Canard cycles for predator–prey systems with Holling types of functional response

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\textbf{A B S T R A C T}

By using the singular perturbation theory developed by Dumortier and Roussarie and recent work of De Maesschalck and Dumortier, we study the canard phenomenon for predator–prey systems with response functions of Holling types. We first develop a formula for computing the slow divergence integrals. By using the formula we prove that for the systems with the response function of Holling types III and IV the cyclicity of any limit periodic set is at most two, that is at most two families of hyperbolic limit cycles or at most one family of limit cycles with multiplicity two can bifurcate from the limit periodic set by small perturbations. We also indicate the regions in parameter space where the corresponding limit periodic set has cyclicity at most one or at most two.

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\textbf{1. Introduction}

The classical predator–prey systems with a response function $p(x)$ of Holling type can be written in the form...
Table 1
Holling types of functional responses and their generalizations.

<table>
<thead>
<tr>
<th>Holling type</th>
<th>Definition</th>
<th>Generalized form</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>( p(x) = mx )</td>
<td>( p(x) = \frac{mx}{ax^2 + bx + 1} ) (( b &gt; -2\sqrt{a} ))</td>
</tr>
<tr>
<td>II</td>
<td>( p(x) = \frac{mx}{ax^2} )</td>
<td>( p(x) = \frac{mx^2}{ax^2 + bx + 1} ) (( b &gt; -2\sqrt{a} ))</td>
</tr>
<tr>
<td>III</td>
<td>( p(x) = \frac{mx^2}{ax^2} )</td>
<td>( p(x) = \frac{mx^2}{ax^2 + bx + 1} ) (( b &gt; -2\sqrt{a} ))</td>
</tr>
<tr>
<td>IV</td>
<td>( p(x) = \frac{mx}{ax^2} )</td>
<td>( p(x) = \frac{mx^2}{ax^2 + bx + 1} ) (( b &gt; -2\sqrt{a} ))</td>
</tr>
</tbody>
</table>

![Fig. 1. Holling type of functional responses.](image-url)

where \( x \) and \( y \geq 0 \) denote the number or density of the prey and predator populations respectively, \( r, K, d \) and \( c \) are positive constants, and we denote \( F(x) = \frac{rx(1 - x/K)}{p(x)} \).

For predator and prey, a functional response is the intake rate of a predator as a function of predator density, it is usually associated with the numerical response, which is the reproduction rate of a predator as a function of prey density. Following Holling [15,16], the functional responses were originally classified into four types, which are called Holling’s type I, II, III and IV, see Table 1 and graphic illustration in Fig. 1.

The most often used functional response is of Holling type II, associated with Monod and Michaelis–Menten. This is an increasing function that saturates, i.e., it has a finite positive limit as \( x \) approaches infinity. In the literature, except for the original Holling type response functions of type I, II and III, the following Holling type IV functional response was proposed

\[
p(x) = \frac{mx}{ax^2 + bx + 1},
\]

where \( a \) and \( m \) are positive constants, and \( b > -2\sqrt{a} \) (so that \( ax^2 + bx + 1 > 0 \) for all \( x \geq 0 \) and hence \( p(x) > 0 \) for all \( x > 0 \)). This Holling type IV functional response (1.2) was associated with Monod–Haldane (see Andrews [1]), as shown in Fig. 2(b), the function increases to a maximum and then decreases, approaching zero as \( x \) approaches infinity, is used to model the situation where the prey can better defend or disguise themselves when their population becomes large enough, a phenomenon called group defence [14].

For the generalized response function of Holling type III,

\[
p(x) = \frac{mx^2}{ax^2 + bx + 1},
\]

the parameters satisfy the same assumption as in (1.2). In this case, for \( x \) sufficiently large the behavior of \( p(x) \) resembles feature of Holling type II, the effect of inhibition is seen (Fig. 2(a)).
If $-2\sqrt{a} < b < 0$ and $a$ nonnegative, $p(x)$ remains nonnegative, the inhibition effect is still observed for large $x$; however, for $x$ small the behavior of $p(x)$ resembles the properties of functional response of Holling type IV (Fig. 2(b)).

The predator–prey system (1.1) with Holling type of functional responses has been extensively studied by many authors, including the studies by May [22], Kasarinoff and van der Deiesch [18], Wolkowicz [26], Wrzosek [27], Rothe and Shafer [23], Ruan and Xiao [25], Zhu, Campbell and Wolkowicz [30], Xiao and Zhu [28] and recent work by Lamontagne, Coutu and Rousseau [20]. The readers can find an extended list of references in the papers [30,28]. In all of these studies, the existence and number of limit cycles are important topics in the bifurcation study of the predator–prey systems for a better understanding of many real world oscillatory phenomena in nature [22,2,24]. To our knowledge, for the results of system (1.1) with Holling types of response functions, only the case of Holling type II was completed solved. Cheng [3] and Chen and Jing [4] proved that the system has at most one limit cycle, see also the book by Ma and Zhou [21] (Example 2 on page 165). Hence, in this paper we will restrict our study to the system (1.1) with response function of Holling types III and IV.

For the predator–prey systems, it is well-known that the existence of limit cycles is related to the existence and bifurcation of a positive equilibrium, homoclinic bifurcation and saddle node bifurcation of limit cycles. Typically in a positively invariant region, if there exists a unique positive equilibrium which is unstable, then there must exist at least one limit cycle according to the theory of Poincaré–Bendixson. On the other hand, if the unique positive equilibrium of a predator–prey system is locally stable but not hyperbolic, there might be more than one limit cycles bifurcated through Hopf bifurcation(s). Hofbauer and So [17] verified numerically that there can exist at least two limit cycles for some two dimensional predator–prey systems of the form (1.1). The results on the Hopf bifurcations of codimension two by Xiao and Zhu [28] for system (1.1) with response function of Holling type IV and Lamontagne, Coutu and Rousseau for system (1.1) with response function of Holling type III [20] indicate that there can be at most two limit cycles from the Hopf bifurcations in the systems respectively in [28] and [20].

For the study of predator–prey system (1.1), usually we assume that all the parameters are positive constants, there is not much study of the system with small parameters. It is interesting and important to study the bifurcation of limit periodic set of the predator–prey system in the case when $d = 0$ and $d = c = 0$. In biology and ecology, it is also interesting to investigate the dynamics of the predator–prey systems when the death rate of predator or conversion rate remains at a very low level. For example, the larger animals like lions, they may only eat once every two or three days due to low successful hunting and unpredictable scavenging, and their average life span can reach over 16 years, therefore it is reasonable to study the predator–prey system with small parameters and consider the dynamics as a perturbation of a degenerate predator–prey system. It is also well known that when one does numerical computations to plot the trajectory of the system (1.1), it takes much longer time when the trajectory passes by the vertical $y$-axis, it becomes much longer when the death rate of the predator becomes small, it behaves like the usual relaxation oscillation phenomena observed in fast–slow systems. There are plenty of examples in ecosystems involving fast–slow process which also calls for the study of the limit case of the system (1.1).
Let us start with the case of $d = c = 0$. Then the system (1.1) is reduced to

\[
\begin{align*}
\dot{x} &= p(x)(F(x) - y), \\
\dot{y} &= 0.
\end{align*}
\]  

(1.4)

Therefore, along any orbit $y$ remains as constant, and the equilibria of the system are

\[
\{ (x, y) \mid x = 0, \ y > 0 \text{ or } 0 \leq x \leq K, \ y = F(x) \}.
\]

In other words, system (1.1) has infinite number of equilibria, which are composed of two curves: the nonnegative $y$-axis, and the curve segment $y = F(x)$ with $0 \leq x \leq K$, see Fig. 3(a). There are different types of limit periodic sets which can generate limit cycles under perturbation inside the predator-prey systems. The family of limit periodic sets and their bifurcations depends on the properties of the function $F(x)$, or in other words depends on the type of response function used in the systems. For one type of the limit periodic set which is composed of horizontal regular orbit and a part of the curve segment on $y = F(x)$, among the family of equilibria on the curve segment, there is one degenerate equilibrium with two zero eigenvalues which we will call a “turning point” on the limit periodic set. Due to the existence of the turning point, the bifurcation study of such limit periodic set is technically very difficult [13,29].

Thanks to the recent progress in the field of bifurcation theory, in particularly the singular perturbation theory developed by Dumortier and Roussarie [9–11] and the recent work of De Maesschalck and Dumortier [6–8], by which we will develop criteria to precisely study the limit cycles bifurcated from the limit periodic sets of the predator-prey systems, as well as the multiplicity of such limit cycles. By using the criteria we study the canard cycles for predator-prey systems with response functions of Holling types in the case when $d$ and $c$ are small.

The bifurcation for the system near $d = 0$ is in fact the bifurcation of degenerate graphics, like the problem in the finiteness part of Hilbert’s 16th problem for quadratic vector fields [13], see Fig. 3(b). Here, since we study the problem in the predator-prey system, the invariance of $x$ and $y$-axis may make it simpler. Due to the technical difficulties and intrinsic difference, we leave this case of $d = 0$ for a separate work.

In Section 2, we first introduce the limit periodic set and its cyclicity using a planar system of a special form, from which we define the settings and terminology to be used in the study. The so-called slow divergence integral will be introduced and used to characterize the cyclicity of the limit periodic set.

We then apply them to study the canard phenomenon in the predator-prey systems (1.1) with the functional responses of Holling types IV and III respectively in Sections 3 and 5, and put some technical proof in Section 4. We will prove that the cyclicity of limit periodic sets is at most two, and we indicate the regions in parameter space where the corresponding limit periodic set has cyclicity at most one or at most two.

Fig. 3. The limit periodic set (slow–fast cycle).
2. Cyclicity of limit periodic set and basic lemmas

We consider the system

\[
X_{\varepsilon, \lambda}: \begin{cases} 
    \frac{du}{dt} = \Phi(v, \lambda) - G(u, \lambda), \\
    \frac{dv}{dt} = \varepsilon^2(\varepsilon B_0 - u g(u, \lambda)),
\end{cases}
\]

where \(\varepsilon \geq 0, B_0 \sim 0\) are small parameters, \(\lambda\) is a multi-dimensional parameter in a compact subset of \(\mathbb{R}^p\), \(\Phi, G\) and \(g\) are \(C^\infty\) functions with the following properties:

\[\begin{array}{ll}
(H_1) \quad & \Phi(0, \lambda) = 0, \quad \frac{\partial \Phi}{\partial v}(v, \lambda) > 0 \text{ for all } v \text{ and } \lambda. \\
(H_2) \quad & G(0, \lambda) = \frac{\partial G}{\partial u}(0, \lambda) = 0, \quad \frac{\partial^2 G}{\partial u^2}(0, \lambda) > 0, \quad u \frac{\partial G}{\partial u}(u, \lambda) > 0 \text{ for } u \in [m_-, m_+] \setminus \{0\} \text{ and for all } \lambda, \text{ where } m_- < 0 < m_+. \\
(H_3) \quad & g(u, \lambda) > 0 \text{ for } u \in [m_-, m_+] \text{ and for all } \lambda.
\end{array}\]

Since \(\frac{\partial \Phi}{\partial v}(v, \lambda) > 0\), we denote the inverse function of \(\Phi\) with respect to \(v\) by \(\Phi^{-1}\). For system (2.1) with \(\varepsilon = 0\), the curve \(C_\lambda\): \([u, v]: v = \Phi^{-1}(G(u, \lambda))\) consists of singular points, it is called the slow manifold or critical curve. Note that the movements of (2.1), for \(\varepsilon > 0\) small, that are close to \(C_\lambda\) will be slow (with speed of order \(O(\varepsilon)\)) while movements not close to \(C_\lambda\) will be fast (of order \(O(1)\)).

By (H1) and (H2) the origin is the unique minimum point of \(C_\lambda\) for \(u \in (m_-, m_+)\), at each point of the slow manifold on the left side of the origin the fast dynamics is hyperbolically repelling while on the right side is hyperbolically attracting, the origin is called a canard point.

For each \(V > 0\), there is a fast orbit of the system \(X_{0, \lambda}\) through \((0, V)\), which has the \(\omega\)-limit point at \((\omega_{V, \lambda}, V)\) and the \(\alpha\)-limit point at \((\alpha_{V, \lambda}, V)\) on the slow manifold \(C_\lambda\). Here we suppose that \([\alpha_{V, \lambda}, \omega_{V, \lambda}] \subset [m_-, m_+]\) (otherwise, decreasing \(V\) will meet this assumption). The loop of this fast orbit combining with the part of slow manifold, namely \(\Gamma_V = \{(u, v): \alpha_{V, \lambda} \leq u \leq \omega_{V, \lambda}, \ v = V\} \cup \{(u, v) \in C_\lambda, \ \alpha_{V, \lambda} \leq u \leq \omega_{V, \lambda}\}\), is called a limit periodic set or a slow–fast cycle of canard type, shown in Fig. 4(a).

**Definition 2.1.** For fixed \(\lambda_0\) and \(V_0 > 0\), if there are \(\sigma > 0\) and \(\varepsilon_0 > 0\) such that for each \(\varepsilon \in (0, \varepsilon_0)\) the system \(X_{\varepsilon, \lambda_0}\) has a limit cycle \(\gamma_{\varepsilon_0}^{\lambda_0}\) in the \(\sigma\)-neighborhood of the canard limit periodic set \(\Gamma_{V_0}\) and \(\gamma_{\varepsilon_0}^{\lambda_0} \to \Gamma_{V_0}\) (in Hausdorff sense) as \(\varepsilon \to 0\), then \(\gamma_{\varepsilon_0}^{\lambda_0}\) is called a canard cycle, bifurcating from \(\Gamma_{V_0}\).

The maximal number of such canard cycles, taking into account of their multiplicities, is called the cyclicity of \(\Gamma_{V_0}\) for \(X_{\varepsilon, \lambda}\) at \((\varepsilon, \lambda) = (0, \lambda_0)\) and is denoted by \(\text{Cycl}(X_{\varepsilon, \lambda}, \Gamma_{V_0}, (0, \lambda_0))\).

By using the theory and results by Dumortier and Roussarie [9–11] and by De Maesschalck and Dumortier [6–8], we obtain the following lemma.

---

**Fig. 4.** The limit periodic set (slow–fast cycle) \(\Gamma_V\).
Lemma 2.2. Suppose that the hypotheses (H1)–(H3) are satisfied. We consider so-called slow divergence integral
\[
I(V, \lambda) = \int_{\alpha_V, \lambda}^{\beta_V, \lambda} \frac{\partial G(u, \lambda)}{\partial u} \left( \Phi^{-1}(G(u, \lambda)) \right)^2 du.
\]  
(2.2)

The following statements hold:
(A) If \( I(V_0, \lambda_0) \neq 0 \), then \( \text{Cycl}(X_{\varepsilon, \lambda}, \Gamma_V, (0, \lambda_0)) \leq 1 \).
(B) If \( I(V_0, \lambda_0) = 0 \) and \( \frac{\partial}{\partial V} (V_0, \lambda_0) \neq 0 \), then \( \text{Cycl}(X_{\varepsilon, \lambda}, \Gamma_V, (0, \lambda_0)) \leq 2 \).
(C) If \( I(V_0, \lambda_0) = 0 \) and \((V_0, \lambda_0)\) is a zero point of \( \frac{\partial}{\partial V} \) with multiplicity \( m \), then \( \text{Cycl}(X_{\varepsilon, \lambda}, \Gamma_V, (0, \lambda_0)) \leq 2 + m \).

Proof. By using the first equation of system (2.1) we have that along the slow manifold
\[
\frac{\partial \Phi}{\partial V} (V) V' = \frac{\partial G}{\partial u} (u, \lambda) u', \quad \frac{d}{d \tau} = \frac{1}{\varepsilon^2} \frac{d}{dt} |_{\varepsilon = 0}.
\]

Hence, by using the second equation of system (2.1) we obtain that the differential equation on slow manifold is given by
\[
u' = -\frac{u G(u, \lambda) \frac{\partial \Phi}{\partial V}(\Phi^{-1}(G(u, \lambda)))}{\frac{\partial G}{\partial u}(u, \lambda)}.
\]
(2.3)

By the hypotheses (H1)–(H3), the right-hand side of (2.3) is negative, and by Theorem 4, Theorems 1 and 2 of [7] for a suitable choice of \( \gamma_0 = B_0(\varepsilon, \lambda) \) there is a family of canard cycles \( \{ \gamma \} \), bifurcating from \( I_V \). The divergence integral (see (2.15)–(2.17) of [8], for example) is given by (2.2). The statements (A)–(C) follow from Theorems 2.22 and 2.23 of [8]. □

We need to change the integral (2.2) to a new form. Under condition (H2) we can define a function \( \tilde{u} = \tilde{u}(u, \lambda) \) by
\[
G(u, \lambda) = G(\tilde{u}, \lambda), \quad m_- < u < 0 < \tilde{u} < m_+.
\]
(2.4)

Hence
\[
\frac{\partial \tilde{u}}{\partial u} = \frac{\partial G}{\partial u} / \frac{\partial G}{\partial \tilde{u}} < 0, \quad u \in (m_-, 0).
\]
(2.5)

By the hypotheses (H1) and (H2) it is easy to see that \( G(u, \lambda) = G(\tilde{u}, \lambda) \) is equivalent to \( \Phi^{-1}[G(u, \lambda)] = \Phi^{-1}[G(\tilde{u}, \lambda)] \), thus \( \omega_{V, \lambda} = \tilde{u}(\alpha_{V, \lambda}) \). Let
\[
h(u, \lambda) = \frac{\partial G}{\partial u}(u, \lambda) \frac{\partial \Phi}{\partial V}(\Phi^{-1}(G(u, \lambda)))
\]
(2.6)

Lemma 2.3. Under hypotheses (H1)–(H3) the slow divergence integral (2.2) can be written as
\[
I(V, \lambda) = \int_{0}^{V} [h(u) - h(\tilde{u})]_{\tilde{u} = \tilde{u}(u, \lambda), u = G^{-1}(w, \lambda)} dw,
\]
(2.7)

where \( u = G^{-1}(w, \lambda) \) is the inverse function of \( w = G(u, \lambda) \) for \( u \in (m_-, 0) \).
Proof. We first rewrite (2.2) as
\[ I(V, \lambda) = \int_0^{\alpha_{V,\lambda}} \frac{\partial G(u, \lambda)}{\partial u} \left( \frac{\partial G(u, \lambda)}{\partial u} \right)^2 du - \int_0^{\alpha_{V,\lambda}} \frac{\partial G(z, \lambda)}{\partial z} \left( \frac{\partial G(z, \lambda)}{\partial z} \right)^2 dz. \]

Then changing variable by \( z = \tilde{u}(u) \) in the second integral and using (2.5), we have
\[ I(V, \lambda) = \int_0^{\alpha_{V,\lambda}} \frac{\partial G(u, \lambda)}{\partial u} \left( \frac{\partial G(u, \lambda)}{\partial u} \right)^2 - \tilde{u}(\tilde{u}, \lambda) \frac{\partial G(\tilde{u}, \lambda)}{\partial \tilde{u}} \left( \frac{\partial G(\tilde{u}, \lambda)}{\partial \tilde{u}} \right)^2 du, \]
that is
\[ I(V, \lambda) = \int_0^{\alpha_{V,\lambda}} \left[ h(u) - h(\tilde{u}(u)) \right] \frac{\partial G}{\partial u}(u, \lambda) du. \]

Changing the integration variable from \( u \) to \( w = G(u, \lambda) \) for \( u \in (\alpha_{V,\lambda}, 0) \), we obtain (2.7). \( \square \)

We remark that in some cases, the slow manifold is S-shaped, i.e. it has a simple minimum point and a simple maximum point, hence a limit periodic set \( \Gamma' \) may consist of two pieces of slow orbits and two pieces of fast orbits, shown in Fig. 4(b) where the minimum point and the maximum point are located at \((0, 0)\) and \((u_{M,\lambda}, v_{M,\lambda})\) respectively. The two types of limit periodic sets, like (a) and (b) of Fig. 4, are often referred to as canard slow–fast cycle without a head and respectively canard slow–fast cycle with a head, see [9] or [19] for example.

In the case of canard slow–fast cycle with a head we suppose that the system has the form (2.1) in the whole region, and the condition \( u \frac{\partial G}{\partial u} > 0 \) for \( u \in [m_-, m_+] \setminus \{0\} \) in (H2) should be replaced by
\[ \frac{\partial G}{\partial u} > 0 \quad \text{for} \quad u \in (0, u_{M,\lambda}); \quad \frac{\partial G}{\partial u} < 0 \quad \text{for} \quad u < 0 \quad \text{and} \quad u > u_{M,\lambda}. \]

The Definition 2.1 and Lemmas 2.2 and 2.3 are naturally generalized to the case of canard slow–fast cycle with a head, but the formula of slow divergence integral (2.2) and (2.7) need to be replaced respectively by
\[ I(V, \lambda) = \int_0^{\alpha_{V,\lambda}} \int_{\alpha_{V,\lambda}}^{\alpha_{V,\lambda}} \frac{\partial G(u, \lambda)}{\partial u} \left( \frac{\partial G(u, \lambda)}{\partial u} \right)^2 \]
\[ + \int_{\alpha_{V,\lambda}}^{u_{M,\lambda}} \frac{\partial G(u, \lambda)}{\partial u} \left( \frac{\partial G(u, \lambda)}{\partial u} \right)^2 du, \]
and
\[ I(V, \lambda) = \int_0^V \left[ h(u) - h(\tilde{u}(u)) \right] dw + \int_{V}^{V_{M,\lambda}} \left[ h(u) - h(\tilde{u}(u)) \right] dw, \]
where \( u = G^{-1}(w, \lambda) \) is the inverse function of \( w = G(u, \lambda) \) for \( u \in [\alpha_{M,\lambda}, 0) \), \( \tilde{u} = \tilde{u}(u, \lambda) \) and \( \tilde{u} = \tilde{u}(u, \lambda) \) is defined by
\[ G(u, \lambda) = G(\tilde{u}, \lambda) = G(\hat{u}, \lambda) \quad \text{for} \quad \alpha_{M,\lambda} < u < 0 < \tilde{u} < u_{M,\lambda} < \hat{u}. \]
Now we consider the predator–prey system (1.1). Suppose that the graph of the function \( y = F(x) \) has a simple maximum (or minimum) point at \( x = x_0 \). In certain neighborhood of \( x_0 \), we want to transform (1.1) to the form (2.1). Note that we consider (1.1) for \( x > 0 \), \( y > 0 \), and \( p(x) > 0 \) for \( x > 0 \).

**Lemma 2.4.** Suppose that there is an \( x_0 > 0 \) such that \( F'(x_0) = 0, F''(x_0) \neq 0 \), and both \( c \) and \( d \) are small parameters. Then we can choose suitable \( c \) and \( d \) such that in a neighborhood of \( x_0 \) the system (1.1) can be transformed to the form

\[
X_{\varepsilon, \lambda} : \begin{cases}
du{dt} = \Phi(v, \lambda) - G(u, \lambda), \\
dv{dt} = \varepsilon^2 (\varepsilon B_0 - u g(u, \lambda)),
\end{cases}
\tag{2.11}
\]

where

\[
g(u, \lambda) = -\frac{1}{u} \left( \frac{1}{p(u + x_0)} - \frac{1}{p(x_0)} \right), \quad \varepsilon B_0 = \frac{1}{p(x_0)} - \frac{c}{d}. \tag{2.12}
\]

Moreover, if \( x_0 \) corresponds to a local maximum point of the function \( F \), then

\[
\Phi(v, \lambda) = F(x_0)(1 - e^{-v}), \quad G(u, \lambda) = F(x_0) - F(u + x_0);
\tag{2.13}
\]

and if \( x_0 \) corresponds to a local minimum point of the function \( F \), then

\[
\Phi(v, \lambda) = F(x_0)\left(e^v - 1\right), \quad G(u, \lambda) = F(u + x_0) - F(x_0). \tag{2.14}
\]

**Proof.** We first divide the two equations of (1.1) by \( p(x) \), and still use \( dt \) for \( p(x) \, dt \). If \( x_0 \) corresponds to a maximum point, we let \( u = x - x_0 \) and \( w = -(y - F(x_0)) \), then the system becomes

\[
\begin{cases}
du{dt} = w - G(u, \lambda), \\
dw{dt} = -(F(x_0) - w)\left(c - \frac{d}{p(u + x_0)}\right).
\end{cases}
\tag{2.15}
\]

Let \( w = F(x_0)(1 - e^{-v}) \), then system (2.15) is transformed to

\[
\begin{cases}
du{dt} = \Phi(v, \lambda) - G(u, \lambda), \\
dv{dt} = d, \\
dw{dt} = \frac{d}{p(u + x_0)} - c,
\end{cases}
\tag{2.16}
\]

where \( \Phi \) and \( G \) are given by (2.13). Now we choose

\[
d = \varepsilon^2, \quad c = \varepsilon^2 \left( \frac{1}{p(x_0)} - \varepsilon B_0 \right). \tag{2.17}
\]

then the system (2.16) take the form (2.11).

The proof for \( x_0 \) corresponding to a minimum point is similar, the differences are that we let \( w = y - F(x_0) \) in the first step, and let \( w = F(x_0)(e^v - 1) \) in the second step, then change \( t \to -t \). \( \square \)
3. Canard cycles of the system with response function of Holling type IV

Suppose that the response function is taken as Holling type IV, i.e. in the system (1.1) we have
\[ p(x) = \frac{mx}{ax^2 + bx + 1}, \quad F(x) = \frac{r}{m} \left( ax^2 + bx + 1 \right) \left( 1 - \frac{x}{K} \right), \quad b > -2\sqrt{a}, K > 0. \]

As mentioned in [30], it is possible to eliminate 3 parameters by rescaling the phase variables and time. Different from the choices of [28,30], we eliminate \( a, r \) and \( m \). For this purpose, we let
\[ (x, y, t) = \left( \frac{1}{\sqrt{a}} \tilde{x}, \frac{\tilde{r}}{\sqrt{a}} \tilde{y}, \frac{\sqrt{a}}{m} \tilde{t} \right), \]
and
\[ (r, K, b, d, c) = \left( \frac{m}{\sqrt{a}} \tilde{r}, \frac{1}{\sqrt{a}} \tilde{K}, \sqrt{a} \tilde{b}, \frac{m}{\sqrt{a}} \tilde{d}, \tilde{c} \right). \]

Then still use \((x, y, t)\) and \((r, K, b, d, c)\) to express \((\tilde{x}, \tilde{y}, \tilde{t})\) and \((\tilde{r}, \tilde{K}, \tilde{b}, \tilde{d}, \tilde{c})\) respectively, the form of system (1.1) keeps the same, but with \( a = r = m = 1 \), i.e. system (1.1) becomes
\[ \dot{x} = p(x)(F(x) - y), \quad \dot{y} = y(-d + cp(x)), \quad (3.1) \]
where
\[ p(x) = \frac{x}{x^2 + bx + 1}, \quad F(x) = \left( x^2 + bx + 1 \right) \left( 1 - \frac{x}{K} \right), \quad b > -2, K > 0. \quad (3.2) \]

The graph of the function \( y = F(x) \), denoted by \( C_F \), has at most one minimum point and at most one maximum point, we denote such an extreme point by \( x_0 \), then
\[ 3x_0^2 + 2(b - K)x_0 + 1 - Kb = 0, \quad (3.3) \]
and the minimum point \( x_0^- \) and maximum point \( x_0^+ \) are given by
\[ x_0^\pm = \frac{K - b \pm \sqrt{M}}{3}, \quad (3.4) \]
where \( M = K^2 + bK + b^2 - 3 \geq 0 \). By Lemma 2.4 we find
\[ g(u, \lambda) = -\frac{1}{u} \left( \frac{1}{p(u + x_0)} - \frac{1}{p(x_0)} \right) = \frac{1 - x_0(u + x_0)}{x_0(u + x_0)}, \quad (3.5) \]
where \( \lambda = (b, K) \). Hence
\[ g(0, \lambda) = \frac{1 - x_0^2}{x_0^2}, \quad g'(0, \lambda) = -\frac{1}{x_0^2}. \quad (3.6) \]

Therefore, to keep the validity of the hypothesis (H3), we need a condition \( 0 < x_0 < 1 \). According to the results of the paper by Zhu, Campbell and Wolkowicz [30], this condition means that we restrict
Fig. 5. The relative position of the curves.

the parameters \((b, K)\) in some regions of the \((b, K)\)-plane, see the proof of Lemma 4.1 for details. We let

\[
M = K^2 + bK + b^2 - 3, \quad N = (K + b)^2 - 4, \tag{3.7}
\]

and define the following curves and straight lines (see Fig. 5):

\[
C_0 = \left\{ (4b^2 - 5bK + 4K^2 - 30) \sqrt{M} + (b - K)(4b^2 + bK + 4K^2 - 18) = 0, \ -1 < b < 1 \right\};
\]

\[
C_1 = \{ M = 0, \ -1 < b < 1 \}; \quad C_2 = \{ bK = 1, \ 0 < b < +\infty \};
\]

\[
C_2^{(1)} = \{ bK = 1, \ 0 < b < 1 \}; \quad C_2^{(2)} = \{ bK = 1, \ 1 < b < +\infty \};
\]

\[
C_3 = \{ bK + 2 = 0, \ -1 < b < 1 - \sqrt{3} \};
\]

\[
C_4 = \{ 2\sqrt{M} + (1 - bK)\sqrt{N} + (bK + 1)(K - b) = 0, \ -\sqrt{2} < b < 1 - \sqrt{3} \};
\]

\[
C_5 = \left\{ (K - b)(bK + 2) - (5 - 2bK)\sqrt{M} = 0, \ -1 < b < \frac{\sqrt{22} - \sqrt{6}}{4} \right\};
\]

\[
C_6 = \left\{ 2\sqrt{M} + (1 - bK)\sqrt{N} - (bK + 1)(K - b) = 0, \ \frac{\sqrt{22} - \sqrt{6}}{4} < b < +\infty \right\};
\]

\[
L_1^{(1)} = \{ K = 2, \ -2 < b < -1 \}; \quad L_1^{(2)} = \{ K = 2, \ -1 < b < +\infty \};
\]

\[
L_2 = \{ b = -2, \ K \geq 2 \}; \quad L_3 = \{ K + b = 2, -2 < b < 1 \};
\]

\[
L_3^{(1)} = \{ K + b = 2, 1 < \sqrt{3} < b < 1 \}; \quad L_3^{(2)} = \{ K + b = 2, -2 < b < 1 - \sqrt{3} \}.
\]

**Remark.** It is obvious that the curve \(C_1\), joining points \(P(-1, 2)\) and \(Q(1, 1)\), is a part of the ellipse \(E: \{ M = 0 \}\); each of \(C_2^{(1)}, C_2^{(2)}\) and \(C_3\) is a part of a branch of hyperbola. In Section 4 we will prove that \(C_0\) defines a smooth curve, which is entirely located in the region bounded by \(C_1, C_3\) and \(L_3^{(1)}\) and can be expressed as \(K = \mu_0(b)\) for \(-1 < b < 1\), satisfying \(\lim_{b \to -1^+} \mu_0(b) = 2\) and \(\lim_{b \to 1^-} \mu_0(b) = 1\),
For convenience we will use the notation $C_i$ to refer to the curves $C$. Fig. 5 can be studied similarly. Note that the curves $C_i$ of $X$ in Fig. 6 shows the relation of this manifold with the slow manifold of system (1.1), given by

$$\{C_i\} = \left\{ \frac{F(u + x_0^-)}{F(x_0^+)} \right\} \text{ and } \left\{ \frac{F(u + x_0^+)}{F(x_0^-)} \right\}$$

Proof. If $M > 0$ then $x_0^\pm$, defined in (3.4), exist and $x_0^- < x_0^+$. By Lemma 2.4 we can transform system (1.1) to the form (2.11) with $x_0 = x_0^-$ or $x_0 = x_0^+$, i.e. to $X_{x_0^-}$ or $X_{x_0^+}$ respectively. By (2.12), (2.14) (resp. (2.13)), the conditions (H1) and (H2) in Section 2 are satisfied. If in addition $x_0^- < 1$ (resp. $x_0^+ < 1$), then the condition (H3) is also satisfied, see (3.6). Note that by (2.14) (resp. (2.13)) the slow manifold of $X_{x_0^-}$ (resp. of $X_{x_0^+}$) is given by

$$v = \ln \frac{F(u + x_0^-)}{F(x_0^+)}$$

Fig. 6 shows the relation of this manifold with the slow manifold of system (1.1), given by $y = F(x)$. For convenience we will use $y = F(x)$ to express the limit periodic set, and will say “slow manifold of $x_0^-$” or “slow manifold of $x_0^+$” respectively, to distinguish the two cases in Fig. 6.
Thus, the existence of a limit periodic set for $X_{0,\lambda}^-$ depends on 3 conditions: (1) $M > 0$; (2) $x_0^- > 0$, i.e. the minimum point $(x_0^-, F(x_0^-))$ is located in the first quadrant; (3) $x_0^- < 1$.

Geometrically, the condition (1) shows that $(b, K)$ is located outside the ellipse $E : \{M = 0\} \supset C_1$, see Fig. 5. By (3.4) the condition (2) implies $K - b > 0$ and $bK - 1 < 0$. Note that the ellipse $E$ is tangent to the hyperbola $(bK - 1 = 0)$ at the point $Q(1, 1)$, and the straight line $(K - b = 0)$ passes this point. At last, condition (3) gives $K - b - 3 \leq 0$, or $K - b - 3 > 0$ with $(b + 2)(K - 2) > 0$. Note that the straight line $(K - b - 3 = 0)$ passes the point $P(-1, 2)$, which is the tangent point of the ellipse $E$ and the straight line $(K = 2)$. Combining the 3 conditions, we find that $(b, K)$ must be located in the region $R_1$, see Fig. 5 and Fig. 8(i).

Similarly, the existence of a limit periodic set for $X_{0,\lambda}^+$ containing $(0, 0)$ depends on 3 conditions: (a) $M > 0$, (b) $x_0^+ > 0$ and (c) $x_0^+ < 1$. The condition (a) is the same as (1). Condition (b) gives $K - b \geq 0$, or $K - b < 0$ and $bK - 1 > 0$. Condition (c) implies $K - b - 3 \leq 0$ and $(b + 2)(K - 2) < 0$. Therefore $(b, K) \in R_2$, see Fig. 5 and Fig. 8(ii).

Suppose $(b, K) \in R_1$, a limit periodic set $\Gamma_Y$ of $X_{0,\lambda}^-$ has some critical cases. First, if $F(0) > F(x_0^+)$, then the possible minimum value of x-axis of $\Gamma_Y$ is $x_1^-$, satisfying $F(x_1^-) = F(x_0^+)$, see Fig. 7(i). If $F(0) < F(x_0^+)$, then the possible maximum value of x-axis of $\Gamma_Y$ is $x_2^-$, satisfying $F(x_2^-) = F(0)$, see Fig. 7(ii). Another fact is the possibility of existence of a saddle point on the slow manifold. From (2.3) we see that this possibility comes from a zero point of $g(u, \lambda)$ for $u = \bar{u} \neq 0$, and such $\bar{u}$ gives $m_+ \in$ in hypothesis (H3). If fact, $g(u, \lambda) > 0$ for $u \in (m_-, m_+)$ and $g_u'(u, \lambda) < 0$, by (3.5), imply $\bar{u} > 0$.

Since $u = x - x_0$, using (3.5) again we have

$$
\tilde{g}^\pm(x, \lambda) = g^\pm (x - x_0^\pm, \lambda) = \frac{1 - x_0^\pm x}{x_0^\pm x}.
$$

(3.8)
It is clear that the saddle point is given by $\hat{x} = \frac{1}{x_0} > x_0^-$, since $x_0^- \in (0, 1)$ for $(b, K) \in R_1$. The discussion for $X^+_{0, \lambda}$ with $(b, K) \in R_2$ is similar.

Thus, we have the following results.

**Lemma 3.2.** The following statements hold for $(b, K) \in R_1$:

1. There exists $\hat{x}_1 < x_0^-$, such that $F(\hat{x}_1) = F(x_0^+)$ and $\hat{x}_1 = 0$ (resp. $> 0$ or $< 0$), if and only if $(b, K) \in \bar{L}_3 = \bar{L}_3^{(1)} \cup \bar{L}_3^{(2)}$ (resp. below or above it), see Figs. 7(i) and 8(i).

2. A saddle point on the slow manifold of $x_0^-$ is located at (resp. right or left to) the maximum point $(x_0^+, F(x_0^+))$, if and only if $(b, K)$ is below $L_3$ and $(b, K) \in C_3$ (resp. left or right to $C_3$). See Figs. 7(i), 8(i) and 1-3 to 1-5 of Fig. 9.

3. A saddle point on the slow manifold of $x_0^-$ is located at (resp. right or left to) the point $(\hat{x}_2, F(\hat{x}_2))$, where $\hat{x}_2 \in (x_0^-, x_0^+)$ and $F(\hat{x}_2) = F(0)$, if and only if $(b, K)$ is above $L_2$ and $(b, K) \in C_4$ (resp. right or left to $C_4$). See Figs. 7(ii), 8(i) and 1-2 to 1-11 of Fig. 9.

**Proof.** We first find the expression of $\hat{x}_1$. Observing from Fig. 7(i) and (3.2) we have

$$F(x) - F(\hat{x}_1) = -\frac{1}{K}(x - \hat{x}_1)(x - x_0^+)^2.$$ 

Making derivative on above equality with respect to $x$, taking $x = x_0^-$ and using (3.4) we obtain

$$\hat{x}_1 = x_0^- - \frac{1}{2}(x_0^+ - x_0^-) = \frac{K - b - 2\sqrt{M}}{3} = \frac{-N}{K - b + 2\sqrt{M}},$$ (3.9)

where $M$ and $N$ are given in (3.7). For $\lambda = (b, K) \in R_1$ we have $K - b > 0$ and $K + b + 2 > 0$, and the statement 1 follows immediately from (3.9).

Secondly, from statement 1 we see if $\hat{x}_1 < 0$, i.e. $F(x_0^+) > F(0)$, then the point $(x_0^+, F(x_0^+))$ never be located in a limit periodic set for slow manifold of $x_0^-$ for $x > 0$. So we consider the case $\hat{x}_1 > 0$ (the case $\hat{x}_1 = 0$ is critical). From (3.8) and above discussion we know that a saddle point on the slow manifold of $x_0^-$ is located at (resp. right or left to) the maximum point $(x_0^+, F(x_0^+))$ if and only if

$$\hat{g}^{-}(x_0^+, \lambda) = 0 \quad (> 0 \text{ or } < 0) \Leftrightarrow \frac{1}{x_0^+} - x_0^- = 0 \quad (> 0 \text{ or } < 0).$$ (3.10)
That is
\[
\frac{K - b + \sqrt{M}}{1 - bK} - \frac{K - b + \sqrt{M}}{3} = \frac{(bK + 2)(K - b + \sqrt{M})}{3(1 - bK)} = 0 \quad (>0 \text{ or } <0).
\]

In region $R_1$ we have $K - b + \sqrt{M} > 0$ and $1 - bK > 0$, and from above expression we get $bK + 2 = 0$ ($>0$ or $<0$), $-1 < b < 0$. The statement 2 is proved.

At last, the existence of $\hat{x}_2$ implies $\hat{x}_1 < 0$, i.e. $(b, K)$ is above $L_3$, which implies $N > 0$. From $F(\hat{x}_2) = F(0) = F(x_0^*)$ for $\hat{x}_2 < x_0^* < x_0^*$ (Figs. 7(ii) and 7(iv)) we find
\[
\hat{x}_2 = \frac{K - b - \sqrt{N}}{2}, \quad x_0^* = \frac{K - b + \sqrt{N}}{2}.
\]

Hence, by (3.8), $g^- (\hat{x}_2, \lambda) = 0$ ($>0$ or $<0$) is equivalent to
\[
\frac{1}{x_0^*} - \hat{x}_2 = \frac{2\sqrt{M} + (1 - bK)\sqrt{N} + (bK + 1)(K - b)}{2(1 - bK)} = 0 \quad (>0 \text{ or } <0).
\]

For $(b, K) \in R_1$ one has $1 - bK > 0$, $-2 < b < 1$ and $K > 1$, and $2\sqrt{M} + (1 - bK)\sqrt{N} + (bK + 1)(K - b) = 0$ defines a strictly monotonic curve $C_4$, which tends to the point $A = L_3 \cap C_3$ and is tangent to $L_3$ at $A$ for $b \to (1 - \sqrt{3})^-$, and has a vortical asymptotic line $\{b = -\sqrt{2}\}$ for $b \to -\sqrt{2}^+$, see Fig. 5. Hence, the statement 3 is proved. \(\Box\)

By the same way we can prove the following results, we omit the details. Note that $x_2^*$ is given in (3.11), and by the same way of computing $\hat{x}_1$ we find $x_1^* = x_0^* + \frac{1}{2}(x_0^* - x_0^*)$, see Fig. 7(iii).

**Lemma 3.3.** The following statements hold for $(b, K) \in R_2$:

1. A saddle point on the slow manifold of $x_0^+$ is located at (resp. left or right to) the point $(x_1^*, F(x_1^*))$, where $x_1^* > x_0^*$ and $F(x_1^*) = F(x_0^*)$, if and only if $(b, K)$ is left to $C_2^{(1)}$ and $(b, K) \in C_5$ (resp. above or below $C_5$), see Figs. 7(iii), 8(ii) and II-3 to II-5 of Fig. 9.

2. A saddle point on the slow manifold of $x_0^+$ is located at (resp. left or right to) the point $(x_2^*, F(x_2^*))$, where $x_2^* > x_0^*$ and $F(x_2^*) = F(0)$, if and only if $(b, K)$ is right to $C_2^{(1)}$ and $(b, K) \in C_6$ (resp. above or below $C_6$), see Figs. 7(iv), 8(ii) and II-9 to II-11 of Fig. 9.

Now we divide $R_1, R_2$ and $\Delta = R_1 \cap R_2$ into some sub-regions and define a region $D$ as follows, see Fig. 8 (also see the global picture in Fig. 5):

- $R_1$ is divided into $R_1^1$ ($i = 1 - 4$) by $L_3 = L_3^{(1)} \cup L_3^{(2)}, C_3$ and $C_4$.
- $R_2$ is divided into $R_2^1$ ($i = 1 - 4$) by $C_2^{(1)} = C_2^{(1a)} \cup C_2^{(1b)}, C_5$ and $C_6$.
- $\Delta$ is divided into $\Delta^1$ ($i = 1 - 4$) by $C_5$ and a part of $L_3^{(1)}$.
- $D$ is the moon-like region bounded by $C_0$ and a curve $\gamma_1$. We will show in the proof of next theorem that $\gamma_1 = \{(b, K): K = K_1(b), -1 < b < 1\}$, where $K_1(b)$ is a smooth function, $\lim_{b \to -1} K(b) = 2$ and $\lim_{b \to 1} K(b) = 1$, i.e. $\gamma_1$ joins the points $P$ and $Q$. Besides, $\gamma_1$ is entirely located in the region bounded by $C_0, C_3$ and $L_3^{(1)}$.

We list all possible limit periodic sets in Fig. 9. For $\lambda \in R_1$, the generic case is shown in type I-2. Its “minimum size” is I-1 (it shrinks to the singular point); its “maximum size” depends on two facts: the relative positions of the saddle point and the maximum point $(x_0^+, F(x_0^+))$ on the slow manifold, and the relative values of $F(0)$ and $F(x_0^+)$. For example, if $\lambda = (b, K)$ takes value from $R_1^1$ passing $C_3$ to $R_1^2$.
(see Fig. 8(ii)) then the possible maximum size of the limit periodic set is I-3, I-4 and I-5 respectively. Similarly, it is I-6, I-7 and I-8 if ‚ for Lemma 2.4 and 3.1, around the local minimum point of position of a saddle point on the slow manifold (see Lemmas 3.2 and 3.3).

Suppose that c and d are chosen as in Theorem 3.4. Hence we give the following results only for the generic cases.

For the types I-1 and II-1, some techniques, like in the work by Dumortier and Roussarie [12], are needed.

III-2 (resp. III-3) is possible. can be a jump point and the saddle point can be excluded from the limit periodic set, so only type III-2 (resp. III-3) is possible.

To study the cyclicity of the limit periodic sets for the critical cases one needs some additional arguments, besides Lemmas 2.2 and 2.3. For example, to study the cyclicity of limit periodic sets of the types I-1 and II-1, some techniques, like in the work by Dumortier and Roussarie [12], are needed. Hence we give the following results only for the generic cases.

**Theorem 3.4.** Suppose that c and d are chosen as in (2.17), where ε > 0 small. For the system (3.1) the following statements hold, depending on the value of B₀ = B₀(ε, b, K):

(A) The limit periodic set of type I-2 exists if ‚ ∈ R₁, its cyclicity is at most one for ‚ ∈ R₁ \ D, and is at most two for ‚ ∈ D.

(B) The limit periodic set of type II-2 exists if ‚ ∈ R₂, its cyclicity is at most one for all ‚ ∈ R₂.

(C) (i) The limit periodic set of type III-1 exists if ‚ ∈ Δ₁, and its cyclicity is at most one for all ‚ ∈ Δ₁;

(ii) The limit periodic set of type II-2 exists if ‚ ∈ Δ and left to L³, its cyclicity is at most one if ‚ is located on or below γ₁, and is at most two if ‚ above γ₁;

(iii) The limit periodic set of type III-3 exists if ‚ ∈ Δ and below C₅, its cyclicity is at most one for all possible ‚.

**Proof.** The existence of different types of limit periodic sets follow from Lemmas 3.1 to 3.3. We study their cycilities. By Lemmas 2.4 and 3.4, around the local minimum point of C₅ at x₀ = x₀⁺ (or maximum point at x₀ = x₀⁻), the system (3.1) can be transformed to the form (2.11), where u = x - x₀, ‚ = (b, K) ∈ R₁ (or R₂), and

\[g(u, ‚) = \frac{1 - x₀(u + x₀)}{x₀(u + x₀)},\]

\[Φ(ν, ‚) = ±F(x₀)(e^{±ν} - 1) \quad \text{if } x₀ = x₀⁺,\]

\[G(u, ‚) = ±[F(u + x₀, ‚) - F(x₀, ‚)] = ±\frac{1}{K}[(K - b - 3x₀) - u]u² \quad \text{for } x₀ = x₀⁻. \quad (3.12)\]

It is easy to check that the hypotheses (H₁) and (H₂) are satisfied, where m₋ = -x₀, m₊ = x₀⁺ - x₀⁻ = \(\frac{2}{\sqrt{K}}\) if x₀ = x₀⁺, and m₋ = max(-(-x₀⁺ - x₀⁻), -x₀⁻), m₊ = K - x₀⁺ if x₀ = x₀⁻; the condition (H₃) is also satisfied, but in some cases we need to decrease the value of m₊ according to the position of a saddle point on the slow manifold (see Lemmas 3.2 and 3.3).

Since Φ(ν, ‚) = F(x₀)ν + ⋯ for ν ~ 0, and G(u, ‚) is a cubic polynomial in u, as discussed in Section 2, a family of canard cycles may exist by perturbing the limit periodic set, see the fundamental work [9].

To study the cyclicity of the limit periodic sets, we need to consider the slow divergence integral (2.2) or (2.7) for ‚ ∈ R₁ or R₂, and (2.9) or (2.10) for ‚ ∈ Δ = R₁ \ R₂. We first consider the case x₀ = x₀⁺. By (2.6) and (3.12) we have
\[ h(u, \lambda) = \frac{x_0[2(K-b) - 3(u + 2x_0)](u + x_0)}{K[1 - x_0(u + x_0)][\frac{d\Phi}{dy}(\Phi^{-1}(G(u, \lambda)))]} = \frac{k(\zeta - 3x)x}{(1 - x_0x)F(x, \lambda)}, \]

where \( x = u + x_0, k = x_0/K, \zeta = 2(K-b) - 3x_0 \).

Let \( \bar{x} = \bar{x}(x, \lambda) \) be defined by \( F(x, \lambda) = F(\bar{x}, \lambda) \) for \( 0 < x < x_0 < \bar{x} < x_0^+ \). Since \( G(u, \lambda) = F(x, \lambda) - F(x_0, \lambda) \), see (2.14), the definition of \( \bar{x} = \bar{x}(x, \lambda) \) corresponds to the definition of \( \bar{u} = \bar{u}(u, \lambda) \) in (2.4) for \( -x_0 < u < 0 < \bar{u} < x_0^* - x_0^* \). Note that \( F(x, \lambda) > 0 \) for \( 0 < x < x_0 \), hence by the formula (2.7) we have

\[ I(Y, \lambda) = \int_{Y_0}^{Y} \frac{k(x - \bar{x})(3x_0x\bar{x} - 3(x + \bar{x}) + \zeta)}{F(x, \lambda)(1 - x_0x)(1 - x_0\bar{x})} \bigg|_{\bar{x}=\bar{x}(x, \lambda), x=F^{-1}(y, \lambda)} dy, \quad (3.13) \]

where \( Y_0 = F(x_0, \lambda) > 0 \). Note that \( (x_0, F(x_0)) \) is the minimum point of \( F \), hence \( Y > Y_0 \), and \( F^{-1}(y, \lambda) \) is the inverse function of \( y = F(x, \lambda) \) for \( x \in (0, x_0) \).

We rewrite (3.13) as

\[ I(Y, \lambda) = \int_{Y_0}^{Y} \varphi(x(y), \lambda)\psi(x(y), \lambda) dy, \quad (3.14) \]

where \( x(y) = F^{-1}(y, \lambda) \in (0, x_0) \) for \( y \in (Y_0, Y) \), and

\[ \varphi(x, \lambda) = \frac{k(x - \bar{x}(x, \lambda))}{F(x, \lambda)(1 - x_0x)(1 - x_0\bar{x}(x, \lambda))}, \]

\[ \psi(x, \lambda) = 3x_0x\bar{x}(x, \lambda) - 3(x + \bar{x}(x, \lambda)) + \zeta. \quad (3.15) \]

Since \( \varphi(x, \lambda) < 0 \) for \( x \in (0, x_0) \) and all admissible \( \lambda \), we can use the Mean Value Theorem for integrals to (3.14) and obtain

\[ I(Y, \lambda) = \psi(x(\theta), \lambda) \int_{Y_0}^{Y} \varphi(x(y), \lambda) dy, \quad (3.16) \]

where \( \theta \in (Y_0, Y) \).

We will show the existence of the curve \( \gamma_1 \), and prove that for any fixed \( \lambda = (b, K) \in D \), there is a unique \( Y = Y(\lambda) > Y_0 \) such that \( I(Y(\lambda), \lambda) = 0 \) and \( \frac{\partial I}{\partial Y}(Y(\lambda), \lambda) \neq 0 \). Moreover, \( Y(\lambda) \to Y_0 \) when the point \( \lambda = (b, K) \) tends to the lower boundary \( C_0 \) from \( D \) and \( Y(\lambda) \to Y_M \), the “maximal level” of \( Y \), when \( \lambda \) tends to the upper boundary \( \gamma_1 \) from \( D \). On the other hand, for any \( \lambda \in R_1 \setminus D \) and \( Y \in (Y_0, Y_M) \) we will prove \( I(Y, \lambda) \neq 0 \). By Lemma 2.2, this gives the statement (A) of the theorem.

We split our proof into two steps.

**Step 1.** The study of the function \( \psi(x, \lambda) \).

For simplicity of the notations, we will often use \( F(x) \) and \( \bar{x}(x) \) instead of \( F(x, \lambda) \) and \( \bar{x}(x, \lambda) \), if there is no confusion. By using \( F(x) = F(\bar{x}) \) and \( x < \bar{x} \) we find

\[ (\bar{x}^2 + x^2) + (b - K)(\bar{x} + x) + \bar{x}x + 1 - bK = 0. \quad (3.17) \]
We let
\[ \xi = \tilde{x} + x, \quad \eta = \tilde{x}x. \quad (3.18) \]

Then (3.17) gives
\[ \eta = \xi^2 + (b - K)\xi + 1 - bK. \quad (3.19) \]

Substituting (3.19) in the second expression of (3.15), we find
\[ \dot{\psi} (\xi, \lambda) = \dot{\psi} (x(\xi), \lambda) = 3x_0\xi^2 + 3[(b - K) - 1]\xi + 2K - b - 3x_0bK, \quad (3.20) \]

where \( x_0 = x_0^- \) and \( x = x(\xi) \) is the inverse function of \( \xi = x + \tilde{x}(x) \), which is globally monotone for \( x \in (0, x_0^-) \), because for \( 0 < x < x_0^- < \tilde{x} < x_0^+ \) we have \( F'(x) < 0 < F'(\tilde{x}) \) and
\[ \xi_1 = 1 + \tilde{x}'(x) = \frac{F'(x) + F'(\tilde{x})}{F'\tilde{x}}. \]

Let us prove that the numerator is negative. By using (3.2) and (3.19) we have
\[
F'(x) + F'(\tilde{x}) = -3(x^2 + \tilde{x}^2) + 2(K - b)(x + \tilde{x}) + 2(bK - 1)
= 6\eta - 3\xi^2 + 2(K - b)\xi + 2(bK - 1)
= 3\xi^2 + 4b - K)\xi + 4(1 - bK).
\]

Denote the above expression by \( \omega(\xi) \), which is a quadratic polynomial in \( \xi \). We let
\[ \xi_m(\lambda) = 2x_0^- = \frac{2(K - b - \sqrt{M})}{3} \quad < \xi_M(\lambda) = \hat{x}_1 + x_0^+ = \frac{2(K - b) - \sqrt{M}}{3}. \quad (3.21) \]

For \( \lambda \in D \) we have \( F(0) > F(x_0^+) \), i.e. \( \hat{x}_1 > 0 \) (see Fig. 7(i) and (3.9)), and by calculation we obtain
\[
\omega(0) = 4(1 - bK) > 0, \quad \omega(\xi_m(\lambda)) = 0, \quad \omega(\xi_M(\lambda)) = -M < 0.
\]

hence \( \omega(\xi) < 0 \) for all \( \xi \in (\xi_m(\lambda), \xi_M(\lambda)) \), i.e. \( \xi_1 < 0 \) for all \( x \in (\hat{x}_1, x_0^-) \). Note that \( \xi \to \xi_m(\lambda) \) as \( x \to x_0^- \) and \( \xi \to \xi_M(\lambda) \) as \( x \to \hat{x}_1 \), where \( \xi_m(\lambda) \) and \( \xi_M(\lambda) \) are the minimum and maximum values of \( \xi \). On the other hand, \( y \to Y_0 \) \( (x \to x_0^- - 0) \) and \( y \to Y_M \) \( (x \to \hat{x}_1 + 0) \) correspond to the minimum and maximum values of \( y \) for a possible limit periodic set \( \Gamma_y \), since \( x_y < 0 \) for \( x \in (\hat{x}_1, x_0^-) \subset (0, x_0^-) \) where \( y = F(x, \lambda) \). See Fig. 7(i) and Lemma 3.2.

**Assertion 1.** For \( \lambda = (b, K) \in R_1 \) we have that
(a) \( \psi(\xi_m(\lambda), \lambda) = 0 \) (resp. \( < 0 \) or \( > 0 \)) if \( \lambda \in C_0 \) (resp. below or above \( C_0 \));
(b) \( \psi(\xi_M(\lambda), \lambda) = 0 \) (resp. \( < 0 \) or \( > 0 \)) if \( \lambda \in \bar{C} = C_3 \cup \{ A \} \cup L_2^{(1)} \) (resp. below or above \( \bar{C} \));
(c) \( \psi(\xi_m(\lambda), \lambda) < 0 < \psi(\xi_M(\lambda), \lambda) \) if \( \lambda \in G \), bounded by \( C_0 \) and \( \bar{C} \);
(d) For each \( \lambda \in G \), \( \psi(\xi, \lambda) = 0 \) has a unique solution \( \xi = \xi^{*\lambda} \in (\xi_m(\lambda), \xi_M(\lambda)) \), continuous in \( \lambda \) and satisfying
\[ (\xi - \xi^{*\lambda}) \psi(\xi, \lambda) < 0 \quad \text{for} \quad \xi \in (\xi_m(\lambda), \xi_M(\lambda)) \setminus \{ \xi^{*\lambda} \}. \quad (3.22) \]
To verify this assertion, we first substitute $\xi = \xi_m(\lambda) = 2x_0^\prime$ in (3.20), and using (3.3) and (3.4) we obtain

$$\tilde{\psi}(\xi_m(\lambda), \lambda) = \frac{1}{9} \left[ (5bK - 4b^2 - 4K^2 + 30) \sqrt{M} + (K - b)(4b^2 + Kb + 4K^2 - 18) \right].$$

In Section 4 we will prove that $\tilde{\psi}(\xi_m(\lambda), \lambda) = 0$ defines a unique curve $C_0$ in the region $R_1$, it can be expressed as $\{(b, \mu_0(b)) : -1 < b < 1\}$ and joints the points $P$ (as $b \to -1$) and $Q$ (as $b \to 1$) (the same endpoints of the curve $C_1$). Moreover, $C_0$ is located above $C_1$ and below $\tilde{C}$ for $-1 < b < 1$, see Fig. 5 and Fig. 8(iv). It is clear now that $\psi(\xi_m(\lambda), \lambda) = 0$ (resp. $< 0$ or $> 0$) if and only if $\lambda \in C_0$ (resp. below or above $C_0$).

On the other hand, substituting $\xi = \xi_m(\lambda)$ in (3.20), we obtain

$$\tilde{\psi}(\xi_m(\lambda), \lambda) = -\frac{1}{3}(bK + 2)(K - b - 2\sqrt{M}),$$

which is zero (resp. $< 0$ or $> 0$) if and only if $\lambda \in \tilde{C}$ (resp. below or above $\tilde{C}$), because it is easy to check that $K - b - 2\sqrt{M} = 0$ is equivalent to $b + K = 2$ for $(b, K) \in R_1$.

Thus the Assertion 1(a) and (b) are proved. The statement (c) follows immediately from (a) and (b), and (d) follows from (c) and the fact that $\tilde{\psi}(\xi, \lambda)$ is a quadratic polynomial in $\xi$ for each fixed $\lambda = (b, K) \in G$, see Fig. 10(i).

From the statements (a) and (b) of Assertion 1 we see that $\psi(\xi, \lambda) < 0$ (resp. $> 0$) if $\lambda \in R_1$ and below $C_0$ (resp. above $\tilde{C}$), hence by (3.14) and (3.15) we have $I(Y, \lambda) > 0$ (resp. $< 0$) for $Y \in (Y_0, Y_M)$. The result certainly can be extended to $\lambda \in C_0$ (resp. $\lambda \in \tilde{C}$), because in this case $\psi(\xi, \lambda)$ is zero only at the end point $\xi = \xi_m(\lambda)$ (resp. $\xi = \xi_M(\lambda)$). Therefore we have $I(Y, \lambda) \neq 0$ for $\lambda \in R_1 \setminus G$. Below we study the case $\lambda \in G$, and will extend $I(Y, \lambda) \neq 0$ from $\lambda \in R_1 \setminus G$ to $\lambda \in R_1 \setminus D$ in the statement (c) of Assertion 3.

For any $\lambda \in G \subset R_1^1$, we have $\dot{x}_1 > 0$ (i.e. $F(0) > F(x_0^+)$), hence $\xi$ takes values from $\xi_m(\lambda)$ to $\xi_M(\lambda)$. Since $\xi'(x) < 0$ and $x'(y) < 0$, where $y = F(x)$ and $x \in (0, x_0)$, we have $\xi = \xi(y)$ with $\xi'(y) > 0$, and the inverse function $y = y(\xi) = F(x(\xi))$ satisfies $y(\xi_m(\lambda)) = Y_0(\lambda) = F(x_0^+)$, $y(\xi_M(\lambda)) = Y_M(\lambda) = F(x_M^+)$ and $Y_\lambda^* = y(\xi_\lambda^*) \in (Y_0(\lambda), Y_M(\lambda))$. Hence we could transform the results for $\xi_\lambda^*$ in Assertion 1 to $Y_\lambda^*$ as follows.

**Assertion 2.** For each $\lambda \in G$ there is a unique $Y_\lambda^* \in (Y_0(\lambda), Y_M(\lambda))$, continuous in $\lambda$ and satisfying

$$y - Y_\lambda^* \psi(x(y), \lambda) < 0 \quad \text{for} \quad y \in (Y_0(\lambda), Y_M(\lambda)) \setminus \{Y_\lambda^*\}. \tag{3.23}$$

Moreover, $\lim_{K \to \mu_0(b)} Y_\lambda^* = Y_0(\lambda)$ and $\lim_{K \to \mu_1(b)} Y_\lambda^* = Y_M(\lambda)$, here $(b, \mu_0(b)) \in C_0$ and $(b, \mu_1(b)) \in C_3 \cup \{A\} \cup L_3^{(1)}$ for $b \in (-1, 1)$.
By direct computations we have that for $\lambda = (b, K) \in G$

$$\sigma(\lambda) := Y_M(\lambda) - Y_0(\lambda) = \frac{4\sqrt{M^2}}{27K} > 0,$$

$$\frac{\partial \sigma(\lambda)}{\partial K} = \frac{2\sqrt{M(4K^2 + bK - 2b^2 + 6)}}{27K^2} > 0. \quad (3.24)$$

**Step 2.** The study of the slow divergence integral $I(Y, \lambda)$.

Let $Y_\ell(\lambda) = Y_0(\lambda) + \epsilon(Y_M(\lambda) - Y_0(\lambda))$, $\ell \in (0, 1)$. Recall that $Y_0$ and $Y_M$ are the minimum and maximum values of $Y$ for a possible limit periodic set $G$ if $\lambda \in G$.

**Assertion 3.** The following statements hold:

(a) For any fixed $b \in (-1, 1)$ and $\ell \in (0, 1)$, there is a unique $K = K_\ell(b) \in ([0, b], (0, b))$ such that for $\lambda = (b, K_\ell(b))$ the cyclicity of the limit periodic set $\Gamma_{Y_\ell(\lambda)}$ is at most two, and if $K = K_\ell(b)$ then the cyclicity of the limit periodic set $\Gamma_{Y_\ell(\lambda)}$ for $\lambda = (b, K)$ is at most one.

(b) For any fixed $\ell \in (0, 1)$, the set $Y_\ell = \{\lambda \in G : \lambda = (b, K_\ell(b)), -1 < b < 1\}$ forms a curve in $G$, and $\lim_{b \to -1+0}(b, K_\ell(b)) = P$ and $\lim_{b \to 1-0}(b, K_\ell(b)) = Q$ for all $\ell \in (0, 1)$. Moreover, $K_0(b) = \lim_{\ell \to 0+0}K_\ell(b) = \mu_0(b)$, i.e., $\gamma_0 = 0$. and $K_1(b) = \lim_{\ell \to 1-0}K_\ell(b) \in ([0, b], (0, b))$, i.e., $\gamma_1 = \{(b, K_1(b)) : -1 < b < 1\}$ is located strictly between $C_0$ and $L_3 \cup \{A\} \cup L_3(1)$, see Fig. 8(iv).

(c) The curves $\{\gamma_\ell : 0 \leq \ell \leq 1\}$ give a foliation of the region $C_0 \cup D \cup \gamma_1$. For any $\lambda \in D$, the cyclicity of $\Gamma_{Y_\ell(\lambda)}$ is at most two, where $\Gamma_{Y_\ell(\lambda)}$ has type 1-2, it tends to type 1-1 or 1-3 if $\lambda \to C_0$ or $\gamma_1$ (i.e., $Y_\ell \to Y_0$ or $Y_M$) respectively, see Fig. 9. Moreover, if $\lambda \in G \setminus D$, then $I(Y, \lambda) \neq 0$ for any $Y \in (Y_0, Y_M)$.

To verify Assertion 3(a), we first choose $K \in ([0, b], (0, b))$ to be very close to $\mu_1(b)$, such that $Y_\lambda^+ > Y_\ell(\lambda)$. By Assertion 2 this is possible, and by formula (3.14) and the property (3.23), we have $I(Y_\ell(\lambda), \lambda) < 0$, since $\varphi < 0$ and $\psi > 0$ for $y \in (Y_0, Y_\ell)$. We next decreases $K$ along the segment $([0, b], (0, b))$, such that $Y_\lambda^+ < Y_\ell(\lambda)$ (by Assertion 2, this is also possible), then

$$I(Y_\ell(\lambda), \lambda) = \int_{Y_0(\lambda)}^{Y_\lambda^+} \varphi(x, y, \lambda) \psi(x, y, \lambda) dy + \int_{Y_\lambda^+}^{Y_\ell(\lambda)} \varphi(x, y, \lambda) \psi(x, y, \lambda) dy.$$

The first integral is negative while the second positive, see (3.23). If we continuously decrease $K$ along the segment $([0, b], (0, b))$, then $Y_\lambda \to Y_0(\lambda)$ as $K \to 0(b)$, hence the first integral goes to zero. On the other hand, for $K \in ([0, b), (0, b))$ by (3.24) we have $(Y_\ell - Y_0)|_{K=0(b, K)} > 0$. Therefore, $I(Y_\ell, \lambda) > 0$ for $K, K_0(b) \ll 1$, hence $I(Y, \lambda)$ has an odd number of zeros for $K \in ([0, b), (0, b))$.

We suppose that $K_\ell(b) \in ([0, b], (0, b))$ is such a zero point, i.e. $Y = Y_\ell(\lambda)$ is a zero of $I(Y, \lambda) = 0$ for $\lambda = (b, K_\ell(b))$. Let us show that it must be a simple zero. In fact, by using formula (3.16) we know that there is a $\theta \in (Y_0(\lambda), Y_\ell(\lambda))$ such that $\psi(x(\theta), \lambda) \neq 0$, here $\lambda = (b, K_\ell(b))$. By Assertion 1(d), for this $\lambda$, $x(\theta)$ is the only zero point of $\psi$, hence $\psi(x(Y_{\ell}(\lambda)), \lambda) \neq 0$. Using formula (3.14), Assertion 2 and $\xi(x)(x') \neq 0$ we find

$$\frac{\partial I(Y, \lambda)}{\partial Y} = \psi(x(Y_{\ell}(\lambda)), \lambda) \psi(x(Y_{\ell}(\lambda)), \lambda) \neq 0, \quad \lambda = (b, K_\ell(b)).$$

By Lemma 2.2, the cyclicity of $\Gamma_{Y_\ell(\lambda)}$ is at most two, where $\lambda = (b, K_\ell(b))$. 


Since the system is smooth enough for all variables and parameters, the locus of \((\ell, K_\ell(b))\) for any fixed \(b \in (-1, 1)\) is continuous. By (3.16), Assertion 2 and the fact that the function \(\varphi(x(y), \lambda)\) is always negative, for each fixed \(K\) there is at most one \(\ell \in (0, 1)\), such that \(I(Y_\ell, \lambda) = 0\), and \(K_\ell(b) \to \mu_0(b)\) implies \(\ell \to 0\). This means that in \((\ell, K)\)-plane the locus of \((\ell, K_\ell(b))\), for any fixed \(b \in (-1, 1)\), starts from the point \((0, \mu_0(b))\), and for each fixed value \(C \in (\mu_0(b), \mu_1(b))\) the straight line \(\{K = C, 0 < \ell < 1\}\) cut the locus at most at one point. On the other hand, if for some constant \(c \in (0, 1)\) the straight line \([\ell = c, \mu_0(b) < K < \mu_1(b)]\) cuts the locus at least at two points, i.e. there are \(K^{(1)}_\ell(b) \neq K^{(2)}_\ell(b)\), such that \(K^{(j)}_\ell(b) \in (\mu_0(b), \mu_1(b))\) and \(I(Y_\ell, \lambda) = 0\) for \(\lambda = (b, K^{(j)}_\ell(b)), j = 1, 2\). Then we would find \(\bar{\ell} \in (0, 1)\) and \(\bar{K}(b) \in (\mu_0(b), \mu_1(b))\) such that \(Y = Y_{\bar{\ell}}\) is a multiple zero of \(I(Y, \lambda)\) for \(\bar{\lambda} = (b, \bar{K}(b))\), see Fig. 11(i), here \(Y = Y_{\bar{\ell}}(\bar{\lambda}) + \bar{\ell}(Y_M(\bar{\lambda}) - Y_{0}(\bar{\lambda}))\). This contradicts the fact that the zero of \(I(Y, \lambda)\) must be simple with respect to \(Y\), as we proved above. Therefore, \(K = K_\ell(b)\) is a unique-valued function and monotone with respect to \(\ell \in (0, 1)\) for any fixed \(b \in (-1, 1)\), see Fig. 11(ii). Thus, if \(K \neq K_\ell(b)\) then \(I(Y_\ell, \lambda) \neq 0\) for \(\lambda = (b, K)\), hence by Lemma 2.2 the cyclicity of the limit periodic set \(\Gamma_{Y_\ell(\lambda)}\) is at most one.

As we mentioned above that \(\lim_{\ell \to 0} K_\ell(b) = \mu_0(b)\), i.e. \(\gamma_0 = C_0\). It is obvious that \(K_\ell(b) > \mu_0(b)\), it remains to show \(K_1(b) < \mu_1(b)\) for all \(b \in (-1, 1)\). We suppose the contrary: \(\exists \lambda^* = (b, K_1(b)) \in C_3 \cup \{A\} \cup L^{(1)}_3\), i.e. \(I(Y_M(\lambda^*), \lambda^*) = 0\). By (3.16),

\[
I(Y_M(\lambda^*), \lambda^*) = \psi(x(\theta), \lambda^*) \int_{Y_0} \varphi(x(y), \lambda^*) dy,
\]

where \(\theta < Y_M(\lambda^*)\) and \(\varphi(x(y), \lambda^*) < 0\) for all \(y\). By Assertion 2, \(Y_M(\lambda^*)\) is the unique zero of \(\psi\), hence \(\psi(x(\theta), \lambda^*) \neq 0\), and this is a contradiction. We remark that if \(\lambda \in C_3 \cup \{A\}\) then a saddle point appear at \((x_0^+, F(x_0^+))\), see I-4 and I-7 of Fig. 9. In this case the integral \(\int_{Y_0} \psi(x(y), \lambda) dy\) is still convergent, since \((1 - x_0^\xi(x(y), \lambda))^{-1}\) has order \(|y - Y_M|^{-\frac{1}{2}}\) as \(y \to Y_M\). A numerical computation shows that for \(b = 0\), \(K_1(0) \approx 1.84725 \in (\mu_0(0), \mu_1(0)) \approx (1.80977, 2)\). Thus, the statements (a) and (b) of Assertion 3 are proved, and statement (c) is a consequence of (a) and (b). The statement (A) of the theorem is proved.

Now we turn to the case \(x_0 = x_0^+\). The form of the slow divergence integral (2.7) keeps the same form (3.13), since \(h(u, \lambda)\) with a different sign and the upper and lower limits of the integral are exchanged, here \(x_0 = x_0^+\) and \(\lambda \in R_2\). Note that in this case \(\xi_M(\lambda) = 2x_0^+ = \frac{2(\sqrt{b} + b\sqrt{M})}{3}\), and \(\xi_0(\lambda) = x_0^+ + x_1^+ = \frac{2(\sqrt{b} + b\sqrt{M})}{3}\) if \(\lambda \in R_2\) and left to \(C^{(1)}_2\) and \(\xi_M(\lambda) = 0 + x_2^+ = \frac{K - b + b\sqrt{M}}{3}\) if \(\lambda \in R_2\) and right to \(C^{(1)}_2\). From (3.20) with \(x_0 = x_0^+\) we have that

\[
\tilde{\psi}(0, \lambda) = 2(K - b) - 3bKx_0^+ = (K - b)(2 - bK) - bK\sqrt{M} < 0, \quad \lambda \in R_2.
\]

In fact, if \((K - b)(2 - bK) > 0\) then
(K - b)^2(2 - bK)^2 - b^2K^2M = -(bK - 1)[4(K - b)^2 + 3b^2K^2] < 0,

since \( bK - 1 > 0 \) for \( \lambda \in R_2 \). On the other hand, in Section 4 we will prove that for \( \lambda \in R_2 \)

\[
\tilde{\psi}(\xi_M(\lambda), \lambda) = \frac{1}{9} \left[ (4K^2 + 4b^2 - 5Kb - 30)\sqrt{M} - (b - K)(4K^2 + Kb + 4b^2 - 18) \right] < 0.
\]

(3.25)

Hence \( \tilde{\psi}(\xi, \lambda) < 0 \) for all \( \xi \in (\xi_M(\lambda), \xi_M(\lambda)) \) and \( \lambda \in R_2 \), because \( \tilde{\psi}(\xi, \lambda) \) is a quadratic polynomial in \( \xi \), \( \xi_M(\lambda) \in (0, \xi_M(\lambda)) \) and the coefficient of the quadratic term is positive, see Fig. 10(ii). We already have \( \varphi(x, \lambda) < 0 \) for all \( max(0, x_0) < x < x_0^+ \) and \( \lambda \in R_2 \). Thus, by (3.14), \( I(Y, \lambda) > 0 \) for all possible \( Y \) and \( \lambda \in R_2 \). Therefore, by Lemma 2.2 the cycllicity of the limit periodic set of type II-2 is at most one, the statement (B) of the theorem is proved.

At last we consider the case \( \lambda \in \Delta \). We move the minimum point \( (x_0^-, F(x_0^-)) \) to the origin, and suppose that the system has the form (2.1) globally, and the maximum point \( (x_0^+, F(x_0^+)) \) takes a coordinate \( (u_M, v_M) \). Comparing with the limit periodic limit \( I_Y \) of Fig. 4(b), the types III-1 and III-2 can be expressed by \( I_Y \) and \( I_Y \) for \( V \in (0, v_M) \) respectively. Since the saddle point is to the limit periodic set, we always have \( g(u, \lambda) = \Phi^{-1}(G(u, \lambda)) > 0 \). By the condition (2.8) and formula (2.6) it is easy to see that \( h(u) - h(\hat{u}(u)) > 0 \) for \( u < 0 \), hence, by formula (2.10), \( I(0, \lambda) > 0 \) for \( \lambda \in \Delta \), the cyclicity of type III-1 is at most one, i.e. the statement (C)(i) of the theorem is proved.

From Assertion 3 we know that, using the notations there, \( I(Y_M, \lambda) = 0 \) (resp. \( > 0 \) or \( < 0 \)) if and only if \( \lambda \in \gamma_1 \) (resp. below or above the curve \( \gamma_1 \)). Hence, by the notations here, for \( I_{V_M} \) we have \( I(v_M, \lambda) = 0 \) (resp. \( > 0 \) or \( < 0 \)) if \( \lambda \in \gamma_1 \) (resp. \( \lambda \) is below or above the curve \( \gamma_1 \)). Using the condition (2.8), formula (2.10) and the construction of \( I_Y \) in Fig. 4(b), we see that if \( V \) decreases from \( v_M \) then the integral \( I(V, \lambda) \) monotonically increases. Hence if \( \lambda \) is on or below the curve \( \gamma_1 \) then \( I(V, \lambda) > 0 \) for all \( V \in (0, v_M) \); and if \( \lambda \) is above the curve \( \gamma_1 \), then \( I(v_M, \lambda) < 0 \) and \( I(0, \lambda) > 0 \), hence there exists a unique \( V^* \in (0, v_M) \) such that \( I(V^*, \lambda) = 0 \). From formula (2.10) it is easy to find that

\[
I(V^* + \Delta V, \lambda) - I(V^*, \lambda) = \int_{V^*}^{V^* + \Delta V} [h(\hat{u}(u)) - h(\hat{u}(u))] dw,
\]

where \( u = G^{-1}(w, \lambda) \in (\alpha_M, 0) \). Since \( h(\hat{u}(u)) > 0 < h(\hat{u}(u)) \) for \( u < 0 \) (see the proof of statement (C)(i)), hence \( \frac{\partial}{\partial V}(V^*, \lambda) < 0 \), and the statement (C)(ii) of the theorem is proved.

To study the cycllicity of type III-3, we need to change the formula (2.10) to

\[
I(V, \lambda) = \int_{0}^{V} [h(u) - h(\hat{u}(u))] dw + \int_{V}^{v_M} [h(\hat{u}(u)) - h(\hat{u}(u))] dw,
\]

where \( u = G^{-1}(w, \lambda) \in (\alpha_M, 0) \). Since \( h(\hat{u}(u)) < 0 < h(u) \) for \( u < 0 \), the first integral is positive, the second is also positive (see the proof of statement (B)), hence \( I(V, \lambda) > 0 \) for all possible \( V \) and \( \lambda \), the statement (C)(iii) of the theorem is proved. \( \square \)

4. The study of curve \( C_0 \) and the proof of (3.25) for \( \lambda \in R_2 \)

For \( \lambda = (b, K) \in R_1 \cup R_2 \) we let

\[
\alpha(\lambda) = 4b^2 - 5bK + 4K^2 - 30, \quad \beta(\lambda) = 4b^2 + bK + 4K^2 - 18, \quad J_\pm(\lambda) = \alpha(\lambda)\sqrt{M(\lambda)} \pm (b - K)\beta(\lambda), \quad M(\lambda) = K^2 + b^2 + bK - 3 > 0.
\]

(4.1)
We will prove that $J_+ (\lambda) = 0$ define a unique curve $C_0 : \{K = \mu_0 (b) : b \in (-1, 1)\}$ for $\lambda \in R_1$ with $\mu_0 (b)$ smooth, $C_0$ is located strictly between $C_1$ and $C_3 \cup \{A\} \cup L_1^{(1)}$ for $b \in (-1, 1)$ and tends to points $P$ and $Q$ as $b \to -1$ and $b \to 1$ respectively; and $J_- (\lambda) < 0$ for $\lambda \in R_2$, which gives (3.25).

Let $J^* (\lambda) := \alpha^2 (\lambda) M(\lambda) - (b - K)^2 \beta^2 (\lambda)$, then calculation shows that $J^* (\lambda) = 0$ is given by

$$16(K^4 + b^4) + 8(bK - 18)(K^2 + b^2) - 3b^3 K^3 + 15b^2 K^2 - 72bK + 300 = 0. \quad (4.2)$$

Note that $J_+ (\lambda) = 0$ in $R_1$ is equivalent to $J^* (\lambda) = 0$ and $\alpha (\lambda) \beta (\lambda) > 0$, since $b - K < 0$ for $\lambda \in R_1$; and $J_- (\lambda) = 0$ in $R_2$ is equivalent to $J^* (\lambda) = 0$ with $\alpha (\lambda) \beta (\lambda) > 0$ if $b - K > 0$, or with $\alpha (\lambda) \beta (\lambda) < 0$ if $b - K < 0$.

In $(b, K)$-plane, the ellipses $E$: $M(\lambda) = 0$ ($C_1 \subset E$, $\alpha (\lambda) = 0$ and $\beta (\lambda) = 0$, and the locus of $J^* (\lambda) = 0$ are symmetric with respect to $\{K = b\}$, and pass through the same points $P(-1, 2)$ and $P'(2, -1)$; the hyperbola $C_2$: $bK = 1$ is also symmetric with respect to $\{K = b\}$, and is tangent to $E$ at the point $Q(1, 1)$, see Fig. 12(ii).

In polar coordinates $b = \rho \cos \theta$, $K = \rho \sin \theta$, if we let $\tau = \sin \theta \cos \theta$ and $r = \rho^2$, then $J^* (\lambda) = 0$ is equivalent to

$$h(\tau, r) = 3\tau^3 r^3 + (17\tau^2 - 8\tau - 16)r^2 + 72(\tau + 2)r - 300 = 0. \quad (4.3)$$

Hence, on each semi-lane $(\theta, \rho) : \theta = c \in [0, 2\pi)$, $\rho \in (0, +\infty)$, Eq. (4.3) has at most 3 real solutions for $\rho$, including multiplicities. Eliminating $r$ from $h(\tau, r) = 0$ and $h_r(\tau, r) = 0$, we find

$$\tau^3 (2\tau - 1)(5\tau + 2)^2 (101\tau^2 + 88\tau + 32) = 0,$$

which has real roots $\tau = \frac{1}{2}, 0$ and $-\frac{2}{5}$, corresponding respectively to $\theta = \frac{\pi}{4}, \frac{\pi}{2}$ and $\left(\pi - \arcsin\left(\frac{2}{\sqrt{5}}\right)\right)$ for $\theta \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$ (the behavior in other 3 quadrants can be obtained by symmetry). On $\theta = \frac{\pi}{4}$ the equation $h(\tau, r) = 0$ has a double positive solution for $r$, corresponding to the point $S(\sqrt{10}, \sqrt{10})$, plus a smaller positive solution, corresponding to the point $Q \left(1, 1\right)$; when $\theta$ increases and passes $\frac{\pi}{2}$, one of 3 positive solutions changes from $+\infty$ to $-\infty$ and become negative; on $\theta = \pi - \arcsin\left(\frac{2}{\sqrt{5}}\right)$ there is a double positive solution, corresponding to the cusp point at $P(-1, 2)$; and when $\theta \in \left(\pi - \arcsin\left(\frac{2}{\sqrt{5}}\right), \frac{3\pi}{4}\right)$, there is no positive solution in $\rho$. Thus it is easy to obtain the global behavior of $A$, shown in Fig. 12(i). We denote the part of $A$ from $P$ to $Q$ by $C_0 = C_0$ and the part above $C_0$ and left to $\{K = b\}$ by $C_2$, the symmetry parts with respect to $\{K = b\}$ by $C_0$ and $C_4$ respectively.

Fig. 12. The relative position of $A$ with $C_1$, $C_2$, $\alpha (\lambda) = 0$ and $\beta (\lambda) = 0$. 

(i) the behavior of $A$: $\{J^* (\lambda) = 0\}$ (ii) the relative positions of curves.
To study the relative positions of $\Lambda$ with $C_1$, $\alpha(\lambda) = 0$ and $\beta(\lambda) = 0$, we also let $b = \rho \cos \theta$, $K = \rho \sin \theta$, then let $\tau = \sin \theta \cos \theta$ and $r = \rho^2$. The equations of these 3 ellipses are respectively

$$h_1(\tau, r) = (1 + \tau)r - 3 = 0,$$

$$h_\alpha(\tau, r) = (4 - 5\tau)r - 30 = 0,$$

$$h_\beta(\tau, r) = (4 + \tau)r - 18 = 0.$$

It is easy to find that the solution of $\{h(\tau, r) = 0, h_1(\tau, r) = 0\}$ consists of $(-\frac{2}{5}, 5)$ and $(\frac{1}{7}, 2)$, corresponding to the points $P$ and $Q$ respectively; the solution of $\{h(\tau, r) = 0, h_\alpha(\tau, r) = 0\}$ consists of $(-\frac{2}{5}, 5)$ and $(\frac{1}{2}, 20)$, corresponding to the points $P$ and $S$ respectively; the solution of $\{h(\tau, r) = 0, h_\beta(\tau, r) = 0\}$ consists of only $(-\frac{2}{5}, 5)$, corresponding to the point $P$. Here we restrict the correspondence for $\theta \in [\frac{\pi}{2}, \frac{3\pi}{4}]$.

Therefore, the relative positions of $\Lambda$ with $C_1$, $\alpha(\lambda) = 0$ and $\beta(\lambda) = 0$ are shown in Fig. 12(ii). Comparing this figure with Fig. 8, we obtain that $\Lambda \cap R_1$ consists of $C_0^1 = C_0$ and a part of $C_0^2$, but only $C_0$ is located in the region that $\alpha(\lambda)\beta(\lambda) > 0$; $\Lambda \cap R_2$ consists of $C_0^4$ and a part of $C_0^5$, but none of them is located in the region that $\alpha(\lambda)\beta(\lambda) > 0$ for $b - K > 0$ or $\alpha(\lambda)\beta(\lambda) < 0$ for $b - K < 0$. That is: $J_+(\lambda) = 0$ defines only a curve $C_0$ for $\lambda \in R_1$, it jolts the points $P$ and $Q$, and is located strictly above $C_1$ for $b \in (-1, 1)$; $J_-(\lambda)$ has a fixed sign in $R_2$, and computation gives $J_-(2, 1) = -44$, hence $J_-(\lambda) < 0$ for $\lambda \in R_2$.

From (4.1) we find that for $b \in (-1, 1)$

$$J_+(\lambda)|_{C_1} = J_+(b, 2 - b) = (b - 1)(b^2 - 2b + 10) < 0, \quad J_+(\lambda)|_A = -12\sqrt{3} < 0,$$

$$J_+(\lambda)|_{C_3} = J_+(\frac{b}{2} - b) = \frac{4}{b^2}(1 - b^2)^2(4 - b^2)\sqrt{(1 - b^2)(4 - b^2) - (b^2 + 2)} < 0.$$

Hence $C_0$ is located strictly below $C_3 \cup \{A\} \cup L_{31}^{(1)}$ for $b \in (-1, 1)$. At last, by using (4.2) we eliminate $K$ from $(J^*(\lambda) = 0, J_+^*(\lambda) = 0)$ and obtain

$$(81b^4 + 640b^2 - 4096)(b^2 - 10)(b^2 - 1)(b^2 - 4) = 0,$$

which has no real solution for $b \in (-1, 1)$. This means that $C_0$ can be expressed in the form $K = \mu_0(b)$ with $\mu_0(b)$ smooth for $b \in (-1, 1)$.

The desired results thus are proved.

**Remark.** It is not surprising that $C_0$ coincides with the curve of Hopf bifurcation of order 2, obtained in [28] and [30]. In fact, removing the square root from $J_+(\lambda) = 0$, defining $C_0$ for $b \in (-1, 1)$, we obtain (4.2), which is exactly the second equation in formula (3.12) of [28]. For the canard cycle, we also need condition (2.17). For $\varepsilon \to 0$ from (2.17) we obtain

$$\frac{d}{c} = p(x_0) = \frac{x_0}{x_0^2 + bx_0 + 1} = \frac{bK^2 + 2(b^2 - 2)K + b + (bK + 2)\sqrt{M}}{(b^2 - 4)(K^2 + bK + 1)}.$$

This is the first equation in formula (3.12) of [28] ($c = 1$ and $d = d_-(b, K)$). Note that in the region $R_1$ Eq. (4.2) gives an extra branch $C_0^2$, as we mentioned above and shown in Fig. 12(i), hence the expression $C_0$ is more precise for Hopf bifurcation of order 2.
5. Canard cycles of the system with response function of Holling type III

Suppose that the response function is taken as Holling type III, i.e. in the system (1.1) we have

\[ p(x) = \frac{mx^2}{ax^2 + bx + 1}, \quad F(x) = \frac{r}{mx}(ax^2 + bx + 1) \left( 1 - \frac{x}{K} \right), \quad b > -2\sqrt{a}, \ K > 0. \]

Different from the choice of [20], we eliminate \( a, m \) and \( r \) by scaling of phase variables, time and parameters. For this purpose, we let

\[ (x, y, t) = \left( \frac{1}{\sqrt{a}}, \pi, \frac{\sqrt{a}}{m} \bar{r} \right), \]

and

\[ (r, K, b, d, c) = \left( \frac{m}{\sqrt{a}} \bar{r}, \frac{1}{\sqrt{a}}, \sqrt{\bar{a}b}, \frac{m}{\sqrt{a}} \bar{d}, \sqrt{\bar{a}r} \bar{c} \right). \]

Skipping all bar’s, we obtain the same form (1.1) with \( a = m = r = 1 \), i.e. system (1.1) becomes

\[ \dot{x} = p(x)(F(x) - y), \quad \dot{y} = y(-d + cp(x)). \tag{5.1} \]

where

\[ p(x) = \frac{x^2}{x^2 + bx + 1}, \quad F(x) = \frac{1}{x}(x^2 + bx + 1) \left( 1 - \frac{x}{K} \right), \quad b > -2, \ K > 0. \tag{5.2} \]

Any extreme point \( x_0 \) of the function \( F \) satisfies the equation

\[ \chi(x_0, \lambda) = 2x_0^3 + (b - K)x_0^2 + K = 0, \tag{5.3} \]

where \( \chi(x, \lambda) = -Kx^2F_x(x, \lambda) \). Removing \( x_0 \) from \( \chi(x_0, \lambda) = 0 \) and \( \frac{\partial \chi}{\partial x_0} = 0 \) we find \( K[(K - b)^3 - 27K] = 0 \). It is not hard to show that in \( (b, K) \)-plane for \( K > 0 \) the curve

\[ C_1: \quad C_1(\lambda) = (K - b)^3 - 27K = 0 \tag{5.4} \]

is tangent to the straight line \( L: \{b = -2\} \) at the point \( A(-2, 1) \), see Fig. 13(i). If \( \lambda = (b, K) \) is located right to \( C_1 \), then (5.3) has only a negative real root, hence \( F(x, \lambda) \) is monotone for \( x > 0 \), there is no canard limit cycle. If \( \lambda \) is located in the narrow region below \( C_1 \), right to \( L \) and above \( \{K = 0\} \), then both of two positive zeros of (5.3) are greater than \( K \), hence \( F \) is negative at these values, we do not need to consider. It is clear that only in the region

\[ U: \quad \{\lambda = (b, K) \mid C_1(\lambda) > 0, \ b + 2 > 0 \text{ and } K > 1\}. \]

Eq. (5.3) has two zeros \( x_0^- \) and \( x_0^+ \) satisfying \( 0 < x_0^- < x_0^+ < K \).

Note that \( \chi_x(x, \lambda) = 2[3x - (K - b)]x \) and \( \chi(0) = \chi \left( \frac{K - b}{2} \right) = K > 0 \) (see Fig. 13(ii)), and from (5.4) it is obvious that \( K > b \) for \( \lambda \in U \), we have that

\[ 0 < x_0^- < \frac{K - b}{3} < x_0^+ < \frac{K - b}{2}, \quad \lambda \in U. \tag{5.5} \]
Besides, $F$ takes a local minimum at $x_0^-$ and a local maximum at $x_0^+$, see Fig. 13(iii).

By Lemma 2.4, around the local minimum point at $x_0 = x_0^-$ (or maximum point at $x_0 = x_0^+$), the system (5.1) can be transformed to the form (2.11), where $\lambda = (b, K) \in U$, and

$$g(u, \lambda) = \frac{(1 + bx_0)(u + x_0) + x_0}{x_0^2(u + x_0)^2}.$$  

$$\Phi(v, \lambda) = \pm F(x_0)(e^{\pm v} - 1) \quad \text{if} \quad x_0 = x_0^\pm,$$

$$G(u, \lambda) = \pm \left[ F(u + x_0, \lambda) - F(x_0, \lambda) \right] = \pm \frac{(K - b - 3x_0 - u)u^2}{K(u + x_0)} \quad \text{if} \quad x_0 = x_0^\pm. \quad (5.6)$$

To find the last expression above we need to use (5.3), and using it again we obtain

$$\frac{\partial G}{\partial u}(u, \lambda) = \pm \frac{-2(u + x_0)^2 + (K - b - 2x_0)(u + 2x_0)]u}{K(u + x_0)^2}$$

$$= \pm \frac{-2x_0^2(u + x_0)^2 + K(u + 2x_0)]u}{Kx_0^2(u + x_0)^2} \quad \text{if} \quad x_0 = x_0^\pm. \quad (5.7)$$

It is easy to check that the hypotheses (H1) and (H2) are satisfied, where $m_- = -x_0$, $m_+ = x_0^+ - x_0^-$ if $x_0 = x_0^-$, and $m_- = -(x_0^+ - x_0^-)$, $m_+ = K - x_0^+$ if $x_0 = x_0^+$; the condition (H3) is also satisfied, but in some cases we need to decrease the value of $m_+$ according to the position of a saddle on the slow manifold, let us consider these cases. To keep $g(u, \lambda) > 0$, a necessary condition is

$$(1 + bx_0)x_0 + x_0 = (2 + bx_0)x_0 > 0, \quad \text{for} \quad \lambda \in U.$$

If $b \geq 0$ we certainly have $(2 + bx_0) > 0$; if $b \in (-2, 0)$, then calculation from (5.3) gives

$$\chi \left( -\frac{2}{b}, \lambda \right) = \frac{(b - 2)(b + 2)(Kb + 4)}{b^3}, \quad b \in (-2, 0).$$

From (5.4) we find

$$C_1(\lambda)\big|_{K = -\frac{4}{b}} = -\frac{(b^2 + 16)(b^2 - 2)}{b^3}.$$  

We define a curve

$$C_2: \quad C_2(\lambda) = bK + 4 = 0 \quad \text{for} \quad b \in (-2, 0).$$
Then $C_2$ is tangent to $C_1$ at the point $P(-\sqrt{2}, 2\sqrt{2})$, and divides $U$ into $U_1$, $U_2$ and $U_3$, see Fig. 14(i).

We denote the part of $C_2$ below $P$ by $C'_2$ and the part above $P$ still by $C_2$. It is easy to check that $(2 + bx_0^-) > 0$ for $\lambda \in U_2$, and $(2 + bx_0^+)$ becomes negative when $\lambda$ crosses $C_2$ into $U_1$, and $(2 + bx_0^-)$ becomes negative when $\lambda$ crosses $C'_2$ into $U_3$. Now let

$$V_1 = U_1 \cup C_2 \cup U_2, \quad V_2 = U_2 \subset V_1.$$  

Then a limit periodic set containing $(x_0^-, F(x_0^-))$ appears only if $\lambda \in V_1$, and containing $(x_0^+, F(x_0^+))$ only if $\lambda \in V_2$.

We next consider the possibility that a saddle point appears on a corner of the limit periodic set. The critical case for $x^-_0$ and $x^+_0$ is respectively (see Fig. 13(iii))

$$\begin{align*}
(1 + bx^-_0)x^+_0 + x^-_0 &= 0, \quad b \in (-2, 0) \quad (5.8) \\
(1 + bx^+_0)x^+_0 + x^+_0 &= 0, \quad b \in (-\sqrt{2}, 0). \quad (5.9)
\end{align*}$$

From $F(x) - F(x^+_0) = -\frac{(x - x_0^+)(x - \hat{x})}{Kx}$ and $F'(x^-_0) = 0$ we find $\hat{x}$, and similarly find $x^+$:

$$\begin{align*}
\hat{x} &= \frac{2(x^-_0)^2}{x^+_0 + x^-_0}, \\
x^+ &= \frac{2(x^+_0)^2}{x^+_0 + x^-_0}. \quad (5.10)
\end{align*}$$

Eliminating $x^-_0$ from (5.8) and (5.3) with $x_0 = x^-_0$, and eliminating $x^+_0$ from the resulting equation and (5.3) with $x_0 = x^+_0$, we obtain

$$K^3(b - 2)(b + 2)(Kb + 4)((b^2 - 1)bk + b^2 + 2)^2 = 0.$$  

Let

$$S^-: \{\lambda \in V_1 \mid S^-(\lambda) := (b^2 - 1)bk + b^2 + 2 = 0\}.$$

which defines a unique curve for $b \in (-\sqrt{2}, -1)$. Since $S^-(\lambda)|_{K = -\frac{b}{2}} = 3(2 - b^2)$, $S^-$ is located entirely left to $C_2$, and $\lim_{b \to -\sqrt{2}} S^- \cap C_2 = \{P\}$, see Fig. 14(ii).
By using (5.10) we write (5.9) as
\[ S(\lambda) := x_0^- + 3x_0^2 + 2(bx_0^+)^2, \quad b \in (-\sqrt{2}, 0). \] (5.11)

We will show that

(a) For each \( b \in (-\sqrt{2}, 0) \) there is at least one \( K = K^+(b) \) such that \((b, K^+(b))\) is located above \( C_1 \) and below \( C_2 \), satisfying \( S(b, K^+(b)) = 0 \).

(b) For each \( b \in (-\sqrt{2}, 0) \) there is at most one \( K = K^+(b) \) such that \( S(b, K^+(b)) = 0 \).

Claims (a) and (b) immediately imply that \( S(\lambda) = 0 \) defines a unique curve for \( b \in (-\sqrt{2}, 0) \)
\[ \{ \lambda \in V_2 \mid K = K^+(b) \}, \]
which is located strictly between \( C_2 \) and \( C_1^* = \{(b, K) \in C_1, \ b \in (-\sqrt{2}, 0)\} \), and \( S^+ \) has the same end point at \( P \), see Fig. 14(ii).

In fact, along \( C_2 : K = -4/b \), we find \( x_0^- = \frac{-b^2 + 16 - b}{4b} \) and \( x_0^+ = -\frac{2}{b} \), hence
\[ S(\lambda)|_{C_2} = \frac{8 - b^2 + b\sqrt{b^2 + 16}}{4b} < 0 \quad \text{for} \ b \in (-\sqrt{2}, 0). \]

On the other hand, along \( C_1 \) we have \( x_0^+ = x_0^- = x_0 \), hence
\[ S(\lambda)|_{C_1^*} = 2x_0(2 + bx_0) > 0 \quad \text{for} \ \lambda \in C_1^*. \]

The claim (a) is proved. To verify claim (b), we first rewrite \( S(\lambda) = 0 \) as
\[ x_0^- = (-2b)x_0^+ \left( \frac{x_0^+ + 3}{b} \right), \quad b \in (-\sqrt{2}, 0). \] (5.12)

Since \( x_0^-, x_0^+ > 1 \) for \( \lambda \in V_2 \) and from (5.3) and (5.5) we obtain
\[ \frac{\partial x_0^-}{\partial K} = \frac{(x_0^-)^2 - 1}{2(3x_0^- - (K - b))x_0^-} < 0, \quad \frac{\partial x_0^+}{\partial K} = \frac{(x_0^+)^2 - 1}{2(3x_0^+ - (K - b))x_0^+} > 0. \]

Thus, for any fixed \( b \in (-\sqrt{2}, 0) \) and for \( K \) increasing, the left side of (5.12) is a decreasing function of \( K \) while the right side is increasing. Therefore, Eq. (5.12), i.e. \( S(\lambda) = 0 \), has at most one solution \( K = K^+(b) \) for any fixed \( b \in (-\sqrt{2}, 0) \).

To give the precise expressions of \( S^+ \), we eliminate \( x_0^- \) from (5.9) and (5.3) with \( x_0 = x_0^- \), then eliminate \( x_0^+ \) from the resulting equality and (5.3) with \( x_0 = x_0^+ \), and finally obtain
\[
9b^2(b^2 - 1)(4b^2 - 9)K^5 - 3b(36b^6 - 86b^4 + 95b^2 - 180)K^4
+ (108b^8 - 2479b^6 + 8966b^4 - 8899b^2 + 900)K^3
- 2b(18b^8 - 129b^6 + 4483b^4 - 23999b^2 + 33280)K^2
+ (-117b^8 + 285b^6 - 8899b^4 + 66560b^2 - 119164)K - 9b^3(3b^2 - 10)^2 = 0,
\]
which defines the curve \( S^+ \) for \( \lambda \in (-\sqrt{2}, 0) \), as well as some extra curves.
Now we denote the two sub-regions of $V_1$ (resp. of $V_2$) divided by $S^-$ (resp. by $S^+$) by $V^1_1$ and $V^2_1$ (resp. $V^1_2$ and $V^2_2$), see Fig. 14(ii). We will show in the proof of next theorem that there are 2 more curves: $C_0$ and $\gamma$, located in between $C_2$ and $S^-$, having the unique common point at $P$, and we denote the region between $C_0$ and $\gamma$ by $W$, see Fig. 14(iii). We list all possible limit periodic sets in Fig. 15, where the type I-2 takes its possible “maximal size” I-3, I-4 and I-5 if $\lambda$ moves from $V^1_1$, passing $S^-$ to $V^2_1$; and the type II-2 takes its possible “maximal size” II-3, II-4 and II-5 if $\lambda$ moves from $V^1_2$, passing $S^+$ to $V^2_2$.

**Theorem 5.1.** Suppose that $c$ and $d$ are chosen as in (2.17), where $\varepsilon > 0$ small. For system (5.1) the following statements hold, depending on the value of $B_0 = B_0(\varepsilon, b, K)$:

(A) The limit periodic set of type I-2 exists if $\lambda \in V_1$, its cyclicity is at most one for $\lambda \in V_1 \setminus W$, and is at most two for $\lambda \in W$.

(B) The limit periodic set of type II-2 exists if $\lambda \in V_2$, its cyclicity is at most one for all $\lambda \in V_2$.

(C) The limit periodic set of types III-1 and III-2 (resp. III-3) exists if $\lambda \in V^1_2$ (resp. $\lambda \in V_2$), and the cyclicity is at most one for all possible $\lambda$.

**Proof.** The proof is similar to the proof of Theorem 3.4, we only indicate some differences for the existence of limit periodic sets and the study of their cyclicities. We first consider the canard cycle around $(x_0^-, F(x_0^-))$. Using (2.6), (5.6) and (5.7) with $x_0 = x_0^-$ we obtain

$$h(u, \lambda) = \frac{-2x_0^2x^2 + K(x + x_0)}{K[(1 + bx_0)x + x_0]F(x, \lambda)},$$

where $x = u + x_0$. Similarly to (3.14) we have

$$I(Y, \lambda) = \int_{Y_0}^Y \varphi(x(y), \lambda)\psi(x(y), \lambda)\;dy,$$

where $x(y) = F^{-1}(y, \lambda) \in (0, x_0)$ for $y \in (Y_0, Y_M)$, and

$$\varphi(x, y) = \frac{x_0^2(x - \tilde{x})}{KF(x, \lambda)[(1 + bx_0)x + x_0][1 + (1 + bx_0)\tilde{x} + x_0]} < 0,$$

$$\psi(x, \lambda) = -2(bx_0 + 1)x\tilde{x} - 2x_0(x + \tilde{x}) - bK,$$

(5.13)
where \( \tilde{x} = \tilde{x}(x, \lambda) \in (x_0^-, x_0^+) \) is defined by \( F(x, \lambda) = F(\tilde{x}, \lambda) \) for \( x \in (\tilde{x}, x_0^-) \) as in Section 4, and let \( \xi = \tilde{x} + x \) and \( \eta = \tilde{x}x \), then from \( F(\tilde{x}, \lambda) = F(x, \lambda) \) we find

\[
\eta = \frac{K}{K - b - \xi}.
\]  

(5.14)

From (5.5) we know that \( K - b - \xi > 0 \) for \( x \) and \( \tilde{x} \) near \( x_0 \), hence \( K - b - \xi > 0 \) for all possible \( x \) and \( \tilde{x} \), because \( \eta > 0 \).

Let us prove that \( \xi' = 1 + \tilde{x}'(x) = \frac{F'(x) + F'(\tilde{x})}{F(\tilde{x})} < 0 \) for \( x \in (\tilde{x}, x_0^-) \). Thus, the maximal interval of \( \xi \) is \((\xi_m, \xi_M)\), where

\[
\xi_m = 2x_0^-,
\]

\[
\xi_M = \hat{x} + x_0^+ = \frac{(x_0^+)^2 + x_0^+x_0^- + 2(x_0^-)^2}{x_0^+ + x_0^-}.
\]  

(5.15)

Since \( F'(\tilde{x}) > 0 \), we need to show for \( x \in (\tilde{x}, x_0^-) \)

\[
F'(x) + F'(\tilde{x}) = \frac{[2(K - b) - 2(x + \tilde{x})](x\tilde{x})^2 - K(x^2 + \tilde{x}^2)}{(xx)\tilde{x}} < 0.
\]

By (5.14), this is equivalent to show

\[
\omega(\xi) = \xi^3 - (K - b)\xi^2 + 4K < 0 \quad \text{for} \quad \xi \in (\xi_m, \xi_M),
\]

which follows from the facts:

(i) \( \omega'(\xi) = \xi[3\xi - 2(K - b)] < 0 \) for \( \xi \in (0, \frac{2(K - b)}{3}) \);

(ii) \( 0 < \xi_m < \xi_M \) and \( \omega(\xi_m) = 4[2x_0^+ + (b - K)x_0^- + K] = 0 \) (see (5.3)); and

(iii) \( \xi_M < \frac{2(K - b)}{3} \).

To verify fact (iii), by using (5.15), eliminating \( x_0^- \) from \( \xi_M - \frac{2(K - b)}{3} = 0 \) and (5.3) with \( x_0 = x_0^- \), then eliminating \( x_0^+ \) from the resulting equality and (5.3) with \( x_0 = x_0^+ \), we obtain \( K[(K - b)^2 - 27K] = 0 \), which is impossible for \( \lambda \in V_1 \), see (5.4). Therefore, fact (iii) can be checked by choosing a special value \( \lambda \in V_1 \): \( \xi_M \approx 3.545 \) and \( \frac{2(K - b)}{3} = 4.2 \) if \( (b, K) = (-0.3, 6) \).

Substituting (5.14) in the second equality of (5.13), we find

\[
\tilde{\psi}(\xi, \lambda) = \psi(x(\xi), \lambda) = \frac{\psi_1(\xi, \lambda)}{K - b - \xi},
\]

where \( K - b - \xi > 0 \) and

\[
\psi_1(\xi, \lambda) = 2x_0^-\xi^2 + [2x_0^-(b - K) + bK]\xi + K[b(b - K) - 2(1 + bx_0^-)].
\]  

(5.16)

Now we study the behavior of \( \psi_1 \). First, from (5.16), (5.15) and (5.3) with \( x_0 = x_0^- \) we obtain

\[
\psi_1(\xi_m, \lambda) = 8(x_0^-)^3 + 4(b - K)(x_0^-)^2 + K(b^2 - Kb - 2) = K(b^2 - bK - 6).
\]  

(5.17)

We define a curve

\[
C_0: \{\lambda \in V_1 \mid b^2 - bK - 6 = 0\}.
\]  

(5.18)
It is not hard to check that the curve $C_0$ is located strictly between $S^-$ and $C_2$ for $K > 2\sqrt{2}$ and has the same endpoint at $P$, see Fig. 14(iii). Thus, we have the following result.

**Lemma 5.2.** $\psi_1(\xi_m, \lambda) > 0$ (resp. $= 0$ or $< 0$) if $\lambda \in V_1$ is located left (resp. at or right) to $C_0$.

Next, from (5.16) and (5.15) we find $\psi_1(\xi_M, \lambda) = \frac{\psi_2(\lambda)}{(x_0^+ + x_0^-)^2}$, where

$$\psi_2(\lambda) = 2x_0^- [4(x_0^-)^4 + 4(x_0^-)^3 x_0^+ + 5(x_0^-)^2 (x_0^+)^2 + 2x_0^- (x_0^+)^3 + (x_0^+)^4]$$

$$+ 2(b - K)[2(x_0^-)^4 + 3(x_0^-)^3 x_0^+ + 2(x_0^-)^2 (x_0^+)^2 + x_0^- (x_0^+)^3]$$

$$+ K(b^2 - bK - 2)(x_0^- + x_0^+)^2 - bK (x_0^-)^2 x_0^+.$$  \hfill (5.19)

Eliminating $x_0^-$ from (5.19) and (5.3) with $x_0 = x_0^-$, then eliminating $x_0^+$ from the resulting equality and (5.3) with $x_0 = x_0^+$, we obtain

$$(b^2 - bK - 6)[(K - b)(K^3 - 27K^2)(b^3 - b)(b^2 + 2)] = 0.$$  

This means that a necessary condition for $\psi_2(\lambda) = 0$ is $\lambda \in C_0 \cup C_1 \cup S^-$. It can be checked that $\psi_2(\lambda) < 0$ if $\lambda \in C_0 \cup C_1$ and $\psi_2(\lambda) = 0$ only if $\lambda \in S^-$, so we have the following result.

**Lemma 5.3.** $\psi_1(\xi_m, \lambda) = 0$ if $\lambda \in V_1 \cap S^-$, and $\psi_1(\xi_M, \lambda) < 0$ if $\lambda$ is located left to $S^-$.  

Since $\psi_1(\xi, \lambda)$ is a quadratic polynomial in $\xi$, by Lemmas 5.2 and 5.3 we immediately obtain that for each $\lambda$ between $S^-$ and $C_0$, there is a unique $\xi^*_\lambda \in (\xi_m(\lambda), \xi_M(\lambda))$, continuous in $\lambda$, such that $\psi_1(\xi^*_\lambda, \lambda) = 0$, implying $\bar{\psi}(\xi^*_\lambda, \lambda) = 0$, and

$$(\xi - \xi^*_\lambda)\bar{\psi}(\xi, \lambda) < 0, \quad \text{for } \xi \in (\xi_m(\lambda), \xi_M(\lambda)) \setminus \{\xi^*_\lambda\}.$$  

Moreover, $\lim_{\lambda \to C_0} \xi^*_\lambda = \xi_m$, $\lim_{\lambda \to S^-} \xi^*_\lambda = \xi_M$, i.e. we have the same results as Assertion 1 in the proof of Theorem 3.4, and we can finish the remaining proof of statement (A) by the same way.

We next consider the canard cycle around $(x_0^+, F(x_0^+))$. As we explained before, the slow divergence integral keeps the same form, $x_0 = x_0^+$ in (5.13), where $x \in (x_0^+, x_0^-)$ and $\bar{x} \in (x_0^+, x^+)$. We have the same form (5.16) for $\psi_1(\xi, \lambda)$, $\lambda \in V_2$, $\xi \in (\xi_m, \xi_M)$, but

$$\xi_m = x_0^- + x^a = \frac{(x_0^-)^2 + x_0^- x_0^+ + 2(x_0^+)^2}{x_0^+ + x_0^-}, \quad \xi_M = 2x_0^+.$$  

Similarly to obtain (5.17) we find

$$\psi_1(\xi_M, \lambda) = \psi_1(2x_0^+, \lambda) = K(b^2 - bK - 6).$$  \hfill (5.20)

Since $V_2$ is located entirely right to the curve $C_0$, we have $\psi_1(\xi_M, \lambda) < 0$ for all $\lambda \in V_2$. Let us show that $\psi_1(\xi_m, \lambda) < 0$ for all $\lambda \in V_2$, hence $\psi_1(\xi, \lambda) < 0$ for all $\xi \in (\xi_m, \xi_M)$ and $\lambda \in V_2$, because $\psi_1$ is a quadratic polynomial in $\xi$ and the coefficient of $\xi^2$ is positive, and this implies that $I(Y, \lambda) > 0$ for all $Y \in (Y_0(\lambda), Y_M(\lambda))$ and $\lambda \in V_2$, i.e. the cyclicity is at most one.

The expression of $\psi_1(\xi_m, \lambda)$ is the same as the right-hand side of (5.19), exchanging $x_0^-$ and $x_0^+$. Hence, by the same procedure we obtain that $\psi_1(\xi_m, \lambda) \neq 0$ for $\lambda \in V_2$, because $V_2$ is located right to $C_0$ and left to $C_1$. Choosing a special value $\lambda \in V_2$, we find that $\psi_1(\xi_m, \lambda) < 0$ for $\lambda \in V_2$. This finishes the proof of statement (B).
The proof of statement (C) is similar to the proof of Theorem 3.4(C), and it is more simple, because the region W does not meet \( V_2 \), where the limit periodic sets with types III-1 to III-3 may exist. We omit the details. □

**Remark.** By a change of variables \((x, y) \mapsto (Kx, \frac{1}{x} y)\), the system (1.1) with Holling type III and \( a = r = m = 1 \) is transformed to system (1.3) of [20] with \( \alpha = K_2, \beta = bK \). Hence the point \( P \) with \((b, K) = (-\sqrt{2}, 2, \sqrt{2})\) corresponds to the point \( S_d \) with \((\alpha, \beta) = (8, -4)\) of [20], at this point an attracting Bogdanov–Takens bifurcation of codimension 3 occurs (see Theorem 5.2 of the paper), and the equation of \( C_0 : b^2 = bK + 6 \) becomes \( \beta^2 = \alpha (\beta + 6) \), which is Eq. (6.15) of [20], corresponding to the Hopf bifurcation of order 2. In [5] Chen and Zhang proved the uniqueness of limit cycles in the spatial case \( b = 0 \). From Fig. 14 we can see, that the line \((b, K): b = 0\) does not meet the region \( W \), hence by Theorem 5.1 the cyclicity of any limit periodic sets is at most one for \( b = 0 \).

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**References**


