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Products of Conjugacy Classes in Algebraic Groups, I

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Here we consider algebraic varieties which are closures of products of conjugacy classes in algebraic groups. Estimates for the dimension of such varieties are obtained. Moreover, these estimates are used in some questions of the Invariant Theory. Also, the structure of the monoid generated by the semisimple conjugacy classes in $SL_n(K)$ is described.

INTRODUCTION

The investigation of products of conjugacy classes in groups is rather popular in the Group Theory ([1]). The most advanced problem there is one of the representation of a group G as C^n, $\bigcup_{i \le m} C^i, C_1C_2 \ldots C_k$, where $C, C_1, \ldots, C_k$ are some conjugacy classes of G. One of the questions is to point out the smallest integers n, m, k (if it is possible) such that $G = C^n$, $G = \bigcup_{i \le m} C^i$, $G = C_1C_2 \ldots C_k$ for any conjugacy classes $C, C_1, \ldots, C_k$ from some fixed set S. For instance, such a set S may contain only one fixed conjugacy class of G or all non-central classes, etc. Another question is an existence of a conjugacy class C satisfying the equality $C^d = G$ for a fixed integer d (J. Thompson conjectured that such a class exists for $d = 2$, if G is a finite simple group. This conjecture is proved for some classes of finite simple groups [1, 10]).

The most considerable achievements in this direction were obtained when G is a finite or linear group. We have no aim to give here any survey of this results. We only mention that any element of $O_n(K)$ can be represented as a product of k reflections, where $k \le n$.

Our interest is to consider algebraic varieties which are the closures (with respect to the Zariski topology) of conjugacy classes in algebraic groups. The source of such interest is connected with the following questions. Suppose G is an algebraic group acting regularily on an algebraic variety X. It is important to know the number $\min(G, X) =$

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$\min\{\dim X - \dim X^g | g \in G,\ g \neq 1\}$ (here X^g is the set of g-invariant points), or, at any rate, to estimate this number. For instance, such estimates can be used in the Invariant Theory ([6]). Sometimes the number

$$\operatorname{corank} g = \dim X - \dim X^g$$

is known for some $g \in G$. If, in addition, we know that $C^k = G$ for any conjugacy class C except, maybe, classes which a priori can not contain elements of the minimal corank, then

$$\min(G, X) \geq \operatorname{corank} g/k.$$

But in many cases it is enough to know the integer k such that the closure $\overline{C^k}$ of C^k coincides with G. It should be noted that the estimates of such k are rather easier than for the case $C^k = G$. This gives us the stimulus to study these algebraic varieties. Moreover, such varieties are connected with some stratifications of G which are Int G-invariant.

In this paper we deal with algebraic groups over an algebraically closed field of characteristic 0.

In the first part we collect the simplest properties of algebraic varieties

$$\overline{M}_k = \overline{C_1 C_2 \dots C_k},$$

where $C_1, \dots, C_k$ are some conjugacy classes of an algebraic group G. The second part is devoted to estimates of the numbers k, e, such that $G = \overline{C}^k$, $G = \overline{C_1 C_2 \dots C_e}$, where G is a simple algebraic group and $C, C_1, \dots, C_e$ are any non-central conjugacy classes of G. In particular, we show that $k \leq 2\operatorname{rank}(G)$, $e \leq 2\operatorname{rank}(G) + 1$. For k such an estimate is sharp if we want to have the general one for all types of groups. It follows, for instance, from the result on reflections in $O_n(K)$. In the third part the applications of such estimates in the direction mentioned above is given. These results in most parts deal with the dimension of considered varieties. But it would be interesting to describe them more precisely from the geometrical point of view, say, in the spirit of the work [11].

In the last part we give some examples of the behaviour of such varieties in the case $G = SL_n(K)$. The most interesting thing here is the structure of the monoid generated by the semisimple conjugacy classes. This structure is induced by some multiplication on the set of partitions of all $m \leq n$. Many questions are closely connected with manipulations under partitions (representations, Schubert varieties, etc.). Presumably, products of conjugacy classes play the defined role for these kind of questions.

1. Multiclasses of Algebraic Groups

1.1. Let G be a group. By a k-conjugacy class of G (or simply k-class) we mean a product

$$M_k = C_1C_2\dots C_k = \{g_1g_2\dots g_k | g_i \in C_i\},$$

where $C_1,\dots,C_k$ are conjugacy classes of G. (If $k > 1$, we will suppose that $C_i \not\subset Z(G)$ for all i.) Every k-class of G is also called a multiclass of G.

The product of two multiclasses is again a multiclass, and we have a commutative monoid $M(G)$ of multiclasses. If $G \subset M(G)$, then the group G is the zero of $M(G)$ (by zero we mean the element $0 \in M(G)$ such that $0 \cdot m = 0$ for every $m \in M(G)$), and in this case we can speak about the ideal of nilpotent elements:

$$NM(G) = \{m \in M(G) | m^d = 0 \text{ for some integer } d\}.$$

The *covering number* of G (denoted $cn(G)$) is the smallest integer n such that $m^n = 0$ for all $m \in NM(G)$ (see [1]). The *extended covering number* of G (denoted $\mathrm{ecn}(G)$) is the smallest integer n such that $m_1m_2\dots m_n = 0$ for every $m_1, m_2, \dots, m_n \in NM(G)$.

1.2. Let G be an algebraic group. For every subset $X \subset G$ by $\overline{X}$ we will denote the closure of X in G. If C is a conjugacy class of G, then C is an open subset of $\overline{C}$. Moreover, if G is a connected group, then C is an irreducible variety.

Let $M_k = C_1C_2\dots C_k$ be a multiclass of G. It is easy to show that:

a. $\overline{C}_1\overline{C}_2\dots\overline{C}_k \subset \overline{M}_k$;

b. $\overline{C_1C_2\dots C_i}\ \overline{C_{i+1}\dots C_k} \subset \overline{M}_k$ for every $i = 1,\dots,k$;

c. $\overline{M}_k$ is irreducible, if G is connected;

d. M_k contains an open subset of $\overline{M}_k$;

Question. When is M_k open in $\overline{M}_k$?

Let $\overline{M}(G)$ be the set of closures of all multiclasses of G. We define

$$\overline{m}_1 \cdot \overline{m}_2 = \overline{m_1 \cdot m_2}$$

for every $m_1, m_2 \in M(G)$. Thus we have an algebraic operation on $\overline{M}(G)$ which is associative and commutative.

We will denote by $\overline{\mathrm{cn}}(G)$ (respectively by $\overline{\mathrm{ecn}}(G)$) the smallest integer c (respectively e) such that

$$\overline{m}^c = 0 \ (\text{respectively } \overline{m_1m_2\dots m_e} = 0)$$

for every $m \in NM(G)$ (respectively $m_1, \dots, m_e \in NM(G)$). The integers $\overline{\mathrm{cn}}(G)$, $\overline{\mathrm{ecn}}(G)$ will be also called a covering number and an extended covering number.

PROPOSITION 1. *Let $f\colon M(G) \to \overline{M}(G)$ be the natural homomorphism ($f(m) = \overline{m}$). Then the set $f(NM(G))$ coincides with the ideal of all nilpotent elements of $\overline{M}(G)$. Moreover, $cn(G) \le 2\overline{cn}(G)$,* $\mathrm{ecn}(G) \le 2\,\overline{\mathrm{ecn}}(G)$.

Proof. This follows from the well-known fact that $G = UV$, if U, V are open in G and $\overline{U} = \overline{V} = G$. ∎

Now we will suppose that G is connected and $G = [G, G]$. We denote the unipotent radical of G by $R_u(G)$.

PROPOSITION 2. *Let $C_1, \dots, C_k$ be conjugacy classes of G. Suppose that the image of each C_i in $G/R_u(G)$ generates the group G/R_u. If $k \ge 2\dim G$, then $C_1 C_2 \dots C_k = G$.*

Proof. Let $M_l = C_1 \dots C_l$, $l < k$, $\overline{M_l} \ne G$. We will show

$$\dim \overline{M_l C_{l+1}} > \dim \overline{M_l}. \tag{1}$$

Suppose $\dim \overline{M_l C_{l+1}} = \dim \overline{M_l}$. Since $g\overline{M_l} \subset \overline{M_l C_{l+1}}$ for every $g \in C_{l+1}$, and $g\overline{M_l}, \overline{M_l C_{l+1}}$ are irreducible, then $g\overline{M_l} = \overline{M_l C_{l+1}}$ (because $\dim \overline{M_l C_{l+1}} = \dim \overline{M_l}$). Thus, $g_1 \overline{M_l} = g_2 \overline{M_l}$ for every $g_1, g_2 \in C_{l+1}$ and therefore $g_2^{-1} g_1 \overline{M_l} = \overline{M_l}$. Let $H = \langle g_2^{-1} g_1 | g_1, g_2 \in C_{l+1} \rangle$. We have $H\overline{M_l} = \overline{M_l}$ and consequently $\overline{H}\overline{M_l} = \overline{M_l}$. The group $G/R_u(G)$ is semisimple (we have $G = [G, G]$), and the image of C_{l+1} in $G/R_u(G)$ generates the group $G/R_u(G)$. Hence $\overline{H} = G$. Since $\overline{H}\overline{M_l} = \overline{M_l}$, then $\overline{M_l} = G$. It is a contradiction. Thus we have proved (1). Now our assertion follows from (1) and Proposition 1. ∎

PROPOSITION 3. *The natural homomorphism*

$$\varphi\colon NM(G) \to NM(G/R_u(G))$$

is surjective.

Proof. Let $H = G/R_u(G)$; $H_1, \dots, H_m$ be the simple components of the semisimple group H (here $G = [G, G]$). Every conjugacy class of H is a product of some conjugacy classes of $H_1, \dots, H_m$. Hence each multiclass of H is the product of some conjugacy classes of $H_1, \dots, H_m$. If $m \in NM(H)$, then $\langle m \rangle = H$ and therefore in the decomposition of m into the product of conjugacy classes of $H_1, \dots, H_m$ we can find classes of each component H_i. Hence $m = cm^*$, where $\langle c \rangle = H$, $m^* \in M(H)$. Let C be the conjugacy class of G such that $\varphi(C) = c$ and M^* be the multiclass of G, $\varphi(M^*) = m^*$. From proposition 2 we obtain that $C \in NM(G)$. Hence

$CM^* \in NM(G)$. Thus we have the element $M = CM^* \in NM(G)$ such that $\varphi(M) = m$. ∎

2. Covering Numbers of Simple Algebraic Groups

Here G is a simple algebraic group over an algebraically closed field K (char $K = 0$); T, B, W are a maximal torus, a Borel subgroup containing it, the Weyl group of G; $r = \operatorname{rank} G$; $\langle \alpha_1, \ldots, \alpha_r \rangle$ is a simple root system.

By Int G we denote the group of inner automorphisms of G. The group Int G operates naturally on the algebra of regular functions $K[G]$. The subalgebra of invariant functions

$$C[G] = K[G]^{\operatorname{Int} G}$$

is the subalgebra of regular functions on G which are constant on the conjugacy classes ([3, Chap. II, Sect. 3]). Let

$$G/\operatorname{Int} G = \operatorname{Specm} C[G],$$

and let

$$\pi_G : G \to G/\operatorname{Int} G$$

be the quotient morphism corresponding to the injection $C[G] \to K[G]$ ([2, Chap. II, Sect. 3]).

Theorem 1. *Let $C_1 = Cg_1, \ldots, C_k = Cg_k$ be conjugacy classes of $g_1, \ldots, g_k \in G \setminus Z(G)$; $M_k = C_1 C_2 \ldots C_k$; let $g_1 = t_1 u_1, \ldots, g_k = t_k u_k$ be the Jordan decomposition, where $t_1, \ldots, t_k \in T$. Then*

I. $\dim \pi_G(\overline{M}_k) \geq \min(k - 1, r)$, *if $t_1, \ldots, t_k \notin Z(G)$ and G is not a group of the type B_r, F_4, G_2, or each element $t_1, \ldots, t_k$ is not in the intersection of the kernels of all the long roots of G*:

II. $\dim \pi_G(\overline{M}_k) \geq \min([k/2], r)$, *if the number of g_i such that $t_i \in Z(G)$ or the number of g_j such that $t_j \notin Z(G)$ is even*:

III. $\dim \pi_G(\overline{M}_k) \geq \min([(k-1)/2], r)$ *in all cases.*

Corollary 1. *Let $k \geq r + 1$ in case* I, *or $k \geq 2r$ in case* II, *or $k \geq 2r + 1$ in the general case* III. *Then $\overline{M}_k = G$.*

Proof of the Corollary. The variety $G/\operatorname{Int} G$ is irreducible and $\dim G/\operatorname{Int} G = \operatorname{rank} G$ ([3, E, Chap. II, Sect. 3]). Since $\overline{M}_k$ is closed and Int G-invariant, then $\pi_G(\overline{M}_k)$ is closed in $G/\operatorname{Int} G$ ([2, Chap. II, Sect. 3]). If $k \geq r + 1$, or $k \geq 2r$, or $k \geq 2r + 1$ in the corresponding cases, then $\dim \pi_G(\overline{M}_k) = r$ according to theorem 1. Therefore $T \subset \overline{M}_k$ and consequently $\overline{M}_k = G$, as $\overline{M}_k$ is closed and Int G-invariant. ∎

COROLLARY 2. *Let $C \subset G \setminus Z(G)$ be a conjugacy class of G. Then* $\dim \pi_G(\overline{C^k}) \geq \min([k/2], r)$. *If $k \geq 2r$, then $\overline{C^k} = G$.*

Proof of the Corollary. When $C_1 = C_2 = \cdots = C_k = C$, the number of $t_i \in Z(G)$, or the number of $t_j \notin Z(G)$ is zero. Thus the condition II holds. The statements follow from theorem 1. ∎

COROLLARY 3. *Let H be a semisimple algebraic group and let r be the maximum of ranks of all simple components of G. If $k \geq 2r$, then $\overline{m}^k = H$ for every $m \in NM(H)$.*

Proof of the Corollary. Let $G_1, \ldots, G_s$ be all the simple components of H and $r_1, \ldots, r_s$ their ranks. It is easy to see that $m = m_1 m_2 \ldots m_3$, where $m_i \in NM(G)$. Since $\overline{m}_i^{2r_i} = G$ according to corollary 2, we have $\overline{m}^{2r} = G$. ∎

Proof of Theorem 1.

LEMMA 1. *Let $S_1, \ldots, S_m$ be closed and W-invariant subsets of T. If* $\dim S_i \geq 1$ *for every $i = 1, \ldots, m$, and $m \leq \dim T$, then*

$$\dim \overline{S_1 S_2 \cdots S_m} \geq m.$$

Proof of the Lemma. It is sufficient to prove that $\dim \overline{M_1 M_2} > \dim M_1$, if M_1, M_2 are closed and W-invariant subsets of T, $M_1 \neq T$, $\dim M_2 > 1$. Suppose $\dim \overline{M_1 M_2} = \dim M_1$. Then an irreducible component of the dimension $\dim M_1$ in $M_1 t$ coincides with an irreducible component of the dimension $\dim \overline{M_1 M_2}$ in $\overline{M_1 M_2}$ for every $t \in M_2$. Since $\dim M_2 \geq 1$, there exists an infinite set of elements $t = t_1 t_2^{-1}$ such that $t_1, t_2 \in M_2$, $M_1^* t = M_1^*$, where M_1^* is the union of all irreducible components of the dimension $\dim M_1$ in M_1. Let $H \subset T$ be the group generated by the elements $\omega(t)$, where $\omega \in W$. Since M_1^* is closed and W-invariant, then $M_1^* \overline{H} = M_1^*$. But $\overline{H}$ is infinite and W-invariant. Therefore $\overline{H} = T$ and $M_1^* T = M_1^*$. Hence we have $M_1^* = T$. There is a contradiction. ∎

LEMMA 2. *Let $t_i, t_j \in Z(G)$ (here we use the notations of theorem* 1). *Then* $\dim \pi_G(\overline{C_i C_j}) \geq 1$.

Proof of the Lemma. We denote by C_i^*, C_j^* the conjugacy classes of u_i, u_j. It is obvious that $\dim \pi_G(\overline{C_i C_j}) = \dim \pi_G(\overline{C_i^* C_j^*})$.

Let $\mathfrak{G}$ be the Lie algebra of G. Since the centre of G is unimportant in this consideration, we can suppose $G \leq GL(\mathfrak{G})$. Let tr be the trace in $\mathrm{End}(\mathfrak{G})$. If we prove that $\mathrm{tr}(C_i^* C_j^*) \neq \dim \mathfrak{G}$, then we obtain the inequality $\dim \pi_G(C_i^* C_j^*) \geq 1$. Really, if $x \in C_i^*$, $y \in C_j^*$ and x, y are in the same Borel subgroup of G, then $\mathrm{tr}(xy) = \dim \mathfrak{G}$. Therefore $\pi_G(\overline{C_i^* C_j^*})$ is not one point, if $\mathrm{tr}(C_i^* C_j^*) \neq \dim \mathfrak{G}$. Since $\overline{C_i^* C_j^*}$ is irreducible, then $\dim \pi_G(\overline{C_i^* C_j^*}) \geq 1$.

Thus we have to prove that $\mathrm{tr}(C_i^* C_j^*) \neq \dim \mathfrak{G}$. Let $u_i = 1 + n_i$, $u_j = 1 + n_j$, where n_i, n_j, are nilpotent elements of $\mathrm{End}(\mathfrak{G})$. We have

$$\mathrm{tr}\left(\sigma u_i \sigma^{-1} u_j\right) = \mathrm{tr}(1) + \mathrm{tr}\left(\sigma n_i \sigma^{-1}\right) + \mathrm{tr}(n_j) + \mathrm{tr}\left(\sigma n_i \sigma^{-1} n_j\right)$$
$$= \dim \mathfrak{G} + \mathrm{tr}\left(\sigma n_i \sigma^{-1} n_j\right)$$

for every $\sigma \in G$. We need to prove that $\mathrm{tr}(\sigma n_i \sigma^{-1} n_j) \neq 0$ for some $\sigma \in G$.

Let $u_i = \exp(\bar{n}_i)$, where $\bar{n}_i \in \mathfrak{G}$ (we suppose $\mathfrak{G} \subset \mathrm{End}(\mathfrak{G})$):

$$d \overset{\mathrm{def}}{=} \min\left\{n \in N \mid \bar{n}_i^n = 0\right\};$$

$2 = p_1 < p_2 < \cdots < p_e$ be all primes $\leq d - 1$; $s_1 = -1$, $s_2 = \sqrt[3]{1}, \ldots,$ $s_e = \sqrt[p_e]{1}$. There exist elements $\sigma_1, \ldots, \sigma_e \in G$ such that $\sigma_m \bar{n}_i \sigma_m^{-1} = s_m \bar{n}_i$ for every $m = 1, \ldots, e$. Actually we can find such elements if we embed n_i in $sl_2(K) \subset \mathfrak{G}$ (the theorem of Jacobson and Morosov) and consider the elements of G which correspond to $sl_2(K)$. Let

$$x_1 = n_i - \sigma_1 n_i \sigma_1^{-1},\ x_2 = x_1 - \sigma_2 x \sigma_2^{-1}, \ldots, x_e = x_{e-1} - \sigma_e x_{e-1} \sigma_e^{-1}.$$

Each element x_m is a polynomial $c_1 \bar{n}_i + c_2 \bar{n}_i^2 + \cdots + c_{d-1} \bar{n}_i^{d-1}$, where $c_1, \ldots, c_{d-1} \in K$, $c_1 \neq 0$. Moreover, if the prime $p \leq m$ and $p | a$, then $c_a = 0$. Thus, $x_e = \alpha \bar{n}_i$; where $\alpha \in K^*$. Let $L \subset \mathrm{End}(\mathfrak{G})$ be the space generated by the elements $\sigma n_i \sigma^{-1}$, $\sigma \in G$. Since $x_e \in L$, then $\mathfrak{G} \subseteq L$, as $\mathfrak{G}$ is an irreducible G-module. If $\mathrm{tr}(\sigma n_i \sigma^{-1} n_j) = 0$ for every $\sigma \in G$, then n_j is orthogonal to $\mathfrak{G}$ (with respect to form tr). It is well-known that $\mathfrak{G} \oplus \mathfrak{G}^{\perp} = \mathrm{End}(\mathfrak{G})$. If $n_j \in \mathfrak{G}^{\perp}$, then the space generated by $\{\tau n_j \tau^{-1} | \tau \in G\}$ belongs to $\mathfrak{G}^{\perp}$. We have proved above that the space generated by $\{\sigma n_i \sigma^{-1} | \sigma \in G\}$ contains $\mathfrak{G}$. The same is true for n_j. Hence $\mathrm{tr}(\sigma n_i \sigma^{-1} n_j) \neq 0$ for some $\sigma \in G$ (the form tr is not degenerated on $\mathfrak{G}$). Thus we have proved our assertion. ∎

LEMMA 3. *Let $\alpha: T \to K^*$ be a root of G; $t \in T$, $t \notin Z(G)$. Suppose that α is a short root, if G is of the type B_r, F_4, G_2, or $\alpha = \pm\varepsilon_i \pm \varepsilon_j$, if G is of the type C_r and $r > 2$. Then there exists an element $w \in W$ such that $\alpha(w(t)) \neq 1$.*

Proof of the Lemma. The lattice generated by the W-orbit of α coincides with the lattice generated by all roots ([4, Tables I–IX]). This proves our statement. ∎

LEMMA 4. *Let $L \leq G$ be a closed reductive subgroup of G; $H = L/Z(L)$; π_H: $H \to H/\operatorname{Int} H$ the quotient morphism. Let X be a closed and* Int *G-invariant subset of G; Y be a closed and* Int *L-invariant subset of $X \cap L$: Y^* be the image of Y in H. Then*

$$\dim \pi_G(X) \geq \dim \pi_H(\overline{Y^*}).$$

Proof of the Lemma. We can suppose that $T^* \subset T$, where T^* is a maximal torus of L. Since $Y \subset X$, then

$$\dim X \cap T \geq \dim Y \cap T^*. \tag{2}$$

Let $T^{**} = T^*/Z(L)$, then

$$\dim Y \cap T^* \geq \dim Y^* \cap T^{**}. \tag{3}$$

Since $X \subset G$ is closed and Int G-invariant, then

$$\dim X \cap T = \dim \pi_G(X) \tag{4}$$

([3, Chap. II, Sect. 3]). The semisimple part of each element of Y^* is conjugate to some element of $Y^* \cap T^{**}$, because Y is closed in L and Int$L$-invariant. Hence

$$\dim Y^* \cap T^{**} = \dim \pi_H(Y^*) = \dim \pi_H(\overline{Y^*}). \tag{5}$$

Now our statement follows from (2)–(5). ∎

LEMMA 5. *Let $t_i, t_j \notin Z(G)$ (here we use the notations of theorem* 1). *Then* $\dim \pi_G(\overline{C_i C_j}) \geq 1$.

Proof of the Lemma. We can choose a root α such that $\alpha \in \{\alpha_1, \dots, \alpha_r\}$ and α satisfies the conditions of lemma 3. Therefore we can suppose $\alpha(t_i), \alpha(t_j) \neq 1$. Let σ_α be the reflection corresponding to α; $P_\alpha = B\langle\sigma_\alpha\rangle B$ be the parabolic subgroup of G; $L = L_\alpha$ be the Levi subgroup of P_α. We denote by Y the closure of the set $\{\sigma t_i \sigma^{-1} \tau t_j \tau^{-1} | \sigma, \tau \in L\}$ in G and by X the closure of $C_{t_i} C_{t_j}$ in G, where C_{t_i}, C_{t_j} are the conjugacy classes of t_i, t_j. Since $\alpha(t_i)$, $\alpha(t_j) \neq 1$, then the images of t_i, t_j in $L/Z(L)$ are not trivial. Moreover, $L/Z(L) \cong PSL_2(K)$. Hence the image Y^* of Y in $L/Z(L)$ is dense in $L/Z(L)$ (the product of two non-trivial classes in $PSL_2(K)$ is dense in $PSL_2(K)$). Therefore $\dim \pi_H(\overline{Y^*}) \geq 1$, where $H = L/Z(L)$. Using lemma 4, we obtain $\dim \pi_G(X) \geq 1$. But $X = \overline{C_{t_i} C_{t_j}} \subset \overline{C_i C_j}$, because $t_i \in \overline{C}_i$, $t_j \in \overline{C}_j$ ([3, Chap. II, Sect. 3]), and consequently $\dim \pi_G(\overline{C_i C_j}) \geq 1$. ∎

Now we can prove II and III. Suppose that the conditions II hold. If k is odd, we can take away one class. Thus we have an even number of those

g_j, for which $t_j \notin Z(G)$, and an even number of those g_j, for which $t_j \notin Z(G)$. Then we may distribute the classes into pairs, where both elements C_i, C_j in each pair satisfy the condition: $t_i, t_j \in Z(G)$, or $t_i, t_j \notin Z(G)$. We have $\dim \pi_G(\overline{C_i C_j}) \geq 1$, according to Lemmas 2, 5. Hence $\dim \overline{C_i C_j} \cap T \geq 1$. Using Lemma I, we obtain

$$\dim M_k \cap T \geq \min\{\text{the number of our pairs}, r = \operatorname{rank} G\}.$$

But the number of our pairs $= [k/2]$. Hence we have proved II. When the number of classes, for which $t_i \in Z(G)$, and the number of classes, for which $t_j \notin Z(G)$, are odd, we can remove one class and obtain the conditions II. Thus we have III.

LEMMA 6. *Let $M \subset G$ be closed, irreducible and* Int *G-invariant. If $M \neq G$ and M contains a regular element x of G (i.e., the* Int *G-orbit of x has dimension* $\dim G - \operatorname{rank} G$), *then*

$$\dim \pi_G(gM) > \dim \pi_G(M)$$

for every $g \in G \setminus Z(G)$.

Proof of the Lemma. Let $F \subset \hat{F}$ be general fibers of the morphisms $gM \to \pi_G(gM)$, $G \to G/\operatorname{Int} G$. We will show

$$\dim F < \dim \hat{F}. \tag{6}$$

Suppose $\dim F = \dim \hat{F}$. The sets $F = \hat{F} \cap gM$, $\hat{F}$ are closed in G and $\hat{F}$ is irreducible. Thus, from $\dim F = \dim \hat{F}$ we obtain $F = \hat{F}$. Let $U \subset gM$ be the union of all fibers of $gM \to \pi_G(gM)$ which have the dimension $\dim \hat{F}$. All fibers of U coincide with the fibers of π_G, as we have shown above. Since the fibers of π_G are $\operatorname{Int} G$-invariant, the set U is $\operatorname{Int} G$-invariant too. Moreover, U contains an open subset of gM, according to our supposition. Hence $gM = \overline{U}$ is $\operatorname{Int} G$-invariant. We have

$$sgMs^{-1} = sgs^{-1}M = gM$$

for every $s \in G$. Then $hM = M$ for every $h = g^{-1}sgs^{-1}$, where $s \in G$. Since G is a simple algebraic group, the closure of the group H generated by the elements $g^{-1}sgs^{-1}$ coincides with G. Then $HM = GM = M$, because $\overline{H}M = M$ and M is closed. Hence $M = G$. This is a contradiction. We have proved (6).

Let us consider the morphism $M \to \pi_G(M)$. Since the set of all regular elements is open in G ([3, Chap. III, Sect. 1]) and M is closed, the set of regular elements in M is open in M and non-empty, because $x \in M$. Since M is irreducible and $\operatorname{Int} G$-invariant, the dimension of a general fiber of $M \to \pi_G(M)$ is equal to dim $\hat{F}$. Comparing it with (6), we obtain our statement. ∎

Now we need to introduce some notations. Let W_j be the Weyl group of $\langle \alpha_1, \dots, \alpha_j \rangle$; $P_j = BW_jB$ be the parabolic subgroup corresponding to W_j; L_j be the Levi subgroup of P_j, that is, $P_j = L_j R_u(P_j)$, where $R_u(P_j)$ is the unipotent radical of P_j. Let

$$\psi_j \colon P_j \to L_j / Z(L_j)$$

be the natural homomorphism ($P_0 = B$, $L_0 = T$, $\psi_0 \colon B \to 1$).

Suppose that $t_1, \dots, t_k \notin Z(G)$ and G is not of the type B_r, F_4, G_2. In this case we choose the usual numeration for $\alpha_1, \dots, \alpha_r$ ([4, Tables I–IX]), if G has the type A_r, D_r, C_r (in the case C_r we have $r > 2$). If G has the type E_6, E_7, E_8, we take the opposite numeration. It is easy to see that $\langle \alpha_1, \dots, \alpha_j \rangle$ is an irreducible root system for every j. Moreover, there is no root $\beta \in \langle \alpha_1, \dots, \alpha_j, \alpha_{j+1} \rangle$ such that $\beta \perp \alpha_1$, $\beta \perp \alpha_2, \dots, \beta \perp \alpha_j$. Since G is not B_r, F_4, G_2 and $\alpha_1 = \varepsilon_1 - \varepsilon_2$, if G is C_r, the conditions of Lemma 3 hold. Therefore we can assume that $t_1, \dots, t_k \notin Z(L)$ (i.e. $\alpha_1(t_i) \neq 1$ for every t_i). Hence

$$\psi_j(t_i) \neq 1$$

form every i, j. Let us put

$$t_{ij} = \psi_j(t_i).$$

We denote by the symbol C_{ij} the conjugacy class of t_{ij} in $L_j / Z(L_j)$. Let

$$M_{m,j} = C_{1j} C_{2j} \dots C_{mj}.$$

LEMMA 7. *If the set $M_{n-1,j}$ contains a regular element of the group $L_j / Z(L_j)$ and $\overline{M}_{n-1,j-1} = L_{j-1} / Z(L_{j-1})$, then*

$$\overline{M}_{n,j} = L_j / Z(L_j).$$

Proof of the Lemma. Let $\hat{G} = L_j / Z(L_j)$, $L = L_{j-1} / Z(L_j)$, $H = L / Z(L) = L_{j-1} / Z(L_{j-1})$, $X = \overline{M}_{n-1,j}$. Let Y be the closure of the set $\{(\sigma_1 \psi_j(t_1) \sigma_1^{-1})(\sigma_2 \psi_j(t_2) \sigma_2^{-1}) \dots (\sigma_{n-1} \psi_j(t_{n-1}) \sigma_{n-1}^{-1}) | \sigma_1, \dots, \sigma_{n-1} \in L\}$ in L. We have $Y \subset X \cap L$. Moreover, the image Y^* of Y in $H = L_{j-1} / Z(L_{j-1})$ contains the set $M_{n-1,j-1}$. Hence $\overline{Y}^* = H$ and $\dim \pi_{\mathrm{H}}(\overline{Y}^*) = \operatorname{rank} H = \operatorname{rank} \hat{G} - 1$. Since $\hat{G}$ is a simple algebraic group, we can use Lemma 4 for $\hat{G}, L, X, Y, Y^*$ instead of G, L, X, Y, Y^*. Thus

$$\dim \pi_{\hat{G}}\left(\overline{M}_{n-1,j}\right) = \dim \pi_{\hat{G}}(X) \geq \operatorname{rank} \hat{G} - 1. \tag{7}$$

The set $\overline{M}_{n-1,j} \subset \hat{G} = L_j/Z(L_j)$ contains a regular element of $\hat{G}$ according to the condition of the lemma. Therefore we can use Lemma 6 for $M = \overline{M}_{n-1,j} \subset G$ and $g \in C_{nj}$. We have

$$\dim \pi_{\hat{G}}\left(\overline{M}_{n,j}\right) = \dim \pi_{\hat{G}}\left(g\overline{M}_{n-1,j}\right) > \dim \pi_{\hat{G}}\left(\overline{M}_{n-1,j}\right).$$

or $\overline{M}_{n-1,j} = \hat{G}$. In both cases,

$$\dim \pi_{\hat{G}}\left(\overline{M}_{n,j}\right) = \operatorname{rank} \hat{G}$$

according to (7). Hence $\overline{M}_{n,j} = \hat{G} = L_j/Z(L_j)$. ∎

LEMMA 8. *If $t_1, \dots, t_k \notin Z(G)$ and G is not of the type B_r, F_4, G_2, then the conditions of Lemma 7 hold for every $\overline{M}_{j,j}, \overline{M}_{j,j-1}, j = 1, \dots, r$. Moreover, the set $\overline{M}_{j,j}$ contains a semisimple regular element of $L_j/Z(L_j)$.*

Proof of the Lemma. Let $j = 1$. Then $\overline{M}_{1,1} = C_{11}$ is the conjugacy class of the non-trivial semisimple element $t_{11} = \psi_1(t_1)$ in the group of rank one $L_1/Z(L_1)$. Thus t_{11} is a semisimple regular element in $\overline{M}_{1,1}$. Since $\overline{M}_{1,0} = C_{1,0} = 1 = L_0/Z(L_0)$, the second condition holds too.

Suppose that our statement is true for every $j' \leq j$. We can suppose $j = r$ and $G = L_j/Z(L_j)$, because $L_j/Z(L_j)$ is a semisimple algebraic group (the center of G is unimportant in our consideration). Using our supposition for $j = r-1, r-2$ and lemma 7, we obtain $\overline{M}_{r,r-1} = L_{r-1}/Z(L_{r-1})$. Thus, we need to prove only that the set $\overline{M}_{r,r}$ contains a semisimple regular element of G. Let $F \leq G$ be the subgroup generated by the root subgroups of the root system $\langle \alpha_1, \dots, \alpha_{r-1} \rangle$; $T_F = \langle h_{\alpha_1}(t), h_{\alpha_2}(t), \dots, h_{\alpha_{r-1}}(t) \rangle \leq T$ be the maximal torus of F; $Z_F = T \cap C_G(F)$. Then we have

$$T = T_F Z_F,$$

and each element $t_i \in C_i \cap T$ is represented as $t_i = \bar{t}_i z_i$ where $\bar{t}_i \in T_F$, $z_i \in Z_F$. Let us consider the sets

$$S = \left\{(\sigma_1 t_1 \sigma_1^{-1})(\sigma_2 t_2 \sigma_2^{-1}) \dots (\sigma_r t_r \sigma_r^{-1}) | \sigma_1, \dots, \sigma_r \in F\right\},$$
$$S_1 = \left\{(\sigma_1 \bar{t}_1 \sigma_1^{-1})(\sigma_2 \bar{t}_2 \sigma_2^{-1}) \dots (\sigma_r \bar{t}_r \sigma_r^{-1}) | \sigma_1, \dots, \sigma_r \in F\right\}.$$

We will show

$$\overline{S}_1 = F. \tag{8}$$

In fact, $FZ_F = L_{r-1}$, $Z_F \subset Z(L_{r-1})$. Thus, the image of S_1 in $L_{r-1}/Z(L_{r-1})$ coincides with $M_{r,r-1}$. Since $\overline{M}_{r,r-1} = L_{r-1}/Z(L_{r-1})$ and $\overline{S}_1$ is closed and Int G-invariant, then we obtain (8). According to our construction we have $S = S_1 z$, where $z = z_1 z_2 \dots z_r$. Using (8), we obtain

$$T_F z \subset \overline{S}_1 z \subset \overline{S} \subset \overline{M}_{r,r}. \tag{9}$$

Now, we will show that for every root $\beta \in \langle \alpha_1, \dots, \alpha_r \rangle$

$$\dim(T_\beta \cap T_F z) < r - 1, \tag{10}$$

where $T_\beta = \operatorname{Ker} \beta$. If $\dim(T_\beta \cap T_F z) = r - 1$ for any root β, then $T_\beta \cap T_F z = T_F z$, because $T_F z$ is closed and irreducible. Thus $T_F z = T_\beta^0 t$ for some $t \in T$, where T_β^0 is the identity component of T_β. But T_F, T_β^0 are subgroups in T. Hence $T_F = T_\beta^0$ and consequently the root subgroups $h_{\alpha_1}(t), \dots, h_{\alpha_{r-1}}(t)$ belong to the kernel of β: $T \to K^*$. Thus we obtain $\beta \perp \alpha_1$, $\beta \perp \alpha_2, \dots, \beta \perp \alpha_{r-1}$. This is impossible, when G is not B_r, F_4, G_2. Now we have proved (10). Then, there exists $s \in T_F z$, $s \notin \operatorname{Ker} \beta$ for any $\beta \in \langle \alpha_1, \dots, \alpha_r \rangle$. The element s is regular ([3, Chap. II, Sect. 4]) and, according to (9), $s \in \overline{M}_{r,r}$. ∎

If G is not B_r, F_4, G_2 and $k \le r + 1$, then we have $\overline{M}_{k,k-1} = L_{k-1}/Z(L_{k-1})$ according to lemmas 7, 8. Let $C_{t_1} \dots C_{t_k}$ be the conjugacy classes of $t_1, \dots, t_k$ in G. Then $C_{t_1} C_{t_2} \dots C_{t_k} \subset \overline{M}_k = \overline{C_1 C_2 \cdots C_k}$, because $C_{t_i} \subset \overline{C}_i$ ([3, Chap. II, Sect. 3]). Using Lemma 4, we obtain

$$\dim \pi_G(\overline{M}_k) \ge \dim \pi_G(\overline{C_{t_1} C_{t_2} \cdots C_{t_k}}) \ge k - 1.$$

Thus we have I.

Suppose G is a group of the type B_r. Let $\alpha_1 = \varepsilon_1 - \varepsilon_2, \dots, \alpha_r = \varepsilon_r$ be a standard numeration of the roots ([4, Table II]). Since $t_1, \dots, t_k \notin \bigcap_{i,j} \operatorname{Ker}(\pm\varepsilon_i \pm \varepsilon_j)$, we may suppose that $\psi_j(t_i) \ne 1$ for every i, j. If $r > 2$, then there is no such $\beta \in \langle \alpha_1, \dots, \alpha_j, \alpha_{j+1} \rangle$ that $\beta \perp \alpha_1, \dots, \beta \perp \alpha_j$. In this case we can use the proof of lemma 8. Therefore, we need to consider the case $r = 2$ and to prove $\dim \pi_G(\overline{C_1 C_2}) \ge 1$, $\dim \pi_G(\overline{C_1 C_2 C_3}) = 2$ (here C_1, C_2, C_3 are conjugacy classes of g_1, g_2, g_3). The first inequality follows from the proof of Lemma 8. The second equality will follow from the existence of regular element in $\overline{C_1 C_2}$ and Lemma 6. An element $t \in \overline{C_1 C_2} \cap T$ is regular, if $t \notin \operatorname{Ker} \gamma$ for every $\gamma \in \langle \alpha_1, \alpha_2 \rangle$. Let

$$D_1 = \operatorname{Ker} \alpha_1 \cup \operatorname{Ker}(\varepsilon_1 + \varepsilon_2), \qquad D_2 = \operatorname{Ker} \alpha_2 \cup \operatorname{Ker} \varepsilon_1.$$

It is sufficient to prove that

$$\overline{C_1 C_2} \cap T \not\subset D_1 \cup D_2. \tag{11}$$

Since $\pi_G(\overline{C_1 C_2})$ is irreducible and $\pi_G(\overline{C_1 C_2}) = \pi_G(\overline{C_1 C_2} \cap T)$, then (11) will follow from

$$\overline{C_1C_2} \cap T \not\subset D_1, \tag{12}$$

$$\overline{C_1C_2} \cap T \not\subset D_2. \tag{13}$$

Let $\beta = \varepsilon_1 + \varepsilon_2$; $G_\alpha = \langle X_{\alpha_1}, X_{-\alpha_1} \rangle$, $G_\beta = \langle X_\beta, X_{-\beta} \rangle \leq G$ be the subgroups of rank one generated by the root subgroups; $T_1 = \langle h_{\alpha_1}(t) \rangle \leq G_{\alpha_1}$, $T_2 = \langle h_\beta(t) \rangle \leq G_\beta$ be their maximal tori. Then $T = T_1T_2$ and $\sigma\tau = \tau\sigma$ for every $\sigma \in G_\alpha$, $\tau \in G_\beta$. We can choose $t_1 \in \overline{C}_1 \cap T$, $t_2 \in \overline{C}_2 \cap T$ such that $\alpha_1(t_1) \neq 1$, $\beta(t_2) \neq 1$. (This is possible according to supposition I) If $t_1 = t_{11}t_{12}$, $t_2 = t_{21}t_{22}$, where $t_{i,j} \in T_j$, then

$$\sigma t_1 \sigma^{-1} \tau t_2 \tau^{-1} = \left(\sigma t_{11} \sigma^{-1} t_{21}\right)\left(t_{12} \tau t_{22} \tau^{-1}\right) \tag{14}$$

for every $\sigma \in G_{\alpha_1}$, $\tau \in G_\beta$. Since $\alpha_1(t_1) \neq 1$, $\beta(t_2) \neq 1$, then $t_{11} \notin Z(G_\alpha)$, $t_{22} \notin Z(G_\beta)$. Using (14), we can see that there exists $t \in \overline{C_1C_2} \cap T$, $\alpha_1(t) \neq 1$, $\beta(t) \neq 1$. Hence we have (12). If we write T_F for T_1 in the proof of Lemma 8, we obtain the inclusion $T_1 z \subset \overline{C_1C_2} \cap T$ for some $z \in T$. Moreover, if $T_1 z \subset \operatorname{Ker} \gamma$ for some $\gamma \in \langle \alpha_1, \alpha_2 \rangle$, then $\gamma \perp \alpha_1$. There are only two roots: $\pm\beta$ which are orthogonal to α. Hence $T_1 z \not\subset D_2$ and we have (13).

Thus, we have proved I in the case, when G is a group of the type B_r.

Let G be a group of the type F_4 or G_2. There exists a simple algebraic subgroup $L \leq G$ of the type D_4 or A_2. We have supposed that $t_1, \dots, t_k$ are not in the intersection of the kernels of all long roots. Therefore we can choose $t_1, \dots, t_k \in L \setminus Z(L)$. If we change $C_1, \dots, C_k$ for the conjugacy classes of $t_1, \dots, t_k$ in L and use lemma 4 and our results for D_r and A_r, we obtain I.

Now theorem I is completely proved. ∎

3. Applications

3.1. Let G be an algebraic group which acts on a variety X. For $g \in G$ we denote the number

$$\dim X - \dim X^g$$

(where $X^g = \{x \in X | g(x) = x\}$) by the symbol corank g. Then we define

$$\begin{aligned} \min(G, X) &= \min\{\operatorname{corank} g | g \in G,\ g \notin Z(G)\}, \\ \max(G, X) &= \max\{\operatorname{corank} g | g \in G\}, \\ \min_s(G, X) &= \min\{\operatorname{corank} g | g \in G,\ g \notin Z(G),\ g = g_s\} \end{aligned}$$

(here g_s is the semisimple part of g), for $n \in Z^+$

$$G[n] = \{g \in G | \text{corank } g \leq n\}.$$

We will deal with an action which satisfies the following conditions:

(a) corank $g_1 g_2, \ldots, g_m \leq \sum_{i=1}^m$ corank g_i for every $g_1, g_2, \ldots, g_m \in G$;

(b) $G[n]$ is a closed subset of G. $(*)$

3.2. THEOREM 2. *Let G be a simple algebraic group and let the action G on X satisfies the conditions $(*)$. Then*

I. $\min(G, X) \geq \max(G, X)/2 \operatorname{rank} G$;

II. $\min_s(G, X) \geq \max(G, X)/(\operatorname{rank} G + 1)$, *or G is a group of the type B_r, F_4, G_2 and* $\min_s(G, X) = \operatorname{corank} t$, *where $t \in T$ is an element of a maximal torus which belongs to the intersection of the kernels of all the long roots.*

Proof. Let $g \in G \setminus Z(G)$, corank $g = \min(G, X)$, and let C_g be the conjugacy class of g. We have

$$\overline{C}_g^{2r} = G$$

according to Corollary 1 (here $r = \operatorname{rank} G$). Let $n = \max(G, X) - 1$. If $C_g^{2r} \subset G[n]$, then $\overline{C_g^{2r}} = G \subset G[n]$, as $G[n]$ is closed. This is in contradiction with the definition of $\max(G, X)$. Hence there exists an element $h \in C_g^{2r} \setminus G[n]$. Therefore

$$\operatorname{corank} h = \max(G, X),$$
$$h = g_1 g_2 \ldots g_{2r},$$

where $g_1, g_2, \ldots, g_{2r} \in C$. Using condition $(*, \text{a.})$, we obtain

$$\operatorname{corank} h = \max(G, X) \leq \sum_{i=1}^{2r} \operatorname{corank} g_i.$$

Since the elements $g_1, \ldots, g_{2r}$ belong to the same conjugacy class, their coranks are equal to each other. Hence

$$2r \operatorname{corank} g = 2r \min(G, X) \geq \max(G, X).$$

Let $t \in T \setminus Z(G)$, corank $t = \min_s(G, X)$, and C_t be the conjugacy class of t. If G is not B_r, F_4, G_2, or t is not in the intersection of kernels of all long roots, then

$$\overline{C_t^{r+1}} = G$$

according to Corollary I. Now the proof can be obtained as above. ∎

3.3. Now we describe three cases when conditions $(*)$ hold.

1. $X = A_k^n$ is the affine space and the action of G on X is equivalent to a linear action.

2. $X = P(V)$, where V is a rational $\hat{G}$-module and $\hat{G}$ is a central extension of G.

3. X is an algebraic group, $G \leq X$ is a closed subgroup which acts on G by inner automorphisms.

Let us check $(*)$. Let $g_1, \dots, g_m \in G$, $\sigma_1 = g_1 g_2 \cdots g_{m-1}$, $\sigma_2 = g_m$, $X_1 = \bigcap_{i=1}^{m-1} X^{g_i}$, $X_2 = X^{g_m}$. Suppose that

$$\operatorname{corank} \sigma_1 \leq \sum_{i=1}^{m-1} \operatorname{corank} g_i. \tag{15}$$

It is easy to see, that in all our cases

$$\dim X_1 \cap X_2 \geq \dim X_1 + \dim X_2 - \dim X. \tag{16}$$

Using (15) and (16), we obtain

$$\begin{aligned}\operatorname{corank} \sigma_1\sigma_2 = \dim X - \dim X^{\sigma_1\sigma_2} &\leq \dim X - \dim X_1 \cap X_2 \\ &\leq (\dim X - \dim X_1) + (\dim X - \dim X_2) \\ &\leq \operatorname{corank} \sigma_1 + \operatorname{corank} \sigma_2 \leq \sum_{i=1}^{m} \operatorname{corank} g_i.\end{aligned}$$

Thus we have checked a.

Let us check b. In case 1 we can suppose that X is a rational G-module. Then the set $G[n]$ is the intersection of two closed subsets of End X, namely, G and $L_n = \{f \in \operatorname{End} X \mid \operatorname{rank}(f - 1) \leq n\}$. This implies b.

In case 2 we consider the subset of End V

$$\hat{L}_n = \{f \in \operatorname{End} V \mid \operatorname{rank}(f - \lambda \cdot 1) \leq n \text{ for some } \lambda \in K\}$$

(here V is a linear space over K). Let $d = \dim V$, $A = (a_{ij})$ be a matrix of the size $d \times d$, where $\{a_{ij}\}$ are algebraically independent over K, and let x be a transcendental element over the field $K(\{a_{ij}\})$. If $n \leq d$, the $(n+1) \times (n+1)$ minors of $A - xE$ are polynomials with respect to x. The coefficients of such polynomials lie in $K(\{a_{ij}\})$. Let $f_1(x), f_2(x), \dots, f_k(x)$

be the set of such polynomials and let $\{c_{pq}\}$ be the set of coefficients of $f_p(x)$. There exists the set of polynomials

$$\phi_1, \dots, \phi_m \in K[\{c_{pq}\}] \subset K[\{a_{ij}\}]$$

such that for every homomorphism

$$\varepsilon\colon K[\{c_{pq}\}] \to K$$

the set $\varepsilon(f_1(x)), \varepsilon(f_2(x)), \dots, \varepsilon(f_k(x))$ has a common root if and only if $\varepsilon(\phi_1) = \varepsilon(\phi_2) = \cdots = \varepsilon(\phi_m) = 0$ (here we also use the fact that among $f_1(x), \dots, f_m(x)$ there is the polynomial

$$\begin{vmatrix} a_{11} - x & a_{12} & \cdots & a_{1n+1} \\ a_{21} & a_{22} - x & \cdots & a_{2n+1} \\ \vdots & & & \\ a_{n1} & \cdots & & a_{n+1} - x \end{vmatrix} = (-1)^{n+1} x^{n+1} + \cdots$$

which has the coefficient at $x^{n+1} \neq 0$ ([5, Chap. IV, Sect. 5). If we consider $\phi_1, \dots, \phi_m$ as polynomials from $K[\{a_{ij}\}]$, then $\hat{L}_n$ is the algebraic set corresponding to $\phi_1, \dots, \phi_m$. Hence, $\hat{L}_n$ is closed in $\operatorname{End} V$. We can suppose $\hat{G} \subset SL(V)$. Therefore the kernel of the homomorphism

$$\vartheta\colon \hat{G} \to G \subset PGL(V)$$

is finite. This implies that the image of the closed subset $\hat{L}_n \cap \hat{G}$ of $\hat{G}$ is closed in G but $\vartheta(\hat{L}_n \cap \hat{G}) = G[n]$ and, consequently, we have b.

If X is an algebraic group and $G \leq X$, then $g \in G[n]$ if and only if the conjugacy class of g in X has dimension $\leq n$. Let $X(n)$ be the set of all $x \in X$ such that the conjugacy class of x has dimension $\leq n$. It is known that $X(n)$ is closed in X ([2, Chap. II, Sect. 2]). Hence $G[n] = G \cap X(n)$ is closed in G.

3.4. Let V be a faithful rational G-module. The numbers $\min(G, V)$ and $\min(G, P(V))$ we call a class and a projective class of the linear group $G \leq GL(V)$ and denote it by $\operatorname{cl}(G)$ and $\operatorname{pr.cl}(G)$ ([6]). Using the estimates of Zarhin ([7]) which give lower bounds of ranks of operators of simple Lie algebras, we have obtained in [6] the inequality $\operatorname{pr.cl}(G) \geq \dim V/6r$, where G is a simple algebraic group and V is an irreducible rational G-module. Here we can improve this inequality using Theorem 2.

THEOREM 3. *Let G be a simple algebraic group and let V be a faithful irreducible rational G-module. Then*

I. $\mathrm{cl}(G) \geq \mathrm{pr.cl}(G) \geq \dim V/3r$;

II. $\mathrm{cl}(G) \geq (\dim V - \dim V^T)/2r$.

Proof. Let $V = \Sigma_\chi V_\chi$ be the decomposition of V into the sum of weight subspaces. Then

$$\dim V_0 \leq \tfrac{1}{3}\dim V. \tag{17}$$

([7], lemma 1). If $\chi \neq 0$, then ([7, Lemma 2])

$$\dim V_\chi \leq \frac{1}{r+1}\dim V. \tag{18}$$

If we exclude the case $r = 1$, $\dim V = 2$ which obviously satisfies the statement of theorem 3, we can obtain from (17), (18)

$$\dim V_\chi \leq \tfrac{1}{3}\dim V.$$

Hence for every weight χ

$$\dim V - \dim V_\chi \geq \tfrac{2}{3}\dim V \tag{19}$$

(here $r > 1$ or $\dim V > 2$). If we have the finite set of homomorphisms $\{\chi_i\colon T \to K\}$, we can find an element $t \in T$ such that $\chi_i(t) \neq \chi_j(t)$ for every $\chi_i \neq \chi_j$. Using (19), we obtain

$$\max(G, P(V)) \geq \tfrac{2}{3}\dim V. \tag{20}$$

Now I follows from (20) and theorem 2. Using the same arguments we obtain II. ∎

Remark. The inequalities (17), (18) follow from the theory of weights. To obtain the inequality like (20), we can use also the theory of characters for representations of compact groups. But in this case we can obtain only $\max(G, P(V)) \geq \frac{1}{2}\dim V$. In fact, we expect that there are the estimates which are stronger than (17), (18) with respect to the asymptotic behaviour. The inequality II of theorem 3 gives the possibility to improve the estimates for $\mathrm{cl}(G)$.

The classes of linear groups play an important role in the Invariant Theory ([6]). But in the case of reductive groups it is more important to know the numbers

$$\mathrm{cl}_s(G) = \min_s(G, V),$$
$$\mathrm{pr.cl}_s(G) = \min_s(G, P(V)),$$

because slice-subgroups of reductive groups are reductive too and the analysis of such subgroups uses the information about coranks of semisimple elements ([6]). ∎

THEOREM 4. *Let G be a semisimple algebraic group and let V be a faithful irreducible rational G-module. Then*

$$\mathrm{cl}_s(G) \geq \mathrm{pr.cl}_s(G) \geq \tfrac{2}{3} \dim V/(r+1)$$

and

$$\mathrm{cl}_s(G) \geq \left(\dim V - \dim V^T\right)/(r+1),$$

or G is a group of the type B_r, F_4, G_2 and $cl_s(G) = \operatorname{corank} t$, where $t \in T$ lies in the intersection of kernels of all long roots.

Proof. This follows from the same arguments as Theorem 3. ∎

Remark. In the case when G is a semisimple group (not necessarily connected), we have obtained in [6] the inequality $\mathrm{pr.cl}(G) \geq \frac{1}{12} \dim V/[G:G^0]r$ (here G^0 is the identity component of G). Using Theorem 3, it is possible to obtain

$$\mathrm{pr.cl}(G) \geq \tfrac{1}{6} \dim V/\left[G:G^0\right]r.$$

Details and applications of the improvement of the codimension inequality of invariant algebras will be done in another work. ∎

3.5. THEOREM 5. *Let G be a simple algebraic group and let T be a maximal torus. Suppose that G is embedded into an algebraic group F as a closed subgroup. Let $g \in G \setminus Z(G)$ and C_g be the conjugacy class of g in F. Then*

$$\dim C_g \geq (\dim F - \dim C_F(T))/(\operatorname{rank} G + 1).$$

If, in addition, $g \in T$ and g is not in the intersection of kernels of all long roots of G, when G is of the type B_r, F_4, G_2, then

$$\dim C_g \geq (\dim F - \dim C_F(T))/(\operatorname{rank} G + 1).$$

Proof. Let L be the Lie algebra of F. We consider the adjoint action G on L. There exists an element $t \in T$ such that $L^T = L^{\langle t \rangle}$. Hence $\dim C_F(t) = \dim L^{\langle t \rangle} = \dim L^T = \dim C_F(T)$ and therefore

$$\max(G, F) \geq \dim F - \dim C_F(t) = \dim F - \dim C_F(T).$$

Now our assertion follows from Theorem 2. ∎

4. EXAMPLES; $SL_n(K)$

4.1. Here we give some examples of multiclasses and their behaviour in $G = SL(k)$ (K is algebraically closed and char $K = 0$).

PROPOSITION 4. *Let $a, b \in G$ be semisimple matrices with different eigenvalues and let C_a, C_b be their conjugacy classes. Then*

$$G \setminus Z(G) \subset C_a C_b.$$

Besides, there exists only one conjugacy class C such that $C^2 = G$. This is the class of the matrix

$$d = \operatorname{diag}(\alpha, \alpha\varepsilon, \alpha\varepsilon^2, \ldots, \alpha\varepsilon^{n-1}),$$

where $\varepsilon = \exp(2\pi i/n)$ and $\alpha^n \varepsilon^{n(n-1)/2} = 1$.

COROLLARY 4. $\overline{C_a C_b} = G$.

Proof of the Corollary. Obvious.

Proof of the Proposition. We omit the proof of the next assertion which is an exercise in linear algebra and can be done by manipulations with transvections.

LEMMA 9. *Let Δ_i: $SL_n(K) \to K^*$ be the function which is the principal i-minor, that is,*

$$\Delta_i\left(\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ & \vdots & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}\right) = \det\begin{pmatrix} a_{11} & \cdots & a_{1i} \\ & \vdots & \\ a_{i1} & \cdots & a_{ii} \end{pmatrix}.$$

Let $\delta_1 \neq 0, \delta_2 \neq 0, \ldots, \delta_{n-1} \neq 0 \in K^*$ and let

$$g = \begin{pmatrix} a_{11} & a_{12} & 0 & & \cdots & & 0 \\ 0 & a_{22} & a_{23} & 0 & \cdots & & 0 \\ 0 & 0 & a_{33} & a_{34} & 0 & \cdots & 0 \\ \vdots & & & & & & \\ 0 & & \cdots & & 0 & a_{nn} & \end{pmatrix}$$
$$\in SL_n(K) \setminus Z(SL_n(K)).$$

Then there exists a matrix $s \in SL_n(K)$ such that

$$\Delta_1(sgs^{-1}) = \delta_1, \Delta_2(sgs^{-1}) = \delta_2, \ldots, \Delta_{n-1}(sgs^{-1}) = \delta_{n-1}.$$

Now we can prove the first part of the theorem. Let

$$\delta_1 = \Delta_1(ab), \delta_2 = \Delta_2(ab), \ldots, \delta_{n-1} = \Delta_{n-1}(ab).$$

For any $x \in G \setminus Z(G)$ we can find a conjugate matrix which looks like g. According to lemma 9, $\Delta_1(x') = \delta_1$, $\Delta_2(x') = \delta_2, \ldots, \Delta_{n-1}(x') = \delta_{n-1}$ for some x' conjugate to x. The Gauss decomposition gives

$$x' = u_1 abu_2. \tag{21}$$

where $u_1 \in U^-$, $u_2 \in U$ (here U^-, U are the groups of lower and upper triangular unipotent matrices in $SL_n(K)$). Since a, b have different eigenvalues, the maps

$$f_a: U^- \to U^-, \qquad f_b: U \to U,$$

which are defined by the formulas

$$f_a(u) = uau^{-1}a^{-1}, \qquad f_b(u) = b^{-1}ubu^{-1}$$

($u \in U^-$ or $u \in U$), are injections. Moreover, $f_a(U^-)$, $f_b(U)$ are closed in U^- or U ([8, Chap. VII, Sect. 18]). Hence f_a, f_b are bijections. Therefore

$$u_1 a = \left(u_1' a u_1'^{-1} a^{-1}\right) a = u_1' a u_1'^{-1} \in C_a,$$

$$bu_2 = b\left(b^{-1} u_2' b u_2'^{-1}\right) = u_2' b u_2'^{-1} \in C_b$$

for some $u_1' \in U^-$, $u_2' \in U$. From (21) we obtain now

$$x' \in C_a C_b.$$

This implies our assertion, because $C_a C_b$ is Int G-invariant.

Let $C^2 = G$ for some conjugacy class C (not necessarily semisimple). Then

$$\varepsilon = g\sigma g\sigma^{-1}, \qquad 1 = g\tau g\tau^{-1} \tag{22}$$

for some $g \in C$, $\sigma, \tau \in G$ (here ε, 1 are scalar matrices). Since $\varepsilon, 1 \in Z(G)$ we can suppose $g, \sigma g\sigma^{-1}, \tau g\tau^{-1} \in B$ (here B is the group of upper triangular matrices). Let $r_1, \ldots, r_n$ be the eigenvalues of g. It follows from (22) that

$$r_1 r_{\pi(1)} = \varepsilon, r_2 r_{\pi(2)} = \varepsilon, \ldots, r_n r_{\pi(n)} = \varepsilon$$

$$r_1 r_{\phi(1)} = 1, r_2 r_{\phi(2)} = 1, \ldots, r_n r_{\phi(n)} = 1 \tag{23}$$

for some substitutions $\pi, \phi \in S_n$. Let

$$t = \mathrm{diag}(r_1, r_2, \ldots, r_n).$$

It follows from (23) that there exist elements $w_1, w_2 \in W$ such that

$$w_1 t w_1^{-1} = t^{-1}\varepsilon, \qquad w_2 t w_2^{-1} = t^{-1}.$$

Let $w = w_2 w_1$. Then

$$wtw^{-1} = t\varepsilon. \tag{24}$$

We obtain from (24)

$$r_2 = r_1\varepsilon,\ r_3 = r_1\varepsilon^2, \ldots, r_n = r_1\varepsilon^{n-1} \tag{25}$$

(under a proper numeration of the eigenvalues). Since t is the semisimple part of g and all eigenvalues of t are different (according to (25)) then $g = t$. Moreover, $r_1^n \varepsilon^{n(n-1)/2} = 1$ and all diagonal matrices which satisfy (25) are conjugate. Hence the equality $C^2 = G$ is possible only for the single conjugacy class C. Furthermore, $G \setminus Z(G) \subset C^2$ according to the first part of the theorem. The incusion $Z(G) \subset C^2$ is obvious. ∎

Remark. The Proposition 4 show that the product of two "general" semisimple conjugacy classes is open in its closure and this closure coincides with the whole group. Moreover, the complement has dimension zero. In cases when there is a multiplicity of eigenvalues, we will be able to obtain weaker results (up to closures).

PROPOSITION 5. *Let $C \subset G \setminus Z(G)$ be an arbitrary conjugacy class and let C_a be the class of a semisimple matrix a with different eigenvalues. Then*

$$\overline{CC_a} = G.$$

Proof. Let $g \in C$ and let C^{-1} be the conjugacy class of g^{-1}. According to Proposition 5, we have $C^{-1} \subset C_a C_b$ for any conjugacy class C_b of a semisimple matrix b with different eigenvalues. Then $\sigma g^{-1} \sigma^{-1} = \tau a \tau^{-1} b$ for some $\sigma, \tau \in G$ and $b^{-1} = \sigma g \sigma^{-1} \tau a \tau^{-1} \in CC_a$. Therefore, any semisimple matrix with different eigenvalues belongs to CC_a. Hence $\overline{CC_a} = G$. ∎

Remark. It may occur that the set $G \setminus CC_a$ would be rather big. For instance, let $n \geq 4$, $g = \mathrm{diag}(1, \varepsilon, \ldots, \varepsilon)$, where $\varepsilon^{n-1} = 1$, $a = \mathrm{diag}(\alpha_1, \ldots, \alpha_n)$, where $\alpha_i \neq \alpha_j$ for any $i \neq j$, and $G_3 = \{g \in G \mid \mathrm{rank}(g - \lambda \cdot 1) \leq n - 3\}$. One can see that $G_3 \subset G \setminus CC_a$, G_3 is closed in G and $\dim G_3 = n^2 - 9$.

PROPOSITION 6. *Let $C_1, \dots, C_n \subset G \setminus Z(G)$ be conjugacy classes. Then*

$$\overline{C_1 C_2 \dots C_n} = G.$$

Proof. It is obvious for $n = 2$. Suppose this is true for $n - 1$. Let $P = \{(a_{ij}) \in G | a_{nj} = 0 \text{ for } j = 1, \dots, n-1\}$, $L = \{(a_{ij}) \in P | a_{in} = 0 \text{ for } i = 1, \dots, n-1\}$. Using the Jordan decomposition, we can choose elements $g_1 \in C_1 \cap P, \dots, g_n \in C_n \cap P$ such that their images $\bar{g}_1, \dots, \bar{g}_n$ in $L/Z(L) \cong PSL_{n-1}(K)$ under the natural homomorphism

$$P \to L/Z(L)$$

are not trivial. We have

$$\overline{C_{\bar{g}_1} C_{\bar{g}_2} \dots C_{\bar{g}_{n-1}}} = L/Z(L). \tag{26}$$

where $C_{\bar{g}_i}$ is the conjugacy class of $\bar{g}_i$. It follows from (26) that among the elements

$$(\sigma_1 g_1 \sigma_1^{-1})(\sigma_2 g_2 \sigma_2^{-1}) \cdots (\sigma_{n-1} g_{n-1} \sigma_{n-1}^{-1}),$$

where $\sigma_1, \dots, \sigma_{n-1} \in L$, there exists a matrix a with different eigenvalues. The statement follows now from Proposition 5. ∎

COROLLARY 5. $\overline{\text{cn}}(SL_n(K)) = \overline{\text{ecn}}(SL_n(K)) = n$.

Proof. According to Proposition 6,

$$\overline{\text{cn}}(SL_n(K)) \le \overline{\text{ecn}}(SL_n(K)) \le n.$$

On the other hand, there exists a conjugacy class C such that $\overline{C^{n-1}} \neq G$ (for instance, the class of $a = \text{diag}(1, \varepsilon, \dots, \varepsilon)$, $\varepsilon^{n-1} = 1$). Hence $\overline{\text{cn}}(SL_n(K)) \ge n$. ∎

Remark. The result of Proposition 6 is better than the one of Theorem 1 when we deal with non-semisimple conjugacy classes of groups which have the type A_r.

4.2. Now we consider the closures of multiclasses generated by semisimple conjugacy classes.

Let $a = \text{diag}(a_1, \dots, a_n)$. By the symbol $a(i)$ we denote

$$\text{diag}(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n) \in GL_{n-1}(K).$$

The next result is very useful for our considerations.

LEMMA 10. *Let* $a = \text{diag}(a_1, \ldots, a_n)$, $b = \text{diag}(b_1, \ldots, b_n)$. *Suppose that there exist pairs* $(i, j), (i', j')$ *such that*

$$\dim \overline{C_{a(i)}C_{b(j)}} = (n-1)^2 - 1$$
$$\dim \overline{C_{a(i')}C_{b(j')}} = (n-1)^2 - 1 \tag{27}$$

$$a_i b_j \neq a_{i'} b_{j'} \tag{28}$$

then

$$\overline{C_a C_b} = G,$$

where $C_{a(i)}, C_{b(j)}, C_{a(i')}, C_{b(j')} \subset GL_{n-1}(K)$, $C_a, C_b \subset SL_n(K)$ *are the corresponding conjugacy classes.*

Proof. We can suppose that $i' = j' = n$. Let us consider the set

$$S' = \{g_1 a g_1^{-1} g_2 b g_2^{-1} | g_1, g_2 \in SL_{n-1}(K)\}$$

(here $SL_{n-1}(K)$ is embedded into $SL_n(K)$ in the natural way). It follows from (27) that

$$\dim \bar{S} \cap T = n - 2.$$

Hence

$$\dim \overline{C_a C_b} \cap T \geq n - 2.$$

Suppose $\dim \overline{C_a C_b} \cap T = n - 2$. Then the affine variety $\overline{C_a C_b} \cap T$ has an irreducible component which is

$$T_n = T \cap \{\text{diag}(x_1, \ldots, x_n) | x_n = a_n b_n\}.$$

Since $\pi_G(\overline{C_a C_b})$ is irreducible (see 2), then $\overline{C_a C_b} \cap T = WT_n$. Now we can consider in the same way the case $i = j = n$. Thus, we will obtain a contradiction with (28). Hence

$$\dim \overline{C_a C_b} \cap T = n - 1.$$

Therefore $\overline{C_a C_b} = G$. ∎

Now we need to introduce some new notations. Let

$$\lambda_m = (m_1, \ldots, m_k),\ m_1 \geq m_2 \geq \cdots \geq m_k$$

be a partition of $m = m_1 + m_2 + \cdots + m_k$. By the symbol $\mathfrak{U}(\lambda_m)$ we denote the union of all sequences

$$a = \Big(\underbrace{a_1, \dots, a_1}_{m_1}, \underbrace{a_2, \dots, a_2}_{m_2}, \dots, \underbrace{a_k, \dots, a_k}_{m_k}\Big),$$

where $a_i \in K^*$, $a_i \neq a_j$ for every $i \neq j$, and by the symbol $\mathfrak{U}$ we define

$$\Big(\bigcup_{\lambda_m, m \leq n} \mathfrak{U}(\lambda_m)\Big) \cup \{(0)\}$$

(here (0) is only a symbol). Then, for every $m = 2k$ we introduce the subset $\mathfrak{B}_m \subset \mathfrak{U}$ which consists of elements

$$a = \Big(\underbrace{a_1, \dots, a_1}_{k}, \underbrace{a_2, \dots, a_2}_{k}\Big)$$

and the set

$$\mathfrak{B} = \bigcup_{m=2k \leq n} \mathfrak{B}_m .$$

Let $m < n$ and let $n - m$ be even. Then for every partition λ_m and for every

$$a = (a_1, \dots, a_1, a_2, \dots, a_2, \dots, a_k, \dots, a_k) \in \mathfrak{U}(\lambda_m)$$

we introduce symbols $\langle a, \delta \rangle$, where $\delta \in K^*$ and

$$\delta^{(m-n)/2} = \prod_{i=1}^{k} a_i^{m_i}.$$

Thus, for every such a there are $(n - m)/2$ symbols $\langle a, \delta \rangle$. In the case $n = 2k$, $m = 0$, we take for δ any kth root of 1.

Now we connect the elements of $\mathfrak{U}$ and the symbols $\langle a, \delta \rangle$ with closed and Int G-invariant subsets of G. Let

$$a = (a_1, \dots, a_1, a_2, \dots, a_2, \dots, a_k, \dots, a_k) \in \mathfrak{U}(\lambda_m).$$

We define

$$T[a] = \Big\{\operatorname{diag}(x_1, \dots, x_n) | x_1 = \cdots = x_{m_1} = a_1, x_{m_1+1} = \cdots = x_{m_1+m_2} = a_2, \dots, x_{m-m_k+1} = \cdots = x_m = a_k, \prod_{i=1}^{n} x_i = 1\Big\}.$$

If there exists the symbol $\langle a, \delta \rangle$, we put

$$T[\langle a, \delta \rangle] = T[a] \cap \{\operatorname{diag}(x_1, \dots, x_n) | x_{n-m+1}x_{n-m+2} = \delta. \\ x_{n-m+3}x_{n-m+4} = \delta, \dots, x_{n-1}x_n = \delta\}.$$

Let $M[a], M[\langle a, \delta \rangle]$ be the closures in G of the sets $\operatorname{Int} GT[a]$. $\operatorname{Int} GT[\langle a, \delta \rangle]$ (by $\operatorname{Int} GX$ for every $X \subset G$ we mean the set of all elements conjugated to elements of X).

PROPOSITION 7. *Let* $a \in \mathfrak{U}(\lambda_m)$, $\lambda_m = (m_1, \dots, m_k)$. *Then* $M[a] = \{g \in G | \operatorname{rank}(g - a_i 1) \leq n - m_i$ *for every* $i = 1, \dots, k\}$.
If, in addition, $n - m = 21$, *then*

$$M[\langle a, \delta \rangle] = M[a] \cap \left\{ g \in G | \det(g - x \cdot 1) = (-1)^n \prod_{i=1}^{k} (x - a_i)^{m_i} \right. \\ \left. \times \prod_{j=1}^{l} \left(x^2 + \varepsilon_j x + \delta \right) \text{ for arbitrary } \varepsilon_j \in K \right\}.$$

Proof. It is easy to show that the right sides of both equalities are closed and contain $\operatorname{Int} GT[a]$ or $\operatorname{Int} GT[\langle a, \delta \rangle]$ as a dense subset. This implies our assertions. ∎

Now we define an operation on the set $\mathfrak{U}$. Let

$$a = (a_1, \dots, a_1, a_2, \dots, a_2, \dots, a_k, \dots, a_k) \in \mathfrak{U}(\lambda_p),$$
$$\lambda_p = (p_1, p_2, \dots, p_k),$$
$$b = (b_1, \dots, b_1, b_2, \dots, b_2, \dots, b_e, \dots, b_e) \in \mathfrak{U}(\lambda_q),$$
$$\gamma_q = (q_1, q_2, \dots, q_e).$$

Suppose $p_1 \geq q_1$. In all cases, except $k = e = 2$, $p = q = n$, we define

$$a \circ b = (c_1, \dots, c_1, c_2, \dots, c_2, \dots, c_t, \dots, c_t) \in \mathfrak{U}(\alpha_g), \tag{29}$$

where $\alpha_s = (s_1, s_2, \dots, s_t)$ is the partition of $s = s_1 + \cdots + s_t$ and

$$s_i = p_1 + q_i - n, \text{ if } p_1 + q_i - n > 0 \tag{30}$$

(if $p_1 + q_1 - n \leq 0$, then we put $a \circ b = (0)$), the elements $c_1, \dots, c_t$ must be computed according to the formulas

$$c_1 = a_1 b_1, c_2 = a_1 b_2, \dots, c_t = a_1 b_t.$$

If $k = e = 2$, $p = q = n$, then we put

$$a \circ b = \langle c, \delta \rangle. \tag{31}$$

where c is obtained by the same rule as above and $\delta = a_1 a_2 b_1 b_2$.

THEOREM 6. *Let $M_k = C_1 C_2 \dots C_k$ be a multiclass of G. If $C_1, \dots, C_k$ are semisimple, then there exists an element $a \in \mathfrak{U}$ such that $\overline{M}_k = M[a]$ or $\overline{M}_k = M[\langle a, \delta \rangle]$. The multiplication of such multiclasses in the monoid $\overline{M}(G)$ (see 1) can be defined by the following rules*:

$$M[a]M[b] = M[a \circ b],$$
$$M[\langle a, \delta \rangle] M[b] = M[a \circ b],$$
$$M[\langle a, \delta \rangle] M[\langle b, \varepsilon \rangle] = M[a \circ b].$$

Proof. For every $m \le n$ we put

$$\mathfrak{U}_m = \bigcup_{\lambda_m} \mathfrak{U}(\lambda_m) \subset \mathfrak{U}.$$

One can observe that the set

$$\{M[a] \| a \in \mathfrak{U}_m\}$$

coincides with the set of semisimple conjugacy classes of $GL_m(K)$. If, in addition, $\prod a_i^{m_i} = 1$, we obtain the semisimple conjugacy classes of $SL_m(K)$. Therefore, it is sufficient to prove the rules of multiplication.

For $a \in \mathfrak{U}_m$ we will denote the conjugacy class of a in $GL_m(K)$ by the symbol C_a.

Let $a \in \mathfrak{U}(\lambda_m)$ and $\lambda = (m_1, \dots, m_k)$. We define

$$h(a) = k, l(a) = m_1.$$

We will say that a pair $a, b \in \mathfrak{U}_m$ is *concerted* if $l(a) + l(b) \le m$, $h(a), h(b) \ge 2$ and either $h(a)$ or $h(b) \ge 3$.

LEMMA 11. *Suppose $m \ge 4$ and $a, b \in \mathfrak{U}_m$ is a concerted pair. Then one can construct concerted pairs $a', b' \in \mathfrak{U}_{m-1}$ and $a'', b'' \in \mathfrak{U}_{m-1}$ by taking away one element from a and one from b such that $a' = a''$, $b' \ne b''$ or $a' \ne a''$, $b' = b''$.*

This lemma can be proved by induction and a routine consideration of cases.

LEMMA 12. *Suppose $n \geq 3$ and $a, b \in \mathfrak{U}_n$ is a concerted pair. Then*

$$\dim \overline{C_a C_b} = n^2 - 1.$$

Proof. Let $n = 3$. Then $a = (a_1, a_2, a_3)$ or $b = (b_1, b_2, b_3)$. If we use Proposition 5, we obtain our assertion. Now we carry out the proof by induction using Lemmas 10, 11. ∎

Let $m = 2k$ and

$$a = (a_1, \dots, a_1, a_2, \dots, a_2) \in \mathfrak{B}_m,$$
$$b = (b_1, \dots, b_1, b_2, \dots, b_2) \in \mathfrak{B}_m.$$

Let V be the space of the natural representation of $GL_m(K)$, $g_1 \in C_a$, $g_2 \in C_b$, $H = \langle g_1, g_2 \rangle$. There exists a chain $V \supset V_1 \supset \cdots \supset V_d = 0$ of H-modules such that $\dim V_i / V_{i+1} \leq 2$ ([9]). This implies that $g = g_1 g_2$ is conjugate to a matrix

$$\operatorname{diag}\left(\alpha_1, \dots, \alpha_k, \delta\alpha_1^{-1}, \dots, \delta\alpha_k^{-1}\right), \tag{32}$$

where $\delta = a_1 a_2 b_1 b_2$, if g is semisimple.

Now we can prove the rule of multiplication for two conjugacy classes. We exclude the trivial case when one of the classes belongs to the center of G.

Let $a, b \in \mathfrak{U}_n$, $n \geq 2$; $h(a), h(b) \geq 2$; $l(a) \geq l(b)$. Let c be the element of $\mathfrak{U}(\alpha_s)$ constructed according to the rule (29) (here $\alpha_s = (s_1, \dots, s_t)$ is the partition of $s = s_1 + \cdots + s_t$ constructed according to the rule (30)). Let $g_1 \in C_a$, $g_2 \in C_b$ and $g = g_1 g_2$ be semisimple. The matrices g_1, g_2, g act naturally on the linear space V of the dimension n. There is a subspace

$$U = V_{s_1} \oplus V_{s_2} \oplus \cdots \oplus V_{s_t} \subset V$$

such that $\dim V_{s_i} = s_i$ and the restriction of g_1, g_2, g on V_{s_i} are the scalar matrices $a_1 \cdot 1, b_i \cdot 1, a_1 b_i \cdot 1$ (this follows from the definitions and the dimension formula for intersections). Let $\bar{g}_1, \bar{g}_2$ be the images g_1, g_2 in $GL(V/U)$ and let $\bar{a}, \bar{b} \in \mathfrak{U}_{n-s}$ be the corresponding sets of eigenvalues of $\bar{g}_1, \bar{g}_2$. There are two cases: $\bar{a}, \bar{b}$ is a concerted pair or $\bar{a}, \bar{b} \in \mathfrak{B}_{n-s}$. In the first case we use Lemma 12 and obtain

$$\overline{C_a C_b} \cap T = WT[c] \tag{33}$$

(here W is the Weyl group of $G = SL_n(K)$). In the second case we use

(32) and obtain

$$\overline{C_a C_b} \cap T = WT[\langle c, \delta \rangle]. \tag{34}$$

The inclusions $\overline{C_a C_b} \supset M[c]$ or $\overline{C_a C_b} \supset M[\langle c, \delta \rangle]$ follow now from (33) or (34). The contrary inclusions follow from the proposition 7. Thus

$$\overline{C_a C_b} = M[a]M[b] = M[a \circ b].$$

Now we can prove the general case. Let $a \in \mathfrak{U}_p$, $b \in \mathfrak{U}_q$. It is possible to extend a and b up to $\hat{a} \in \mathfrak{U}_n$, $\hat{b} \in \mathfrak{U}_n$ adding different elements of K^*. According to the definitions, we have $a \circ b = \hat{a} \circ \hat{b}$. Since $M[\hat{a}]$, $M[\hat{b}]$ are the conjugacy classes and $M[\hat{a}] \subset M[a]$, $M[\hat{b}] \subset M[b]$, we have

$$M[a \circ b] \subset M[a]M[b].$$

If we take any conjugacy class C_x from $M[a]$ and any one C_y from $M[b]$, the product $\overline{C_x C_y}$ will lie in $M[x \circ y] \subset M[a \circ b]$ as x, y are extensions of a, b. Therefore

$$M[a \circ b] = M[a]M[b].$$

The cases of multiclasses $M[\langle a, \delta \rangle]$ are easily reduced to the case $M[a]$. Really, there exists a semisimple element $g \in M[\langle a, \delta \rangle]$ whose eigenvalues with multiplicity coincide with elements of a. Now we can use the same arguments as above. ∎

Remark. It is easy to see that the analogous construction will describe the case $G = GL_n(K)$.

4.3. We denote by the symbol $\overline{M}_s(G)$ the monoid generated by the semisimple conjugacy classes of G in $\overline{M}(G)$. Consider some properties of $\overline{M}_s(G)$ which easily follow from Proposition 7 and Theorem 6.

I. If $M[a]$ is not a conjugacy class, then $a \in \mathfrak{U}(\lambda_m)$ for some $\lambda_m = (m_1, \ldots, m_k)$, where $m \leq n - 2$. In this case

$$\dim M[a] = n^2 - \sum_{i=1}^{k} m_i^2 - 1.$$

II. If $M[\langle a, \delta \rangle]$ is not a multiclass of the type $M[x]$, then $a \in \mathfrak{U}(\lambda_m)$, where $m \leq n - 4$. In this case

$$\dim M[\langle a, \delta \rangle] = n^2 - \sum_{i=1}^{k} m_i^2 - (n - m)/2.$$

III. If $a \in \mathfrak{U}(\lambda_m)$ and $m \leq n-4$, the multiclass $M[\langle a, \delta\rangle]$ may occur only among 2-classes.

IV. If $a = (a_1, \ldots, a_1, a_2, \ldots, a_2, \ldots, a_k, \ldots, a_k)$, $\hat{a}_1 = (a_1, \ldots, a_1)$, $\hat{a}_2 = (a_2, \ldots, a_2), \ldots, \hat{a}_k = (a_k, \ldots, a_k)$, then

$$M[a] = M[\hat{a}_1] \cap M[\hat{a}_2] \cap \cdots \cap M[\hat{a}_k].$$

V. $M[(0)] = G$.

VI. For an integer $i \leq n$ we define

$$\overline{M}_s^i(G) = \{M[a], M[\langle a, \delta\rangle] | l(a) = n - i\}.$$

If $i > n$, we put

$$\overline{M}_s^i(G) = G.$$

The multiplication in $\overline{M}_s(G)$ induces the maps

$$\overline{M}_s^i(G) \times \overline{M}_s^j(G) \to \overline{M}_s^{i+j}(G).$$

Thus, $\overline{M}_s(G)$ is a graded monoid. Moreover, the set

$$\overline{M}_s(G, m) = \bigcup_{i \geq m} \overline{M}_s^i(G)$$

is an ideal of $\overline{M}_s(G)$. Note that $\overline{M}_s^i(G)$ belongs to the discriminant variety of the morphism π_G: $G \to G/\operatorname{Int} G$ if and only if $i \leq n-2$.

VII. Let Λ_n be the set of all partitions of all positive integers $\leq n$. We also suppose that $(0) \in \Lambda_n$. For the symbol @ we put

$$\lambda_m^{@} = \{\langle \lambda, @\rangle | \lambda \in \Lambda_m\}.$$

The set $\Delta_n = \Lambda_n \cup \Lambda_{n-2}^{@}$ is a commutative monoid with respect to the operation defined by the rules corresponding to (30), (31) and theorem 6. The map

$$f: \overline{M}_s(G) \to \Delta_n,$$

where $f(M[a]) = \lambda_m$ for $a \in \mathfrak{U}(\lambda_m)$ and $f(M[\langle a, \delta\rangle) = \langle \lambda_m, @\rangle$ for all δ, is a homomorphism. The image of this homomorphism is the submonoid of Δ_n generated by all partition of the number n.

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