JOURNAL OF DIFFERENTIAL EQUATIONS **127**, 295–319 (1996)
ARTICLE NO. 0071

Elliptic Eigenvalue Problems and Unbounded Continua of Positive Solutions of a Semilinear Elliptic Equation

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Received November 8, 1994; revised April 27, 1995

We derive a result on the limit of certain sequences of principal eigenvalues associated with some elliptic eigenvalue problems. This result is then used to give a complete description of the global structure of the curves of positive steady states of a parameter dependent diffusive version of the classical logistic equation. In particular, we characterize the bifurcation values from infinity to positive steady states. The stability of the positive steady states as well as the asymptotic behaviour of positive solutions is also discussed. © 1996 Academic Press, Inc.

1. Introduction

In many reaction–diffusion systems used to model a great diversity of phenomena in the applied sciences the single densities of the species, if uncoupled, obey some type of logistic growth law (cf. [16] Chapter III, Section 1; [3]; [11]; [12]). This means that they satisfy an evolution problem of the form

$$\begin{cases} \partial_t u + \mathscr{L}(x, D)\, u = m(x)\, u - a(x)\, h(x, u)\, u & \text{in} \quad \Omega \times (0, \infty), \\ \mathscr{B}(x, D)\, u = 0 & \text{on} \quad \partial\Omega \times (0, \infty), \\ u(\cdot, 0) = u_0 & \text{in } \Omega. \end{cases} \tag{1.1}$$

0022-0396/96 $18.00

where Ω is a smooth bounded domain of $\mathbb{R}^N$, $N \geqslant 1$, $\mathscr{L}(x, D)$ is an uniformly elliptic differential operator of second order, and the boundary operator $\mathscr{B}(x, D)$ is of Dirichlet, Neumann or Robin type. The coefficients m and a are smooth functions in Ω such that a is nonnegative, while m may change sign in Ω. The function h is smooth, increasing in u, and satisfies $h(\cdot, 0) \equiv 0$. For a qualitative analysis of the above mentioned systems a precise understanding of the single equation is imperative. In this paper we shall deal with the question of the existence and stability of positive steady state solutions of (1.1). We will also study the parameter dependent elliptic problem

$$\begin{cases} \mathscr{L}(x, D)\, w = \lambda m(x)\, w - a(x)\, wh(x, w) & \text{in } \Omega, \\ \mathscr{B}(x, D)\, w = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where the parameter λ varies over all of $\mathbb{R}$. Here we adress the problem of the existence of curves (λ, w_λ) of positive solutions. We shall see how in some instances bifurcation to positive solutions from the trivial branch $(\lambda, 0)$ occurs and how in other cases global smooth curves of positive solutions that are bounded away from the trivial branch can exist.

We now describe the contents of this work. In Section 2 we give a new result on linear elliptic eigenvalue problems. More precisely, we consider the sequence σ_k of principal eigenvalues associated with the eigenvalue problems

$$\begin{cases} \mathscr{L}(x, D)\, \varphi + q_k \varphi = \sigma\varphi & \text{in } \Omega, \\ \mathscr{B}(x, D)\, \varphi = 0 & \text{on } \partial\Omega, \end{cases}$$

where each function q_k is assumed to be nonnegative and to satisfy $q_k(x) = 0$ for all $x \in \Omega_0$ and $k \geqslant 1$, where Ω_0 is a smooth subdomain of Ω. We show that if for every compact subset K of $\bar{\Omega} \backslash \Omega_0$ the following holds

$$\lim_{k \to \infty} \min_{x \in K} q_k(x) = \infty$$

then the limit

$$\lim_{k \to \infty} \sigma_k$$

exists and equals the principal eigenvalue of the following eigenvalue problem

$$\begin{cases} \mathscr{L}(x, D)\, \varphi = \sigma\varphi & \text{in } \Omega_0, \\ \varphi = 0 & \text{on } \partial\Omega_0. \end{cases} \tag{1.3}$$

In Section 3 we characterize the existence of positive steady states of (1.1) and show their uniqueness and linear stability. We also give a complete description of the asymptotic behaviour of positive solutions of (1.1). Namely, we show that if the trivial solution is linearly stable or neutrally stable, then it is globally asymptotically stable with respect to positive initial data. Therefore, no positive steady state exists in this case. Moreover, we show that if (1.1) possesses a positive steady-state, then it is a global attractor with respect to positive initial data. Furthermore, we show that if the trivial solution is linearly unstable and (1.1) does not admit a positive steady state, then any solution to the evolutionary parabolic model starting at a positive initial state grows to infinity as time increases. The results of Section 2 provide us with the key to construct explicit examples exhibiting each of the above mentioned behaviours. This constructions will be given in Section 4, where we analyze and solve the problem of the existence of curves of positive solutions of the parameter dependent problem (1.2) and determine their shape. In particular, we consider the case where the coefficient a vanishes identically on a subset of Ω. It seems that this case has not been analyzed previously. This coefficient models the limiting effects of crowding of the population and it can be regarded as a sort of damping term. To be precise, we show that if $a(x)$ vanishes on some region with non-empty interior then bifurcation to positive solution from infinity occurs. Moreover, if the set of $x \in \Omega$ at which a vanishes is the closure of a sufficiently regular subdomain Ω_0 of Ω, then the value of λ at which bifurcation to positive solutions from infinity occurs is given by the principal eigenvalue of (1.3). This seems to be a new phenomenon for the logistic equation. Moreover, this phenomenology can be shown to occur for more general families of semilinear elliptic and periodic-parabolic problems, on bounded or on unbounded domains. This phenomenom of bifurcation from infinity has a natural biological interpretation: If the population grows exponentially in some subdomain of the habitat, then it may reach arbitrarily large values in that subregion, which entails bifurcation from infinity. We should point out that the existing abstract theorems about bifurcation from infinity do not apply stright away to our problem.

2. Principal Eigenvalues for Linear Boundary Value Problems and Stabilization

Let Ω be a bounded domain of $\mathbb{R}^N$, $N \geqslant 1$, of class $C^{2+\mu}$ for some $\mu \in (0, 1)$ and consider the following strongly uniformly elliptic operator of second order

$$\mathscr{L}(x, D) := -\sum_{i,j=1}^{N} a_{ij}(x)\, D_i D_j + \sum_{i=1}^{N} a_i(x)\, D_i + a_0(x),$$

where a_{ij}, a_i, $a_0 \in C^\mu(\bar{\Omega})$, $i, j = 1, ..., N$. By $\mathscr{D}$ and $\mathscr{R} = \mathscr{R}(x, D)$ we shall denote boundary operators given by

$$\mathscr{D}u := u \qquad \text{and} \qquad \mathscr{R}u := \partial_\nu u + b_0(x)\, u,$$

where $\nu \in C^{1+\mu}(\partial\Omega, \mathbb{R}^N)$ is an outward pointing nowhere tangential vector-field on $\partial\Omega$ and $b_0 \in C^{1+\mu}(\partial\Omega)$ is non-negative. Hence, $\mathscr{D}$ is the *Dirichlet* boundary operator and $\mathscr{R}$ is either the *Neumann* boundary operator, if $b_0 = 0$, or the *Robin* boundary operator, if $b_0 \neq 0$. Throughout this work we denote by $\mathscr{B}$ any of these boundary operators. Given $q \in C^\mu(\bar{\Omega})$ and a $C^{2+\mu}$ subdomain Ω_0 of Ω we shall consider the linear elliptic eigenvalue problem

$$\begin{cases} \mathscr{L}(x, D)\, \varphi + q(x)\, \varphi = \lambda\varphi & \text{in } \Omega_0, \\ \mathscr{B}(x, D)\, \varphi = 0 & \text{on } \partial\Omega_0. \end{cases} \tag{2.1}$$

It is well known that (2.1) admits a unique eigenvalue $\sigma_1^{\Omega_0}[\mathscr{L} + q, \mathscr{B}] \in \mathbb{R}$ associated with a positive eigenfunction φ_1. Moreover, φ_1 is unique up to scalar multiplication, $\varphi_1(x) > 0$ for all $x \in \Omega_0$ and $\partial_\nu \varphi_1(x) < 0$ for all $x \in \partial\Omega_0$ in the Dirichlet case. As usual, we shall call $\sigma_1^{\Omega_0}[\mathscr{L} + q, \mathscr{B}]$ the *principal eigenvalue* of (2.1) and φ_1 its associated *principal eigenfunction.* In [2] the existence of a principal eigenvalue is shown for a general class of elliptic operators on arbitrary domains.

Remark 2.1. We now state the properties of the principal eigenvalue used in this paper. In the sequel $q, q_1, q_2, ...$ stand for functions of $C^\mu(\bar{\Omega})$ and $\Omega_0, \Omega_1, ...$ are subdomains of Ω of class $C^{2+\mu}$.

(a) Monotonicity with respect to the potential: $\sigma_1^{\Omega_0}[\mathscr{L} + q_1, \mathscr{B}] < \sigma_1^{\Omega_0}[\mathscr{L} + q_2, \mathscr{B}]$, whenever $q_1 < q_2$.

(b) Continuous dependence with respect to the potential: $\sigma_1^{\Omega_0}[\mathscr{L} + q_n, \mathscr{B}] \to \sigma_1^{\Omega_0}[\mathscr{L} + q, \mathscr{B}]$, whenever $q_n \to q$ in $C(\bar{\Omega})$.

(c) Concavity: The mapping $\lambda \mapsto \sigma_1^{\Omega_0}[\mathscr{L} + \lambda q, \mathscr{B}]: \mathbb{R} \to \mathbb{R}$ is concave and analytic.

(d) Domination of Dirichlet over Robin and Neumann eigenvalues:

$$\sigma_1^{\Omega_0}[\mathscr{L} + q, \mathscr{R}] < \sigma_1^{\Omega_0}[\mathscr{L} + q, \mathscr{D}].$$

(e) Monotonicity with respect to the domain for Dirichlet eigenvalues: $\sigma_1^{\Omega_0}[\mathscr{L} + q, \mathscr{D}] < \sigma_1^{\Omega_1}[\mathscr{L} + q, \mathscr{D}]$, whenever Ω_1 is a proper subdomain of Ω_0.

(f) For each $M > 0$ there exists $\varepsilon > 0$ such that $\sigma_1^{\Omega_0}[\mathscr{L} + q, \mathscr{D}] > M$ if $|\Omega_0| < \varepsilon$, where $|\cdot|$ denotes the Lebesgue measure of $\mathbb{R}^N$. In particular, the strong maximum principle holds provided $|\Omega_0|$ is sufficiently small.

(g) Continuous variation of the principal eigenvalue with respect to the domain for Dirichlet boundary conditions: We say that $\lim_{k\to\infty}\Omega_k=\Omega_0$ if the following two conditions are satisfied:

(i) There exists a sequence Ω_k^I, $k\geqslant 1$, of subdomains of Ω with boundaries of class $C^{2+\nu}$ such that

$$\Omega_k^I\subset\Omega_{k+1}^I,\qquad \Omega_k^I\subset\Omega_0\cap\Omega_k,\qquad k\geqslant 1,$$

and

$$\bigcup_{k=1}^{\infty}\Omega_k^I=\Omega_0.$$

(ii) There exists a sequence Ω_k^E, $k\geqslant 1$, of subdomains of Ω with boundaries of class $C^{2+\nu}$ such that

$$\Omega_{k+1}^E\subset\Omega_k^E,\qquad \Omega_0\cup\Omega_k\subset\Omega_k^E,\qquad k\geqslant 1,$$

and

$$\bigcap_{k=1}^{\infty}\bar{\Omega}_k^E\subset\bar{\Omega}_0.$$

Suppose $a_{ij}\in C^1(\bar{\Omega})$ and $a_j\in C^1(\bar{\Omega})$ for all i,j. Assume in addition that

$$\lim_{k\to\infty}\Omega_k=\Omega_0.$$

Then,

$$\lim_{k\to\infty}\sigma_1^{\Omega_k}[\mathscr{L},\mathscr{D}]=\sigma_1^{\Omega_0}[\mathscr{L},\mathscr{D}].$$

Properties (a)–(e) are well known and can be found in [6]. Property (f) can be found in [2]. A proof of (g) can be found in [10]. When $\mathscr{L}$ is selfadjoint then the continuous variation of the principal eigenvalue with respect to the domain is a classical property which goes back to Courant and Hilbert, [4].

The principal eigenvalue of (2.1) is intimately connected with the stability properties of the zero solution of the parabolic problem

$$\begin{cases}\partial_t u+\mathscr{L}(x,D)\,u+q(x)\,u=0 & \text{in}\quad \Omega_0\times(0,\infty),\\ \mathscr{B}(x,D)\,u=0 & \text{on}\quad \partial\Omega_0\times(0,\infty).\end{cases}\tag{2.2}$$

Let $1<p<\infty$ be and let A_q denote the $L_p(\Omega_0)$-realization of the elliptic boundary value problem $(\mathscr{L}+q, \mathscr{B})$. Recall that $-A_q$ is the infinitesimal generator of an analytic C_0-semigroup of strongly positive compact operators on $L_p(\Omega_0)$ and that its *exponential type* $\omega(A_q)$ is given by

$$\omega(A_q) := \sup\{\omega \in \mathbb{R} \mid \exists M>0\colon \|e^{-tA_q}\|_{\mathscr{L}(L_p)} \leqslant Me^{-t\omega} \text{ for } t \geqslant 0\}.$$

So, $(e^{-tA_q})_{t\geqslant 0}$ is exponentially stable if and only if $\omega(A_q)>0$. The exponential type of the semigroup $(e^{-tA_q})_{t\geqslant 0}$ may be characterized as

$$\omega(A_q) = \sigma_1^{\Omega_0}[\mathscr{L}+q, \mathscr{B}]. \tag{2.3}$$

Therefore, $\sigma_1^{\Omega_0}[\mathscr{L}+q, \mathscr{B}]$ measures the degree of stability of $(e^{-tA_q})_{t\geqslant 0}$. As a consequence of (2.3), the following result characterizes the exponential stability of $(e^{-tA_q})_{t\geqslant 0}$ by means of supersolutions.

LEMMA 2.2. *The following statements are equivalent.*

(a) $\sigma_1^{\Omega_0}[\mathscr{L}+q, \mathscr{B}]>0$.

(b) *There exists a function* $\bar{u} \in C^{2+\mu}(\bar{\Omega}_0)$, $\bar{u}>0$, *such that*

$$\begin{cases} \mathscr{L}(x, D)\,\bar{u} + q(x)\,\bar{u} \geqslant 0 & \text{in } \Omega_0, \\ \mathscr{B}(x, D)\,\bar{u} \geqslant 0 & \text{on } \partial\Omega_0, \end{cases} \tag{2.4}$$

with at least one of these inequalities strict. In other words, $\bar{u}$ *is a positive strict supersolution of*

$$\begin{cases} \mathscr{L}(x, D)\,u + q(x)\,u = 0 & \text{in } \Omega_0, \\ \mathscr{B}(x, D)\,u = 0 & \text{on } \partial\Omega_0. \end{cases} \tag{2.5}$$

Proof. If (a) holds, then the principal eigenfunction $\bar{u}$ associated with $\sigma_1^{\Omega_0}[\mathscr{L}+q, \mathscr{B}]$ satisfies (2.4). Now, assume (b) and let $\sigma \leqslant 0$ be arbitrary. Then, for any $\alpha>0$ the function $\alpha\bar{u}$ is a strict positive supersolution of

$$\begin{cases} \mathscr{L}(x, D)\,u + q(x)\,u = \sigma u & \text{in } \Omega_0, \\ \mathscr{B}(x, D)\,u = 0 & \text{on } \partial\Omega_0. \end{cases} \tag{2.6}$$

By Serrin's sweeping principle, (2.6) does not admit a positive solution. Therefore, $\sigma_1^{\Omega_0}[\mathscr{L}+q, \mathscr{B}]>0$. ∎

Remark 2.3. A similar argument shows that $\sigma_1^{\Omega_0}[\mathscr{L}+q, \mathscr{B}]<0$ if and only if (2.5) admits a positive strict subsolution. A subsolution is defined as a supersolution by just reversing the inequality signs in (2.4).

Now, suppose $q_k \in C^\mu(\bar{\Omega})$, $k \geqslant 1$, is an increasing sequence of nonnegative functions. By monotonicity, the mapping

$$k \mapsto \sigma_1^{\Omega_0}[\mathscr{L} + q_k, \mathscr{B}]$$

is also increasing and therefore the degree of stability of the semigroups $(e^{-tA_{q_k}})_{t \geqslant 0}$ increases with k. We now analyze how stable these semigroups can get, generalizing some previous results in [9].

THEOREM 2.4. *Let (q_k) be an increasing sequence of nonnegative functions in $C^\mu(\bar{\Omega})$ and $n \geqslant 1$ an integer number. Assume that $\Omega_1, ..., \Omega_n$ are $C^{2+\mu}$-subdomains of Ω such that $\bar{\Omega}_1, ..., \bar{\Omega}_n$ are pair-wise disjoint and contained in Ω. Moreover, suppose that*

$$q_k \equiv 0 \qquad on \quad \bigcup_{i=1}^{n} \Omega_i \tag{2.7}$$

and that

$$\lim_{k \to \infty} \min_{x \in K} q_k(x) = \infty, \tag{2.8}$$

for any compact subset K of $\bar{\Omega} \setminus \bigcup_{i=1}^n \Omega_i$. Then,

$$\sigma_1^{\Omega_0}[\mathscr{L} + q_k, \mathscr{B}] \nearrow \min_{1 \leqslant i \leqslant n} \sigma_1^{\Omega_i}[\mathscr{L}, \mathscr{D}] \tag{2.9}$$

as k tends to infinity.

Proof. To keep the notation whithin reasonable bounds we only prove the case $n = 2$. Without loss of generality we can assume that

$$\sigma_1^{\Omega_1}[\mathscr{L}, \mathscr{D}] \leqslant \sigma_1^{\Omega_2}[\mathscr{L}, \mathscr{D}].$$

By the various properties of the principal eigenvalue listed in Remark 2.1 and by (2.7) we have

$$\sigma_1^{\Omega}[\mathscr{L} + q_k, \mathscr{B}] \leqslant \sigma_1^{\Omega}[\mathscr{L} + q_k, \mathscr{D}] \leqslant \sigma_1^{\Omega_1}[\mathscr{L} + q_k, \mathscr{D}] = \sigma_1^{\Omega_1}[\mathscr{L}, \mathscr{D}].$$

Thus, $\lim_{k \to \infty} \sigma_1^{\Omega}[\mathscr{L} + q_k, \mathscr{B}]$ exists and lies below $\sigma_1^{\Omega_1}[\mathscr{L}, \mathscr{D}]$. It suffices to show that for any $\varepsilon > 0$ there exists $k_0 \geqslant 1$ such that

$$0 \leqslant \sigma_1^{\Omega_1}[\mathscr{L}, \mathscr{D}] - \sigma_1^{\Omega}[\mathscr{L} + q_k, \mathscr{B}] < \varepsilon \tag{2.10}$$

for all $k \geqslant k_0$. Fix $\varepsilon > 0$. By the continuous domain dependence and domain monotonicity of the Dirichlet principal eigenvalue for $i = 1, 2$ there exist $C^{2+\mu}$-subdomains Ω_i^ε containing Ω_i such that

$$\begin{gathered} \bar{\Omega}_1^\varepsilon \cup \bar{\Omega}_2^\varepsilon \subset \Omega, \qquad \bar{\Omega}_1^\varepsilon \cap \bar{\Omega}_2^\varepsilon = \emptyset, \\ \sigma_1^{\Omega_1^\varepsilon}[\mathscr{L}, \mathscr{D}] \leqslant \sigma_1^{\Omega_2^\varepsilon}[\mathscr{L}, \mathscr{D}], \end{gathered} \tag{2.11}$$

and

$$\sigma_1^{\Omega_i^\varepsilon}[\mathscr{L}, \mathscr{D}] \leqslant \sigma_1^{\Omega_i}[\mathscr{L}, \mathscr{D}] \leqslant \sigma_1^{\Omega_i^\varepsilon}[\mathscr{L}, \mathscr{D}] + \varepsilon, \tag{2.12}$$

for $i = 1, 2$. Let φ_i be the principal eigenfunction associated with $\sigma_1^{\Omega_i^\varepsilon}[\mathscr{L}, \mathscr{D}]$, which is unique up to positive multiplicative constants. By definition,

$$\begin{cases} \mathscr{L}(x, D)\, \varphi_i = \sigma_1^{\Omega_i^\varepsilon}[\mathscr{L}, \mathscr{D}]\, \varphi_i & \text{in } \Omega_i^\varepsilon, \\ \varphi_i = 0 & \text{on } \partial\Omega_i^\varepsilon, \end{cases} \tag{2.13}$$

for $i = 1, 2$. We now choose two $C^{2+\mu}$-subdomains, Ω_1^* and Ω_2^*, such that

$$\bar{\Omega}_i \subset \Omega_i^* \subset \bar{\Omega}_i^* \subset \Omega_i^\varepsilon$$

for $i = 1, 2$ and take any strictly positive function $\bar{u} \in C^{2+\mu}(\bar{\Omega})$ satisfying

$$\bar{u} = \varphi_i \qquad \text{in } \Omega_i^* \tag{2.14}$$

for $i = 1, 2$, and

$$\mathscr{B}(x, D)\, \bar{u} > 0 \qquad \text{on } \partial\Omega. \tag{2.15}$$

As $\bar{u} > 0$, it follows from (2.12) that

$$\mathscr{L}(x, D)\, \bar{u} + q_k(x)\, \bar{u} + (\varepsilon - \sigma_1^{\Omega_1}[\mathscr{L}, \mathscr{D}])\, \bar{u} > f_k(x) \qquad \text{in } \Omega, \tag{2.16}$$

where

$$f_k(x) := (\mathscr{L}(x, D) - \sigma_1^{\Omega_1^\varepsilon}[\mathscr{L}, \mathscr{D}])\, \bar{u}(x) + q_k(x)\, \bar{u}(x),$$

for $x \in \bar{\Omega}$ and $k \geqslant 0$. Moreover, since $q_k > 0$ we find from (2.13) and (2.14) that

$$f_k \geqslant 0 \qquad \text{in} \quad \Omega_1^* \cup \Omega_2^*, \qquad k \geqslant 1.$$

On the other hand, (2.8) implies that there exists $k_0 \geqslant 1$ such that

$$f_k \geqslant 0 \qquad \text{in} \quad K := \bar{\Omega} \backslash (\Omega_1^* \cup \Omega_2^*)$$

for all $k \geqslant 1$. Thus, $f_k \geqslant 0$ in $\bar{\Omega}$ for any $k \geqslant k_0$ and hence

$$\begin{cases} \mathscr{L}(x, D)\, \bar{u} + q_k(x)\, \bar{u} + (\varepsilon - \sigma_1^{\Omega_1}[\mathscr{L}, \mathscr{D}])\, \bar{u} > 0 & \text{in } \Omega, \\ \mathscr{B}(x, D)\, \bar{u} > 0 & \text{on } \partial\Omega, \end{cases}$$

for all $k \geqslant k_0$. Finally, it follows from Lemma 2.2 that

$$\sigma_1^{\Omega}[\mathscr{L} + q_k + \varepsilon - \sigma_1^{\Omega_1}[\mathscr{L}, \mathscr{D}], \mathscr{B}] > 0, \qquad k \geqslant k_0.$$

From this relation (2.9) follows readily. The proof is completed. ▮

In the previous theorem assumption (2.8) does not allow q_k to vanish somewhere on $\partial\Omega$. The next result shows that (2.8) can be weakened to cover the case where q_k may vanish on $\partial\Omega$; at least when dealing with Dirichlet boundary conditions.

Corollary 2.5. *Theorem* 2.4 *remains valid if* $\mathscr{B} = \mathscr{D}$ *and we require that* (2.8) *holds for compact subsets* K *of* $\Omega \setminus \bigcup_{i=1}^{n} \Omega_i$. *In particular,* q_k *may vanish on* $\partial\Omega$.

Proof. We keep the notations of the proof of Theorem 2.4 and without loss of generality we restrict ourselves to the case $N = 2$. It is easy to see that it suffices to prove the theorem in the case when $q_k \equiv 0$ on $\partial\Omega$ for k sufficiently large. Note that we can extend, if necessary, the coefficients of $\mathscr{L}$ to the whole of $\mathbb{R}^N$ so that $\mathscr{L}(x, D)$ be defined for all $x \in \mathbb{R}^N$. Let Ω_1^e and Ω_2^e be two bounded $C^{2+\mu}$-subdomains of $\mathbb{R}^N$ such that

$$\bar{\Omega} \subset \Omega_1^e \subset \bar{\Omega}_1^e \subset \Omega_2^e$$

and

$$\sigma_1^{\Omega_3}[\mathscr{L}, \mathscr{D}] \geqslant \sigma_1^{\Omega_1}[\mathscr{L}, \mathscr{D}],$$

where $\Omega_3 := \Omega_1^e \setminus \bar{\Omega}$. This can be accomplished by taking Ω_1^e sufficiently close to Ω so that the Lebesgue measure of Ω_3 is sufficiently small.

After choosing a suitable extension we may assume that $q_k \in C^{\mu}(\bar{\Omega}_2^e)$, that $q_k \equiv 0$ on Ω_3 and that (2.8) holds for any compact subset K of $\bar{\Omega}_2^e \setminus (\Omega_1 \cup \Omega_2 \cup \Omega_3)$. Hence, we can apply Theorem 2.4 to obtain

$$\sigma_1^{\Omega_2^e}[\mathscr{L} + q_k, \mathscr{D}] \nearrow \sigma_1^{\Omega_1}[\mathscr{L}, \mathscr{D}],$$

as $k \to \infty$. On the other hand,

$$\sigma_1^{\Omega_2^e}[\mathscr{L} + q_k, \mathscr{D}] \leqslant \sigma_1^{\Omega}[\mathscr{L} + q_k, \mathscr{D}]$$

for all $k \geqslant 1$. The proof is completed. ▮

The following theorem complements the two previous results.

THEOREM 2.6. *Let (q_k) be a sequence of nonnegative functions in $C^{\mu}(\bar{\Omega})$ such that*

$$\lim_{k \to \infty} \min_{x \in \bar{\Omega}} q_k(x) = \infty. \tag{2.17}$$

Then,

$$\sigma_1^{\Omega}[\mathscr{L} + q_k, \mathscr{B}] \nearrow \infty$$

as k tends to infinity. In case of Dirichlet boundary conditions we may replace (2.17) *by*

$$\lim_{k \to \infty} \min_{x \in K} q_k(x) = \infty \tag{2.18}$$

for any compact subset K of Ω.

Proof. Set $c_k := \min_{x \in \bar{\Omega}} q_k(x)$. Then $c_k \nearrow \infty$ and

$$\sigma_1^{\Omega}[\mathscr{L} + q_k, \mathscr{B}] \geqslant \sigma_1^{\Omega}[\mathscr{L} + c_k, \mathscr{B}] = \sigma_1^{\Omega}[\mathscr{L}, \mathscr{B}] + c_k,$$

which yields the assertion. When condition (2.18) holds we may proceed as in the proof of Corollary 2.5. ▮

3. POSITIVE SOLUTIONS OF THE LOGISTIC EQUATION

Let Ω, $\mathscr{L}$ and $\mathscr{B}(x, D)$ be as in Section 2. In this section we study the problem of existence, uniqueness and stability of positive solutions of

$$\begin{cases} \mathscr{L}(x, D)\, w = m(x)\, w - a(x)\, wh(x, w) & \text{in } \Omega, \\ \mathscr{B}(x, D)\, w = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where we assume

$$a, m \in C^{\mu}(\bar{\Omega}),\ a \geqslant 0 \qquad \text{and} \qquad a \neq 0. \tag{A1}$$

The function $h\colon \bar{\Omega} \times [0, \infty) \to \mathbb{R}$ is of class $C^{\mu, 1+\mu}$ and satisfies $h(x, 0) \equiv 0$ and $h(x, w) > 0$, $\partial_w h(x, w) > 0$ for all $w > 0$, $x \in \Omega$. Moreover, $\lim_{\xi \to \infty} h(x, \xi) = \infty$ for each $x \in \Omega$. (A2)

Under these assumptions any solution of (3.1) lies in the Banach space

$$X := \{ w \in C^{2+\mu}(\bar{\Omega}) \mid \mathscr{B}(x, D)\, w \equiv 0 \}.$$

If $w \in X \setminus \{0\}$ is a nonnegative solution of (3.1), then it follows from the maximum principle that w lies in the interior of the cone X^+ of non-negative functions in X. The next result shows the uniqueness and stability of positive solutions of (3.1).

LEMMA 3.1. *Problem* (3.1) *admits at most one positive solution. Moreover, any positive solution* w *of* (3.1) *satisfies*

$$\sigma_1^\Omega[\mathscr{L} - m + aw\partial_w h(\cdot, w) + ah(\cdot, w), \mathscr{B}] > 0.$$

In particular, any positive solution of (3.1) *is linearly stable and therefore non-degenerate.*

Proof. Let w be a positive solution of (3.1). Then, we find from the uniqueness of the principal eigenvalue that

$$\sigma_1^\Omega[\mathscr{L} - m + ah(\cdot, w), \mathscr{B}] = 0.$$

Moreover, it follows from (A2) that $aw\partial_w h(\cdot, w) > 0$ and hence

$$0 = \sigma_1^\Omega[\mathscr{L} - m + ah(\cdot, w), \mathscr{B}] < \sigma_1^\Omega[\mathscr{L} - m + aw\partial_w h(\cdot, w) + ah(\cdot, w), \mathscr{B}].$$

This relation entails the linear stability of w.

We now show the uniqueness of positive solutions. We will argue by contradiction. Let $w_1 \neq w_2$ be two positive solutions of (3.1). Then, it follows easily that $w_1 + w_2$ is a supersolution of (3.1) lying above w_1 and w_2 and hence the method of sub and supersolutions shows that we can assume without loss of generality that $w_1 < w_2$. On the other hand, since the mapping $w \mapsto h(\cdot, w)$ is increasing we find that

$$0 = \sigma_1^\Omega[\mathscr{L} - m + ah(\cdot, w_1), \mathscr{B}] < \sigma_1^\Omega[\mathscr{L} - m + ah(\cdot, w_2), \mathscr{B}] = 0.$$

This contradiction completes the proof of the uniqueness. ▮

The next result provides us with a necessary condition for the existence of positive solutions of (3.1). We will show that, in general, this condition is not sufficient.

LEMMA 3.2. *If* $\sigma_1^\Omega[\mathscr{L} - m, \mathscr{B}] \geqslant 0$, *then* (3.1) *does not admit a positive solution.*

Proof. Suppose (3.1) admits a positive solution w. Then, it follows from $ah(\cdot, w) > 0$ that

$$\sigma_1^\Omega[\mathscr{L} - m, \mathscr{B}] < \sigma_1^\Omega[\mathscr{L} - m + ah(\cdot, w), \mathscr{B}] = 0,$$

which is the contrary of the assumption. The proof is completed. ▮

We now show that (3.1) admits arbitrarily small positive subsolutions if $\sigma_1^{\Omega}[\mathcal{L}-m, \mathcal{B}]<0$.

LEMMA 3.3. *If the zero solution of* (3.1) *is linearly unstable, i.e. if*

$$\sigma_1^{\Omega}[\mathcal{L}-m, \mathcal{B}]<0,$$

then (3.1) *admits arbitrarily small subsolutions in* $int(X^+)$.

Proof. Let φ_1 be the principal eigenfunction corresponding to $\sigma_1^{\Omega}[\mathcal{L}-m, \mathcal{B}]$. It is easy to check that $\underline{w}_\varepsilon := \varepsilon\varphi$ is a subsolution of (3.1) if $\varepsilon>0$ is sufficiently small. ∎

The following lemma characterizes in various ways the existence of positive solutions of (3.1).

LEMMA 3.4. *Suppose* $\sigma_1^{\Omega}[\mathcal{L}-m, \mathcal{B}]<0$. *Then, the following statements are equvalent.*

(a) *Problem* (3.1) *admits a positive solution.*

(b) *Problem* (3.1) *admits a positive supersolution.*

(c) *Problem* (3.1) *admits arbitrarily large positive supersolutions.*

(d) *There exists* $v\in X$, $v>0$, *such that* $\sigma_1^{\Omega}[\mathcal{L}-m+ah(\cdot, v), \mathcal{B}]\geqslant 0$.

Proof. The equivalence of (a) and (b) is evident in the light of Lemma 3.3. The equivalence of (b) and (c) follows from the fact that if $\bar{w}$ is a supersolution, then for each $\kappa>1$ the function $\kappa\bar{w}$ is also a supersolution. Since for every positive solution w we have $\sigma_1^{\Omega}[\mathcal{L}-m+ah(\cdot, w), \mathcal{B}]=0$, (a) implies (d). It remains to show that (d) implies (c). Suppose d) holds and let φ_1 be the principal eigenfunction associated with $\sigma_1^{\Omega}[\mathcal{L}-m+ah(\cdot, v), \mathcal{B}]$. If $\kappa>0$ is large enough so that $\kappa\varphi_1\geqslant v$, then $\kappa\varphi_1$ is easily seen to be a supersolution of (3.1). This implies (c) and completes the proof. ∎

Before giving our main existence theorem, we make the following structural assumption on the damping coefficient $a(x)$, which we assume to hold henceforth.

Ω_0 is a possibly void $C^{2+\mu}$-subdomain of Ω such that $\bar{\Omega}_0\subset\Omega$ and $\bar{\Omega}_0=\{x\in\bar{\Omega}\mid a(x)=0\}$. In case of Dirichlet boundary conditions, i.e. when $\mathcal{B}=\mathcal{D}$, we only require $\Omega_0\subset\Omega$. Hence, in this case we allow a to vanish on $\partial\Omega$. (A3)

THEOREM 3.5. *Suppose* (A1)–(A3). *Then,* (3.1) *admits a positive solution if and only if*

$$\sigma_1^{\Omega}[\mathscr{L}-m, \mathscr{B}]<0<\sigma_1^{\Omega_0}[\mathscr{L}-m, \mathscr{D}], \tag{3.2}$$

where we set $\sigma_1^{\Omega_0}[\mathscr{L}-m, \mathscr{D}]=\infty$ *if* $\Omega_0=\varnothing$.

Proof. We have already seen in Lemma 3.2 that condition $\sigma_1^{\Omega}[\mathscr{L}-m, \mathscr{B}]<0$ is necessary for the existence of a positive solution. Suppose now that w is a positive solution of (3.1). Then, it follows from Remark 2.1 that

$$\begin{aligned} 0=\sigma_1^{\Omega}[\mathscr{L}-m+ah(\cdot, w), \mathscr{B}] &\leqslant \sigma_1^{\Omega}[\mathscr{L}-m+ah(\cdot, w), \mathscr{D}] \\ &<\sigma_1^{\Omega_0}[\mathscr{L}-m+ah(\cdot, w), \mathscr{D}], \end{aligned}$$

since Ω_0 is a proper subdomain of Ω, because $a\neq 0$. Moreover, $a\equiv 0$ in Ω_0. Therefore,

$$0<\sigma_1^{\Omega_0}[\mathscr{L}-m, \mathscr{D}].$$

This shows that (3.2) is necessary for the existence of a positive solution. We finally show that (3.2) is sufficient for the existence of a positive solution. Suppose (3.2). Then, by (A3) we may choose a sequence $v_k\in X$, $v_k>0$, such that

$$\lim_{k\to\infty} \min_{x\in K} a(x)\, h(x, v_k(x))=\infty$$

for every compact subset $K\subset\bar{\Omega}\backslash\Omega_0$, in the general case, or $K\subset\Omega\backslash\Omega_0$ in the Dirichlet case. Now, by Theorem 2.4 or Corollary 2.5 we obtain

$$\sigma_1^{\Omega}[\mathscr{L}-m+a(x)\, h(x, v_k(x)), \mathscr{B}]\nearrow\sigma_1^{\Omega_0}[\mathscr{L}-m, \mathscr{D}]$$

as $k\to\infty$. Since $\sigma_1^{\Omega_0}[\mathscr{L}-m, \mathscr{D}]>0$, we find that for k large enough

$$\sigma_1^{\Omega}[\mathscr{L}-m+a(x)\, h(x, v_k(x)), \mathscr{B}]>0.$$

The equivalence of (a) and (d) in Lemma 3.4 yields the existence of a positive solution of (3.1). The proof is completed. ∎

Remark 3.6. If we allow the damping coefficient function a to vanish in several $C^{2+\mu}$-subdomains $\Omega_1, ..., \Omega_n$ of Ω, as in Theorem 2.4 and Corollary 2.5, then Theorem 3.5 remains valid if we replace (3.2) by

$$\sigma_1^{\Omega}[\mathscr{L}-m, \mathscr{B}]<0<\min_{1\leqslant i\leqslant n} \sigma_1^{\Omega_i}[\mathscr{L}-m, \mathscr{D}].$$

We now give a complete description of the longtime behaviour of the positive solutions of the parabolic evolutionary problem

$$\begin{cases} \partial_t w + \mathscr{L} w = m(x)\, w - a(x)\, wh(x, w) & \text{in} \quad \Omega \times (0, \infty), \\ \mathscr{B} w = 0 & \text{on} \quad \partial\Omega \times (0, \infty), \\ w(\cdot, 0) = u_0 & \text{on } \bar{\Omega}. \end{cases} \tag{3.3}$$

Given $u_0 \in X^+$ it is well known that (3.3) admits a unique classical solution $w(x, t, u_0)$. A priori this solution exists only for small times, but since the nonlinearity we are dealing with is sublinear we have in fact global solutions in time. We shall say that a positive steady state w_0 of (3.3), i.e. a positive solution of (3.1), is *globally asymptotically stable* if

$$\lim_{t \to \infty} \| w(x, t, u_0) - w_0 \|_{C(\bar{\Omega})} = 0$$

for each $u_0 \in X^+ \backslash \{0\}$. Our result on the longtime behaviour of solutions of (3.3) reads as follows.

THEOREM 3.7. *Suppose* (A1) *and* (A2). *Then, the following assertions are true.*

(a) *If* $\sigma_1^{\Omega}[\mathscr{L} - m, \mathscr{B}] \geqslant 0$, *then the zero solution of* (3.3) *is globally asymptotically stable.*

(b) *If* $\sigma_1^{\Omega}[\mathscr{L} - m, \mathscr{B}] < 0$ *and there exists a positive steady* w_0 *of* (3.3), *then* w_0 *is globally asymptotically stable.*

(c) *If* $\sigma_1^{\Omega}[\mathscr{L} - m, \mathscr{B}] < 0$ *and* (3.3) *does not admit a positive steady state, then*

$$\lim_{t \to \infty} \| w(\cdot, t, u_0) \|_{C(\bar{\Omega})} = \infty$$

for each $u_0 \in X^+ \backslash \{0\}$.

Proof. (a) Let φ_1 be the principal eigenfunction corresponding to $\sigma_1^{\Omega}[\mathscr{L} - m, \mathscr{B}]$. Since $\sigma_1^{\Omega}[\mathscr{L} - m, \mathscr{B}] \geqslant 0$ holds, $\kappa\varphi_1$ is a supersolution of (3.1) if $\kappa > 0$. As zero is the only non-negative steady state of (3.3) and, due to the results in [1] and [15], the function $t \mapsto w(\cdot, t, \kappa\varphi_1)$ is decreasing and converges to a steady state of (3.3), we find that

$$\lim_{t \to \infty} \| w(\cdot, t, \kappa\varphi_1) \|_{C(\bar{\Omega})} = 0. \tag{3.4}$$

Let $u_0 \in X^+ \backslash \{0\}$. Then, there exists $\kappa > 0$ such that $0 \leqslant u_0 < \kappa\varphi_1$. Moreover, for such κ we have $0 \leqslant w(\cdot, t, u_0) \leqslant w(\cdot, t, \kappa\varphi_1)$ for all $t \geqslant 0$, due to the parabolic maximum principle. This estimate together with (3.4) proves assertion (a).

(b) By Lemma 3.3 and Lemma 3.4 we get arbitrarily small subsolutions and arbitrarily large supersolutions of (3.1). The uniqueness of the positive solution of (3.1) combined with a monotonicity argument as in part a) yields the assertion.

(c) By Lemma 3.4 we find that

$$\sigma_1^{\Omega}[\mathscr{L} - m + ah(\cdot, v), \mathscr{B}] < 0$$

for each $v \in X$, $v > 0$. Thus, the semigroup generated by $m - ah(\cdot, v) - \mathscr{L}$ is unstable and hence

$$\lim_{t \to \infty} \|e^{t(m - ah(\cdot, v) - \mathscr{L})} u_0\|_{C(\bar{\Omega})} = \infty \tag{3.5}$$

for any $u_0 \in X^+ \backslash \{0\}$. To complete the proof we argue by contradiction. Suppose there is $u_0 \in X^+ \backslash \{0\}$ such that

$$\|w(\cdot, t, u_0)\|_{C(\bar{\Omega})} \leqslant c_0 < \infty$$

for $t \geqslant 0$. Then, by the results in [14] we find that

$$\|w(\cdot, t, u_0)\|_{C^1(\bar{\Omega})} \leqslant c_1 < \infty$$

for all $t \geqslant 0$, where $c_1 > 0$ is constant. This estimate implies the existence of $v \in \operatorname{int}(X^+)$ such that

$$0 \leqslant w(\cdot, t, u_0) \leqslant v$$

for $t \geqslant 0$. Thus,

$$\begin{aligned} \partial_t w(\cdot, \cdot, u_0) &+ \mathscr{L} w(\cdot, \cdot, u_0) \\ &= m w(\cdot, \cdot, u_0) - ah(\cdot, w(\cdot, \cdot, u_0))\, w(\cdot, \cdot, u_0) \\ &\geqslant m w(\cdot, \cdot, u_0) - ah(\cdot, v)\, w(\cdot, \cdot, u_0) \qquad \text{in} \quad \Omega \times (0, \infty), \end{aligned}$$

together with the parabolic comparison principle imply that

$$e^{t(m - ah(\cdot, v) - \mathscr{L})} u_0 \leqslant w(\cdot, t, u_0)$$

for $t \geqslant 0$. Therefore, due to (3.5) we obtain

$$\lim_{t \to \infty} \|w(\cdot, t, u_0)\|_{C(\bar{\Omega})} = \infty.$$

This contradiction completes the proof. ∎

4. Unbounded Curves of Positive Solutions

We now study the parameter dependent elliptic logistic equation

$$\begin{cases} \mathscr{L}(x, D)\, w = \lambda m(x)\, w - a(x)\, wh(x, w) & \text{in } \Omega, \\ \mathscr{B}(x, D)\, w = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where Ω, $\mathscr{L}(x, D)$, $\mathscr{B}(x, D)$ are as in the previous section and a, m and h satisfy (A1)–(A3). We assume that the parameter λ varies over the real axis. Set

$$\Lambda := \{\lambda \in \mathbb{R} \mid (4.1) \text{ admits a positive solution}\}.$$

By the results in Section 3 for each $\lambda \in \Lambda$ problem (4.1) has a unique positive solution, which we shall denote by w_λ. Set

$$\Sigma := \{(\lambda, w_\lambda) \mid \lambda \in \Lambda\}.$$

The main goal of this section is to clarify the structure of Σ. We first introduce some notation. Given a $C^{2+\mu}$-subdomain $\tilde{\Omega}$ of Ω and a boundary operator $\tilde{\mathscr{B}}$ we shall set

$$\sigma_{\tilde{\Omega}, \tilde{\mathscr{B}}}(\lambda) := \sigma_1^{\tilde{\Omega}}[\mathscr{L} - \lambda m, \tilde{\mathscr{B}}]$$

for any $\lambda \in \mathbb{R}$, with the understanding that $\sigma_{\tilde{\Omega}, \tilde{\mathscr{B}}} \equiv \infty$ if $\tilde{\Omega} = \emptyset$ (this is consistent with the notation used in Theorem 3.5).

Recall that Ω_0 is the proper subdomain of Ω defined in (A3). By Remark 2.1 we have

$$\sigma_{\Omega, \mathscr{B}}(\lambda) < \sigma_{\Omega_0, \mathscr{D}}(\lambda) \tag{4.2}$$

for all $\lambda \in \mathbb{R}$. The importance of $\sigma_{\Omega, \mathscr{B}}$ and $\sigma_{\Omega_0, \mathscr{D}}$ is made evident by the following proposition, which is an immediate consequence of Theorem 3.5 and Lemma 3.1.

Proposition 4.1. *The set Λ is given by*

$$\Lambda = \{\lambda \in \mathbb{R} \mid \sigma_{\Omega, \mathscr{B}}(\lambda) < 0 < \sigma_{\Omega_0, \mathscr{D}}(\lambda)\}$$

and the mapping $\lambda \mapsto w_\lambda$ is C^1 from Λ into X.

This result shows that the structure of Λ will be revealed by the graphs of $\sigma_{\Omega, \mathscr{B}}$ and $\sigma_{\Omega_0, \mathscr{D}}$. The qualitative behaviour of these functions is well understood (see [8] and [6] Section 15).

Remark 4.2. If $m>0$ then it follows from the maximum principle that the mapping $\lambda \mapsto \|w_\lambda\|_\infty : \Lambda \to \mathbb{R}^+$ is increasing. This mapping is decreasing if $m<0$.

LEMMA 4.3. *For any $C^{2+\mu}$-subdomain $\tilde{\Omega}$ of Ω and any boundary operator $\tilde{\mathscr{B}}$ on $\partial\tilde{\Omega}$ the function $\sigma_{\tilde{\Omega}, \tilde{\mathscr{B}}} : \mathbb{R} \to \mathbb{R}$ is analytic and concave. Moreover, setting*

$$\mathscr{N}(\tilde{\Omega}) := \inf_{x \in \tilde{\Omega}} m(x) \qquad \textit{and} \qquad \mathscr{P}(\tilde{\Omega}) := \sup_{x \in \tilde{\Omega}} m(x)$$

we have

(a) $\mathscr{P}(\tilde{\Omega}) > 0 \Rightarrow \lim_{\lambda \to \infty} \sigma_{\tilde{\Omega}, \tilde{\mathscr{B}}}(\lambda) = -\infty$.

(b) $\mathscr{N}(\tilde{\Omega}) < 0 \Rightarrow \lim_{\lambda \to -\infty} \sigma_{\tilde{\Omega}, \tilde{\mathscr{B}}}(\lambda) = -\infty$.

(c) $\sigma_{\tilde{\Omega}, \tilde{\mathscr{B}}}(\lambda) \geqslant \sigma_{\tilde{\Omega}, \tilde{\mathscr{B}}}(0) - \lambda \mathscr{P}(\tilde{\Omega})$ *for* $\lambda > 0$.

(d) $\sigma_{\tilde{\Omega}, \tilde{\mathscr{B}}}(\lambda) \geqslant \sigma_{\tilde{\Omega}, \tilde{\mathscr{B}}}(0) - \lambda \mathscr{N}(\tilde{\Omega})$ *for* $\lambda < 0$.

From the concavity of $\sigma_{\Omega, \mathscr{B}}$ and $\sigma_{\Omega_0, \mathscr{D}}$ as well as (4.2) we obtain the following result on the structure of Λ.

PROPOSITION 4.4. *One of the following alternatives holds*:

(a) $\Lambda = \emptyset$.

(b) $\Lambda = (\underline{\lambda}, \bar{\lambda})$, *where* $-\infty \leqslant \underline{\lambda} < \bar{\lambda} \leqslant \infty$.

(c) $\Lambda = (\underline{\lambda}_1, \bar{\lambda}_1) \cup (\underline{\lambda}_2, \bar{\lambda}_2)$, *where* $-\infty \leqslant \underline{\lambda}_1 < \bar{\lambda}_1 \leqslant \underline{\lambda}_2 < \bar{\lambda}_2 \leqslant \infty$.

Note that if $\lambda_0 \in \partial\Lambda$ then either $\sigma_{\Omega, \mathscr{B}}(\lambda_0) = 0$ or $\sigma_{\Omega_0, \mathscr{D}}(\lambda_0) = 0$. The next theorem shows that if $\sigma_{\Omega, \mathscr{B}}(\lambda_0) = 0$, then $(\lambda_0, 0)$ is a bifurcation point to positive solutions from the branch $(\lambda, 0)$ of trivial solutions of (4.1) and that if $\sigma_{\Omega_0, \mathscr{D}}(\lambda_0) = 0$, then λ_0 is a bifurcation point from infinity.

Remark 4.5. Suppose there is $\lambda_0 \in \mathbb{R}$ such that either $\sigma_{\Omega, \mathscr{B}}(\lambda_0) = 0$ or $\sigma_{\Omega_0, \mathscr{D}}(\lambda_0) = 0$. Then, $\Lambda \cap U \neq \emptyset$ for any neighbourhood U of λ_0. In particular, $\Lambda \neq \emptyset$. This fact follows easily from Theorem 3.5 using the concavity of $\sigma_{\Omega, \mathscr{B}}$ and $\sigma_{\Omega_0, \mathscr{D}}$ and taking (4.2) into account.

THEOREM 4.6. *Suppose $\Lambda \neq \emptyset$. Let $\lambda_0 \in \partial\Lambda$. Then, one of the following alternatives holds*:

(a) $\sigma_{\Omega, \mathscr{B}}(\lambda_0) = 0$ *and (necessarily)* $\lim_{\lambda \to \lambda_0} \|w_\lambda\|_X = 0$;

(b) $\sigma_{\Omega_0, \mathscr{D}}(\lambda_0) = 0$ *and (necessarily)* $\lim_{\lambda \to \lambda_0} \|w_\lambda\|_\infty = \infty$.

Moreover, if $\sup \Lambda = \infty$, *then*

$$\lim_{\lambda \to \infty} \|w_\lambda\|_\infty = \begin{cases} c \geqslant 0 & \text{if} \quad m < 0 \\ \infty & \text{if} \quad m \nless 0, \end{cases}$$

and if $\inf \Lambda = -\infty$, *then*

$$\lim_{\lambda \to -\infty} \|w_\lambda\|_\infty = \begin{cases} c \geqslant 0 & \text{if} \quad m > 0 \\ \infty & \text{if} \quad m \ngtr 0. \end{cases}$$

Proof. Since $\lambda_0 \in \partial\Lambda$, it follows from Theorem 3.5 that either $\sigma_{\Omega, \mathscr{B}}(\lambda_0) = 0$ or $\sigma_{\Omega_0, \mathscr{D}}(\lambda_0) = 0$.

(a) We first assume that

$$\sigma_{\Omega, \mathscr{B}}(\lambda_0) = 0.$$

Without loss of generality we can assume that

$$\sigma_{\Omega, \mathscr{B}}(\lambda) < 0 \qquad \text{for} \quad \lambda > \lambda_0.$$

The other case can be shown by a similar argument. It suffices to prove that there exist $\varepsilon > 0$ and $c > 0$ such that

$$\|w_\lambda\|_X \leqslant c \tag{4.3}$$

for all $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$. Indeed, if (4.3) holds, then by using a standard compactness argument we find that there exists a non-negative solution of (4.1), say $\tilde{w}$, such that

$$\lim_{\lambda \to \lambda_0} w_\lambda = \tilde{w} \qquad \text{in } X.$$

But, due to Theorem 3.5, problem (4.1) does not admit a positive solution if $\lambda = \lambda_0$. Thus, $\tilde{w} \equiv 0$, which gives the assertion. To show (4.3) we argue as follows. Let $v \in X^+$, $v > 0$. For this choice of v we have $ah(\cdot, v) > 0$. Hence,

$$\sigma_1^\Omega[\mathscr{L} - \lambda_0 m + ah(\cdot, v), \mathscr{B}] > 0.$$

By continuity, there exists $\varepsilon_1 > 0$ such that

$$\sigma_1^\Omega[\mathscr{L} - \lambda m + ah(\cdot, v), \mathscr{B}] > 0$$

for all $\lambda \in [\lambda_0, \lambda_0 + \varepsilon_1]$. Let φ be the principal eigenfunction associated with

$$\mu_0 := \sigma_1^\Omega[\mathscr{L} - \lambda_0 m + ah(\cdot, v), \mathscr{B}],$$

i.e.,

$$\mathscr{L}\varphi = \lambda_0 m\varphi - ah(\cdot, v)\,\varphi + \mu_0\varphi \qquad \text{in } \Omega.$$

Since $\mu_0 > 0$, there exists $\varepsilon_2 \in (0, \varepsilon_1)$ such that

$$\lambda_0 m + \mu_0 \geqslant \lambda m$$

for all $\lambda \in I_0 := [\lambda_0, \lambda_0 + \varepsilon_2]$. Let $\kappa > 0$ be such that $\kappa\varphi > v$. Then, it follows easily that the function $\bar{u}$ defined by $\bar{u} := \kappa\varphi$ satisfies

$$\mathscr{L}\bar{u} \geqslant \lambda m\bar{u} - ah(\cdot, \bar{u})\,\bar{u} \qquad \text{in } \Omega$$

for all $\lambda \in I_0$ and hence $\bar{u}$ is a supersolution of (4.1) for each $\lambda \in I_0$. Thus, there exists a constant $c > 0$ such that $\|w_\lambda\|_\infty \leqslant c$ for all $\lambda \in I_0$. Hence, there exists $N > 0$ such that for all $p \in (0, \infty)$ we have

$$\|-a_0 w_\lambda + \lambda m w_\lambda - w_\lambda h(\cdot, w_\lambda)\|_{L^p} \leqslant N$$

for all $\lambda \in I_0$. By the L^p estimates of Agmon, Douglis and Nirenberg, there exists a constant $C_1 > 0$, depending on Ω and the coefficients of $\mathscr{L}$, such that

$$\|w_\lambda\|_{W_0^{1,p}} \leqslant C_1 N$$

for all $\lambda \in I_0$. Choose p large enough such that

$$v < 1 - \frac{N}{p}.$$

Then, we have the continuous imbedding $W_0^{1,p}(\Omega) \subset C^v(\bar{\Omega})$ and therefore there exists a constant $C_2 > 0$ such that

$$\|w_\lambda\|_v \leqslant C_1 C_2 N$$

for all $\lambda \in I_0$. Finally, it follows from Shauder's estimates that

$$\|w_\lambda\|_{2,v} \leqslant C_1 C_2 C_3 N$$

for all $\lambda \in I_0$, where $C_3 > 0$ is a constant depending on Ω and the C^v-norms of the coefficients of $\mathscr{L}$. This shows (4.3), concluding the proof of part (a).

(b) We now assume that $\sigma_{\Omega_0, \mathscr{D}}(\lambda_0) = 0$. To show that λ_0 is a bifurcation point from infinity to positive solutions we shall argue by contradiction. Suppose there exist a constant $c > 0$ and a sequence $\lambda_n \in \Lambda$ such that $\lambda_n \to \lambda_0$ and

$$\|w_{\lambda_n}\|_\infty \leqslant c, \qquad n \in \mathbb{N}.$$

Then, the same argument as in the proof of part a) shows that

$$\lim_{n\to\infty} \|w_{\lambda_n}\|_\infty = 0$$

and hence

$$0 = \sigma_1^\Omega[\mathscr{L} - \lambda_n m + ah(\cdot, w_{\lambda_n}), \mathscr{B}] \to \sigma_1^\Omega[\mathscr{L} - \lambda_0 m, \mathscr{B}]$$

as $n \to \infty$. Thus,

$$0 = \sigma_1^\Omega[\mathscr{L} - \lambda_0 m, \mathscr{B}] < \sigma_1^{\Omega_0}[\mathscr{L} - \lambda_0 m, \mathscr{D}] = 0,$$

which yields a contradiction. Therefore,

$$\lim_{\lambda\to\lambda_0} \|w_\lambda\|_\infty = \infty.$$

The proof of part (b) is completed.

Finally, we consider the case when $\sup \Lambda = \infty$. The case $\inf \Lambda = -\infty$ can be treated in a similar way. If $m < 0$ we have seen in Remark 4.2 that the mapping $\lambda \to \|w_\lambda\|_\infty$ is decreasing. Hence, $\|w_\lambda\|_\infty \searrow c$ as $\lambda \to \infty$ for some constant $c \geqslant 0$. Now, suppose $m \nleqslant 0$ and there exist $R > 0$ and a sequence $\lambda_n \nearrow \infty$ such that

$$\|w_{\lambda_n}\|_\infty < R$$

for all $n \geqslant 1$. Then,

$$\begin{aligned}\sigma_{\Omega, \mathscr{B}}(\lambda_n) &:= \sigma_1^\Omega[\mathscr{L} - \lambda_n m + ah(\cdot, R), \mathscr{B}] \\ &\geqslant \sigma_1^\Omega[\mathscr{L} - \lambda_n m + ah(\cdot, w_{\lambda_n}), \mathscr{B}] = 0.\end{aligned}$$

On the other hand, as $\sup_{x\in\Omega} m(x) > 0$ it follows from Lemma 4.3(a) that

$$\lim_{n\to\infty} \sigma_{\Omega, \mathscr{B}}(\lambda_n) = -\infty.$$

This contradiction completes the proof. ∎

We shall now interpret Proposition 4.4 and Theorem 4.6 distinguishing the cases where m changes sign and the case $m > 0$. The case $m < 0$ can be reduced to the case $m > 0$ by changing the sign of λ. For notational convenience we set

$$Z_\Omega := \{ \lambda \in \mathbb{R} \mid \sigma_{\Omega, \mathscr{B}}(\lambda) = 0 \}.$$

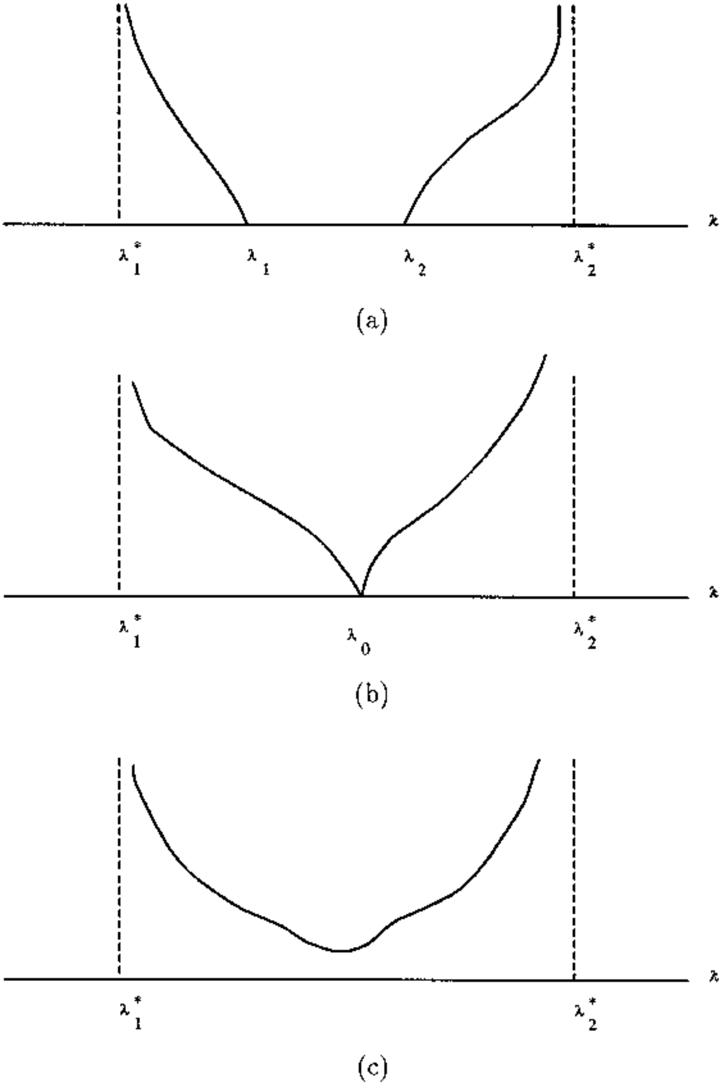

FIG. 1. Bifurcation diagrams for sign indefinited weights.

The Indefinite Case. Suppose m changes sign in Ω. Then, by Lemma 4.3(a) and (b) the function $\sigma_{\Omega, \mathscr{B}}(\lambda)$ may have either two distinct zeroes $\lambda_1 < \lambda_2$, or exactly one zero λ_0, which coincides with the point where $\sigma_{\Omega, \mathscr{B}}$ attains its maximum, or no zero at all. In other words, any of the following cases may arise

$$Z_\Omega = \{\lambda_1, \lambda_2\}, \qquad Z_\Omega = \{\lambda_0\} \qquad \text{or} \qquad Z_\Omega = \varnothing.$$

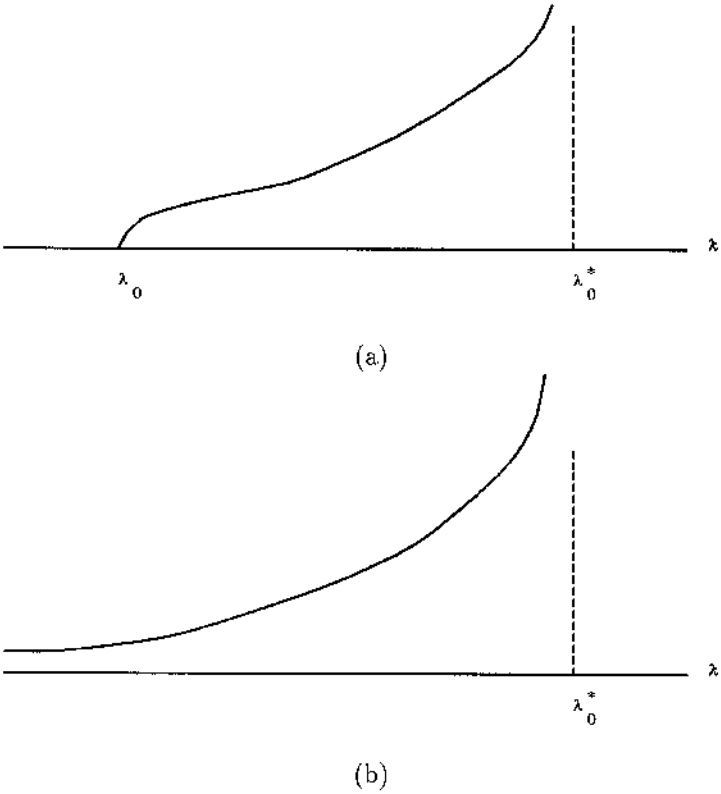

FIG. 2. Bifurcation diagrams for sign definited weights.

By Theorem 4.6 the points λ_0, λ_1 and λ_2 are of bifurcation to positive solutions from the trivial branch $(\lambda, w) = (\lambda, 0)$ of (4.1).

Suppose $\Lambda \neq \emptyset$. Set $\lambda_1^* := \inf \Lambda$ and $\lambda_2^* := \sup \Lambda$. Note that both λ_1^* and λ_2^* are finite only if they are zeroes of $\sigma_{\Omega_0, \mathscr{D}}$. By Theorem 4.6 we have

$$\lim_{\lambda \to \lambda_i^*} \|w_\lambda\|_\infty = \infty$$

for $i = 1, 2$. From the above discussion we infer that only the previous three qualitatively different diagrams depicting Σ in Fig. 1 may occur.

The Definite Case. Assume $m > 0$. Then, $\sigma_{\Omega, \mathscr{B}}(\lambda)$ is decreasing and admits either one zero, say λ_0, or none. Thus,

$$Z_\Omega = \{\lambda_0\} \qquad \text{or} \qquad Z_\Omega = \emptyset.$$

Assume that $Z_\Omega = \{\lambda_0\}$. Then, it follows from Theorem 4.6 that $(\lambda_0, 0)$ is a bifurcation point to positive solutions from the line of trivial solutions of (4.1). Necessarily, the bifurcation is supercritical. Suppose $\Lambda \neq \emptyset$ and set $\lambda_0^* := \sup \Lambda$. Then, thanks to Theorem 4.6 we have

$$\lim_{\lambda \to \lambda_0^*} \|w_\lambda\|_\infty = \infty$$

and therefore only two bifurcation diagrams for Σ can occur (see Fig. 2).

4. Concluding Remarks

(a) If $a_0 \equiv 0$, then large constants are strict positive supersolutions of (4.1) and Lemma 2.2 implies that $\sigma_1^\Omega[\mathscr{L}, \mathscr{B}] > 0$. Thus, it follows from Lemma 4.3 that $\sigma_{\Omega, \mathscr{B}}(\lambda)$ admits a zero and therefore bifurcation to positive solutions from the trivial branch $(\lambda, 0)$ always occurs. In particular, the diagrams shown in Figure 1(c) and Figure 2(b) are not possible.

(b) It was shown by Remark 4.5 and Theorem 4.6 that if $\sigma_{\Omega, \mathscr{B}}(\lambda_0) = 0$, then $(\lambda_0, 0)$ is a bifurcation point to positive solutions from the trivial branch. Observe that the zeroes of $\sigma_{\Omega, \mathscr{B}}(\lambda)$ are the principal eigenvalues of the weighted linear elliptic boundary value problem

$$\begin{cases} \mathscr{L}(x, D)\, \varphi = \lambda m \varphi & \text{in } \Omega, \\ \mathscr{B}(x, D)\, \varphi = 0 & \text{on } \partial\Omega. \end{cases}$$

If m has definite sign and $Z_\Omega = \{\lambda_0\}$ it is straightforward to see that λ_0 is an M-simple eigenvalues of the operator $L\colon X \to Y$, where $Y := C^\mu(\bar{\Omega})$, $Lu := \mathscr{L}(\cdot, D)\, u$ for $u \in X$, and $M\colon Y \to Y$ stands for the multiplication operator induced by m. If m changes sign and $Z_\Omega = \{\lambda_1, \lambda_2\}$, then the fact that λ_1 and λ_2 are M-simple eigenvalues of L is far from being immediate and was shown in [7]. M-simplicity is equivalent to the *transversality condition* of Crandall and Rabinowitz in [5] and therefore in any of these cases the local bifurcation theorem of [5] applies giving rise to a local curve of positive solutions bifurcating from $(\lambda, 0)$. However, if m changes sign and $Z_\Omega = \{\lambda_0\}$, then λ_0 is no longer M-simple, precluding us from using the theorem of Crandall and Rabinowitz. However, as we have seen, bifurcation to positive solutions from $(\lambda, 0)$ at $(\lambda_0, 0)$ still occurs.

(c) In the present situation it is interesting to note that, being zeroes of $\sigma_{\Omega_0, \mathscr{D}}$, the points of bifurcation from infinity are the principal eigenvalues of a linear elliptic eigenvalue problem with respect to the weight m. Namely, they are the principal eigenvalues of

$$\begin{cases} \mathscr{L}(x, D)\,\varphi = \lambda m \varphi & \text{in } \Omega_0, \\ \varphi = 0 & \text{on } \partial\Omega_0. \end{cases}$$

This fact follows from Theorem 2.4 and Lemma 3.4.

(d) Bifurcation from infinity can also be obtained if instead of (A2) we assume that there exists $C > 0$ such that $h(x, \xi) \leqslant C$ for all $(x, \xi) \in \bar{\Omega} \times \mathbb{R}^+$ and that $\mathscr{P}(\Omega)$, defined in Lemma 4.3, is strictly positive. Indeed, in that case for any $v \in X^+$ we have

$$\sigma_1^\Omega[\mathscr{L} - \lambda m + ah(\cdot, v)] \leqslant \sigma_1^\Omega[\mathscr{L} - \lambda m + aC].$$

Moreover, by Lemma 4.3, it follows that

$$\lim_{\lambda \to \infty} \sigma_1^\Omega[\mathscr{L} - \lambda m + aC] = -\infty.$$

Since for any positive solution w of (4.1) we have

$$\sigma_1^\Omega[\mathscr{L} - \lambda m + ah(\cdot, w)] = 0,$$

we conclude that (4.1) does not admit a positive solution for large values of λ. It is now clear that if a branch of positive solutions emanates supercritically from $(\lambda, 0)$ at some value λ_0 then there exists $\lambda^* \in (\lambda_0, \infty)$ such that bifurcation to positive solutions from infinity occurs at λ^*.

Acknowledgments

J. L-G is on leave from Complutense University, Madrid. This work was initiated during a visit of J. L-G to the Institut für Mathematik der Universität Zürich in June 1993 and continued in this institution during the summer semester of 1994. It is a pleasure for J. L-G to thank Prof. Dr. H. Amann for the invitation to Zürich. This work was partially supported by DGYCIT PB93-0465 and the European Network *Reaction Diffusion Equations* under contract ERBCHRXCT93-0409.

References

1. H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, *SIAM Rev.* **18** (1976), 620–709.
2. H. Berestycki, L. Nirenberg, and S. R. S. Varadhan, The principal eigenvalue and maximum principle for second order elliptic operators in general domains, *Comm. Pure Appl. Math.* **XLVII** (1994), 47–92.

3. K. J. Brown, P. C. Dunne, and R. A. Gardner, A semilinear parabolic system arising in the theory of superconductivity, *J. Differential Equations* **40** (1981), 232–252.
4. R. Courant and D. Hilbert, "Methods of Mathematical Physics," Wiley–Interscience, New York, 1962.
5. M. G. Crandall and P. H. Rabinowitz, Bifurcation, perturbation of simple eigenvalues, and linearized stability, *Arch. Rational Mech. Anal.* **52** (1973), 161–180.
6. P. Hess, "Periodic–Parabolic Boundary Value Problems and Positivity," Pitman Research Notes in Mathematics Series 247, Longman, Harlow Essex, 1991.
7. P. Hess and T. Kato, On some linear and nonlinear eigenvalue problems with an indefinite weight function, *Comm. Partial Differential Equations* **5** (1980), 99–1030.
8. T. Kato, Superconvexity of the spectral radius and convexity of the spectral bound and type, *Math. Z.* **180** (1982), 265–273.
9. J. López-Gómez, On linear weighted boundary value problems, "Partial Differential Equations; Models in Physics and Biology," pp. 188–203, Akademic-Verlag, Berlin, 1994.
10. J. López-Gómez, Bifurcation theory, lecture notes, 1994.
11. J. D. Murray, "Mathematical Biology," Biomathematics Texts, Springer, New York, 1989.
12. A. Okubo, "Diffusion and Ecological Problems: Mathematical Models," Springer, New York, 1980.
13. M. H. Protter and H. F. Weinberger, "Maximum Principles in Differential Equations," Prentice–Hall, Englewood Cliffs, NJ, 1967.
14. R. Redlinger, Über die C^2-Kompaktheit der Bahn von Lösungen semilinearer parabolischer Systeme, *Proc. Roy. Soc. Edinburgh Sect. A* **93** (1982), 99–103.
15. D. H. Sattinger, "Topics in Stability and Bifurcation Theory," Lecture Notes in Mathematics, Vol. 309, Springer-Verlag, Berlin/New York, 1973.
16. R. Temam, "Infinite-Dimensional Dynamical Systems in Mechanics and Physics," Series in Applied Mathematical Sciences, Vol. 68, Springer, New York, 1988.