Existence and global attractivity of periodic solution for an impulsive delay differential equation with Allee effect✩

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Received 7 May 2004
Available online 31 March 2005
Submitted by K. Gopalsamy

Abstract

Sufficient conditions are obtained for the existence and global attractivity of positive periodic solution of an impulsive delay differential equation with Allee effect. The results of this paper improve and generalize noticeably the known theorems in the literature.

Keywords: Impulsive delay differential equation; Periodic solution; Global attractivity

1. Introduction and preliminaries

Recently, theory and application of impulsive delay differential equations have developed. Various mathematical models in the study of population dynamics, biology, biotechnology, etc. can be expressed by impulsive delay differential equations. These processes and phenomena, which adequate mathematical models are impulsive delay differential equations, are characterized by the fact that per sudden changing of their state and that the processes under consideration depend on their prehistory at each moment of time. The
existence and the attractivity of periodic solutions of impulsive delay differential equations is a notable research field. However, there are few publications in this area. The purpose of this paper is to study the existence and the global attractivity of positive periodic solution for an impulsive delay Lotka–Volterra type model with Allee effect. For the theory of impulsive differential equation, delay differential equations, and impulsive delay differential equations, we refer to the monographs [2,6,8,9,12] and papers [1,3,4,7,10,11,13–20].

Consider impulsive delay Lotka–Volterra type equation with Allee effect:

\[ y'(t) = y(t) \left[ a(t) + \beta(t)y_p(t - \sigma(t)) - \gamma(t)y_q(t - \sigma(t)) \right] \quad \text{a.e. } t \geq 0, \; t \neq \tau_k, \quad (1.1)_a \]

\[ y(\tau_k^+) - y(\tau_k) = b_k y(\tau_k), \quad k = 1, 2, \ldots \quad (1.1)_b \]

Some special cases of nonimpulsive differential equations of (1.1) \((1.1)_a-(1.1)_b\), that is

\[ y'(t) = y(t) \left[ a + b(t - \sigma) - cy(t - \sigma) \right], \quad (1.2) \]
\[ y'(t) = y(t) \left[ a + by(t - \sigma) - cy^2(t - \sigma) \right], \quad (1.3) \]
\[ y'(t) = y(t) \left[ a(t) + b(t)y(t - m\omega) - c(t)y^2(t - m\omega) \right], \quad \text{and} \quad (1.4) \]
\[ y'(t) = y(t) \left[ a(t) + b(t)y^p(t - m\omega) - c(t)y^q(t - m\omega) \right], \quad (1.5) \]

have been investigated respectively in [7,10,13,15]. Equations (1.2)–(1.5) exhibit the single species population growth models with Allee effect (see [9, p. 143]). This phenomenon occurs when the per capita growth rate increases as density increases and decreases after the density passes a certain critical value. This is certainly not the case in the logistic equation

\[ x'(t) = rx(t) \left[ 1 - x(t)/K \right], \]

where per capita growth rate is a decreasing function of the density.

For easy reference in the sequel, we list the following hypotheses:

\((H_1)\) \(0 < \tau_1 < \tau_2 < \cdots\), are fixed impulsive points with \(\lim_{k \to \infty} \tau_k = \infty\);

\((H_2)\) \(a, \gamma \in ([0, \infty), (0, \infty)), \beta \in ([0, \infty), (\infty, \infty))\) are locally summable functions,

\(\sigma \in ([0, \infty), [0, \infty))\) is Lebesgue measurable function, and \(p, q\) are positive constants and \(q > p\);

\((H_3)\) \(\{b_k\}\) is a real sequence and \(b_k > -1, \; k = 1, 2, \ldots\), \(B(t) = \prod_{0 < \tau_k < t}(1 + b_k)\);

\((H_4)\) \(\alpha, \beta, \gamma, \sigma, \text{ and } B(t)\) are bounded functions on \([0, \infty)\);

\((H_4')\) \(\alpha, \beta, \gamma, \sigma, \text{ and } B(t)\) are periodic functions with common periodic \(\omega > 0\).

Here and in the sequel we assume that a product equals to unit if the number of factors is equal to zero. We consider the solutions of (1.1) with initial condition

\[ y(t) = \phi(t) \quad \text{for } -r \leq t \leq 0, \; \phi \in L([-r, 0], [0, \infty)), \; \phi(0) > 0, \quad (1.6) \]

where \(L([-r, 0], [0, \infty))\) denotes the set of Lebesgue measurable functions on \([-r, 0]\) and \(-r = \inf_{t \geq 0}(t - \sigma(t))\).
Definition. A function $y \in ([-r, \infty), (0, \infty))$ is said to be a solution of (1.1) on $[-r, \infty)$, if

(i) $y(t)$ is absolutely continuous on each interval $[0, \tau_1]$ and $(\tau_k, \tau_{k+1}]$, $k = 1, 2, \ldots$;

(ii) for any $\tau_k$, $k = 1, 2, \ldots$, $y(\tau_k^+)$ and $y(\tau_k^-)$ exist and $y(\tau_k^-) = y(\tau_k)$;

(iii) $y(t)$ satisfies $(1.1)_a$ for almost everywhere (a.e.) in $[0, \infty) \setminus \{\tau_k\}$ and satisfies $(1.1)_b$ for every $t = \tau_k$, $k = 1, 2, \ldots$.

We also consider the nonimpulsive delay differential equation with Allee effect

\begin{equation}
 u'(t) = u(t) \left[ a(t) + b(t)u^p(t - \sigma(t)) - c(t)u^q(t - \sigma(t)) \right], \quad \text{a.e. } t \geq 0 \tag{1.7}
\end{equation}

with initial condition

\begin{equation}
 u(t) = \phi(t) \quad \text{for } -r \leq t \leq 0, \; \phi \in L([-r, 0], [0, \infty)), \; \phi(0) > 0, \tag{1.8}
\end{equation}

where

$$b(t) = B^{-p}(t)\beta(t) \quad \text{and} \quad c(t) = B^{-q}(t)\gamma(t), \quad t \geq 0. \tag{1.9}$$

By a solution $u(t)$ of (1.7) we mean an absolutely continuous function defined on $[-r, \infty)$ satisfies (1.7) a.e. for $t \geq 0$ and $u(t) = \phi(t)$ on $[-r, 0]$.

The following lemmas will be used in the proofs of our results. The proof of the first lemma is similar to that of [16, Theorem 1] and it will be omitted.

Lemma 1.1. Assume that $(H_1)$–$(H_4)$ hold. Then

(i) if $u(t)$ is a solution of (1.7) on $[-r, \infty)$, then $y(t) = B(t)u(t)$ is a solution of (1.1) on $[-r, \infty)$;

(ii) if $y(t)$ is a solution of (1.1) on $[-r, \infty)$, then $u(t) = B^{-1}(t)y(t)$ is a solution of (1.7) on $[-r, \infty)$.

Lemma 1.2. Assume that $(H_1)$–$(H_4)$ hold. Then the solutions of (1.1) are defined on $[-r, \infty)$ and are positive on $[-r, \infty)$.

Proof. Clearly, by Lemma 1.1, we only need to prove that the solutions of (1.7) are defined and positive on $[-r, \infty)$. From (1.7) and (1.8), it is easy to obtain

\begin{equation}
 u(t) = \phi(0) \exp \left[ \int_0^t \left( a(s) + b(s)u^p(s - \sigma(s)) - c(s)u^q(s - \sigma(s)) \right) ds \right]. \tag{1.10}
\end{equation}

The assertion of the lemma follows immediately for all $t \in [0, \infty)$. The proof of Lemma 1.2 is complete. \(\square\)

In this paper, for convenience, we will introduce the following notations. Let $f$ be a bounded Lebesgue measurable function on $[0, \infty)$. We define

\begin{align*}
 f^* &= \sup_{t \geq 0} f(t), \quad f_* = \inf_{t \geq 0} f(t), \\
 f_+(t) &= \max\{ f(t), 0 \}, \quad f_-(t) = \min\{ f(t), 0 \}. \tag{1.10}
\end{align*}
In particular, for constants \( b, b_+ = \max\{b, 0\}, b_- = \min\{b, 0\} \). If \( f \) is \( \omega \)-periodic function, define
\[
f(t)_{av} = \frac{1}{\omega} \int_0^\omega f(s) \, ds.
\]

The paper is organized as follows. In Section 2, we derive quite a refined estimate of the solutions of (1.1) on \([0, \infty)\) and obtain the permanence of (1.1). In Section 3, we study the existence of periodic positive solution of (1.1) by using the continuation theorem of coincidence degree theory. In Section 4, we investigate the uniqueness and the global attractivity of periodic solution of (1.1) and obtain some easily verifiable sufficient conditions for the global attractivity of periodic positive solution of (1.1). In [7,10,13,15], the estimate of the solutions of (1.2)–(1.5) and the global attractivity of either positive equilibriums of (1.2) and (1.3) or unique periodic positive solutions of (1.4) and (1.5) restrict that the conditions of Allee effects \( b \) and \( b(t) \) are nonpositive or other some conditions. In this paper, we will delete these restricted conditions about \( b \) or \( b(t) \).

Remark 1.1. From (H3) and (H4), since \( 1 + b_k > 0 \) and \( B(t) = \prod_{0 < \tau_k < t} (1 + b_k) \) is bounded, it follows that \( \{b_k\} \) may be oscillatory and it does not tend to zero as \( k \) tends to \( \infty \). In (H4'), \( B(t) \) is a periodic function with period \( \omega > 0 \), that is,
\[
B(t) = B(t + \omega) \quad \text{for all } t \geq 0.
\]

Thus from (1.12), \( B(t) \) is periodic if and only if \( \prod_{\tau_k \leq t \leq \tau_k + \omega} (1 + b_k) = 1 \). This fact requires some assumptions of periodicity on \( b_k \) and \( \tau_k, k = 1, 2, \ldots \). See example in Section 3.

2. Estimates of solutions and permanence

In this section, we establish certain upper and lower estimates for the solutions of (1.1) and permanence of (1.1).

The following lemma is a modification of [10, Lemma 2] which will be used repeatedly in the proofs of our results. Its proof is straightforward and will be omitted.

Lemma 2.1. Assume that \( a, c, p, q \) are positive constants with \( q > p > 0 \) and \( b \) is a real number. Set \( g(x) = a + bx^p - cx^q, x \geq 0 \). Then

(i) there exists a unique positive constant \( x_0 \) such that \( g(x_0) = 0 \) and
\[
g(x) > 0 \quad \text{for } 0 \leq x < x_0, \quad g(x) < 0 \quad \text{for } x_0 < x < \infty,
\]
(ii) \( g(x) \) takes maximum at point \( x_M = (pb_+/(qc))^{1/(q-p)} \). Furthermore, \( g(x) \) is increasing for \( 0 \leq x \leq x_M \) and is decreasing for \( x_M \leq x < \infty \).

To establish main result of this section, for bounded measurable functions \( a > 0, c > 0, \) and \( b \), we define two functions as follows:
\[
f_1(u) = a^* + b^* u^p - c^* u^q \quad \text{and} \quad f_2(u) = a^* + b^* u^p - c_* u^q, \quad q > p > 0.
\]
where \( a^*, b^*, c^* \) defined as in (1.10).

From Lemma 2.1, there exist \( u_1 \) and \( u_2 \) such that \( f_1(u_1) = 0, f_2(u_2) = 0 \) with \( 0 < u_1 \leq u_2 \), where \( u_1 \) and \( u_2 \) are unique zero points respectively for \( f_1 \) and \( f_2 \). \( u_1 = u_2 \) if and only if \( a, b, c \) are constants.

Throughout this paper, we assume that the roots of \( f_1(u) = 0 \) and \( f_2(u) = 0 \) are respectively \( u_1 \) and \( u_2 \).

**Theorem 2.1.** Assume that \((H1)\)–\((H4)\) hold. If \( y(t) \) is a solution of (1.1), then there exists \( T > r > 0 \) such that for all \( t \geq T \),

\[
B(t)u_1 e^{-\mu_1 t} \leq y(t) \leq B(t)u_2 e^{\mu_2 t},
\]

where

\[
\mu_2 = \sup_{0 \leq t < \infty} \int_{t-\sigma(t)}^{t} \left[ a(s) + b_+(s) \left( \frac{pb_+(s)}{qc(s)} \right)^{p/(q-p)} - c(s) \left( \frac{pb_+(s)}{qc(s)} \right)^{q/(q-p)} \right] ds,
\]

\[
\mu_1 = \sup_{0 \leq t < \infty} \int_{t-\sigma(t)}^{t} \left[ -a(s) - b_-u_2 e^{\mu_2 s} + c(s)u_2^{q} e^{\mu_2 s} \right] ds.
\]

**Proof.** First, we establish the estimate of the solutions of (1.7). Next, by using Lemma 1.1, we will prove (2.2).

From Lemma 1.2, the solutions of (1.7) are positive on \([0, \infty)\). Suppose that solution \( u(t) \) of (1.7) oscillates about \( u_2 \) and \( \sup_{0 \leq t < \infty} u(t) > u_2 \). Then there exist two sequences \( \{t_n\} \) and \( \{\xi_n\} \) such that

\[
r < t_1 < t_2 < \cdots < t_n < t_{n+1} < \cdots,
\]

\[
\lim_{n \to \infty} t_n = \infty \quad \text{and} \quad u(t_n) = u_2, \quad n = 1, 2, \ldots,
\]

and \( u(\xi_n) \) is the maximum of \( u(t) \) on \((t_n, t_{n+1})\) with \( u(\xi_n) > u_2 \). Then for any enough small \( \varepsilon > 0 \), there exist \( \delta > 0 \) and \( \tilde{\xi}_n \) such that \( \tilde{\xi}_n \in (\xi_n - \delta, \xi_n] \) with \( u' (\tilde{\xi}_n) \geq 0, u(\tilde{\xi}_n) > u_2 \) and

\[
u(\xi_n) - u(\xi_n) < \varepsilon \quad \text{for} \quad n = 1, 2, \ldots.
\]

From (1.7) we obtain

\[
0 \leq u' (\tilde{\xi}_n) < u(\tilde{\xi}_n) \left[ a^* + b^* u^p (\tilde{\xi}_n - \sigma(\tilde{\xi}_n)) - c^* u^q (\tilde{\xi}_n - \sigma(\tilde{\xi}_n)) \right].
\]

Thus

\[
a^* + b^* u^p (\tilde{\xi}_n - \sigma(\tilde{\xi}_n)) - c^* u^q (\tilde{\xi}_n - \sigma(\tilde{\xi}_n)) > 0.
\]

It follows from Lemma 2.1 that \( u(\xi_n - \sigma(\tilde{\xi}_n)) < u_2 \); so let \( \xi_n^0 \) be a zero of \( u(t) - u_2 \) in \((\xi_n - \sigma(\tilde{\xi}_n), \tilde{\xi}_n) \cap [t_n, \tilde{\xi}_n)\). Thus \( u(\xi_n^0) = u_2 \).

In view of Lemma 2.1, for any \( t > r \) we have
\[
\begin{align*}
    u'(t) &= u(t) \left[ a(t) + b(t) u^p(t - \sigma(t)) - c(t) u^q(t - \sigma(t)) \right] \\
    &\leq u(t) \left[ a(t) + b(t) u^p(t - \sigma(t)) - c(t) u^q(t - \sigma(t)) \right] \\
    &\leq u(t) \left[ a(t) + b(t) u^p(t - \sigma(t)) - c(t) u^{(p/(q-p))} - c(t) \left( \frac{pb+(s)}{qc(s)} \right)^{(p/(q-p))} \right].
\end{align*}
\] (2.6)

Integrating (2.6) from \( \xi_n \) to \( \tilde{\xi}_n \), we obtain
\[
0 < \ln \frac{u(\tilde{\xi}_n)}{u(\xi_n)} \leq \tilde{\xi}_n \int_{\xi_n}^{\tilde{\xi}_n} \left[ a(s) + b(s) \left( \frac{pb+(s)}{qc(s)} \right)^{(p/(q-p))} - c(s) \left( \frac{pb+(s)}{qc(s)} \right)^{(q/(q-p))} \right] ds \leq \mu_2, \tag{2.7}
\]

From (2.5) and (2.7), it follows that
\[
0 < \ln \frac{u(\xi_n) - \epsilon}{u(\xi_0)} \leq \mu_2 e^{\mu_2}, \tag{2.8}
\]

Suppose that \( u(t) \) is nonoscillatory about \( u_2 \). We now claim that for any \( \epsilon > 0 \) there is \( T_1 > r \) such that for all \( t \geq T_1 \),
\[
u(t) < u_2 + \epsilon. \tag{2.9}
\]

Otherwise, since \( u(t) > u_2 \), by using Lemma 2.1, \( u'(t) < 0 \) a.e. for \( t \geq \tilde{T} > T_1 \). Thus if \( u(t) > u_2 + \epsilon \), in view of Lemma 2.1, we find
\[
u(t) \leq u(t) \left[ a^* + b^* u^p(t - \sigma(t)) - c^* u^q(t - \sigma(t)) \right] \leq u(t) \left[ a^* + b^* (u_2 + \epsilon)^p - c^* (u_2 + \epsilon)^q \right] < 0, \quad \text{a.e. for } t \geq \tilde{T} + r,
\]

which contradicts \( u(t) > 0 \). Hence (2.9) holds. This fact implies that there exists \( T_2 > T_1 \) such that for \( t \geq T_2 \),
\[
u(t) \leq u_2 e^{\mu_2}. \tag{2.10}
\]

Suppose that \( u(t) \) oscillates about \( u_1 \) and \( \inf_{t \in [t, \infty)} u(t) < u_1 \). Then there exist two sequences of \( \{s_n\} \) and \( \{\eta_n\} \) such that
\[
0 < s_1 < s_2 < \cdots < s_n < s_{n+1} < \cdots,
\]

\[
\lim_{n \to \infty} s_n = \infty \quad \text{and} \quad u(s_n) = u_1, \quad n = 1, 2, \ldots
\]

and \( u(\eta_n) \) is the minimum of \( u(t) \) on \( (s_n, s_{n+1}) \) with \( u(\eta_n) < u_1 \). Then for any enough small \( \epsilon > 0 \) there exist \( \delta > 0 \) and \( \tilde{\eta}_n \in (\eta_n - \delta, \eta_n] \) with \( u'(\tilde{\eta}_n) \leq 0 \), \( u(\tilde{\eta}_n) < u_1 \) and
\[
u(\tilde{\eta}_n) - u(\eta_n) < \epsilon, \quad n = 1, 2, \ldots. \tag{2.11}
\]
From (1.7), we find
\[ 0 \geq u'(\tilde{\eta}_n) > u(\tilde{\eta}_n)[a_* + b_au^p(\tilde{\eta}_n - \sigma(\tilde{\eta}_n)) - c^*u^q(\tilde{\eta}_n - \sigma(\tilde{\eta}_n))], \]
which, in view of Lemma 2.1, implies that \( u(\tilde{\eta}_n - \sigma(\tilde{\eta}_n)) > u_1 \). Therefore, there exists \( \eta_n^0 \in (\tilde{\eta}_n - \sigma(\tilde{\eta}_n), \tilde{\eta}_n) \) such that \( u(\eta_n^0) = u_1, n = 1, 2, \ldots \). Integrating (1.7) from \( \eta_n^0 \) to \( \tilde{\eta}_n \) and using (2.8) and (2.10), we obtain that for \( \eta_n^0 > T_2 \),
\[ 0 > \ln \frac{u(\tilde{\eta}_n)}{u(\eta_n^0)} \geq \int_{\eta_n^0}^{\tilde{\eta}_n} \left[ a(s) + b_-(s)u_2^p e^{p\mu_2} - c(s)u_2^q e^{\mu_2} \right] ds. \]
Thus we have
\[ u(\tilde{\eta}_n) \geq \exp \left( \int_{\eta_n^0}^{\tilde{\eta}_n} \left[ a(s) + b_-(s)u_2^p e^{p\mu_2} - c(s)u_2^q e^{\mu_2} \right] ds \right), \quad (2.12) \]
As \( u_2 e^{\mu_2} > u_2 \), we see
\[ a(t) + b_-(t)u_2^p e^{p\mu_2} - c(t)u_2^q e^{\mu_2} \leq a(t) + b(t)u_2^p e^{p\mu_2} - c(t)u_2^q e^{\mu_2} < 0. \]
Thus (2.12) and (2.4) yield
\[ u(\tilde{\eta}_n) \geq u_1 \exp \left( \int_{\eta_n^0}^{\tilde{\eta}_n} \left[ a(s) + b_-(s)u_2^p e^{p\mu_2} - c(s)u_2^q e^{\mu_2} \right] ds \right) \geq u_1 e^{-\mu_1}, \]
which with (2.11) implies for any \( t > T_2 \), \( u(t) \geq u(\eta_n^0) \geq u(\tilde{\eta}_n) - \epsilon \), that is,
\[ u(t) > u_1 - \epsilon. \quad (2.13) \]
Suppose that \( u(t) \) is nonoscillatory about \( u_1 \) and \( u(t) < u_1 \). We will prove that for any \( \epsilon > 0 \) there exists \( T_3 > T_2 \) such that
\[ u(t) > u_1 - \epsilon. \quad (2.14) \]
Otherwise, as \( u(t) < u_1 \) eventually, by using Lemma 2.1, there exists \( \tilde{T} \geq T_3 \) such that for all \( t \geq \tilde{T}, \)
\[ u'(t) \geq u(t)\left[ a_* + b_au^p(t - \sigma(t)) - c^*u^q(t - \sigma(t)) \right] > 0 \quad \text{a.e.} \]
Hence \( u'(t) > 0 \) a.e. for \( t \geq \tilde{T} \). If \( u(t) \leq u_1 - \epsilon \) eventually, then
\[ u'(t) > u(t)\left[ a_* + b_*(u_1 - \epsilon)^p - c^*(u_1 - \epsilon)^q \right] > 0, \]
which leads to \( u(t) \) to be unbounded. This contradicts (2.10). Thus (2.14) is proved. On the other hand, it follows from Lemma 2.1 that
\[ a(t) + b_-(t)u_2^p e^{p\mu_2} - c(t)u_2^q e^{\mu_2} < 0. \]
Therefore,
\[ \mu_1 = \sup_{0 \leq t < \infty} \int_{t - \sigma(t)}^{t} \left[ -a(s) - b_-(s)u_2^p e^{p\mu_2} + c(s)u_2^q e^{\mu_2} \right] ds > 0, \]
which implies that there exists $T_3 > T_2$ such that for all $T \geq T_3$,

$$u(t) \geq u_1 - \varepsilon \geq u_1 e^{-\mu_1}. \tag{2.15}$$

From (2.8), (2.10), (2.13), and (2.15) we obtain

$$u_1 e^{-\mu_1} \leq u(t) \leq u_2 e^{\mu_2}, \quad \text{for all sufficiently large } t. \tag{2.16}$$

By using Lemma 1.1, (2.2) can be obtained and the proof of Theorem 2.1 is complete. \hfill \Box

**Remark 2.1.** It is easy to see that Theorem 2.1 does not require that $a(t)$, $\beta(t)$, $\gamma(t)$, $\sigma(t)$ are periodic and $b(t)$ is nonpositive in (1.1). Hence Theorem 2.1 improves and generalizes noticeable [13, Theorem 2.3], [15, Theorem 3.2] and corresponding estimates of the solutions of (1.2) and (1.3) in [7] and [10], respectively.

The following corollary is an immediate result of Theorem 2.1.

**Corollary 2.1.** Assume that $(H_1)$–$(H_4)$ hold. Then (1.1) is permanent.

We apply Theorem 2.1 to nonimpulsive delay differential equation (1.3) that is, $b_k \equiv 0$ in (1.1). In this case,

$$a, c, p, q \text{ are positive constants with } q > p \text{ and } b \text{ is real number}, \tag{2.17}$$

and $\tilde{y}$ is unique positive equilibrium of (1.3) satisfying $a + b\tilde{y}^p - c\tilde{y}^q = 0$. Thus from Theorem 2.1 we have the following estimates of the solutions.

**Corollary 2.2.** Assume that (2.17) holds. Then for any solution $y(t)$ of (1.3), there exists $T > \sigma$ such that for all $t \geq T$,

$$\tilde{y} e^{-\mu_1} \leq y(t) \leq \tilde{y} e^{\mu_2},$$

where

$$\tilde{y} \text{ is unique positive equilibrium of (1.3)}, \tag{2.18}$$

$$\mu_2 = \left[ a + b_+ \left( \frac{pb_+}{qc} \right)^{p/(q-p)} - c \left( \frac{pb_+}{qc} \right)^{q/(q-p)} \right] \sigma, \tag{2.19}$$

$$\mu_1 = [-a - b_- (\tilde{y} e^{\mu_2})^p + c (\tilde{y} e^{\mu_2})^q] \sigma. \tag{2.20}$$

By using Theorem 2.1 to (1.5), we have the following result.

**Corollary 2.3.** Assume that $a(t) > 0$, $c(t) > 0$, and $b(t)$ are locally summable periodic functions with common periodic $\omega > 0$ and $p, q$ are positive constants with $q > p$, $m$ is a positive integer. Then for any solution $y(t)$ of (1.5), there exists $T > m \omega$ such that

$$u_1 e^{-\mu_1} \leq y(t) \leq u_2 e^{\mu_2} \quad \text{for all } t \geq T, \tag{2.21}$$

where
$$\mu_2 = \left\{ a_{av} + \left[ b_+(t) \left( \frac{pb_+(t)}{qc(t)} \right)^{p/(q-p)} \right]_{av} - \left[ c(t) \left( \frac{pb_+(t)}{qc(t)} \right)^{q/(q-p)} \right]_{av} \right\} m\omega,$$

$$\mu_1 = \left\{ -a_{av} - \left[ b_-(t) \left( \mu_2 e^{\mu_2} \right)^p \right]_{av} + \left[ c(t) \left( \mu_2 e^{\mu_2} \right)^q \right]_{av} \right\} m\omega. \quad (2.22)$$

Remark 2.2. Corollaries 2.2 and 2.3 improve and generalize [13, Theorem 2.3] and [15, Theorem 3.2], respectively.

3. Existence of positive periodic solutions

In this section, by using the continuation theorem, we show the existence of at least one positive periodic solution of (1.1). We first make some preparations.

Let $$X, Z$$ be normed vector spaces. Assume that $$L : \text{Dom} L \subset X \to Z$$ is a linear mapping, and $$N : X \to Z$$ is a continuous mapping. The mapping $$L$$ will be called a Fredholm mapping of index zero if $$\text{dim} \ker L = \text{condim} \text{Im} L < \infty$$ and $$\text{Im} L$$ is closed in $$Z$$. If $$L$$ is a Fredholm mapping of index zero, there exist continuous projectors: $$P : X \to X$$ and $$Q : Z \to Z$$ such that $$\text{Im} P = \ker L$$, $$\text{Im} L = \ker Q = \text{Im} (I - Q)$$. It follows that $$L \mid_{\text{Dom} \cap \ker P : (I - P)X \to \text{Im} L}$$ is invertible. We denote the inverse of that map by $$K_P$$.

If $$\Omega$$ is an open bounded subset of $$X$$, the mapping $$N$$ will be called $$L$$-compact on $$\overline{\Omega}$$ if $$QN(\overline{\Omega})$$ is bounded and $$K_P (I - Q)N : \overline{\Omega} \to X$$ is compact. Since $$\text{Im} Q$$ is isomorphic to $$\ker L$$, there exist isomorphisms $$J : \text{Im} Q \to \ker L$$.

For convenience, we introduce the continuation theorem as follows.

Continuation Theorem (see [5, p. 40]). Let $$L$$ be Fredholm mapping of index zero and let $$N$$ be $$L$$-compact on $$\overline{\Omega}$$. Assume that

(i) for each $$\lambda \in (0, 1)$$, every solution $$x$$ of $$Lx = \lambda Nx$$ is such that $$x \in \partial \Omega$$;

(ii) $$QN \neq 0$$ for each $$x \in \partial \Omega \cap \ker L$$ and

$$\deg \{ JQN, \Omega \cap \ker L, 0 \} \neq 0.$$

Then equation $$Lx = Nx$$ has at least one solution in $$\text{Dom} L \cap \overline{\Omega}$$. 

Theorem 3.1. Assume that $$(H_1)$$–$$(H_3)$$ and $$(H'_4)$$ hold and $$q > p$$. Then (1.1) has at least one $$\omega$$-periodic positive solution.

Proof. Obviously, by Lemmas 1.1 and 1.2, we only need to prove that (1.7) has at least one $$\omega$$-periodic positive solution. By Lemma 2.1, it is easy to see $$a_{av} + b_{av} x^p - c_{av} x^q = 0$$ has a unique positive solution.

Making the change of variable $$u(t) = \exp x(t)$$, $$t \geq 0$$, (1.3) is reformulated as

$$x'(t) = a(t) + b(t) e^{px(t-\sigma(t))} - c(t) e^{qx(t-\sigma(t))}, \quad \text{a.e. } t \geq 0. \quad (3.1)$$

Let

$$X = Z = \{ x \in C([0, \infty), R), \ x(t + \omega) = x(t) \}. \quad (3.2)$$
For any \( x \in X \) (or \( Z \)), set \( \| x \| = \max_{0 \leq t \leq \omega} |x(t)| \). Then \( X \) and \( Z \) are both Banach spaces when they are endowed with the norm \( \| \cdot \| \). Let

\[
N x = a(t) + b(t)e^{p x(t) - \sigma(t)} - c(t)e^{q x(t) - \sigma(t)}, \quad \text{a.e. } t \geq 0.
\]

(3.3)

\[
L x = x', \quad P x = \frac{1}{\omega} \int_{0}^{\omega} x(t) \, dt, \quad x \in X, \quad Q z = \frac{1}{\omega} \int_{0}^{\omega} z(t) \, dt, \quad z \in Z.
\]

Thus

\[
\text{Ker } L = \{ x \mid x \in X, \ x = h, \ h \in \mathbb{R} \}, \quad \text{Im } L = \left\{ z \mid z \in Z, \ \int_{0}^{\omega} z(t) \, dt = 0 \right\},
\]

and

\[
\dim \text{Ker } L = \text{codim } \text{Im } L = 1,
\]

and \( P, Q \) are continuous projectors such that \( \text{Im } P = \text{Ker } L, \ \text{Ker } Q = \text{Im } L = \text{Im } (I - Q) \). It follows that \( L \) is a Fredholm mapping of index zero. Furthermore, the generalized inverse (to \( L \)) \( K_P : \text{Im } L \rightarrow \text{Ker } P \cap \text{Dom } L \) is given by

\[
K_P(z) = \int_{0}^{t} z(s) \, ds - \frac{1}{\omega} \int_{0}^{\omega} \int_{0}^{t} z(s) \, ds \, dt.
\]

Thus

\[
Q N x = \frac{1}{\omega} \int_{0}^{\omega} \left[ a(t) + b(t)e^{p x(t) - \sigma(t)} - c(t)e^{q x(t) - \sigma(t)} \right] \, dt,
\]

\[
K_P (I - Q) N x = \int_{0}^{t} \left[ a(s) + b(s)e^{p x(s) - \sigma(s)} - c(s)e^{q x(s) - \sigma(s)} \right] \, ds
\]

\[
- \frac{1}{\omega} \int_{0}^{\omega} \int_{0}^{t} \left[ a(s) + b(s)e^{p x(s) - \sigma(s)} - c(s)e^{q x(s) - \sigma(s)} \right] \, ds \, dt
\]

\[
\times \left( \frac{1}{\omega} - \frac{1}{2} \right) \int_{0}^{\omega} \left[ a(s) + b(s)e^{p x(s) - \sigma(s)} - c(s)e^{q x(s) - \sigma(s)} \right] \, ds.
\]

Clearly, \( QN \) and \( K_P (I - Q) N \) are continuous and it is easy to prove that \( K_P (I - Q) N(\bar{\Omega}) \) is compact for any open bounded set \( \Omega \subset X \). Moreover, \( Q N(\bar{\Omega}) \) is bounded. Thus, \( N \) is \( L \)-compact on \( \bar{\Omega} \) with any open bounded set \( \Omega \subset X \). The isomorphism \( J \) of \( \text{Im } Q \) onto \( \text{Ker } L \) can be the identity mapping, since \( \text{Im } Q = \text{Ker } L \).

Corresponding to the operator equation \( L x = \lambda N x, \ \lambda \in (0, 1) \), we have

\[
x'(t) = \lambda \left[ a(t) + b(t)e^{p x(t) - \sigma(t)} - c(t)e^{q x(t) - \sigma(t)} \right], \quad \lambda \in (0, 1), \quad \text{a.e.} \quad (3.4)
\]
Suppose that \( x \in X \) is a solution of (3.4) for a certain \( \lambda \in (0, 1) \). Integrating (3.4) on \([0, \omega]\), we obtain
\[
\int_0^\omega \left[ a(t) + b(t)e^{px(t-\sigma(t))} - c(t)e^{qx(t-\sigma(t))} \right] dt = 0.
\]
Hence
\[
\int_0^\omega a(t) dt = \int_0^\omega \left[ c(t)e^{qx(t-\sigma(t))} - b(t)e^{px(t-\sigma(t))} \right] dt. \tag{3.5}
\]
It follows from (3.5) that there exists a constant \( M_0 \) such that
\[
\int_0^\omega \left| c(t)e^{qx(t-\sigma(t))} - b(t)e^{px(t-\sigma(t))} \right| dt < M_0. \tag{3.6}
\]
Then in view of (3.4) and (3.6),
\[
\int_0^\omega \left| x'(t) \right| dt = \lambda \int_0^\omega \left[ a(t) + b(t)e^{px(t-\sigma(t))} - c(t)e^{qx(t-\sigma(t))} \right] dt
\leq \int_0^\omega a(t) dt + \int_0^\omega \left| c(t)e^{qx(t-\sigma(t))} - b(t)e^{px(t-\sigma(t))} \right| dt
\leq \bar{a} \omega + M_0 := M_1. \tag{3.7}
\]
As \( x \in X \), there exist \( t_0 \in [0, \omega] \) and a constant \( M_2 > 0 \) such that \( x(t_0) < M_2 \). Hence
\[
x(t) \leq x(t_0) + \int_0^\omega \left| x'(t) \right| dt < M_1 + M_2.
\]
On the other hand, there exist \( t_1 \in [0, \omega] \) and a constant \( M_3 > 0 \) such that \( x(t_1) > -M_3 \). Thus we obtain
\[
x(t) \geq x(t_1) - \int_0^\omega \left| x'(t) \right| dt > -(M_3 + M_1).
\]
Clearly, \( M_i, i = 0, 1, 2, 3 \), are independent of \( \lambda \). Set \( H = \sum_{i=0}^4 M_i \) where \( M_4 \) is a sufficiently large positive constant such that the unique solution \( v_0 \) of \( a_{av} + b_{av}e^{pv} - c_{av}e^{qv} = 0 \) satisfies \( \| v_0 \| < M_4 \), then \( \| x \| < H \).

Let \( \Omega = \{ x \in X : \| x \| < H \} \). It is clear that \( \Omega \) verifies condition (i) in Continuation Theorem. When \( x \in \partial \Omega \cap \text{Ker} L = \partial \Omega \cap R, x \) is a constant with \( |x| = H \). Then
\[
QN x = \frac{1}{\omega} \int_0^\omega \left[ a(t) + b(t)e^{px} - c(t)e^{qx} \right] dt = a_{av} + b_{av}e^{px} - c_{av}e^{qx} \neq 0.
\]
Furthermore, let \( J = I \) be identity mapping since \( \text{Im} \, P = \text{Ker} \, L \). Thus condition (ii) is satisfied.

By now we have proved that \( \Omega \) verifies all conditions of Continuation Theorem. Hence (3.1) has at least one solution \( \tilde{x}(t) \) in \( \text{Dom} \, L \cap \overline{\Omega} \). Set \( \tilde{u}(t) = \exp \tilde{x}(t) \), then it is easy to see that \( \tilde{u}(t) \) is a \( \omega \)-periodic positive solution of (1.7). Therefore, by Lemma 1.1, (1.1) has a \( \omega \)-periodic positive solution \( \tilde{y}(t) = \prod_{0 < \tau_k < t} (1 + b_k) \tilde{u}(t) \). The proof of Theorem 3.1 is complete. \( \square \)

**Remark 3.1.** From the proof of Theorem 3.1, one can know that the deviating argument \( \sigma(t) \) has not effect on the existence of positive periodic solution of (1.1). Hence Theorem 3.1 is also true for both advanced type and mixed type impulsive differential equations. Theorem 3.1 is even true for the corresponding impulsive ordinary differential equation, that is,

\[
y'(t) = y(t) \left[ a(t) + \beta(t) y^p(t) - \gamma(t) y^q(t) \right], \quad \text{a.e.} \; t \geq 0, \; t \neq \tau_k, \]

\[
y(\tau_k^+) - y(\tau_k) = b_k y(\tau_k), \quad k = 1, 2, \ldots
\]

and ordinary differential equation

\[
y'(t) = y(t) \left[ a(t) + b(t) y^p(t) - c(t) y^q(t) \right], \quad t \geq 0.
\]

**Example.** Let \( \tau_k = k \pi \), \( b_{2k+1} = 1 \), and \( b_{2(k+1)} = -\frac{1}{2} \), \( k = 1, 2, \ldots \). Consider impulsive differential equation

\[
y'(t) = y(t) \left[ (1 + \sin^2 t) + \cos t y^p(t - \sin t) - (1 + \cos^2 t) y^q(t - \sin t) \right],
\]

\[
t \neq \tau_k, \quad (3.8)_a
\]

\[
y(\tau_k^+) - y(\tau_k) = y(\tau_k), \quad k = 1, 2, \ldots, \quad (3.8)_b
\]

where \( q > p > 0 \). It is easy to verify

\[
B(t) = \prod_{0 < \tau_k < t} (1 + b_k) = \prod_{0 < \tau_k < t + 2\pi} (1 + b_k) = B(t + 2\pi), \quad t \geq 0.
\]

Hence by Theorem 3.1 and Remark 3.1, Eq. (3.8) has at least one \( 2\pi \)-periodic positive solution.

4. Global attractivity of positive periodic solution

In this section, our aim is to obtain an explicit sufficient condition for the global attractivity of positive periodic solution with respect to all other positive solutions of (1.1).

The following two results extracted respectively from [6, p. 21] (also see [17]) and [11] with modifications on Caratheodory type conditions are needed in the proofs of our results of this section. Their proofs are respectively similar to those in [6] and [11] and will be omitted.
Lemma 4.1. Let \( a, \sigma \) satisfy that

(i) \( a, \sigma \in ([0, \infty), [0, \infty)) \), \( a \) is locally summable function, \( \sigma \) is bounded Lebesgue measurable function, and \( \sigma^* = \sup_{0 \leq t < \infty} \sigma(t) \);

(ii) \( \limsup_{t \to \infty} \int_{t}^{t+\sigma^*} a(s) \, ds = \frac{3}{2} \) and \( \liminf_{t \to \infty} \int_{t}^{t+\sigma^*} a(s) \, ds = \mu > 0 \).

Then all nontrivial solutions of

\[ y'(t) + a(t)y(t - \sigma(t)) = 0 \]

satisfy \( \lim_{t \to \infty} y(t) = 0 \).

Lemma 4.2. Assume that \( \sigma \) is a nonnegative constant and \( a \in ([0, \infty), (0, \infty)) \) is a locally summable function. Moreover,

\[ \int_{0}^{\infty} a(t) \, dt = \infty \]

and

\[ \lim_{t \to \infty} \int_{t-\sigma}^{t} a(s) \, ds \quad \text{exists and} \quad \lim_{t \to \infty} \int_{t-\sigma}^{t} a(s) \, ds < \frac{\pi}{2} \]

Then all nontrivial solutions of \( y'(t) + a(t)y(t - \sigma) = 0 \) satisfy \( \lim_{t \to \infty} y(t) = 0 \).

Theorem 4.1. Assume that \( (H_1) - (H_3) \) and \( (H'_4) \) hold. Moreover, let \( \sigma^* = \sup_{0 \leq t < \infty} \sigma(t) \),

\[ \liminf_{t \to \infty} \int_{t}^{t+\sigma^*} \left[ qc(s)(u_1 e^{-\mu_1})^q - pb_+(s)(u_1 e^{-\mu_1})^p \right] \, ds > 0 \] \hspace{1cm} (4.1)

and

\[ \limsup_{t \to \infty} \int_{t}^{t+\sigma^*} \left[ qc(s)(u_2 e^{\mu_2})^q - pb_-(s)(u_2 e^{\mu_2})^p \right] \, ds < \frac{3}{2} \] \hspace{1cm} (4.2)

where \( \mu_1 \) and \( \mu_2 \) are defined respectively in (2.23) and (2.22). Then there exists a unique \( \omega \)-periodic positive solution \( \tilde{y}(t) \) of (1.1) such that all other positive solutions \( y(t) \) of (1.1) satisfy

\[ \lim_{t \to \infty} \left( y(t) - \tilde{y}(t) \right) = 0 \] \hspace{1cm} (4.3)

Proof. Clearly, it is immediate that if \( \tilde{y}(t) \) satisfies (4.3), then the periodic positive solution \( \tilde{y}(t) \) will be unique. Hence to complete the proof of the theorem, it suffices to prove (4.3).
By Lemma 1.1, we only need to show \( \lim_{t \to \infty} (u(t) - \tilde{u}(t)) = 0 \) where \( \tilde{u}(t) \) is the periodic positive solution and \( u(t) \) is an arbitrary positive solution of (1.7).

From Theorem 3.1, there exist periodic positive solution of (1.7) under the hypotheses of Theorem 4.1. Let \( \tilde{u}(t) \) be periodic positive solution of (1.7). Set \( u(t) = \tilde{u}(t)e^{\tilde{x}(t)} \). Then (1.7) reduces to

\[
x'(t) = b(t)\tilde{u}^p(t - \sigma(t)) - c(t)\tilde{u}^q(t - \sigma(t)) - c(t)\tilde{u}^q(t - \sigma(t))e^{\tilde{y}(t)},
\]

a.e. \( t \geq 0 \). (4.4)

Put

\[
G(t, z) = b(t)\tilde{u}^p(t - \sigma(t))e^{pq} - c(t)\tilde{u}^q(t - \sigma(t))e^{pq}.
\]

Then

\[
\partial G(t, z) \frac{\partial}{\partial z} = pb(t)\tilde{u}^p(t - \sigma(t))e^{pq} - qc(t)\tilde{u}^q(t - \sigma(t))e^{pq},
\]

and from (4.4) we obtain

\[
x'(t) = G(t, x(t - \sigma(t))) - G(t, 0).
\]

By the mean value theorem, we can rewrite (4.5) in the form

\[
x'(t) = -F(t)x(t - \sigma(t)),
\]

where

\[
F(t) = -\frac{\partial G(t, z)}{\partial z} \bigg|_{z = \xi(t)} = -pb(t)\tilde{u}^p(t - \sigma(t))e^{pq} + qc(t)\tilde{u}^q(t - \sigma(t))e^{pq} - pb(t)\eta^p(t) + qc(t)\eta^q(t)
\]

and \( \eta(t) \) lies between \( \tilde{u}(t - \sigma(t)) \) and \( u(t - \sigma(t)) \). By using (2.16), we find that for all large \( t \),

\[
-pb_+(t)(u_1e^{-\mu_1})^p + qc(t)(u_1e^{-\mu_1})^q \leq F(t) \leq -pb_-(t)(u(2e^{\mu_2}))^p + qc(t)(u(2e^{\mu_2}))^q.
\]

From Lemma 4.1, in view of (4.1) and (4.2), we conclude that \( \lim_{t \to \infty} x(t) = 0 \) which implies that

\[
\lim_{t \to \infty} [u(t) - \tilde{u}(t)] = \lim_{t \to \infty} \tilde{u}(t)(e^{\tilde{x}(t)} - 1) = 0.
\]

The proof of Theorem 4.1 is complete. \( \Box \)

**Corollary 4.1.** Assume that (2.17) hold and

\[
[qc(\tilde{y}e^{\mu_2})^q - pb_-(\tilde{y}e^{\mu_2})^p] < 3^2
\]

Then all solutions of (1.3) satisfy \( \lim_{t \to \infty} y(t) = \tilde{y} \), where \( \tilde{y} \) and \( \mu_2 \) are defined by (2.18) and (2.19), respectively.
Proof. From Corollary 2.2 we have
\[ \tilde{y}e^{-\mu_1} \leq y(t) \leq \tilde{y}e^{\mu_2} \quad \text{for all } t \geq T, \]
where \( \mu_1 \) is defined by (2.20). In view of (2.19), \( \mu_2 > 0 \), and Lemma 2.1,
\[ a + b_-(\tilde{y}e^{\mu_2})^p - c(\tilde{y}e^{\mu_2})^q \leq a + b(\tilde{y}e^{\mu_2})^p - c(\tilde{y}e^{\mu_2})^q < 0. \]
Therefore
\[ \mu_1 = \left[ -a - b_-(\tilde{y}e^{\mu_2})^p + c(\tilde{y}e^{\mu_2})^q \right] \sigma > 0. \]
Thus \( \tilde{y}e^{-\mu_1} < \tilde{y} \). By using Lemma 2.1 again, we have that \( c(\tilde{y}e^{-\mu_1})^q - b_+(\tilde{y}e^{-\mu_1})^p > 0 \). Hence \( qc(\tilde{y}e^{-\mu_1})^q - pb_+(\tilde{y}e^{-\mu_1})^p > 0 \). In view of (4.7) and Theorem 4.1, the desirable conclusion is obtained. The proof of Corollary 4.1 is complete.

Theorem 4.2. Assume that \((H_1)-(H_3), \ (H'_4)\) hold and \( \sigma = m\omega \), \( m \) is a positive integer. Moreover,
\[ \lim_{t \to \infty} \int_{t-m\omega}^{t} F(s) \, ds \text{ exists and } \left[ q\gamma(t)av\left(u_2e^{\mu_2}\right)^q - pb(t)av\left(u_2e^{\mu_2}\right)^p \right] m\omega < \frac{\pi}{2}, \]
where \( F \) defined by (4.6). Then there exists a unique \( \omega \)-periodic positive solution \( \tilde{y}(t) \) such that all other positive solutions of (1.1) satisfy \( \lim_{t \to \infty} [y(t) - \tilde{y}(t)] = 0 \).

Proof. As the proof of Theorem 4.1, by Theorem 3.1, existence of \( \omega \)-periodic solution of (1.1) is proved. Thus we can obtain (4.4), (4.6), and (4.7). By Lemma 4.2, the proof of Theorem 4.2 can be complete.

Remark 4.1. Theorem 4.2 improves and generalizes [13, Theorem 2.4] and [15, Theorem 3.3].

Acknowledgments

We are grateful to the referees for valuable suggestions.

References