldots, x_{n+1}) \in V^{n+1}$,*

$$\begin{aligned}\lim_{j\longrightarrow\infty} \frac{1}{|2|^j}\varphi_i\big(2^j x_1, x_2, \ldots, x_{n+1}\big) = \cdots &= \lim_{j\longrightarrow\infty} \frac{1}{|2|^j}\varphi_i\big(x_1, \ldots, x_{i-2}, 2^j x_{i-1}, x_i, \ldots, x_{n+1}\big) \\ &= \lim_{j\longrightarrow\infty} \frac{1}{|2|^j}\varphi_i\big(x_1, \ldots, x_{i-1}, 2^j x_i, 2^j x_{i+1}, x_{i+2}, \ldots, x_{n+1}\big) \\ &= \lim_{j\longrightarrow\infty} \frac{1}{|2|^j}\varphi_i\big(x_1, \ldots, x_{i+1}, 2^j x_{i+2}, x_{i+3}, \ldots, x_{n+1}\big) = \cdots \\ &= \lim_{j\longrightarrow\infty} \frac{1}{|2|^j}\varphi_i\big(x_1, \ldots, x_n, 2^j x_{n+1}\big) = 0, \qquad (1)\end{aligned}$$

and the limit

$$\lim_{k\longrightarrow\infty} \max\left\{\frac{1}{|2|^j}\varphi_i\big(x_1, \ldots, x_{i-1}, 2^j x_i, 2^j x_i, x_{i+1}, \ldots, x_n\big):\ 0 \leqslant j < k\right\}, \qquad (2)$$

denoted by $\widetilde{\varphi}_i(x_1, \dots, x_n)$, exists. If $f : V^n \longrightarrow W$ is a function satisfying

$$\begin{aligned} &\left\| f\left(x_1, \dots, x_{i-1}, x_i + x_i', x_{i+1}, \dots, x_n\right) - f(x_1, \dots, x_n) - f\left(x_1, \dots, x_{i-1}, x_i', x_{i+1}, \dots, x_n\right)\right\| \\ &\quad \leqslant \varphi_i\left(x_1, \dots, x_i, x_i', x_{i+1}, \dots, x_n\right), \quad \left(x_1, \dots, x_i, x_i', x_{i+1}, \dots, x_n\right) \in V^{n+1},\ i \in \{1, \dots, n\}, \end{aligned} \tag{3}$$

then for every $i \in \{1, \dots, n\}$ there exists a multi-additive mapping $F_i : V^n \longrightarrow W$ for which

$$\left\| f(x_1, \dots, x_n) - F_i(x_1, \dots, x_n)\right\| \leqslant \frac{1}{|2|}\widetilde{\varphi}_i(x_1, \dots, x_n), \quad (x_1, \dots, x_n) \in V^n. \tag{4}$$

For every $i \in \{1, \dots, n\}$ the function F_i is given by

$$F_i(x_1, \dots, x_n) := \lim_{j \longrightarrow \infty} \frac{1}{2^j} f\left(x_1, \dots, x_{i-1}, 2^j x_i, x_{i+1}, \dots, x_n\right), \quad (x_1, \dots, x_n) \in V^n. \tag{5}$$

If, moreover,

$$\begin{aligned} &\lim_{l \longrightarrow \infty} \lim_{k \longrightarrow \infty} \max\left\{\frac{1}{|2|^j}\varphi_i\left(x_1, \dots, x_{i-1}, 2^j x_i, 2^j x_i, x_{i+1}, \dots, x_n\right):\ l \leqslant j < k + l\right\} = 0, \\ &\quad i \in \{1, \dots, n\},\ (x_1, \dots, x_n) \in V^n, \end{aligned} \tag{6}$$

then for every $i \in \{1, \dots, n\}$, F_i is the unique multi-additive mapping satisfying condition (4)*.*

Proof. Fix $x_1, \dots, x_n \in V$, $j \in \mathbb{N} \cup \{0\}$ and $i \in \{1, \dots, n\}$. Putting $x_i' := x_i$ in (3) we get

$$\left\| \frac{1}{2} f(x_1, \dots, x_{i-1}, 2x_i, x_{i+1}, \dots, x_n) - f(x_1, \dots, x_n)\right\| \leqslant \frac{1}{|2|}\varphi_i(x_1, \dots, x_i, x_i, x_{i+1}, \dots, x_n). \tag{7}$$

Hence

$$\begin{aligned} &\left\| \frac{1}{2^{j+1}} f\left(x_1, \dots, x_{i-1}, 2^{j+1} x_i, x_{i+1}, \dots, x_n\right) - \frac{1}{2^j} f\left(x_1, \dots, x_{i-1}, 2^j x_i, x_{i+1}, \dots, x_n\right)\right\| \\ &\quad \leqslant \frac{1}{|2|^{j+1}}\varphi_i\left(x_1, \dots, x_{i-1}, 2^j x_i, 2^j x_i, x_{i+1}, \dots, x_n\right), \end{aligned}$$

and consequently from (1) it follows that $(\frac{1}{2^j} f(x_1, \dots, x_{i-1}, 2^j x_i, x_{i+1}, \dots, x_n))_{j \in \mathbb{N}}$ is a Cauchy sequence. Since the space W is complete, this sequence is convergent and we define $F_i : V^n \longrightarrow W$ by (5).

Using (7) and induction one can show that for any $k \in \mathbb{N}$ we have

$$\begin{aligned} &\left\| \frac{1}{2^k} f\left(x_1, \dots, x_{i-1}, 2^k x_i, x_{i+1}, \dots, x_n\right) - f(x_1, \dots, x_n)\right\| \\ &\quad \leqslant \frac{1}{|2|} \max\left\{\frac{1}{|2|^j}\varphi_i\left(x_1, \dots, x_{i-1}, 2^j x_i, 2^j x_i, x_{i+1}, \dots, x_n\right):\ 0 \leqslant j < k\right\}. \end{aligned}$$

Letting $k \longrightarrow \infty$ in this inequality and using (2) we see that (4) holds.

Now, fix also $x_i' \in V$ and note that according to (3) we have

$$\begin{aligned} &\left\| \frac{1}{2^j} f\left(x_1, \dots, x_{i-1}, 2^j\left(x_i + x_i'\right), x_{i+1}, \dots, x_n\right) - \frac{1}{2^j} f\left(x_1, \dots, x_{i-1}, 2^j x_i, x_{i+1}, \dots, x_n\right) \right. \\ &\qquad \left. - \frac{1}{2^j} f\left(x_1, \dots, x_{i-1}, 2^j x_i', x_{i+1}, \dots, x_n\right)\right\| \\ &\quad \leqslant \frac{1}{|2|^j}\varphi_i\left(x_1, \dots, x_{i-1}, 2^j x_i, 2^j x_i', x_{i+1}, \dots, x_n\right). \end{aligned}$$

Next, fix $k \in \{1, \dots, n\} \setminus \{i\}$, $x_k' \in V$ and assume that $k < i$ (the same arguments apply to the case where $k > i$). From (3) it follows that

$$\begin{aligned} &\left\| \frac{1}{2^j} f\left(x_1, \dots, x_{k-1}, x_k + x_k', x_{k+1}, \dots, x_{i-1}, 2^j x_i, x_{i+1}, \dots, x_n\right) - \frac{1}{2^j} f\left(x_1, \dots, x_{i-1}, 2^j x_i, x_{i+1}, \dots, x_n\right) \right. \\ &\qquad \left. - \frac{1}{2^j} f\left(x_1, \dots, x_{k-1}, x_k', x_{k+1}, \dots, x_{i-1}, 2^j x_i, x_{i+1}, \dots, x_n\right)\right\| \\ &\quad \leqslant \frac{1}{|2|^j}\varphi_k\left(x_1, \dots, x_k, x_k', x_{k+1}, \dots, x_{i-1}, 2^j x_i, x_{i+1}, \dots, x_n\right). \end{aligned}$$

Letting $j \longrightarrow \infty$ in the above two inequalities and using (1) we see that the mapping F_i is multi-additive.

Let us finally assume, moreover, that (6) holds and let $F_i' : V^n \longrightarrow W$ be multi-additive mapping satisfying condition (4). Then

$$\begin{aligned}
&\|F_i(x_1,\dots,x_n) - F_i'(x_1,\dots,x_n)\| \\
&= \lim_{l\longrightarrow\infty} \frac{1}{|2|^l} \|F_i(x_1,\dots,x_{i-1},2^l x_i,x_{i+1},\dots,x_n) - F_i'(x_1,\dots,x_{i-1},2^l x_i,x_{i+1},\dots,x_n)\| \\
&\leqslant \lim_{l\longrightarrow\infty} \frac{1}{|2|^l} \max\{\|F_i(x_1,\dots,x_{i-1},2^l x_i,x_{i+1},\dots,x_n) - f(x_1,\dots,x_{i-1},2^l x_i,x_{i+1},\dots,x_n)\|, \\
&\quad \|f(x_1,\dots,x_{i-1},2^l x_i,x_{i+1},\dots,x_n) - F_i'(x_1,\dots,x_{i-1},2^l x_i,x_{i+1},\dots,x_n)\|\} \\
&\leqslant \lim_{l\longrightarrow\infty} \frac{1}{|2|^{l+1}} \widetilde{\varphi}_i(x_1,\dots,x_{i-1},2^l x_i,x_{i+1},\dots,x_n) \\
&= \lim_{l\longrightarrow\infty} \frac{1}{|2|} \lim_{k\longrightarrow\infty} \max\left\{\frac{1}{|2|^{j+l}} \varphi_i(x_1,\dots,x_{i-1},2^{j+l}x_i,2^{j+l}x_i,x_{i+1},\dots,x_n)\colon 0 \leqslant j < k\right\} \\
&= \frac{1}{|2|} \lim_{l\longrightarrow\infty} \lim_{k\longrightarrow\infty} \max\left\{\frac{1}{|2|^j} \varphi_i(x_1,\dots,x_{i-1},2^j x_i,2^j x_i,x_{i+1},\dots,x_n)\colon l \leqslant j < k+l\right\} = 0,
\end{aligned}$$

and therefore $F_i' = F_i$. □

Analysis similar to that in the proof of Theorem 1 gives the following:

Theorem 2. *Let V be a linear space and W be a complete non-Archimedean space. Assume also that $n \in \mathbb{N}$ and for every $i \in \{1,\dots,n\}$, $\varphi_i : V^{n+1} \longrightarrow [0,\infty)$ is a mapping such that for each $(x_1,\dots,x_{n+1}) \in V^{n+1}$,*

$$\begin{aligned}
&\lim_{j\longrightarrow\infty} |2|^j \varphi_i\left(\frac{x_1}{2^j},x_2,\dots,x_{n+1}\right) = \cdots \\
&= \lim_{j\longrightarrow\infty} |2|^j \varphi_i\left(x_1,\dots,x_{i-2},\frac{x_{i-1}}{2^j},x_i,\dots,x_{n+1}\right) = \lim_{j\longrightarrow\infty} |2|^j \varphi_i\left(x_1,\dots,x_{i-1},\frac{x_i}{2^j},\frac{x_{i+1}}{2^j},x_{i+2},\dots,x_{n+1}\right) \\
&= \lim_{j\longrightarrow\infty} |2|^j \varphi_i\left(x_1,\dots,x_{i+1},\frac{x_{i+2}}{2^j},x_{i+3},\dots,x_{n+1}\right) = \cdots = \lim_{j\longrightarrow\infty} |2|^j \varphi_i\left(x_1,\dots,x_n,\frac{x_{n+1}}{2^j}\right) = 0,
\end{aligned}$$

and the limit

$$\lim_{k\longrightarrow\infty} \max\left\{|2|^{j+1} \varphi_i\left(x_1,\dots,x_{i-1},\frac{x_i}{2^{j+1}},\frac{x_i}{2^{j+1}},x_{i+1},\dots,x_n\right)\colon 0 \leqslant j < k\right\},$$

denoted by $\widetilde{\varphi}_i(x_1,\dots,x_n)$, exists. If $f : V^n \longrightarrow W$ is a function satisfying (3), then for every $i \in \{1,\dots,n\}$ there exists a multi-additive mapping $F_i : V^n \longrightarrow W$ for which (4) holds. For every $i \in \{1,\dots,n\}$ the function F_i is given by

$$F_i(x_1,\dots,x_n) := \lim_{j\longrightarrow\infty} 2^j f\left(x_1,\dots,x_{i-1},\frac{x_i}{2^j},x_{i+1},\dots,x_n\right), \quad (x_1,\dots,x_n) \in V^n. \tag{8}$$

If, moreover,

$$\begin{aligned}
&\lim_{l\longrightarrow\infty} \lim_{k\longrightarrow\infty} \max\left\{|2|^j \varphi_i\left(x_1,\dots,x_{i-1},\frac{x_i}{2^j},\frac{x_i}{2^j},x_{i+1},\dots,x_n\right)\colon l+1 \leqslant j < k+l+1\right\} = 0, \\
&\quad i \in \{1,\dots,n\},\ (x_1,\dots,x_n) \in V^n,
\end{aligned}$$

then for every $i \in \{1,\dots,n\}$, F_i is the unique multi-additive mapping satisfying condition (4).

Corollary 1. *Let V be a normed space and W be a complete non-Archimedean space over a non-Archimedean field with $|2| < 1$. Assume also that $\delta > 0$, $n \in \mathbb{N}$ and $\rho : [0,\infty) \longrightarrow [0,\infty)$ is a mapping such that $\rho(\frac{1}{|2|}) < \frac{1}{|2|}$ and*

$$\rho\left(\frac{1}{|2|}t\right) \leqslant \rho\left(\frac{1}{|2|}\right)\rho(t), \quad t \in [0,\infty). \tag{9}$$

If $f : V^n \longrightarrow W$ is a function satisfying

$$\begin{aligned}
&\|f(x_1,\dots,x_{i-1},x_i+x_i',x_{i+1},\dots,x_n) - f(x_1,\dots,x_n) - f(x_1,\dots,x_{i-1},x_i',x_{i+1},\dots,x_n)\| \leqslant \delta(\rho(\|x_i\|) + \rho(\|x_i'\|)), \\
&\quad (x_1,\dots,x_i,x_i',x_{i+1},\dots,x_n) \in V^{n+1},\ i \in \{1,\dots,n\},
\end{aligned}$$

then for every $i \in \{1, \dots, n\}$ there exists a unique multi-additive mapping $F_i : V^n \longrightarrow W$ for which

$$\| f(x_1, \dots, x_n) - F_i(x_1, \dots, x_n) \| \leqslant \frac{2}{|2|}\delta\rho(\|x_i\|), \quad (x_1, \dots, x_n) \in V^n.$$

For every $i \in \{1, \dots, n\}$ the function F_i is given by (8)*.*

Proof. For every $i \in \{1, \dots, n\}$ put

$$\varphi_i(x_1, \dots, x_{n+1}) := \delta(\rho(\|x_i\|) + \rho(\|x_{i+1}\|)), \quad (x_1, \dots, x_{n+1}) \in V^{n+1}.$$

Fix $i \in \{1, \dots, n\}$ and $(x_1, \dots, x_{n+1}) \in V^{n+1}$. Then

$$\begin{aligned}\lim_{j \longrightarrow \infty} |2|^j \varphi_i\left(\frac{x_1}{2^j}, x_2, \dots, x_{n+1}\right) = \dots &= \lim_{j \longrightarrow \infty} |2|^j \varphi_i\left(x_1, \dots, x_{i-2}, \frac{x_{i-1}}{2^j}, x_i, \dots, x_{n+1}\right) \\ &= \lim_{j \longrightarrow \infty} |2|^j \varphi_i\left(x_1, \dots, x_{i+1}, \frac{x_{i+2}}{2^j}, x_{i+3}, \dots, x_{n+1}\right) = \cdots \\ &= \lim_{j \longrightarrow \infty} |2|^j \varphi_i\left(x_1, \dots, x_n, \frac{x_{n+1}}{2^j}\right) = \lim_{j \longrightarrow \infty} |2|^j \varphi_i(x_1, \dots, x_{n+1}) = 0\end{aligned}$$

and

$$\lim_{j \longrightarrow \infty} |2|^j \varphi_i\left(x_1, \dots, x_{i-1}, \frac{x_i}{2^j}, \frac{x_{i+1}}{2^j}, x_{i+2}, \dots, x_{n+1}\right) \leqslant \lim_{j \longrightarrow \infty} \left(|2|\rho\left(\frac{1}{|2|}\right)\right)^j \varphi_i(x_1, \dots, x_{n+1}) = 0.$$

Next, note that the sequence

$$\left(|2|^j \varphi_i\left(x_1, \dots, x_{i-1}, \frac{x_i}{2^j}, \frac{x_i}{2^j}, x_{i+1}, \dots, x_n\right)\right)_{j \in \mathbb{N} \cup \{0\}}$$

is decreasing and therefore

$$\begin{aligned}&\lim_{k \longrightarrow \infty} \max\left\{|2|^{j+1} \varphi_i\left(x_1, \dots, x_{i-1}, \frac{x_i}{2^{j+1}}, \frac{x_i}{2^{j+1}}, x_{i+1}, \dots, x_n\right):\ 0 \leqslant j < k\right\} \\ &\quad = |2|\varphi_i\left(x_1, \dots, x_{i-1}, \frac{x_i}{2}, \frac{x_i}{2}, x_{i+1}, \dots, x_n\right)\end{aligned}$$

and

$$\begin{aligned}&\lim_{l \longrightarrow \infty} \lim_{k \longrightarrow \infty} \max\left\{|2|^j \varphi_i\left(x_1, \dots, x_{i-1}, \frac{x_i}{2^j}, \frac{x_i}{2^j}, x_{i+1}, \dots, x_n\right):\ l+1 \leqslant j < k+l+1\right\} \\ &\quad = \lim_{l \longrightarrow \infty} |2|^{l+1} \varphi_i\left(x_1, \dots, x_{i-1}, \frac{x_i}{2^{l+1}}, \frac{x_i}{2^{l+1}}, x_{i+1}, \dots, x_n\right) = 0.\end{aligned}$$

To get our assertion it is sufficient to use Theorem 2. □

Example 1. Let $|2| < 1$ and $p \in (0, 1)$. Then the mapping $\rho : [0, \infty) \longrightarrow [0, \infty)$ given by

$$\rho(t) := t^p, \quad t \in [0, \infty), \tag{10}$$

satisfies (9) and $\rho(\frac{1}{|2|}) < \frac{1}{|2|}$.

The below proposition reduces the original system of n Cauchy equations to a single functional equation.

Proposition 1. *(See [11].) Assume that $n \in \mathbb{N}$ and let V be a commutative group and W be a linear space. A mapping $f : V^n \longrightarrow W$ is multi-additive if and only if*

$$f(x_{11} + x_{12}, \dots, x_{n1} + x_{n2}) = \sum_{1 \leqslant i_1, \dots, i_n \leqslant 2} f(x_{1i_1}, \dots, x_{ni_n}), \quad (x_{11}, \dots, x_{n1}), (x_{12}, \dots, x_{n2}) \in V^n. \tag{11}$$

Now, we prove the stability of Eq. (11).

Theorem 3. *Let V be a commutative group and W be a complete non-Archimedean space. Assume also that $n \in \mathbb{N}$ and $\varphi : V^{2n} \longrightarrow [0, \infty)$ is a mapping such that for each $(x_{11}, x_{12}, \dots, x_{n1}, x_{n2}) \in V^{2n}$,*

$$\lim_{j \longrightarrow \infty} \frac{1}{|2|^{jn}} \varphi\left(2^j x_{11}, 2^j x_{12}, \dots, 2^j x_{n1}, 2^j x_{n2}\right) = 0 \tag{12}$$

and the limit

$$\lim_{k \longrightarrow \infty} \max\left\{ \frac{1}{|2|^{jn}} \varphi\left(2^j x_{11}, 2^j x_{11}, \dots, 2^j x_{n1}, 2^j x_{n1}\right) \colon 0 \leqslant j < k \right\}, \tag{13}$$

denoted by $\widetilde{\varphi}(x_{11}, \dots, x_{n1})$, exists. If $f : V^n \longrightarrow W$ is a function satisfying

$$\left\| f(x_{11} + x_{12}, \dots, x_{n1} + x_{n2}) - \sum_{1 \leqslant i_1, \dots, i_n \leqslant 2} f(x_{1i_1}, \dots, x_{ni_n}) \right\| \leqslant \varphi(x_{11}, x_{12}, \dots, x_{n1}, x_{n2}),$$
$$(x_{11}, x_{12}, \dots, x_{n1}, x_{n2}) \in V^{2n}, \tag{14}$$

then there exists a multi-additive mapping $F : V^n \longrightarrow W$ for which

$$\left\| f(x_{11}, \dots, x_{n1}) - F(x_{11}, \dots, x_{n1}) \right\| \leqslant \frac{1}{|2|^n} \widetilde{\varphi}(x_{11}, \dots, x_{n1}), \quad (x_{11}, \dots, x_{n1}) \in V^n. \tag{15}$$

The function F is given by

$$F(x_{11}, \dots, x_{n1}) := \lim_{j \longrightarrow \infty} \frac{1}{2^{nj}} f\left(2^j x_{11}, \dots, 2^j x_{n1}\right), \quad (x_{11}, \dots, x_{n1}) \in V^n. \tag{16}$$

If, moreover,

$$\lim_{l \longrightarrow \infty} \lim_{k \longrightarrow \infty} \max\left\{ \frac{1}{|2|^{jn}} \varphi\left(2^j x_{11}, 2^j x_{11}, \dots, 2^j x_{n1}, 2^j x_{n1}\right) \colon l \leqslant j < k + l \right\} = 0, \quad (x_{11}, \dots, x_{n1}) \in V^n, \tag{17}$$

then F is the unique multi-additive mapping satisfying condition (15).

Proof. Fix $(x_{11}, \dots, x_{n1}) \in V^n$ and $j \in \mathbb{N} \cup \{0\}$. Putting $x_{i2} := x_{i1}$ for $i \in \{1, \dots, n\}$ in (14) we get

$$\left\| \frac{1}{2^n} f(2x_{11}, \dots, 2x_{n1}) - f(x_{11}, \dots, x_{n1}) \right\| \leqslant \frac{1}{|2|^n} \varphi(x_{11}, x_{11}, \dots, x_{n1}, x_{n1}). \tag{18}$$

Hence

$$\left\| \frac{1}{2^{n(j+1)}} f\left(2^{j+1} x_{11}, \dots, 2^{j+1} x_{n1}\right) - \frac{1}{2^{nj}} f\left(2^j x_{11}, \dots, 2^j x_{n1}\right) \right\| \leqslant \frac{1}{|2|^{n(j+1)}} \varphi\left(2^j x_{11}, 2^j x_{11}, \dots, 2^j x_{n1}, 2^j x_{n1}\right),$$

and consequently from (12) it follows that $(\frac{1}{2^{nj}} f(2^j x_{11}, \dots, 2^j x_{n1}))_{j \in \mathbb{N}}$ is a Cauchy sequence. Since the space W is complete, this sequence is convergent and we define $F : V^n \longrightarrow W$ by (16).

Using (18) and induction one can show that for any $k \in \mathbb{N}$ we have

$$\left\| \frac{1}{2^{kn}} f\left(2^k x_{11}, \dots, 2^k x_{n1}\right) - f(x_{11}, \dots, x_{n1}) \right\| \leqslant \frac{1}{|2|^n} \max\left\{ \frac{1}{|2|^{jn}} \varphi\left(2^j x_{11}, 2^j x_{11}, \dots, 2^j x_{n1}, 2^j x_{n1}\right) \colon 0 \leqslant j < k \right\}.$$

Letting $k \longrightarrow \infty$ in this inequality and using (13) we see that (15) holds.

Now, fix also $(x_{12}, \dots, x_{n2}) \in V^n$ and note that according to (14) we have

$$\left\| \frac{1}{2^{nj}} f\left(2^j (x_{11} + x_{12}), \dots, 2^j (x_{n1} + x_{n2})\right) - \sum_{1 \leqslant i_1, \dots, i_n \leqslant 2} \frac{1}{2^{nj}} f\left(2^j x_{1i_1}, \dots, 2^j x_{ni_n}\right) \right\|$$
$$\leqslant \frac{1}{|2|^{nj}} \varphi\left(2^j x_{11}, 2^j x_{12}, \dots, 2^j x_{n1}, 2^j x_{n2}\right).$$

Letting $j \longrightarrow \infty$ in the above inequality and using (12) we see that the mapping F satisfies Eq. (11), and Proposition 1 now shows that F is multi-additive.

Let us finally assume, moreover, that (17) holds and let $F' : V^n \longrightarrow W$ be a multi-additive mapping satisfying condition (15). Then

$$\begin{aligned}
&\left\|F(x_{11},\dots,x_{n1}) - F'(x_{11},\dots,x_{n1})\right\| \\
&\quad= \lim_{l\longrightarrow\infty} \frac{1}{|2|^{ln}} \left\|F\left(2^l x_{11},\dots,2^l x_{n1}\right) - F'\left(2^l x_{11},\dots,2^l x_{n1}\right)\right\| \\
&\quad\leqslant \lim_{l\longrightarrow\infty} \frac{1}{|2|^{ln}} \max\left\{\left\|F\left(2^l x_{11},\dots,2^l x_{n1}\right) - f\left(2^l x_{11},\dots,2^l x_{n1}\right)\right\|, \left\|f\left(2^l x_{11},\dots,2^l x_{n1}\right) - F'\left(2^l x_{11},\dots,2^l x_{n1}\right)\right\|\right\} \\
&\quad\leqslant \lim_{l\longrightarrow\infty} \frac{1}{|2|^{(l+1)n}} \widetilde{\varphi}\left(2^l x_{11},\dots,2^l x_{n1}\right) \\
&\quad= \lim_{l\longrightarrow\infty} \frac{1}{|2|^{n}} \lim_{k\longrightarrow\infty} \max\left\{\frac{1}{|2|^{(j+l)n}} \varphi\left(2^{j+l} x_{11}, 2^{j+l} x_{11},\dots,2^{j+l} x_{n1}, 2^{j+l} x_{n1}\right)\colon 0\leqslant j<k\right\} \\
&\quad= \frac{1}{|2|^{n}} \lim_{l\longrightarrow\infty} \lim_{k\longrightarrow\infty} \max\left\{\frac{1}{|2|^{jn}} \varphi\left(2^{j} x_{11}, 2^{j} x_{11},\dots,2^{j} x_{n1}, 2^{j} x_{n1}\right)\colon l\leqslant j<k+l\right\} = 0,
\end{aligned}$$

and therefore $F' = F$. □

Corollary 2. *Let V be a normed space and W be a complete non-Archimedean space. Assume also that $\delta > 0$, $n \in \mathbb{N}$ and $\rho : [0,\infty) \longrightarrow [0,\infty)$ is a mapping such that $\rho(|2|) < |2|^n$ and*

$$\rho\left(|2|t\right) \leqslant \rho\left(|2|\right)\rho(t), \quad t \in [0,\infty). \tag{19}$$

If $f : V^n \longrightarrow W$ is a function satisfying

$$\left\| f(x_{11}+x_{12},\dots,x_{n1}+x_{n2}) - \sum_{1\leqslant i_1,\dots,i_n\leqslant 2} f(x_{1i_1},\dots,x_{ni_n}) \right\| \leqslant \delta \sum_{i=1}^{n}\sum_{j=1}^{2} \rho\left(\|x_{ij}\|\right), \quad (x_{11},x_{12},\dots,x_{n1},x_{n2}) \in V^{2n},$$

then there exists a unique multi-additive mapping $F : V^n \longrightarrow W$ for which

$$\left\| f(x_{11},\dots,x_{n1}) - F(x_{11},\dots,x_{n1}) \right\| \leqslant \frac{2}{|2|^n}\delta \sum_{i=1}^{n} \rho\left(\|x_{i1}\|\right), \quad (x_{11},\dots,x_{n1}) \in V^n.$$

The function F is given by (16).

Proof. Put

$$\varphi(x_{11},x_{12},\dots,x_{n1},x_{n2}) := \delta \sum_{i=1}^{n}\sum_{j=1}^{2} \rho\left(\|x_{ij}\|\right), \quad (x_{11},x_{12},\dots,x_{n1},x_{n2}) \in V^{2n}.$$

Fix $(x_{11},x_{12},\dots,x_{n1},x_{n2}) \in V^{2n}$. Then

$$\lim_{j\longrightarrow\infty} \frac{1}{|2|^{jn}} \varphi\left(2^j x_{11}, 2^j x_{12},\dots,2^j x_{n1}, 2^j x_{n2}\right) \leqslant \lim_{j\longrightarrow\infty} \left(\frac{\rho(|2|)}{|2|^n}\right)^j \varphi(x_{11},x_{12},\dots,x_{n1},x_{n2}) = 0.$$

Next, note that the sequence

$$\left(\frac{1}{|2|^{jn}} \varphi\left(2^j x_{11}, 2^j x_{11},\dots,2^j x_{n1}, 2^j x_{n1}\right)\right)_{j\in\mathbb{N}\cup\{0\}}$$

is decreasing and therefore

$$\lim_{k\longrightarrow\infty} \max\left\{\frac{1}{|2|^{jn}} \varphi\left(2^j x_{11}, 2^j x_{11},\dots,2^j x_{n1}, 2^j x_{n1}\right)\colon 0\leqslant j<k\right\} = \varphi(x_{11},x_{11},\dots,x_{n1},x_{n1})$$

and

$$\begin{aligned}
&\lim_{l\longrightarrow\infty} \lim_{k\longrightarrow\infty} \max\left\{\frac{1}{|2|^{jn}} \varphi\left(2^j x_{11}, 2^j x_{11},\dots,2^j x_{n1}, 2^j x_{n1}\right)\colon l\leqslant j<k+l\right\} \\
&\quad= \lim_{l\longrightarrow\infty} \frac{1}{|2|^{ln}} \varphi\left(2^l x_{11}, 2^l x_{11},\dots,2^l x_{n1}, 2^l x_{n1}\right) = 0.
\end{aligned}$$

To get our assertion it is sufficient to use Theorem 3. □

Example 2. Let $|2| < 1$ and $p > n$. Then the mapping $\rho : [0,\infty) \longrightarrow [0,\infty)$ given by (10) satisfies (19) and $\rho(|2|) < |2|^n$.

References

[1] M. Albert, J. Baker, Functions with bounded nth differences, Ann. Polon. Math. 43 (1983) 93–103.
[2] T. Aoki, On the stability of the linear transformation in Banach spaces, J. Math. Soc. Japan 2 (1950) 64–66.
[3] R. Badora, On the separation with n-additive functions, in: General Inequalities, vol. 7, Oberwolfach, 1995, in: Internat. Ser. Numer. Math., vol. 123, Birkhäuser, Basel, 1997, pp. 219–230.
[4] D.G. Bourgin, Classes of transformations and bordering transformations, Bull. Amer. Math. Soc. 57 (1951) 223–237.
[5] J. Brzdęk, A note on stability of additive mappings, in: Stability of Mappings of Hyers–Ulam Type, in: Hadronic Press Collect. Orig. Artic., Hadronic Press, Palm Harbor, FL, 1994, pp. 19–22.
[6] J. Brzdęk, D. Popa, B. Xu, On nonstability of the linear recurrence of order one, J. Math. Anal. Appl. 367 (2010) 146–153.
[7] J. Chung, Hyers–Ulam–Rassias stability of Cauchy equation in the space of Schwartz distributions, J. Math. Anal. Appl. 300 (2004) 343–350.
[8] K. Ciepliński, On multi-Jensen functions and Jensen difference, Bull. Korean Math. Soc. 45 (2008) 729–737.
[9] K. Ciepliński, Stability of the multi-Jensen equation, J. Math. Anal. Appl. 363 (2010) 249–254.
[10] K. Ciepliński, Stability of multi-Jensen mappings in non-Archimedean normed spaces, in: Functional Equations in Mathematical Analysis, Springer, New York, in press.
[11] K. Ciepliński, Generalized stability of multi-additive mappings, Appl. Math. Lett. 23 (2010) 1291–1294.
[12] V.A. Faĭziev, T. Riedel, Stability of Jensen functional equation on semigroups, J. Math. Anal. Appl. 364 (2010) 341–351.
[13] G.L. Forti, Hyers–Ulam stability of functional equations in several variables, Aequationes Math. 50 (1995) 143–190.
[14] Z. Gajda, Local stability of the functional equation characterizing polynomial functions, Ann. Polon. Math. 52 (1990) 119–137.
[15] Z. Gajda, On stability of additive mappings, Int. J. Math. Sci. 14 (1991) 431–434.
[16] P. Găvruţă, A generalization of the Hyers–Ulam–Rassias stability of approximately additive mappings, J. Math. Anal. Appl. 184 (1994) 431–436.
[17] P. Găvruţă, On a problem of G. Isac and Th.M. Rassias concerning the stability of mappings, J. Math. Anal. Appl. 261 (2001) 543–553.
[18] F.Q. Gouvêa, p-Adic Numbers. An Introduction, Universitext, Springer-Verlag, Berlin, 1997.
[19] M. Eshaghi Gordji, M.B. Savadkouhi, Stability of cubic and quartic functional equations in non-Archimedean spaces, Acta Appl. Math. 110 (2010) 1321–1329.
[20] K. Hensel, Über eine neue Begründung der Theorie der algebraischen Zahlen, Jahresber. Deutsch. Math.-Verein. 6 (1899) 83–88.
[21] D.H. Hyers, On the stability of the linear functional equation, Proc. Natl. Acad. Sci. USA 27 (1941) 222–224.
[22] D.H. Hyers, G. Isac, Th.M. Rassias, Stability of Functional Equations in Several Variables, Progr. Nonlinear Differential Equations Appl., vol. 34, Birkhäuser Boston, Inc., Boston, MA, 1998.
[23] S.-M. Jung, Hyers–Ulam–Rassias Stability of Functional Equations in Mathematical Analysis, Hadronic Press, Palm Harbor, FL, 2001.
[24] A. Khrennikov, Non-Archimedean Analysis: Quantum Paradoxes, Dynamical Systems and Biological Models, Math. Appl., vol. 427, Kluwer Academic Publishers, Dordrecht, 1997.
[25] M. Kuczma, An Introduction to the Theory of Functional Equations and Inequalities. Cauchy's Equation and Jensen's Inequality, Birkhäuser, Basel, 2009.
[26] L. Maligranda, A result of Tosio Aoki about a generalization of Hyers–Ulam stability of additive functions – a question of priority, Aequationes Math. 75 (2008) 289–296.
[27] S. Mazur, W. Orlicz, Grundlegende Eigenschaften der polynomischen Operationen I, II, Studia Math. 5 (1934) 50–68, 179–189 (in German).
[28] D. Miheţ, V. Radu, On the stability of the additive Cauchy functional equation in random normed spaces, J. Math. Anal. Appl. 343 (2008) 567–572.
[29] A.K. Mirmostafaee, M.S. Moslehian, Stability of additive mappings in non-Archimedean fuzzy normed spaces, Fuzzy Sets and Systems 160 (2009) 1643–1652.
[30] M.S. Moslehian, The Jensen functional equation in non-Archimedean normed spaces, J. Funct. Spaces Appl. 7 (2009) 13–24.
[31] M.S. Moslehian, Th.M. Rassias, Stability of functional equations in non-Archimedean spaces, Appl. Anal. Discrete Math. 1 (2007) 325–334.
[32] A. Najati, F. Moradlou, Hyers–Ulam–Rassias stability of the Apollonius type quadratic mapping in non-Archimedean spaces, in: Stability of Functional Equations and Applications, Tamsui Oxf. J. Math. Sci. 24 (2008) 367–380.
[33] W. Prager, J. Schwaiger, Stability of the multi-Jensen equation, Bull. Korean Math. Soc. 45 (2008) 133–142.
[34] Th.M. Rassias, On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. 72 (1978) 297–300.
[35] Th.M. Rassias, On the stability of functional equations and a problem of Ulam, Acta Appl. Math. 62 (2000) 23–130.
[36] Th.M. Rassias, P. Šemrl, On the behavior of mappings which do not satisfy Hyers–Ulam stability, Proc. Amer. Math. Soc. 114 (1992) 989–993.
[37] A.M. Robert, A Course in p-Adic Analysis, Grad. Texts in Math., vol. 198, Springer-Verlag, New York, 2000.