ULTRAFILTERS ON \( \omega \) AND ATOMS IN THE LATTICE OF UNIFORMITIES II

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Received 2 September 1985
Revised 4 August 1987

Interaction between ultrafilters and uniformities on a countable set is investigated. Various ultrafilters are constructed such that atoms in the lattice of uniformities refining the corresponding ultrafilter uniformities have special properties.

AMS (MOS) Subj. Class.: 54E15, 03E50, 05C55, 05C65
ultrafilters uniformities countable set

Introduction

In the first part [11] of the paper, we have investigated basic properties of atoms in the lattice of uniformities on a given set and we related atoms to ultrafilters. We have shown that there are two types of atoms: proximally non-discrete (those inducing a non-trivial proximity) and proximally discrete ones; main results concerned the former type.

In the present part, we investigate proximally discrete atoms; they are characterized [11] as those refining the ultrafilter uniformity \( U_\infty \) for some ultrafilter \( \mathcal{F} \) (cf. Section 1 below). Restricting ourselves to the case of a countable underlying set and assuming the continuum hypothesis, we bring several constructions of ultrafilters with special behaviour of corresponding atoms.

As a technical tool, we develop a general ultrafilter construction (Section 2). Finite combinatorics considerations are involved: our building blocks are finite cubes Section 3 and selective hypergraphs [8] (cf. Section 5.3).
Main results are in Sections 4 and 5. In Section 4, we find ultrafilters with given number of atoms refining the corresponding ultrafilter uniformity; all these atoms are zero-dimensional. In Section 5, we prove the existence of non-zero-dimensional atoms. In fact, we find an ultrafilter such that there are \(2^n\) atoms refining the ultrafilter uniformity and all these atoms are non-zero-dimensional. All the ultrafilters \(\mathcal{F}\) constructed are such that \(\mathcal{F} > \mathcal{G}\) in the Rudin–Keisler order for a given ultrafilter \(\mathcal{G}\).

Some of the results (Sections 2 and 4) appeared in seminar notes [13] and [14].

1. Preliminaries

1.1. Covers and uniformities. As for the basic definitions of the theory of uniform spaces to be used, we refer to the first part [11] of the paper or to [4]. Actually, only the following concepts are needed: a uniformity as a family of covers, a basis for a uniformity, a meet (denoted by \(\wedge\)) of two covers or uniformities, respectively, the relation "to be finer than" (denoted by \(<\)) for covers and uniformities, the uniformly discrete uniformity, a \(C\)-discrete set (where \(C\) is a cover, that is, a set \(A\) such that \(|A \cap C| \leq 1\) for all \(C \in C\)). A uniformity is called zero-dimensional if it has a basis consisting of partitions. A cover \(C\) of a set \(X\) is said to be point-finite if each \(x \in X\) is contained in finitely many members of \(C\) only.

Lemma [15]. Every uniformity on a countable set has a basis consisting of point-finite covers.

We shall work with uniformities induced by ultrafilters, metrics and partitions:

If \(\mathcal{F}\) is a filter on a set \(X\) then the filter uniformity \(\mathcal{U}_{\mathcal{F}}\) has a basis formed by all covers of the form

\[\{F\} \cup \{\{x\}; \ x \in X - F\} \ (F \in \mathcal{F}).\]

If \(\rho\) is a metric and \(\mathcal{P}\) a partition on a set \(X\) then the metric uniformity \(\mathcal{U}_{\rho}\) and the partition uniformity \(\mathcal{U}_{\mathcal{P}}\) have bases consisting of all covers \(\{\{y \in X; \ \sigma(x, y) < \varepsilon\}; \ x \in X\} \ (\varepsilon > 0)\) and of the single cover \(\mathcal{P}\), respectively.

1.2. Ultrafilters. The Rudin–Keisler order for ultrafilters on a set \(X\) is defined by \(\mathcal{F} > \mathcal{G}\) iff \(f\mathcal{F} = \mathcal{G}\) for some map \(f: X \rightarrow X\). If \(f\) is, in addition, finite-to-one (i.e. \(f^{-1}x\) is finite for every \(x \in X\)) then \(\mathcal{F}\) is said to be a finite-to-one lift of \(\mathcal{G}\).

Let \(\mathcal{F}\) be an ultrafilter on \(X\). Then \(\mathcal{F}\) is called selective if for every partition \(\mathcal{P}\) of \(X\) with \(\mathcal{F} \cap \mathcal{P} = \emptyset\), \(\mathcal{F}\) contains a \(\mathcal{P}\)-discrete set. \(\mathcal{F}\) is called rare if the last condition restricted to covers consisting of finite sets holds. \(\mathcal{F}\) is called a \(p\)-point if for every partition \(\mathcal{P}\) of \(X\) with \(\mathcal{F} \cap \mathcal{P} = \emptyset\) there exists \(F \in \mathcal{F}\) such that \(F \cap P\) is finite for all \(P \in \mathcal{P}\).
1.3. **Atoms.** An *atom* in the lattice of uniformities on a given set $X$ is a uniformity $\mathcal{A}$ on $X$ such that the only uniformity on $X$ that is strictly finer than $\mathcal{A}$ is the uniformly discrete one. Notice that for every uniformity $\mathcal{U}$ there exists an atom $\mathcal{A}$ with $\mathcal{A} < \mathcal{U}$ if $\mathcal{U}$ is not uniformly discrete.

We will be concerned with *proximally discrete atoms* (the proximally non-discrete ones were investigated in [11]); they can be characterized without any reference to proximities as follows.

**Proposition** [11]. An atom $\mathcal{A}$ is proximally discrete if $\mathcal{A} < \mathcal{U}_F$ for some ultrafilter $\mathcal{F}$.

**Remark** [11]. In the lattice of uniformities on a countable set (in fact, on a set of non-measurable cardinality), $\mathcal{U}_F$ is an atom iff $\mathcal{F}$ is selective.

2. **The machine**

We present a general ultrafilter construction which is a modification of that from [14]. (Note that a similar construction was used in [18].) Constructions in subsequent sections are its special cases obtained by a suitable choice of parameters.

2.1. The ultrafilter constructions in Sections 4 and 5 have the same general pattern: we will find an ultrafilter $\mathcal{F}$ on $\omega$ such that the collection of atoms refining $\mathcal{U}_F$ has desired properties. Moreover, given any ultrafilter $\mathcal{G}$ on $\omega$, we want to have $\mathcal{F} > \mathcal{G}$ in the Rudin-Keisler order and, in some cases, to have $\mathcal{F}$ as close as possible to $\mathcal{G}$ in the sense that $\mathcal{F}$ is a finite-to-one lift of $\mathcal{G}$.

2.2. **Basic partition.** We start with an appropriate partition $\mathcal{R} = \{R_n; n \in \omega\}$ of $\omega$ into finite sets such that $\lim|R_n| = \infty$. The partition $\mathcal{R}$, called the *basic partition*, is the first parameter of our construction. $\mathcal{R}$ induces a finite-to-one map $q: \omega \to \omega$ where $R_n = q^{-1}(\{n\})(n \in \omega)$.

Then we shall construct a filter $\mathcal{F}'$ such that $q\mathcal{F}'$ is a Fréchet filter, that is, for every $F \in \mathcal{F}'$, $F \cap R_n \neq \emptyset$ for all $n$ but a finite number. Then $\mathcal{F}$ will be an arbitrary ultrafilter extending the family $\mathcal{F}' \cup \{q^{-1}(G); G \in \mathcal{G}\}$; in fact, we shall build $\mathcal{F}'$ in such a way that $\mathcal{F}$ will be the only ultrafilter extending this family.

2.3. **The size.** Naturally, trying to avoid the case that $\sup|F \cap R_n| < \infty$ for some $F \in \mathcal{F}$ (then we would have $\mathcal{F} \sim \mathcal{G}$) we want the cardinalities $|F \cap R_n|$ to tend to the infinity for $n \to \infty$. It turns out that the increasing cardinality of sets $F \cap R_n$ does not suffice for our purposes. We have to introduce, in each of the constructions below, a better measure of being large. We shall call it the size. Crucial properties of the size and the concept of a large set are summarized in the following.
**Definition.** Given a basic partition $\mathcal{R} = \{R_n; n \in \omega\}$, a *size* is an integer valued function $sz$, the domain of which is the set of finite subsets of $\omega$, such that

(i) $0 \leq sz M \leq |M|$ for every finite set $M \subset \omega$,
(ii) $Q \subset M$ implies $sz Q \leq sz M$ for every pair $Q, M$ of finite subsets of $\omega$,
(iii) $\lim sz R_n = \infty$.

A set $F \subset \omega$ is called $(\mathcal{R}, sz)$-large (or simply large) if

$$\lim sz(F \cap R_n) = \infty.$$ 

2.4. **The basic system.** The filter $\mathcal{F}'$ we shall construct will consist of large sets with a special combinatorial behaviour with respect to suitable covers etc. This leads to the following definition.

**Definition.** A *basic system* is a set $\Sigma$, such that each $\mathcal{F} \in \Sigma$ is a family of subsets of $\omega$, such that

(i) $|\Sigma| \leq \omega_1$,
(ii) $S_1 \subset S_2 \in \mathcal{F}$ implies $S_1 \in \mathcal{F}$ ($\mathcal{F} \in \Sigma$).

Given a basic partition $\mathcal{R}$, a size function $sz$ and a basic system $\Sigma$, a set $F \subset \omega$ is called $(\mathcal{R}, sz, \mathcal{F})$-nice, where $\mathcal{F} \in \Sigma$, if it is $(\mathcal{R}, sz)$-large and belongs to $\mathcal{F}$.

A filter $\mathcal{F}$ on $\omega$ is called $(\mathcal{R}, sz, \Sigma)$-nice if for every $\mathcal{F} \in \Sigma$, $\mathcal{F}$ contains an $(\mathcal{R}, sz, \mathcal{F})$-nice set.

2.5. **Theorem.** Let $\mathcal{R}, sz, \Sigma$ be a basic partition, a size and a basic system, respectively. Suppose that for every $\mathcal{F} \in \Sigma$, each $(\mathcal{R}, sz)$-large set contains an $(\mathcal{R}, sz, \mathcal{F})$-nice subset.

Then there exists an $(\mathcal{R}, sz, \Sigma)$-nice filter on $\omega$.

**Proof.** Let $\Sigma = \{\mathcal{F}_\alpha; \alpha < \omega_1\}$. Define sets $F_\alpha (\alpha < \omega_1)$ by transfinite induction. Let $F_0 = \omega$. Suppose $\{F_\beta; \beta < \alpha\}$ have been defined such that every finite intersection $F_\beta_0 \cap \cdots \cap F_\beta_\alpha$ is $(\mathcal{R}, sz)$-large and for each $\beta + 1 < \alpha$, $F_\beta + 1$ is $(\mathcal{R}, sz, \mathcal{F}_\beta)$-nice. Let $\{F_n; n \in \omega\}$ be an enumeration of $\{F_\beta; \beta < \alpha\}$ and let $G_n = \bigcap\{F_i; i \leq n\}$. As each set $G_k (k \in \omega)$ is $(\mathcal{R}, sz)$-large, there exists $n_k \in \omega$ such that $sz(G_k \cap R_m) \geq k$ for all $m \geq n_k$. We may suppose $n_0 < n_1 < \cdots < n_k < \cdots$. Put $P_\alpha = \bigcup\{G_k \cap R_m; k \in \omega, n_k < m < n_{k+1}\}$. Then $sz(P_\alpha \cap R_m) \geq k$ for $m \geq n_k$; hence $\lim sz(P_\alpha \cap R_m) = -\infty$ for $m \to \infty$, that is, the set $P_\alpha$ is $(\mathcal{R}, sz)$-large. If $\alpha$ is limit, put $F_\alpha = P_\alpha$; if $\alpha = \beta + 1$, let $F_\alpha$ be an $(\mathcal{R}, sz, \mathcal{F}_\beta)$-nice subset of $P_\alpha$. We have to show that all finite intersections $F_\alpha \cap F_\beta_0 \cap \cdots \cap F_\beta_\alpha$, equivalently, all $F_\alpha \cap G_k (k \in \omega)$, are $(\mathcal{R}, sz)$-large. But $F_\alpha \cap G_k \cap R_m = F_\alpha \cap R_m$ for sufficiently large $m$, thus $\lim sz(F_\alpha \cap G_k \cap R_m) = \infty$ for $m \to \infty$. Indeed, if $m \geq n_k$, say $n_p \leq m < n_{p+1}$ for some $p > k$, then $(F_\alpha \cap G_k) \cap R_m = (F_\alpha \cap R_m) \cap G_k = (F_\alpha \cap R_m) \cap G_p = F_\alpha \cap R_m$. Having completed the induction, we define $\mathcal{F}$ to be the filter a basis of which is $\{F_\alpha; \alpha < \omega_1\}$. The filter $\mathcal{F}$ is $(\mathcal{R}, sz, \Sigma)$-nice. \(\square\)

2.6. **The nice ultrafilters.** Having an $(\mathcal{R}, sz, \Sigma)$-nice filter $\mathcal{F}'$ we define $\mathcal{F}$ to be an arbitrary ultrafilter refining both $\mathcal{F}'$ and $\{q^{-1}(G); G \in \mathcal{F}\}$ where $q$ is from Section
2.2 and $\mathcal{G}$ is a given ultrafilter on $\omega$. To ensure nice properties of $\mathcal{F}$ we need to check combinatorial properties of its members. So we need to have good evidence for membership of sets in $\mathcal{F}$. This is provided by adding all systems $\mathcal{S}_M = \{A \subseteq \omega; \text{ for every } n \in \omega, \text{ either } A \cap R_n \subseteq M \text{ or } A \cap R_n \subseteq \omega - M \} \quad (M \subseteq \omega)$ to $\Sigma$ and by supposing the following Ramsey property of the size: there exists a function $\varphi : \omega \to \omega$ such that $\text{sz}(M_1 \cup M_2) \geq \varphi(k)$ implies $\max\{\text{sz}(M_1), \text{sz}(M_2)\} \geq k$ for all $k \in \omega$ and all pairs $M_1, M_2$ of finite subsets of $\omega$.

**Lemma.** Let $\mathcal{R}$ be a basic partition and $\text{sz}$ a size function with the Ramsey property. Then each $(\mathcal{R}, \text{sz})$-large set contains an $(\mathcal{R}, \text{sz}, \mathcal{S}_M)$-nice subset, for every $M \subseteq \omega$.

**Proof.** Let $F$ be $(\mathcal{R}, \text{sz})$-large. For every $n \in \omega$, let $F_n$ be the one of the sets $F \cap M \cap R_n, F \cap (\omega - M) \cap R_n$ with maximal value of the size function. It follows easily from the Ramsey property of the size function that $\lim \text{sz}(F \cap R_n) = \infty$ implies $\lim \text{sz} F_n = \infty$. So the set $F' = \bigcup \{F_n; n \in \omega\}$ is $(\mathcal{R}, \text{sz})$-large and clearly $F' \in \mathcal{S}_M$. So $F'$ is an $(\mathcal{R}, \text{sz}, \mathcal{S}_M)$-nice set contained in $F$.  

**Corollary (CH).** Let $\mathcal{R}$, $\text{sz}$, $\Sigma$ be a basic partition, a size and a basic system, respectively. Suppose that $\text{sz}$ has the Ramsey property and that for every $\mathcal{F} \in \Sigma$, each $(\mathcal{R}, \text{sz})$-large set contains an $(\mathcal{R}, \text{sz}, \mathcal{F}_M)$-nice subset. Then there exists an $(\mathcal{R}, \text{sz}, \Sigma)$-nice filter $\mathcal{F}'$ on $\omega$ such that for every ultrafilter $\mathcal{G}$ on $\omega$, the filter generated by the family $\mathcal{F}' \cup \{\bigcup \{R_n; n \in G\}; G \in \mathcal{G}\}$ is an ultrafilter.

**Proof.** Let $\Sigma_0 = \{\mathcal{F}_M; M \subseteq \omega\}$. By the CH, $|\Sigma_0| = \omega_1$ and so $\Sigma \cup \Sigma_0$ is a basic system. By Theorem 2.5, Theorem and by the preceding Lemma, there exists an $(\mathcal{R}, \text{sz}, (\Sigma \cup \Sigma_0))$-nice filter $\mathcal{F}'$ on $\omega$. Let $\mathcal{G}$ be an ultrafilter on $\omega$. Denote $\mathcal{F}$ the filter generated by the family in question; thus $F \in \mathcal{F}$ iff $F \supseteq F' \cap (\bigcup \{R_n; n \in G\})$ for some $G \in \mathcal{G}$ and $F' \in \mathcal{F}'$. To prove that $\mathcal{F}$ is an ultrafilter, consider any partition $\{M_1, M_2\}$ of $\omega$. Choose an $(\mathcal{R}, \text{sz}, \mathcal{F}_M)$-nice set $F' \in \mathcal{F}'$. As $F' \in \mathcal{F}_M$, sets $G_i = \{n \in \omega; F' \cap R_n \subseteq M_i\} (i = 1, 2)$ cover $\omega$ and so one of them belongs to $\mathcal{G}$. If $G_i \in \mathcal{G}$ then $F' \cap (\bigcup \{R_n; n \in G_i\}) \subseteq M_i$, so $M_i \in \mathcal{F}$. Thus $\mathcal{F}$ is an ultrafilter.  

3. A combinatorial lemma

In this section, we state an auxiliary result of finite combinatorial character. We shall need it to form basic partitions, sizes and basic systems in Section 4.
3.1. First, let us introduce some preliminary notions. Let \( A_1, A_2, \ldots, A_s \) be pairwise disjoint sets of cardinality \( n \in \omega \). Then the set \( A_1 \times A_2 \times \cdots \times A_s \) will be called an \( s \)-cube. If \( B_i \subseteq A_i \) for \( i = 1, \ldots, s \), \( |B_i| = \cdots = |B_s| \), then the cube \( B_1 \times B_2 \times \cdots \times B_s \) will be called a subcube of \( A_1 \times A_2 \times \cdots \times A_s \). If no special emphasis on the coordinate set will be needed, we shall use the notation \( Q(n^s) \) for an \( s \)-cube where \( n \) is the cardinality of coordinate sets.

If \( Q(n^s) = F_1 \times F_2 \times \cdots \times F_s \), denote by \( \mathcal{F}_i \) (\( 1 \leq i \leq s \)) the partition of \( Q(n^s) \) consisting of sets of the form

\[
\{f_i\} \times \cdots \times \{f_{i-1}\} \times F_i \times \{f_{i+1}\} \times \cdots \times \{f_s\} \quad \text{where} \quad f_j \in F_j
\]

for \( j = 1, \ldots, i-1, i+1, \ldots, s \).

3.2. Lemma. Let \( s \geq 1 \) be a natural number. Then there exists a function \( \alpha : \omega \to \omega \) with the following property: For every \( r \in \omega \), \( r \geq 1 \), for every \( n \geq \alpha(r) \) and for every cover \( \mathcal{C} \) of the cube \( Q(n^s) \) there exists a subcube \( Q(r^s) \) of \( Q(n^s) \) such that either

(i) \( Q(r^s) \) is \( \mathcal{C} \)-discrete, or

(ii) there exists \( i \in \{1, \ldots, s\} \) such that for every \( T \in \mathcal{F}_i \) there is \( y_T \in T \) with \( Q(r^s) \cap T \subseteq \text{st}^2(y_T, \mathcal{C}) \).

(Recall that \( \text{st}^2(y, \mathcal{C}) = \bigcup \{ \text{st}(z, \mathcal{C}) ; z \in \text{st}(y, \mathcal{C}) \} \) where \( \text{st}(y, \mathcal{C}) = \bigcup \{ C \in \mathcal{C} ; y \in C \} \).)

The rest of Section 3 is devoted to the proof of the Lemma.

3.3. First we shall present a result and some definitions from [7] which are important for the proof of the lemma. Recall that \( [A]^m \) denotes the set of \( m \)-element subsets of a set \( A \).

Claim [7]. Given natural numbers \( s, m, r, v \), there exists a natural number \( n = \gamma(s, m, r, v) \) with the following property:

If \( A_i \) (\( i = 1, \ldots, s \)) are sets with \( |A_i| \geq n \) (\( i = 1, \ldots, s \)) and

\[
c : [A_1]^m \times \cdots \times [A_s]^m \to \{1, \ldots, v\}
\]

is a mapping then there are subsets \( B_i \subseteq A_i, |B_i| = r \) (\( i = 1, \ldots, s \)) such that the mapping \( c \) restricted to the set \( [B_1]^m \times \cdots \times [B_s]^m \) is constant.

Suppose that the sets \( A_i \) are linearly ordered. Let \( < \) be the lexicographic ordering of \( A_1 \times \cdots \times A_s \).

For every pair \( (a, a') \), \( a < a' \) of distinct elements of \( A_1 \times \cdots \times A_s \), define a type \( t(a, a') = (t_1, \ldots, t_s) \) by

\[
t_i = 0 \quad \text{iff} \quad a_i < a'_i,\\
t_i = 1 \quad \text{iff} \quad a_i = a'_i,\\
t_i = 2 \quad \text{iff} \quad a_i > a'_i.
\]
Observe that there are \( \frac{1}{2}(3^3 - 1) = \nu \) different types and suppose that the set of all types is \( \{t^1, t^2, \ldots, t^\nu\} \).

To every pair \( x, x' \in A_1 \times \cdots \times A_s \), \( x = (x_1, \ldots, x_s) \), \( x' = (x'_1, \ldots, x'_s) \), \( x_i < x'_i \) \( i = 1, \ldots, s \) and a type \( t \) assign a pair \( w(x, x', t) = (a, a') \), where \( a = (a_1, \ldots, a_s) \), \( a' = (a'_1, \ldots, a'_s) \) are defined by

\[
\begin{align*}
a_i &= x_i, & a'_i &= x'_i & \text{iff } t_i = 0, \\
a_i &= a'_i = x_i & \text{iff } t_i = 1, \\
a_i &= x'_i, & a'_i &= x_i & \text{iff } t_i = 2.
\end{align*}
\]

3.4. Proof of Lemma 3.2. Let \( s \geq 1 \) be a natural number. For every \( r \in \omega \), \( r \geq 1 \) put \( \alpha(r) = \gamma(s, 2, r + 2, 2^{(3^s - 1)/2}) \). Let \( \mathcal{C} \) be a cover of a cube \( Q(n^s) = A_1 \times \cdots \times A_s \). Consider a fixed linear order on each \( A_i \) and the types \( t^1, \ldots, t^\nu \) where \( \nu = \frac{1}{2}(3^3 - 1) \) as defined in 3.3.

Define a symmetric binary relation \( E \) on \( A_1 \times \cdots \times A_s \) as follows: \( (a, a') \in E \) iff there exists \( C \in \mathcal{C} \) with \( a, a' \in C \). For every pair \( x, x' \in A_1 \times \cdots \times A_s \), \( x = (x_1, \ldots, x_s) \), \( x' = (x'_1, \ldots, x'_s) \), \( x_i < x'_i \) \( (i = 1, \ldots, s) \) define a sequence \( c(x, x') = (\varepsilon_1, \ldots, \varepsilon_s) \) by

\[
\varepsilon_j = 0 \quad \text{if } w(x, x', t^j) = (a, a') \notin E,
\]

\[
\varepsilon_j = 1 \quad \text{if } w(x, x', t^j) = (a, a') \in E.
\]

This defines a mapping \( c: [A_1]^s \times \cdots \times [A_s]^s \to \{0, 1, 2, \ldots, 2^\nu\} \). Using the Claim in Section 3.3, take sets \( B^*_1, [B^*_1]^s \to \{0, 1, \ldots, 2^r\} \) such that \( c \) restricted to the set \( [B^*_1]^s \times \cdots \times [B^*_s]^s \) is constant. Put \( b^*_i = \max B^*_i, \ b^*_s = \min B^*_s, \ b_i = B_i - b^*_i, \ b^*_i \) \( (i = 1, \ldots, s) \).

Suppose first that the value of \( c \) on \( [B^*_1]^s \times \cdots \times [B^*_s]^s \) is \( (0, \ldots, 0) \). Then the subcubes \( B_1 \times \cdots \times B_s \) are \( \mathcal{C} \)-discrete (and thus the case (i) of Lemma 3.2 holds).

Indeed, if \( a, a' \in B_1 \times \cdots \times B_s \), \( a = (a_1, \ldots, a_s) \), \( a' = (a'_1, \ldots, a'_s) \) and if \( I = \{i; a_i = a'_i\} \), put

\[
\begin{align*}
x_i &= \min\{a_i, a'_i\} & i = 1, \ldots, s, \\
x'_i &= \max\{a_i, a'_i\} & i = 1, \ldots, s, \quad i \notin I, \\
x'_i &= b'_i & i = 1, \ldots, s, \quad i \in I.
\end{align*}
\]

Then clearly \( (a, a') = w(x, x', t) \) where \( t \) is the type of \( (a, a') \). As \( c(\{x_1, x'_1\}, \ldots, \{x_s, x'_s\}) = (0, \ldots, 0) \), we have \( (a, a') \notin E \).

Second, suppose the constant value of \( c \) on \( B^*_1 \times \cdots \times B^*_s \) to be distinct from \( (0, \ldots, 0) \). Then there exists a type \( t \) such that for every pair \( a, a' \in B^*_1 \times \cdots \times B^*_s \) of type \( t \) we have \( (a, a') \in E \). Let \( i_0 \) be the first coordinate with \( t_{i_0} \neq 1 \), i.e. \( t_{i_0} = 0 \). Consider the cube \( Q(r^s) = B_1 \times \cdots \times B_s \). For every \( T \in \mathcal{T}_0 \),

\[
T = \{a_1\} \times \cdots \times \{a_{i_0-1}\} \times B_{i_0} \times \{a_{i_0+1}\} \times \cdots \times \{a_s\}
\]
put \( y_T = (a_1, \ldots, a_{i_0-1}, b'_1, a_{i_0+1}, \ldots, a_i) \) and \( c = (c_1, \ldots, c_i) \) where

\[
\begin{align*}
  c_i &= b_i' & \text{if } t_i = 0, \\
  c_i &= a_i & \text{if } t_i = 1, \\
  c_i &= b_i'' & \text{if } t_i = 2.
\end{align*}
\]

Hence the type of \((y_T, c)\) is \( t \). Observe that for any \( b \in T \cap Q(r') \) the type of \((b, c)\) is also \( t \). Hence both \((y_T, c) \in E\) and \((b, c) \in E\) which implies \( b \in \text{st}^2(y_T, c) \). Thus the case (ii) of the lemma holds. This concludes the proof. \( \square \)

3.5. Corollary [2]. For every pair \( m, s \) of natural numbers there exists a natural number \( \beta(m, s) \) such that for every \( n \geq \beta(m, s) \) and for every partition \( \{M_1, M_2\} \) of a cube \( Q(n') \) there exists a subcube \( Q(m') \) that is contained either in \( M_1 \) or in \( M_2 \).

4. Zero-dimensional atoms

In the present section, we construct an ultrafilter \( E \) on \( \omega \) for every \( s < \omega \) such that there are exactly \( s \) atoms \( \mathcal{A} \) with \( \mathcal{A} \ll \mathcal{U}_E \); all these atoms are zero-dimensional. The construction is exploited to obtain an ultrafilter \( E \) such that there are \( 2^s \) atoms refining \( \mathcal{U}_E \).

As the referee kindly pointed out, the atoms in the lattice of zero-dimensional uniformities (in particular: zero-dimensional atoms in the lattice of all uniformities) correspond canonically to maximal elementary submodels of the ultrapower \( N^\omega/E \) where the structure \( N \) is \( \omega \) with all relations and all functions, cf. [17].

Theorem (CH). Let \( 1 \leq s < \omega \) and let \( \mathcal{G} \) be an ultrafilter on \( \omega \). Then there is an ultrafilter \( F \succ \mathcal{G} \) on \( \omega \) such that there are precisely \( s \) distinct atoms refining \( \mathcal{U}_E \). All these atoms are zero-dimensional; in fact, each of them has a basis of the form \( \{ F \wedge P; P \in \mathcal{U}_E \} \) for some partition \( F \) of \( \omega \).

The proof is divided into Sections 4.1.1 to 4.1.8; first, we state two general lemmas (4.1.1 and 4.1.2); then we construct a basic partition, a size function and a basic system (4.1.3-4.1.5) to apply the construction of Section 2. The resulting ultrafilter is shown to have desired properties in 4.1.8.

4.1.1. Lemma (CH). For every ultrafilter \( \mathcal{G} \) on \( \omega \) there exists a rare ultrafilter \( \mathcal{G}' \) on \( \omega \) with \( \mathcal{G}' \succ \mathcal{G} \).

Proof. Fix some partition \( \{A_n; n \in \omega\} \) of \( \omega \) with each \( A_n \) infinite. Let \( \{R_\alpha; \alpha < \omega_1\} \) be a list of all partitions of \( \omega \) into finite sets. Define sets \( U_\alpha \subseteq \omega \) for \( \alpha < \omega_1 \) by induction as follows. Set \( U_{-1} = \omega \). Let \( \beta < \omega_1 \) and let \( U_\alpha \) have been defined for all \( \alpha < \beta \) such that for every finite set \( D \subseteq \beta \), the set \( A_\alpha \cap (\bigcap_{\alpha \in D} U_\alpha) \) is infinite for each \( n \in \omega \), and \( U_\alpha \) is \( R_\alpha \)-discrete. Let \( \{S_\alpha; n \in \omega\} \) be a list of all sets of the form
A_n \cap (\bigcap_{\alpha \in D} U_\alpha) \ (n \in \omega, D \subset \beta, D \text{ finite}) such that each of these sets is listed \omega times. Then define \( U_\beta = \{ x_n; n \in \omega \} \) where the points \( x_n \) are defined by induction: choose \( x_0 \in S_0 \); let \( R_0 \in R_\beta \) be such that \( x_0 \in R_0 \). If \( x_0, \ldots, x_n \) have been chosen with \( x_i \in S_i, i = 0, \ldots, n \) and if \( R_0, \ldots, R_n \in R_\beta \) are such that \( x_i \in R_i, i = 0, \ldots, n \), choose \( x_{n+1} \) to be an arbitrary point of the set \( S_{n+1} - \bigcup_{i=1}^n R_i \). Then for every ultrafilter \( G \) on \( \omega \), the system \( V = \{ U_\alpha; \alpha < \omega_1 \} \cup \{ \bigcup_{n \in \eta} A_n; \eta \in \mathcal{P} \} \) has the finite intersection property and every ultrafilter \( G' \) containing \( V \) is clearly rare and \( G' \sim 3 \).

4.1.2. Lemma. Let \( G \) be a rare ultrafilter on \( \omega \) and \( \mathcal{C} \) a point-finite cover of \( \omega \) consisting of finite sets. Then \( G \) contains a \( \mathcal{C} \)-discrete set.

Proof. Let \( \mathcal{C} = \{ C_n; n \in \omega \} \). Then the sets \( A_i (i \in \omega) \) defined by \( A_0 = C_0, A_{i+1} = \text{st}(A_0 \cup \cdots \cup A_i, \mathcal{C}) - (A_0 \cup \cdots \cup A_i) \) form a partition of \( \omega \) into finite sets and so we can find \( a_i \in A_i (i \in \omega) \) such that the set \( A = \{ a_i; i \in \omega \} \) belongs to \( G \). Clearly \( \text{st}(A_i, \mathcal{C}) \cap A_j = \emptyset \) for \( j > i+2 \) belongs to \( G \). Clearly \( \text{st}(A_i, \mathcal{C}) \cap A_j = \emptyset \) for \( j > i+2 \) and so both \( \{ a_i; i \text{ even} \} \) and \( \{ a_i; i \text{ odd} \} \) are \( \mathcal{C} \)-discrete and one of these sets belongs to \( G \).

4.1.3. The basic partition. Let \( 1 \leq s < \omega \). Let \( \mathcal{R} = \{ R_n; n \in \omega \} \) be a partition of \( \omega \) with \( |R_n| = n^s \). We may and shall assume that each \( R_n \) is an \( s \)-cube \( Q(n^s) \) (cf. 3.1), say \( R_n = F_{n}^{(1)} \times \cdots \times F_{n}^{(s)} \) where \( |F_{n}^{(i)}| = n \) for \( i = 1, \ldots, s \).

4.1.4. The size. Given a finite set \( A \subset \omega \), define \( \text{sz} A \) to be the maximal number \( t \) such that \( A \) contains a cube \( Q(t^s) \), more precisely, such that there are sets \( B_1, \ldots, B_s \) such that \( \sum_{i=1}^s |B_i| = t \) and \( B_1 \times \cdots \times B_s \subset A \cap R_n \) for some \( n \). Clearly, the function \( \text{sz} \) satisfies (i), (ii), (iii) of 2.3. Also, by virtue of the Claim of 3.5, \( \text{sz} \) possesses the Ramsey property.

4.1.5. The basic system. For \( i = 1, \ldots, s \), let \( \mathcal{T}_i \) be the partition of \( \omega \) the trace of which on \( R_n = Q(n^s) \) coincides with \( \mathcal{T}_i \) (cf. 3.1), that is, \( \mathcal{T}_i \) consists of sets of the form \( \{ f_1 \} \times \cdots \times \{ f_{i-1} \} \times F_{n}^{(i)} \times \{ f_{i+1} \} \times \cdots \times \{ f_s \} \). Let \( \{ \mathcal{C}_\alpha; \alpha < \omega_1 \} \) be a list of all point-finite covers of \( \omega \). For \( \alpha < \omega_1 \), let \( \mathcal{F}_\alpha \) be the family of all sets \( Y \subset \omega \) such that the following holds:

(i) for every \( n \in \omega \), either \( Y \cap R_n \) is \( \mathcal{C}_\alpha \)-discrete, or there \( i \leq s \) such that for each \( T \in \mathcal{T}_i, T \subset R_n \), there exists \( y_T \in T \) with \( Y \cap T \subset \text{st}^2(y_T, \mathcal{C}_\alpha) \);

(ii) for every \( C \in \mathcal{C}_\alpha \), the set \( C \cap \bigcup \{ Y \cap R_n; n \in \omega, Y \cap R_n \in \mathcal{C}_\alpha \text{-discrete} \} \) is finite. Then \( \Sigma = \{ \mathcal{F}_\alpha; \alpha < \omega_1 \} \) is our basic system.

4.1.6. Claim. Every \( (\mathcal{R}, \text{sz}) \)-large set \( F \) contains an \( (\mathcal{R}, \text{sz}, \mathcal{F}_\alpha) \)-nice subset \( Y \).

Proof. For every \( n \in \omega \), let \( Y'_n \subset R_n \cap F \) be a maximal cube \( Q(r^s) \) such that either \( Y'_n \) is \( \mathcal{C}_\alpha \)-discrete or there \( i < s \) such that for each \( T \in \mathcal{T}_i, T \subset R_n \), there exists \( y_T \in T \) with \( Y \cap T \subset \text{st}^2(y_T, \mathcal{C}_\alpha) \). Using Lemma 3.2 and the fact that \( F \) is large we get \( \lim \text{sz} Y'_n = \infty \).
Denote $K = \{ n \in \omega ; \ Y'_n \text{ is } C_\alpha \text{-discrete} \}$. If $K$ is finite then the set $Y = \bigcup \{ Y'_n ; n \in \omega \}$ satisfies both (i) and (ii) of 4.1.5 and so $Y \in \mathcal{F}_\alpha$; then $Y$ is $(\mathcal{R}, sz, \mathcal{F}_\alpha)$-nice.

Let $K$ be infinite. Let $\mathcal{C}_\alpha = \{ C_i ; i \in \omega \}$. For every $k \in \omega$, $m \in K$ consider the partition $\{ M_1, M_2 \}$ of $Y'_m$ where $M_1 = (C_1 \cup \cdots \cup C_k) \cap Y'_m$, $M_2 = Y'_m - (C_1 \cup \cdots \cup C_k)$. By Corollary 3.5, if $m$ is large enough, say $m \geq n_k$, then either $M_1$ or $M_2$ contains a cube $Q((k+1)^s)$. But $Y'_m$ is $C_\alpha$-discrete, so $|M_1| \leq k$ and then the cube $Q((k+1)^s)$ is contained in $M_2$; thus $sz M_2 \geq k + 1$. We may assume that numbers $n_k$ form an increasing sequence. For every $m \in K$ put $Y_m = Y'_m - (C_1 \cup \cdots \cup C_k)$ whenever $n_k \leq m < n_{k+1}$. Thus $sz Y_m \geq k + 1$ and so, putting $Y_m = Y'_m$ for $m \in \omega - K$ we have $\lim sz Y_m = \infty$. Then the set $Y = \bigcup \{ Y_m ; m \in \omega \}$ is large, it satisfies (i) of 4.1.5, and finally it satisfies (ii) of 4.1.5 because every $C_k \in \mathcal{C}_\alpha$ does not meet any $Y \cap R_m = (Y'_m)$ with $m > n_k$. Thus, $Y$ is $(\mathcal{R}, sz, \mathcal{F}_\alpha)$-nice.  

4.1.7. Now, we are prepared to construct our ultrafilter $\mathcal{F}$ with $\mathcal{F} > \mathcal{G}$ where $\mathcal{G}$ is a given ultrafilter on $\omega$. By the virtue of 4.1.1, we may assume that $\mathcal{G}$ is rare. The corollary in 2.6 yields an ultrafilter $\mathcal{F}$ generated by the family $\mathcal{F}' \cup \bigcup \{ R_n ; n \in G \}$; $G \in \mathcal{G}$ where $\mathcal{F}'$ is an $(\mathcal{R}, sz, \Sigma)$-nice filter. Clearly, $\mathcal{F} > \mathcal{G}$ viz. $q\mathcal{F} = \mathcal{G}$ where $qm = n$ for all $n \in \omega$ and all $m \in R_n$.

4.1.8. For $i = 1, \ldots, s$, let $\mathcal{A}_i$ be the uniformity with a basis $\{ \mathcal{T}_i \cap \mathcal{P} ; \mathcal{P} \in \mathcal{U}_\mathcal{F} \}$ where $\mathcal{T}_i$ is the partition of 4.1.5.

Claim. $\mathcal{A}_1, \ldots, \mathcal{A}_s$ are pairwise distinct atoms refining $\mathcal{U}_\mathcal{F}$ and there exists no other atom $\mathcal{A} < \mathcal{U}_\mathcal{F}$.

Proof. As $\mathcal{F}'$ consists of large sets, one gets easily that no $\mathcal{A}_i$ is uniformly discrete. On the other hand, $\mathcal{A}_i \wedge \mathcal{A}_j$ is uniformly discrete for $i \neq j$ because $\mathcal{T}_i \cap \mathcal{T}_j = \{ \{ x \} ; x \in \omega \}$. So the $\mathcal{A}_i$'s are pairwise distinct.

Thus it suffices to prove that for every uniformity $\mathcal{U} < \mathcal{U}_\mathcal{F}$ either there exists $i$ with $\mathcal{A}_i < \mathcal{U}$ or $\mathcal{U}$ is uniformly discrete. Suppose $\mathcal{U} > \mathcal{A}_i$ for no $i$. Then there are covers $\mathcal{E}_i \in \mathcal{U}$ with $\mathcal{E}_i \in \mathcal{A}_i$ ($i = 1, \ldots, s$) and for $\mathcal{E} = \bigwedge_{i=1}^s \mathcal{E}_i$ we have $\mathcal{E} \in \mathcal{U}$, $\mathcal{E} \in \mathcal{A}_i$ ($i = 1, \ldots, s$). Choose $\mathcal{E} \in \mathcal{U}$ with $\{ st^2(x, \mathcal{E}) ; x \in \omega \} < \mathcal{E}$. As $\mathcal{U}$ has a basis consisting of point-finite covers (see Lemma 1.1) we may assume $\mathcal{E} = \mathcal{E}_\alpha$ for some $\alpha < \omega_1$, and then there is a set $Z \in \mathcal{F}$ which is $(\mathcal{R}_\alpha, sz, \mathcal{F}_\alpha)$-nice. Consider the cover $\mathcal{P}_Z = \{ \{ x \} ; x \in \omega \}$. Since $\mathcal{P}_Z \in \mathcal{U}_\mathcal{F}$ and $\mathcal{U} < \mathcal{U}_\mathcal{F}$, $\mathcal{P}_Z$ belongs to $\mathcal{U}$. The set $Z$, being in $\mathcal{L}_\alpha$, yields sets $K, A_1, \ldots, A_s$ which cover $\omega$ where $n \in A_i$ ($i = 1, \ldots, s$) iff for each $T \in \mathcal{T}_i$, $T \subset R_n$ one can choose a point $y_T \in T$ such that $Z \cap T \subset st^2(y_T, \mathcal{E}_\alpha)$, and $n \in K$ iff $Z \cap R_n$ is $\mathcal{C}_\alpha$-discrete. Now, one of the sets $K, A_1, \ldots, A_s$ is in $\mathcal{G}$ if $T \in \mathcal{F}_\alpha$ for some $i$ then the cover $\mathcal{L}_i = \{ \{ R_n ; n \in A_i \} \} \cup \{ \{ x \} ; x \in \omega \}$ belongs to $\mathcal{U}_\mathcal{F}$ and so $\mathcal{L}_i \cap \mathcal{T}_i \cap \mathcal{P}_Z \in \mathcal{A}_i$ and the last cover refines $\{ st^2(x, \mathcal{E}) ; x \in \omega \}$ and (consequently) $\mathcal{E}$ which is impossible because $\mathcal{E} \in \mathcal{A}_i$. Hence $K \notin \mathcal{G}$. Put $\mathcal{L}_0 = \{ \{ R_n ; n \in K \} \} \cup \{ \{ x \} ; x \in \omega \}$. Thus $\mathcal{L}_0 \in \mathcal{U}_\mathcal{F} < \mathcal{U}$. The set $Z$, being $(\mathcal{R}_\alpha, sz, \mathcal{F}_\alpha)$-nice, belongs to $\mathcal{F}_\alpha$ and then the condition (ii) of 4.1.5 says just that the cover $\mathcal{L}_0 \cap \mathcal{C}_\alpha \cap \mathcal{P}_Z$ consists of finite
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sets and so does the cover \( \hat{\mathcal{C}} = \{ \hat{\mathcal{C}}_i ; C \in \mathcal{L}_0 \wedge \mathcal{C}_n \wedge \mathcal{P}_Z \} \) where \( \hat{\mathcal{C}} = \{ n \in \omega ; C \wedge R_n \neq \emptyset \} \).
Moreover \( \hat{\mathcal{C}} \) is point finite for \( \mathcal{C}_n \) is. We use the fact that \( \mathcal{C} \) is rare to find a \( \hat{\mathcal{C}} \)-discrete set \( G \in \mathcal{G} \), see 4.1.2. Thus each \( \hat{\mathcal{C}} \in \hat{\mathcal{C}} \) contains at most one \( n \in G \), that is, each \( C \in \mathcal{L}_0 \wedge \mathcal{C}_n \wedge \mathcal{P}_Z \) meets at most one \( R_n \) with \( n \in G \). Moreover, \( |C \wedge R_n \wedge Z| = 1 \) for all \( C \in \mathcal{L}_0 \wedge \mathcal{C}_n \wedge \mathcal{P}_Z \) and \( n \in K \) because \( R_n \wedge Z \) is \( \mathcal{C}_n \) discrete for \( n \in K \) and thus \( |C \cap (\cup \{ R_n \wedge Z ; n \in K \wedge G \})| \leq 1 \) for all \( C \in \mathcal{L}_0 \wedge \mathcal{C}_n \wedge \mathcal{P}_Z \). In other words, the set \( H \wedge Z \in \mathcal{F} \) where \( H = \bigcup \{ R_n ; n \in K \wedge G \} \) is \( \mathcal{L}_0 \wedge \mathcal{C}_n \wedge \mathcal{P}_Z \)-discrete, and so, for \( \mathcal{P}_H = \{ H \} \cup \{ \{ x \} ; x \in \omega \} \in \mathcal{U}_\mathcal{Z} \) we have \( \mathcal{P}_H \wedge \mathcal{L}_0 \wedge \mathcal{P}_Z = \{ \{ x \} ; x \in \omega \} \). As \( \mathcal{P}_H , \mathcal{L}_0 , \mathcal{C}_n , \mathcal{P}_Z \in \mathcal{U} \), the uniformity \( \mathcal{U} \) is uniformly discrete which concludes the proof.  

4.2. The preceding theorem shows that there are ultrafilters \( \mathcal{F} \) on \( \omega \) such that each atom \( \mathcal{A} \subset \mathcal{U}_\mathcal{Z} \) is obtained by adding a single partition to \( \mathcal{U}_\mathcal{Z} \). The next theorem exhibits an ultrafilter of quite dissimilar nature.

**Theorem (CH).** Let \( \mathcal{G} \) be an ultrafilter on \( \omega \). Then there exists an ultrafilter \( \mathcal{F} > \mathcal{G} \) on \( \omega \) such that for each partition \( \mathcal{F} \) of \( \omega \), the uniformity with the basis \( \{ \mathcal{F} \wedge \mathcal{P} ; \mathcal{P} \in \mathcal{U}_\mathcal{Z} \} \) is never an atom. Moreover, there exists exactly one atom \( \mathcal{A} \subset \mathcal{U}_\mathcal{Z} \) and this atom is zero-dimensional.

4.3. **Theorem (CH).** Let \( \mathcal{G} \) be an ultrafilter on \( \omega \). Then there exists an ultrafilter \( \mathcal{F} > \mathcal{G} \) on \( \omega \) such that there are \( 2^\omega \) distinct atoms refining \( \mathcal{U}_\mathcal{Z} \).

(Notice that \( 2^\omega \) is the cardinality of the set of all uniformities on \( \omega \).)

**Proof.** For \( s = 1, 2, \ldots \), let \( \mathcal{F}_s \) be the ultrafilter constructed in 4.1. Then \( \mathcal{F} = \sum_s \mathcal{F}_s \) has the required properties (recall that \( \sum_s \mathcal{F}_s \) is the ultrafilter generated by the family of all sets of the form \( \bigcup \{ F_s ; s \in G \} \) where \( F_s \in \mathcal{F}_s \) for all \( s \in G \) and \( G \in \mathcal{G} \).
Clearly $\mathcal{F} > \mathcal{G}$. To conclude the proof, recall some facts from the proof of Theorem 4.1. For every $s$, $\mathcal{F}_s$ is an ultrafilter on a countable set, say on $K_s$, which is a disjoint union of $s$-cubes $Q(n^s) = F_n^{(1,s)} \times \cdots \times F_n^{(s,s)}$ ($n \in \omega$). For every $Z \subset \{1, \ldots, s\}$, let $\mathcal{F}_{s,Z}$ be the partition of $K_s$ consisting of sets of the form

$$\{(g_1, \ldots, g_s) \in F_n^{(1,s)} \times \cdots \times F_n^{(s,s)} : g_i = f_i \text{ for } i \in \{1, \ldots, s\} - Z\}$$

for all $n \in \omega$ and for various choices $f_i \in F_n^{(i,s)}$ ($i \in \{1, \ldots, s\}$). Then $\mathcal{F}_{s,\{i\}}$ ($i = 1, \ldots, s$) are just the partitions $\mathcal{F}_i$ of 4.1.5 with the property that $\{\mathcal{F}_{s,\{i\}} \land \mathcal{P} : \mathcal{P} \in \mathcal{F}_s\}$ ($i = 1, \ldots, s$) are bases of $s$ distinct atoms refining $\mathcal{U}_\varphi$. Obviously, $\mathcal{F}_{s,Z} \land \mathcal{F}_{s,w} = \mathcal{F}_{s,Z \lor W}$ and $\mathcal{F}_{s,Z} = \{\{x\} : x \in \omega\}$ iff $Z = \emptyset$.

Now $\mathcal{F}$ is an ultrafilter on $\omega$ which is represented as the disjoint union of sets $K_s$. Define $X = \{(s, i) : s \in \omega, s > 0, 1 \leq i \leq s\}$. For $Z \subset X$, put $Z_i = \{(s, i) : s \in \omega, s > 0\}$ and define a partition $\mathcal{F}_Z$ by $\mathcal{F}_Z = \bigcup\{\mathcal{F}_{s,Z} : s \in \omega, s > 0\}$. Now, it is clear that for $Z \subset X$, the uniformity with a basis $\{\mathcal{F}_Z \land \mathcal{P} : \mathcal{P} \in \mathcal{U}_\varphi\}$ is uniformly discrete iff $\{s : Z_s = \emptyset\} \in \mathcal{G}$. Let $\mathcal{H}$ be the filter on $X$ generated by the family $\{\{(s, i) : s \in \mathcal{P}\} : \mathcal{P} \in \mathcal{G}\}$. Let $\mathcal{M}$ be the collection of all ultrafilters $\mathcal{F}$ refining $\mathcal{H}$. Obviously $|\mathcal{M}| = 2^{2^\omega}$.

For every $\mathcal{F} \in \mathcal{M}$, let $\mathcal{U}(\mathcal{F})$ be the uniformity generated by $\{\mathcal{F}_{s,Z} : Z \in \mathcal{I} \cup \mathcal{U}_\varphi\}$. Since $\mathcal{I} \supset \mathcal{H}$ and $\mathcal{I}$ is a filter, $\mathcal{U}(\mathcal{I})$ is not uniformly discrete. Let $\mathcal{A}(\mathcal{I})$ be an atom refining $\mathcal{U}(\mathcal{I})$. If $\mathcal{I}, \mathcal{I}' \in \mathcal{M}$, $\mathcal{I} \neq \mathcal{I}'$ then $\mathcal{A}(\mathcal{I}) \neq \mathcal{A}(\mathcal{I}')$ for one can find $Z \in \mathcal{I}$, $Z' \in \mathcal{I}'$ with $Z \cap Z' = \emptyset$ and so $\mathcal{A}(\mathcal{I}) \land \mathcal{A}(\mathcal{I}')$ contains $\mathcal{F}_Z \land \mathcal{F}_{Z'} = \{\{x\} : x \in \omega\}$, that is, $\mathcal{A}(\mathcal{I}) \land \mathcal{A}(\mathcal{I}')$ is uniformly discrete. Thus there are $2^\omega$ atoms refining $\mathcal{U}_\varphi$. □

5. Non-zero-dimensional atoms

The atoms constructed in the first part of the paper and in Section 4 were zero-dimensional. The existence of a non-zero-dimensional atom was an open problem for some time. The reason was that any construction attempting to get a non-zero-dimensional atom must be complicated. Indeed, if $\mathcal{A}$ is such an atom then there exists a cover $\mathcal{C} \subset \mathcal{A}$ which is refined by no partition $\mathcal{P} \in \mathcal{U}$. Thus for every partition $\mathcal{P} \subset \mathcal{C}$ there is a cover $\mathcal{C}_\mathcal{P} \subset \mathcal{U}$, $\mathcal{C}_\mathcal{P} \subset \mathcal{C}$ which kills $\mathcal{P}$ in the sense that $|\mathcal{C} \cap \mathcal{P}| \leq 1$ for every $C \in \mathcal{C}_\mathcal{P}$ and $\mathcal{P} \in \mathcal{P}$. On the other hand, the $\mathcal{C}_\mathcal{P}$'s must not kill each other. Further, it will be shown that $\mathcal{C}$ can always be chosen to consist of finite sets. Thus investigation of non-zero-dimensional atoms leads to finite combinatorics considerations involving very strong Ramsey type properties of structures to be used.

The first construction of a non-zero-dimensional atom was given in [13]; here we present another construction with more additional properties answering some related questions:

1. Does there exist an ultrafilter $\mathcal{F}$ on $\omega$ such that there is more than one non-zero-dimensional atom refining $\mathcal{U}_\varphi$?
(2) Does there be an atom in the lattice of zero-dimensional uniformities such that there is more than one non-zero-dimensional atom refining it?

(3) Which ultrafilters $\mathcal{U}$ possess a finite-to-one lift $\mathcal{F}$ such that there is a non-zero-dimensional atom refining $\mathcal{U}$? The question was inspired by Theorem 5.2 below implying that each $\mathcal{F}$ with a non-zero-dimensional atom refining $\mathcal{U}$ is a non-trivial finite-to-one lift; so the question weakens the (unsolved) problem of the characterization of those $\mathcal{F}$'s.

(4) Do there exist two distinct atoms $\mathcal{A}_1, \mathcal{A}_2$ with $d\mathcal{A}_1 = d\mathcal{A}_2$ where $d$ stands for the distal modification (see 5.8)? Then $d\mathcal{A}_1 \cap d\mathcal{A}_2 \neq d(\mathcal{A}_1 \cap \mathcal{A}_2)$; the existence of two uniformities with the last property was an open problem [6], [16].

All these questions are answered below. Construct an ultrafilter $\mathcal{F}$ and $2^\omega$ non-zero-dimensional atoms refining the ultrafilter uniformity $\mathcal{U}$. All these atoms have the same distality which is an atom in the lattice of zero-dimensional uniformities. The ultrafilter $\mathcal{F}$ is constructed to be a finite-to-one lift of an arbitrary given ultrafilter $\mathcal{G}$.

5.1. The structure of a non-zero-dimensional atom is given in the following theorem and its consequences.

**Theorem** (Pelant [14]). Every non-zero-dimensional atom on $\omega$ contains a partition into finite sets.

**Proof.** Let $\mathcal{A}$ be a non-zero-dimensional atom. First, let us prove that $\mathcal{A}$ contains a cover consisting of finite sets. As $\mathcal{A}$ is non-zero-dimensional, there exists a cover $\mathcal{C}$ that cannot be refined by any partition $\mathcal{P} \in \mathcal{A}$. By Lemma 1.1, we may assume that $\mathcal{C}$ is point-finite. Choose $\mathcal{V} \in \mathcal{A}$ such that for every $x \in \omega$ there is $C \in \mathcal{C}$ with $\text{st}(x, \mathcal{V}) \subseteq C$. Write $\mathcal{C} = \{C_n; n \in \omega\}$ and define a partition $\mathcal{P} = \{P_n; n \in \omega\}$ by $P_n = \{x \in \omega; \text{st}(x, \mathcal{V}) \subseteq C_n \text{ but } \text{st}(x, \mathcal{V}) \not\subseteq C_m \text{ for all } m < n\}$. As $\mathcal{P} \notin \mathcal{A}$ and $\mathcal{A}$ is an atom, there exists $\mathcal{W} \in \mathcal{A}$ with $\mathcal{W} \cap \mathcal{P} = \{\{x\}; x \in \omega\}$. We claim that $\mathcal{W} \cap \mathcal{V}$ consists of finite sets. Indeed, if $V \in \mathcal{V}$ and $W \in \mathcal{W}$ then $V$ meets finitely many $P_n$'s only (proof: fix an $x \in V$; if $V \cap P_n \neq \emptyset$ then for any $y \in V \cap P_n$ we have $x \in \text{st}(y, \mathcal{V}) \subseteq C_n$ and by the point-finiteness of $\mathcal{C}$, the set $\{n \in \omega; x \in C_n\}$ is finite) and $W$ meets each $P_n$ in at most one point. Thus $V \cap W$ is finite.

We have proved that $\mathcal{A}$ contains a cover $\mathcal{C}$ consisting of finite sets. Again, we may assume that $\mathcal{C}$ is point-finite. Denote $\mathcal{H}$ the set of components of $\mathcal{C}$, choose points $x_K \in K$ for all $K \in \mathcal{H}$ and define sets $H^K_n, M^K_n$ ($n \in \omega, K \in \mathcal{H}$) as follows:

$$M^K_0 = \text{st}(x_K; \mathcal{C}), \quad M^K_{n+1} = \bigcup \{\text{st}(x; \mathcal{C}); x \in M^K_n\}; \quad H^K_0 = M^K_0, \quad H^K_{n+1} = M^K_{n+1} - M^K_n.$$  

By Proposition 1.3, there exists an ultrafilter $\mathcal{F}$ with $\mathcal{A} \prec \mathcal{U}\mathcal{G}$ (recall from [11] that proximally non-discrete atoms are zero-dimensional). Then one of the sets $\bigcup\{H^K_n; K \in \mathcal{H}, n \text{ odd}\}, \bigcup\{H^K_n; K \in \mathcal{H}, n \text{ even}\}$ belongs to $\mathcal{F}$; without loss of generality, let $H$ denote the first of these sets and let $H \in \mathcal{F}$. Denote $\mathcal{H} = \{H\} \cup \{\{x\}; x \in \omega - H\}$. Then $\mathcal{H} \in \mathcal{U}\mathcal{G}$ and so $\mathcal{H} \cap \mathcal{C} \in \mathcal{A}$. It follows easily from the definition of the sets $H^K_n$ that for every $x \in H^K_n$ we have $\text{st}(x, \mathcal{C}) \subseteq H^K_{n-1} \cup H^K_n \cup H^K_{n+1}$ and so $\text{st}(x, \mathcal{H} \cap \mathcal{C}) \subseteq H^K_n$. It follows that the partition $\{H^K_n; K \in \mathcal{H}, n \in \omega\}$ belongs to $\mathcal{A}$. 

As $\mathcal{C}$ is point finite and consists of finite sets, each of the sets $H_n^k$ is finite. The proof is concluded. □

**Corollary.** Every non-zero-dimensional atom on $\omega$ refines some ultrafilter-and-partition uniformity $\mathcal{U}_\mathcal{F} \times \mathcal{H}_\mathcal{F}$ where $\mathcal{R}$ is a partition into finite sets and $\mathcal{F}$ is a nontrivial finite-to-one lift (of its image under any map $q$ with $\mathcal{R} = \{q^{-1}(n); n \in \omega\}$).

**Corollary.** If $\mathcal{F}$ is a rare ultrafilter on $\omega$ then all atoms refining $\mathcal{U}_\mathcal{F}$ are zero-dimensional.

### 5.2. Hypergraphs

To define the basic partition, the size function and the basic system for our construction of a non-zero-dimensional atom, we make use of a sequence of selective hypergraphs of [11]:

Recall that a hypergraph is a couple $(\mathcal{R}, \mathcal{H})$ where $\mathcal{R}$ is a finite set (the set of vertices) and $\mathcal{H}$ is a set of subsets of $\mathcal{R}$ (the set of edges). A cycle of length $s$ in a hypergraph is a sequence $x_1, \ldots, x_s$ of pairwise distinct vertices such that there are pairwise distinct edges $H_1, \ldots, H_s$ with $\{x_i, x_{i+1}\} \subseteq H_i$ for $i = 1, \ldots, s-1$, $\{x_s, x_1\} \subseteq H_s$.

In [11], a sequence $(\mathcal{R}_n, \mathcal{H}_n)$ of hypergraphs and a sequence $\mathcal{R}_n'$ of finite sets is constructed with the following properties.

(i) $\mathcal{R}_0$ is a singleton and $\mathcal{H}_0 = \{\mathcal{R}_0\}$;

(ii) $\mathcal{R}_n' \supseteq \mathcal{R}_n$ and $|\mathcal{R}_n' - \mathcal{R}_n| > n + 3$ suffices for our purposes;

(iii) $(\mathcal{R}_n, \mathcal{H}_n)$ is an $|\mathcal{R}_{n-1}'|$-hypergraph, that is, each of its edges has cardinality $|\mathcal{R}_{n-1}'|$, for all $n \geq 1$;

(iv) $(\mathcal{R}_n, \mathcal{H}_n)$ has no cycles of length $\leq 2|\mathcal{R}_{n-1}'|$; notice that the absence of cycles of length 2 means that $|H \cap H'| \leq 1$ for any pair of distinct edges;

(v) $(\mathcal{R}_n, \mathcal{H}_n)$ is selective in the following sense: if $\mathcal{P}$ is any partition of $\mathcal{R}_n$ then there is an edge $H \in \mathcal{H}_n$ such that either $H$ is $\mathcal{P}$-discrete or $H$ is contained in a member of $\mathcal{P}$.

(vi) Edges of $\mathcal{H}_n$ cover $\mathcal{R}_n$.

Each $H \in \mathcal{H}_n$ will be regarded as a copy of $\mathcal{R}_{n-1}'$; further, $\mathcal{R}_{n-1}'$ contains $\mathcal{R}_{n-1}$ and $\mathcal{R}_{n-1}$ contains copies of $\mathcal{R}_{n-2}'$ etc.; we shall speak about canonical copies of $\mathcal{R}_k$ and of $\mathcal{R}_k'$ in $\mathcal{R}_n$ for $k < n$. More precisely: for every $n \geq 1$ and every $H \in \mathcal{H}_n$ we fix a bijection $\alpha_H : \mathcal{R}_{n-1}' \to H$. Then a canonical copy of $\mathcal{R}_k$ in $\mathcal{R}_n$ where $k < n$ is any set of the form

$$\alpha_{H_1} \alpha_{H_{n-1}} \cdots \alpha_{H_k} [\mathcal{R}_k]$$

where $H_i \in \mathcal{H}_i$ for $i = k + 1, \ldots, n$. Analogously for $\mathcal{R}_k'$.

### 5.3. The basic partition, the size and the basic system

We may and shall assume that the sets $\mathcal{R}_k$ of 5.2 form a partition of $\omega$. $\mathcal{R} = \{ \mathcal{R}_n; n \in \omega \}$ will be our basic partition.

For every finite set $M \subseteq \omega$, define $\text{sz} M$ to be the largest $k \geq 1$ such that $M$ contains a canonical copy of $\mathcal{R}_k$, if any, and $\text{sz} M = 0$ otherwise.
Remark. The size function $sz$ admits the Ramsey property (cf. 2.6).

Proof. Let $sz M_1 \cup M_2 \geq k + 1$. Then $M_1 \cup M_2$ contains a canonical copy of $R_{k+1}$. The traces of $M_1 - M_2$, $M_2 - M_1$, $M_1 \cap M_2$ on this copy yield a partition $\mathcal{P}$ of $R_{k+1}$. The selectivity of $(R_{k+1}, \mathcal{H}_{k+1})$ (cf. 5.2(v)) implies the existence of an $H \in \mathcal{H}_{k+1}$, which is either $\mathcal{P}$-discrete or is contained in a member of $\mathcal{P}$. The former case is impossible because $|H| > 3$ (see 5.2(ii)) while $|\mathcal{P}| \leq 3$. Thus $H \subseteq M_1$ or $H \subseteq M_2$. Hence $sz M_1 \geq k$ or $sz M_2 \geq k$. \qed

For every partition $\mathcal{P}$ of $\omega$, let $\mathcal{P}_\omega$ be the family of all subsets $S$ of $\omega$ such that for every $n \in \omega$, the set $S \cap R_n$ is either $\mathcal{P}$-discrete or is contained in a member of $\mathcal{P}$. Let $\Sigma$ be the family of all these $\mathcal{P}_\omega$. Under the CH, $|\Sigma| = \omega_1$; thus $\Sigma$ is a basic system.

Lemma. For every partition $\mathcal{P}$ of $\omega$, every $(\mathcal{R}, sz)$-large set $F$ contains an $(\mathcal{R}, sz, \mathcal{P}_\omega)$-nice subset $F'$.

Proof. Let $F \subseteq \omega$ be large. By the definition, for every $k \in \omega$ there exists $n_k \in \omega$ such that for each $m \geq n_k$, $sz F \cap R_m \geq k$, that is, $F \cap R_m$ contains a canonical copy of $R_k$. We may assume $n_0 < n_1 < n_2 \ldots$.

Let $k \in \omega$, $k > 1$ and let $n \in \omega$, $n_k \leq n < n_{k+1}$. As $F \cap R_n$ contains a canonical copy of $R_k$, by virtue of selectivity of the hypergraph $(R_k, \mathcal{H}_k)$, there exists $H \in \mathcal{H}_k$ such that either $H$ is $\mathcal{P}$-discrete or $H \subseteq P$ for some $P \in \mathcal{P}$. Denote $F_n$ the canonical copy of $R_{k-1}$ in $R_n$ that is contained in $H$. Then $F' = \bigcup \{F_n; n \in \omega\}$ is clearly large and belongs to $\mathcal{P}_\omega$; that is, it is $(\mathcal{R}, sz, \mathcal{P}_\omega)$-nice. \qed

5.4. The ultrafilter. By 2.6, Corollary, and by 5.3, Remark and Lemma, we have the following.

Proposition (CH). For every ultrafilter $\mathcal{G}$ on $\omega$ there exists an $(\mathcal{R}, sz, \Sigma)$-nice filter $\mathcal{F}'$ such that the family

$$\mathcal{F}' \cup \{\bigcup \{R_n; n \in G\}; G \in \mathcal{G}\}$$

generates an ultrafilter $\mathcal{F}$ that is a finite-to-one lift of $\mathcal{G}$.

5.5. The ultrafilter-and-partition uniformity. Denote $\mathcal{N} = \cup_{\mathcal{F}} \cup_{\mathcal{G}}$ with $\mathcal{F}$ from 5.4 and $\mathcal{R}$ from 5.3. Thus basis of $\mathcal{N}$ consists of covers of the form

$$\{F \cap R_n; n \in \omega\} \cup \{\{n\}; n \in \omega - F\}, \quad F \in \mathcal{F},$$
equivalently,

$$\bigcup \{F' \cap R_n; n \in G\} \cup \{\{x\}; x \in \bigcup \{R_n; n \in \omega - G\}\}$$

$$F' \in \mathcal{F}', \quad G \in \mathcal{G}.$$
Proposition (CH). \( \mathcal{N} \) is an atom in the lattice of zero-dimensional uniformities on \( \omega \).

**Proof.** It suffices to prove that for every partition \( \mathcal{P} \) of \( \omega \) either \( \mathcal{P} \in \mathcal{N} \), or there exists \( \mathcal{Q} \in \mathcal{N} \) such that \( |P \cap Q| \leq 1 \) for every \( P \in \mathcal{P} \) and \( Q \in \mathcal{Q} \). By 5.4, \( \mathcal{F} \) contains an \((\mathcal{R}, sz, \mathcal{S}_Q)\)-nice set \( \mathcal{M} \) and then, for every \( n \in \omega \), either

(a) \( \mathcal{M} \cap R_n \) is \( \mathcal{P} \)-discrete, or

(b) \( \mathcal{M} \cap R_n \subset P \) for some \( P \in \mathcal{P} \).

As \( \mathcal{F} \) is an ultrafilter, we may suppose that all \( n \in \omega \) satisfy the same condition. If it is (a), then \( \mathcal{Q} = \{ \mathcal{M} \cap R_n \mid n \in \omega \} \) has the property that \( |P \cap Q| \leq 1 \) for all \( P \in \mathcal{P} \) and \( Q \in \mathcal{Q} \). If it is (b) then \( \mathcal{Q} \) refines \( \mathcal{P} \) and so \( \mathcal{P} \in \mathcal{N} \). The proof is finished. \( \Box \)

5.6. The metrics on hypergraphs. We shall define a metric \( \rho \) on \( \omega = \bigcup \{ R_n \mid n \in \omega \} \), by induction on \( n \), such that

(i) diam \( R_n = 1 \) \( (n \in \omega, n > 0) \),

(ii) \( \min \{ \rho(x, y) \mid x, y \in R_n, x \neq y \} = 1/n \),

(iii) \( \rho(x, y) = 1 \) whenever \( x \in R_n, y \in R_n, m \neq n \).

First, \( R_0 \) is a singleton and so there is a unique metric on \( R_0 \). Let \( \rho \) be defined on \( R_n \) such that (i) and (ii) hold. Consider the set \( R_n' \supset R_n \) and remember that \( |R_n' - R_n| \geq 2 \) and choose

(iv) \( x_1^{(n+1)}, x_2^{(n+1)} \in R_n' - R_n \)

and extend the metric on \( R_n \) to \( R_n' \) by putting

(v) \( \rho(x_1^{(n+1)}, x_2^{(n+1)}) = 1/(n + 1) \),

(vi) \( \rho(x, y) = 1 \) if \( x \neq y, x, y \in R_n' \) and \( \{x, y\} \not\subset R_n, \{x, y\} \neq \{x_1^{(n+1)}, x_2^{(n+1)}\} \).

Now, \( R_{n+1} \) is covered by edges of \( \mathcal{H}_{n+1} \); each edge is a canonical copy of \( R_n' \). This defines \( \rho(x, y) \) for any pair \( x, y \subset R_{n+1} \) which are in the same canonical copy. More precisely,

(vii) \( \rho(x, y) = \rho(x_1, x_2, \ldots, x_k) \) if \( x \in H \in \mathcal{H}_{n+1} \).

This is correct because there exists at most one such \( H \) for a given couple \( x, y \), see 5.2(iv). Further, let us define \( \rho(x, y) \) for other pairs \( x, y \subset R_{n+1} \) by

(viii) \( \rho(x, y) = \min \{1, \min \{\rho(x_1, x_2) + \cdots + \rho(x_{k-1}, x_k)\} \} \),

the latter minimum being taken over all \( k \)-tuples \( x_1, \ldots, x_k \) (for various \( k \)) with \( x_1 = x, x_k = y \) such that \( \rho(x_1, x_2), \ldots, \rho(x_{k-1}, x_k) \) are defined by (vii). To prove that \( \rho \) is indeed a metric, we have to verify that (vii) coincides with (viii) if \( x, y \in H \in \mathcal{H}_{n+1} \). Suppose the contrary. Then there are \( x_1, \ldots, x_k \in R_{n+1} \) such that the sum in (viii) is smaller than \( \rho(x_1', x_k') \) as defined by (vii). Of course, in that case \( k > 2 \). Moreover, we may assume that \( x_1, \ldots, x_k \) are pairwise distinct. Then \( x_1, \ldots, x_k \) form a cycle in \((R_{n+1}, \mathcal{H}_{n+1})\). By 5.2(iv), \( k > 2 \). Hence the sum in (viii) has more than \( 2(n+1) \) summands and each of them is at least \( 1/(n+1) \) (cf. (iii)). It follows that the sum is more than 2 while \( \rho(x_1, x_k) \), if defined by (vii), is \( \leq 1 \) (see (i)), a contradiction.

We have defined the metric \( \rho \) on \( R_n \) for every \( n \in \omega \). To extend it to all of \( \omega = \bigcup \{ R_n \mid n \in \omega \} \), we just use (iii) as a definition.
Proposition. For the metric \( \rho \) constructed above we have: the uniformity \( \mathcal{U}_\rho = \mathcal{U} \) is strictly finer than \( \mathcal{U} \) but is not uniformly discrete.

Proof. Let us consider a fixed basic cover \( \mathcal{C} \) from 5.5 and its members \( F' \cap R_n \) (\( F' \subset \mathcal{F}' \), \( n \in G \)). As \( G \) is infinite and \( F' \) is large, the function \( sz \) is unbounded on members of \( \mathcal{C} \). It follows that each \( R_n \) admits an isometrical embedding into some member of \( \mathcal{C} \). First, recall that \( \operatorname{diam} R_n = 1 \) (\( n > 0 \)); so \( \mathcal{C} \) cannot refine the cover of \( \rho \)-balls of radius \( 1/2 \); hence \( \mathcal{N} \) does not refine \( \mathcal{U}_\rho \) and so \( \mathcal{N} \) is strictly finer than \( \mathcal{U} \). Second, suppose \( \mathcal{N}_\rho \) to be uniformly discrete. Then there exists \( \mathcal{C} \) as above and \( \epsilon > 0 \) such that \( \rho(x, y) > \epsilon \) for any two distinct points \( x, y \) lying in the same member of \( \mathcal{C} \); but this is impossible because for \( n > 1/\epsilon \), some member of \( \mathcal{C} \) contains an isometric copy of \( R_n \) which admits two points of distance \( 1/n < \epsilon \), a contradiction. □

Lemma. There exists a family \( \mathcal{M} \) of metrics \( \rho \) obtained by the above construction by means of various choices (iv), such that

(a) \( |\mathcal{M}| = 2^\omega \),

(b) \( \mathcal{U}_{\rho_1} \wedge \mathcal{U}_{\rho_2} \) is uniformly discrete for \( \rho_1, \rho_2 \in \mathcal{M}, \rho_1 \neq \rho_2 \).

Proof. (1) Denote \( \varphi_0(n) \) the number of two-element subsets of \( R_n^r - R_n \); suppose that these sets are numbered in a fixed way by \( 1, \ldots, \varphi_0(n) \). Then the choice (iv) \( (n \in \omega) \) can be represented by a function \( \varphi: \omega \to \omega \), where \( \varphi \leq \varphi_0 \).

(2) Let us prove the following: If \( \varphi_1, \varphi_2 \) are functions corresponding to two metrics \( \rho_1, \rho_2 \) and \( \varphi_1(n) \neq \varphi_2(n) \) for all \( n \) but a finite number then \( \mathcal{U}_{\rho_1} \wedge \mathcal{U}_{\rho_2} \) is uniformly discrete. Indeed, let \( n_0 \) be such that \( \varphi_1(n) \neq \varphi_2(n) \) for all \( n > n_0 \). There exists \( \epsilon > 0 \) such that

\[
\rho_1(x, y) + \rho_2(x, y) \geq \epsilon, \quad x, y \in R_0 \cup \cdots \cup R_{n_0}, \quad x \neq y
\]

because the set \( R_0 \cup \cdots \cup R_{n_0} \) is finite. Without loss of generality, \( \epsilon < 1 \). Suppose

(ix) \( \rho_1(x, y) + \rho_2(x, y) \geq \epsilon \), \( x, y \in R_0 \cup \cdots \cup R_n, x \neq y \).

for some \( n \geq n_0 \). Let \( x, y \in R_0 \cup \cdots \cup R_{n+1} \); we may suppose \( x, y \in R_{n+1} \), cf. (iii). If \( x, y \) lie in the same canonical copy of \( R_n^r \), that is, if \( \rho_1(x, y), \rho_2(x, y) \) are defined by (vii), then \( \rho_1(x, y) + \rho_2(x, y) \geq \epsilon \) by the induction assumption, or by (v), (vi) (since \( \varphi_1(n + 1) \neq \varphi_2(n + 2) \)). If not, let \( \rho_1(x, y), \rho_2(x, y) \) be realized by sums

\[
(x) \quad \rho_1(x, y) = \sum_{i=1}^{m-1} \rho_1(x_i, x_{i+1}),
\]

\[
\rho_2(x, y) = \sum_{j=1}^{h-1} \rho_2(y_j, y_{j+1}),
\]

according to (viii) where \( x_1 = y_1 = x, \ x_m = y_h = y \). Then the set \( \{x_1, \ldots, x_m, y_2, \ldots, y_{h-1}\} \) contains a cycle in \( (R_{n+1}, \mathcal{H}_{n+1}) \). So \( m + h - 2 > 2|R_n| \geq 2(n + 3) \) (see 5.2(ii) and (iv)). We may assume that the \( x_i \)'s resp. the \( y_j \)'s are pairwise
distinct and so each of the summands in (x) is \( \geq 1/(n+1) \). It follows that \( \rho_1(x, y) + \rho_2(x, y) > (m + h - 2)/(n + 1) > 2 > \varepsilon \). Hence the inequality (ix) holds for \( x, y \in R_0 \cup \cdots \cup R_{n+1} \), too. By induction, it holds for any couple \( x, y \in \omega, x \neq y \). But then \( \mathcal{U}_{\rho_1} \wedge \mathcal{U}_{\rho_2} \) is uniformly discrete.

(3) Now the assertion follows immediately by the following folklore.

**Lemma.** Let \( \varphi_0: \omega \to \omega \) be an unbounded function. Then there exists a family \( \Phi \) of functions \( \varphi: \omega \to \omega \) such that

(i) \( |\Phi| = 2^\omega \),
(ii) \( \varphi \preceq \varphi_0 \) for all \( \varphi \in \Phi \),
(iii) if \( \varphi_1, \varphi_2 \in \Phi, \varphi_1 \neq \varphi_2 \), then \( \varphi_1(n) \neq \varphi_2(n) \) for all \( n \in \omega \) but a finite number.

5.7. **Theorem** (CH). There exists a uniformity \( \mathcal{N} \), generated by an ultrafilter \( \mathcal{F} \) and a partition \( \mathcal{R} \) into finite sets on \( \omega \), such that

(i) \( \mathcal{N} \) is an atom in the lattice of zero-dimensional uniformities on \( \omega \),
(ii) \( \mathcal{N} \) is not an atom in the lattice of all uniformities on \( \omega \), in fact, there are at least \( 2^\omega \) pairwise uniformly non-homeomorphic atoms finer than \( \mathcal{N} \), and these atoms are non-zero-dimensional.

Moreover, \( \mathcal{F} \) can be constructed to be a finite-to-one lift of any given ultrafilter \( \mathcal{G} \) on \( \omega \).

**Proof.** Let \( \mathcal{N} = \mathcal{U}_{\varphi} \wedge \mathcal{U}_{\mathcal{R}} \) be as in 5.5. Consider the family \( \mathcal{M} \) of metrics on \( \omega \) from 5.6 and corresponding uniformities \( \mathcal{U}_{\rho}, \rho \in \mathcal{M} \). For every \( \rho \in \mathcal{M} \), choose an atom \( A_\rho \) refining \( \mathcal{U}_{\rho} \wedge \mathcal{N} \), see 1.3. By 5.6, Proposition, the atoms \( A_\rho \) are strictly finer than \( \mathcal{N} \), so \( \mathcal{N} \) is not an atom in the lattice of all uniformities on \( \omega \). It follows that the atoms \( A_\rho \) are non-zero-dimensional. By the first lemma in 5.6, these atoms are pairwise distinct and their number is \( 2^\omega \). In fact, the atoms are uniformly non-homeomorphic: any homeomorphism \( h \) between \( A_{\rho_1} \) and \( A_{\rho_2} \) preserves sets that are not uniformly discrete, i.e. \( hF = F \). As for any ultrafilter \( \mathcal{F} \), it follows \( hx = x \) for all \( x \in F \) for some \( F \in \mathcal{F} \). Hence \( A_{\rho_1} = A_{\rho_2} \) and so \( \rho_1 = \rho_2 \). Following the construction of \( \mathcal{F} \) by 5.4, \( \mathcal{F} \) can be made to be a finite-to-one lift of a given ultrafilter \( \mathcal{G} \) on \( \omega \).

5.8. Recall that a uniformity is distal [3, 6] if it admits a basis consisting of covers of finite order, that is, of covers \( \mathcal{C} \) such that each point is contained in at most \( n \) members of \( \mathcal{C} \) for some natural number \( n \) (notice that if \( n \) is the same for all \( \mathcal{C} \) then the uniformity is called finite-dimensional).

**Proposition.** If \( \mathcal{N} \) is an atom in the lattice of zero-dimensional uniformities then it is also an atom in the lattice of distal uniformities.

**Proof.** Let \( \mathcal{C} \) be a cover of a finite order. It is proved in [4] that then \( \mathcal{C} = \bigcup_{i=1}^n \mathcal{C}_i \) for some natural number \( n \) where each \( \mathcal{C}_i \) is a family of pairwise disjoint sets. Let \( \mathcal{C}_i \) be the partition obtained from \( \mathcal{C}_i \) by adding singletons \( \{x\} \) with \( x \in \omega - \bigcup \mathcal{C}_i \);
i = 1, \ldots, n. If \( \mathcal{C}_i \in \mathcal{N} \) then also \( \mathcal{C} \in \mathcal{N} \). On the other hand, if no \( \mathcal{C}_i \) belongs to \( \mathcal{N} \) then there are partitions \( \mathcal{P}_i \in \mathcal{N} \) such that \( |C \cap P| \leq 1 \) for every \( i \), every \( C \in \mathcal{C}_i \) and every \( P \in \mathcal{P}_i \). Then \( |C \cap P| \leq 1 \) also for every \( C \in \mathcal{C} \) and every \( P \in \bigwedge_{i=1}^n \mathcal{P}_i \in \mathcal{N} \). This proves that \( \mathcal{N} \) is an atom in the lattice of distal uniformities.

For any uniformity \( \mathcal{U} \), let \( d\mathcal{U} \) denote the distal modification of \( \mathcal{U} \), that is, the uniformity whose basis consists of all covers of finite order in \( \mathcal{U} \). Then we have, as a corollary, the following proposition which solves the problem of existence of a couple of uniformities \( \mathcal{U}_1, \mathcal{U}_2 \) with the property that \( d\mathcal{U}_1 \wedge d\mathcal{U}_2 \neq d(\mathcal{U}_1 \wedge \mathcal{U}_2) \), formulated in [6] and [16].

**Proposition.** All the non-zero-dimensional uniformity atoms of 5.7 have the same distal modification, viz \( \mathcal{N} \). Thus \( d\mathcal{A}_1 \wedge d\mathcal{A}_2 \neq d(\mathcal{A}_1 \wedge \mathcal{A}_2) \) for any couple \( \mathcal{A}_1, \mathcal{A}_2 \) of these atoms.

**Acknowledgement**

We are indebted to the referee for a lot of remarks and stimulating suggestions.

**References**