J. Differential Equations 252 (2012) 202–235

Contents lists available at SciVerse ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Abstract fractional Cauchy problems with almost sectorial operators ☆

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A R T I C L E   I N F O

Article history:
Received 28 September 2010
Revised 25 August 2011
Available online 13 September 2011

MSC:
26A33
34K37
35K90
47A60
34K30
47A10

Keywords:
Almost sectorial operators
Semigroup of growth $1+\gamma$
Caputo fractional derivative
Fractional Cauchy problems
Mild and classical solutions

A B S T R A C T

Of concern are the Cauchy problems for linear and semilinear time fractional evolution equations involving in the linear part, a linear operator A whose resolvent satisfies the estimate of growth $-\gamma$ $(-1<\gamma<0)$ in a sector of the complex plane, which occurs when one considers, for instance, the partial differential operators in the limit domain of dumb-bell with a thin handle or in the space of Hölder continuous functions. By constructing a pair of families of operators in terms of the generalized Mittag-Leffler-type functions and the resolvent operators associated with A (for the first time), and a deep analysis on the properties for these families, we obtain the existence and uniqueness of mild solutions and classical solutions to the Cauchy problems. Moreover, we present three examples to illustrate the feasibility of our results.

☆ Rong-Nian Wang acknowledges support from the NSF of Jiangxi Province of China (2009GQS0018) and the Youth Foundation of Jiangxi Provincial Education Department of China (GJJ10051). Ti-Jun Xiao acknowledges support from the NSF of China (11071042) and the Research Fund for the Shanghai Key Laboratory for Contemporary Applied Mathematics (08DZ2271900) and the Laboratory of Mathematics for Nonlinear Science at Fudan University.

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0022-0396/$ – see front matter
doi:10.1016/j.jde.2011.08.048

1. Introduction

Let $(X, \|\cdot\|)$ be a complex Banach space. As usual, for a linear operator A, we denote by $D(A)$ the domain of A, by $\sigma(A)$ its spectrum, while $\rho(A) := \mathbb{C} - \sigma(A)$ is the resolvent set of A, and denote by the family $R(z; A) = (zI - A)^{-1}$, $z \in \rho(A)$ of bounded linear operators the resolvent of A. Moreover, we denote by $\mathscr{L}(Y, Z)$ the space of all bounded linear operators between two normed spaces Y and Z with the operator norm $\|\cdot\|_{\mathscr{L}(Y,Z)}$, we abbreviate this notation to $\mathscr{L}(Y)$ when $Y = Z$, and write $\|T\|_{\mathscr{L}(X)}$ as $\|T\|$ for every $T \in \mathscr{L}(X)$ when it has no loss of the clarity.

When dealing with parabolic evolution equations, it is usually assumed that the partial differential operator in the linear part is a sectorial operator, stimulated by the fact that this class of operators appears very often in the applications. For example, one can find from [17,27,37] that many elliptic differential operators equipped with homogeneous boundary conditions are sectorial when they are considered in the Lebesgue spaces (e.g. L^p-spaces) or in the space of continuous functions. We here mention that the operator A_ε in Example 1.1, which acts on a domain of "dumb-bell with a thin handle", is sectorial on V_ϵ^p. However, as presented in Example 1.1 and Example 1.2, though the resolvent set of some partial differential operators considered in some special domains such as the limit "domain" of dumb-bell with a thin handle or in some spaces of more regular functions such as the space of Hölder continuous functions, contains a sector, but for which the resolvent operators do not satisfy the required estimate to be a sectorial operator.

Example 1.1. In this notation the "dumb-bell with a thin handle" has the form

$$\Omega_\varepsilon = D_1 \cup Q_\varepsilon \cup D_2 \quad \big(\varepsilon \in (0,1];\ \text{small}\big),$$

where D_1 and D_2 are mutually disjoint bounded domains in $\mathbb{R}^N$ $(N \geqslant 2)$ with smooth boundaries, joined by a thin channel, Q_ε (which is not required to be cylindrical), which degenerates to a 1-dim line segment Q_0 as ε approaches zero. This implies that passing to the limit as $\varepsilon \to 0$, the limit "domain" of Ω_ε consists of the fixed part D_1, D_2 and the line segment Q_0. Without loss of generality, we may assume that $Q_0 = \{(x, 0, \ldots, 0);\ 0 < x < 1\}$. Let $P_0 = (0, 0, \ldots, 0)$, $P_1 = (1, 0, \ldots, 0)$ be the points where the line segment touches the boundary of D_1 and D_2. Put $\Omega = D_1 \cup D_2$.

Firstly, consider the evolution equation of parabolic type equipped with Neumann boundary condition in the form

$$\begin{cases} u_t - \Delta u + u = f(u), \quad x \in \Omega_\epsilon,\ t > 0, \\ \dfrac{\partial u}{\partial n} = 0, \quad x \in \partial\Omega_\epsilon, \end{cases} \tag{1.1}$$

where Δ stands for the Laplacian operator with respect to the spatial variable $x \in \Omega_\epsilon$, $\partial\Omega_\epsilon$ is the boundary of Ω_ϵ, $\frac{\partial}{\partial n}$ denotes the outward normal derivative on $\partial\Omega_\epsilon$ and $f : \mathbb{R} \to \mathbb{R}$ is a nonlinearity. Let V_ϵ^p $(1 \leqslant p < \infty)$ denote the family of spaces based on $L^p(\Omega_\epsilon)$, equipped with the norm

$$\|u\|_{V_\epsilon^p} = \left(\int_\Omega |u|^p + \frac{1}{\varepsilon^{N-1}} \int_{Q_\epsilon} |u|^p \right)^{\frac{1}{p}}.$$

Define the linear operator $A_\varepsilon : D(A_\varepsilon) \subset V_\epsilon^p \mapsto V_\epsilon^p$ by

$$D(A_\varepsilon) = \left\{ u \in W^{2,p}(\Omega_\epsilon);\ \Delta u \in V_\epsilon^p,\ \frac{\partial u}{\partial n}\bigg|_{\partial\Omega_\epsilon} = 0 \right\},$$
$$A_\varepsilon u = -\Delta u + u, \quad u \in D(A_\varepsilon).$$

It follows from a standard argument that the operator A_ε generates an analytic semigroup on V^p_ϵ. Moreover, the following estimate holds

$$\left\|R(\lambda;-A_\varepsilon)\right\|_{\mathscr{L}(L^p(\Omega_\epsilon))}\leqslant\frac{C}{|\lambda|},\quad \text{for } \lambda\in\Sigma'_\theta,$$

where $\Sigma'_\theta=\{\lambda\in\mathbb{C};\ |\arg(\lambda-1)|\leqslant\theta\}$ with $\theta>\frac{\pi}{2}$, and C is a constant that does not depend on ε (e.g. see [17,37]).

The limit problem of (1.1) as $\varepsilon\to 0$ is the following problem studied in [6]

$$\begin{cases} w_t-\Delta w+w=f(w), & x\in\Omega,\ t>0,\\ \dfrac{\partial w}{\partial n}=0, & x\in\partial\Omega,\\ v_t-\dfrac{1}{g}(gv_x)_x+v=f(v), & x\in Q_0=(0,1),\\ v(0)=w(P_0), & v(1)=w(P_1), \end{cases}$$

where w is a function that lives in Ω and v lives in the line segment Q_0, the function $g:[0,1]\to(0,\infty)$ is a smooth function related to the geometry of the channel Q_ε, more exactly, on the way the channel Q_ε collapses to the segment line Q_0. Observe that the vector (w,v) is continuous in the junction between Ω and Q_0 and the variable w does not depend on the variable v, but v depends on w.

We identify V^p_0 with $L^p(\Omega)\oplus L^p_g(0,1)$ $(1\leqslant p<\infty)$ endowed with the norm $\|(w,v)\|_{V^p_0}=(\int_\Omega|w|^p+\int_0^1 g|v|^p)^{1/p}$. Consider the operator $A_0:D(A_0)\subset V^p_0\mapsto V^p_0$ defined by

$$\begin{aligned} D(A_0)&=\big\{(w,v)\in V^P_0;\ w\in D(\Delta_\Omega),\ v\in L^p_g(0,1),\\ &\qquad w(P_0)=v(0),\ w(P_1)=v(1)\big\},\\ A_0(w,v)&=\left(-\Delta w+w,-\frac{1}{g}\big(gv'\big)'+v\right),\quad (w,v)\in V^P_0, \end{aligned}\tag{1.2}$$

where Δ_Ω is the Laplace operator with homogeneous Neumann boundary conditions in $L^p(\Omega)$ and $D(\Delta_\Omega)=\{u\in W^{2,p}(\Omega);\ \frac{\partial u}{\partial n}|_{\partial\Omega}=0\}$.

As pointed out by [4], the operator A_0 defined by (1.2) is not a sectorial operator. Its spectrum is all real and, therefore, it is contained in a sector but the resolvent estimate is different from the case of sectorial operator. More precisely, the operator A_0 has the following properties (see also [3,5]):

(a) the domain $D(A_0)$ is dense in V^P_0,
(b) if $p>\frac{N}{2}$, then A_0 is a closed operator,
(c) A_0 has compact resolvent, and
(d) for some $\mu\in(0,\frac{\pi}{2})$, $\Sigma_\mu:=\{\lambda\in\mathbb{C}\setminus\{0\};\ |\arg\lambda|\leqslant\pi-\mu\}\cup\{0\}\subset\rho(-A_0)$, and for $\frac{N}{2}<q\leqslant p$ the following estimate holds:

$$\left\|R(\lambda;-A_0)\right\|_{\mathscr{L}(V^q_0,V^p_0)}\leqslant\frac{C}{1+|\lambda|^{\gamma'}},\quad \lambda\in\Sigma_\mu,\tag{1.3}$$

for each $0<\gamma'<1-\frac{N}{2q}-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})<1$, where C is a positive constant.

Remark 1.1. In fact, it is easy to prove that the estimate (1.3) with $p=q>\frac{N}{2}$ is equivalent to $\|R(\lambda;-A_0)\|_{\mathscr{L}(V^p_0)}\leqslant\frac{\widetilde{C}}{|\lambda|^{\gamma'}}$ $(\lambda\in\Sigma_\mu\setminus\{0\})$ for $0<\gamma'<1-\frac{N}{2p}$, where $\widetilde{C}$ is a positive constant.

We refer to [3, Section 2] for a complete and rigorous definition of the dumb-bell domain, and to [2–5,9,16,21] for related studies of partial differential equations involving dumb-bell domain.

Example 1.2. Assume that Ω is a bounded domain in $\mathbb{R}^N$ ($N \geqslant 1$) with boundary $\partial\Omega$ of class C^{4m} ($m \in \mathbb{N}$). Let $C^l(\overline{\Omega})$, $l \in (0,1)$, denote the usual Banach space with norm $\|\cdot\|_l$. Consider the elliptic differential operator $A' : D(A') \subset C^l(\Omega) \mapsto C^l(\Omega)$ in the form

$$D(A') = \{u \in C^{2m+l}(\overline{\Omega});\ D^\beta u\big|_{\partial\Omega} = 0,\ |\beta| \leqslant m-1\},$$

$$A'u = \sum_{|\beta|\leqslant 2m} a_\beta(x) D^\beta u(x), \quad u \in D(A'),$$

where β is a multiindex in $(\mathbb{N}\cup\{0\})^n$, $|\beta| = \sum_{j=1}^n \beta_j$, $D^\beta = \prod_{j=1}^n (-i\frac{\partial}{\partial x_j})^{\beta_j}$. The coefficients $a_\beta : \overline{\Omega} \mapsto \mathbb{C}$ of A' are assumed to satisfy

(i) $a_\beta \in C^l(\overline{\Omega})$ for all $|\beta| \leqslant 2m$,
(ii) $a_\beta(x) \in \mathbb{R}$ for all $x \in \overline{\Omega}$ and $|\beta| = 2m$, and
(iii) there exists a constant $M > 0$ such that

$$M^{-1}|\xi|^2 \leqslant \sum_{|\beta|=2m} a_\beta(x)\xi^\beta \leqslant M|\beta|^2, \quad \text{for all } \xi \in \mathbb{R}^N,\ x \in \overline{\Omega}.$$

Then, the following statements hold.

(a) A' is not densely defined in $C^l(\overline{\Omega})$,
(b) there exist $\nu, \varepsilon > 0$ such that

$$\sigma(A'+\nu) \subset S_{\frac{\pi}{2}-\varepsilon} = \left\{\lambda \in \mathbb{C}\setminus\{0\};\ |\arg\lambda| \leqslant \frac{\pi}{2} - \varepsilon\right\} \cup \{0\},$$

$$\left\|R(\lambda; A'+\nu)\right\|_{\mathscr{L}(C^l(\overline{\Omega}))} \leqslant \frac{C}{|\lambda|^{1-\frac{l}{2m}}}, \quad \lambda \in \mathbb{C}\setminus S_{\frac{\pi}{2}-\varepsilon},$$

(c) the exponent $\frac{l}{2m} - 1 \in (-1,0)$ is sharp. In particular, the operator $A'+\nu$ is not sectorial.

Notice in particular that the Laplace operator satisfies the conditions (a)–(c) in Example 1.2. For more details we refer to [42].

Let us recall the following definition:

Definition 1.1. Let $-1 < \gamma < 0$ and $0 < \omega < \pi/2$. By $\Theta_\omega^\gamma(X)$ we denote the family of all linear closed operators $A : D(A) \subset X \to X$ which satisfy

(1) $\sigma(A) \subset S_\omega = \{z \in \mathbb{C}\setminus\{0\};\ |\arg z| \leqslant \omega\} \cup \{0\}$ and
(2) for every $\omega < \mu < \pi$ there exists a constant C_μ such that

$$\left\|R(z; A)\right\| \leqslant C_\mu |z|^\gamma \quad \text{for all } z \in \mathbb{C}\setminus S_\mu. \tag{1.4}$$

A linear operator A will be called an almost sectorial operator on X if $A \in \Theta_\omega^\gamma(X)$.

Observe that from Example 1.1 and Remark 1.1, if $p > \frac{N}{2}$, then $A_0 \in \Theta_{\mu}^{-\gamma'}(V_0^p)$ for some $\gamma' \in (0, 1 - \frac{N}{2p})$ and $\mu \in (0, \frac{\pi}{2})$, that is, A_0 is an almost sectorial operator on V_0^p. Also, from Example 1.2 one can find that $(A' + \nu) \in \Theta_{\frac{\pi}{2}-\varepsilon}^{\frac{l}{2m}-1}(C^l(\overline{\Omega}))$, which implies that $A' + \nu$ is an almost sectorial operator on $C^l(\overline{\Omega})$.

Remark 1.2. Let $A \in \Theta_{\omega}^{\gamma}(X)$, then the definition implies that $0 \in \rho(A)$.

Remark 1.3. We say that the estimate (1.4) in Definition 1.1 is "deficient" since $\gamma > -1$. From [38], note in particular that if $A \in \Theta_{\omega}^{\gamma}(X)$, then A generates a semigroup $T(t)$ with a singular behavior at $t = 0$ in a sense, called semigroup of growth $1 + \gamma$. Moreover, the semigroup $T(t)$ is analytic in an open sector of the complex plane $\mathbb{C}$, but the strong continuity fails at $t = 0$ for data which are not sufficiently smooth. Hence, it is impossible to apply to A the general results and techniques on generation of strongly continuous operator semigroup, as it is developed in [37].

Examples of almost sectorial operators which are not sectorial were first introduced by W. von Wahl in [42]. Since then, some other examples for such operators were also presented, see [27, Example 3.1.33] and [38]. Recently, the study of evolution equations involving almost sectorial operators has been investigated to a large extent. We would like to mention that F. Periago and B. Straub [38] give a functional calculus for almost sectorial operators, and using the semigroup of growth $1 + \gamma$ which is defined by this functional calculus, obtained the existence and uniqueness of mild solutions and classical solutions for Cauchy problems of abstract evolution equations involving almost sectorial operators, that, by constructing an evolution process of growth $1 + \gamma$, A.N. Carvalho et al. [6] established the existence of mild solutions for Cauchy problem for non-autonomous evolution equation, in which the operator in the linear part depends on time t and for each t, it is almost sectorial, and that J.M. Arrieta et al. [4,5] analyzed the behavior of the asymptotic dynamics of a reaction–diffusion equation in a dumb-bell domain as the channel shrinks to a line segment, where the partial differential operator in equation forms an almost sectorial operator in appropriate space. Moreover, from [12], one can find results on linear abstract Cauchy problem with almost sectorial operators, whenever the part of this operator in the closure of its domain is sectorial. Notice also that most of the previous research concerns the case of derivative of first order (integer order) in time, there has been little regarding the case of derivative of fractional order in time.

On the other hand, starting from some speculations of Leibniz and Euler, followed by the works of other eminent mathematicians including Laplace, Fourier, Abel, Liouville and Riemann, the fractional calculus which allows us to consider integration and differentiation of any order, not necessarily integer, has been the object of extensive study for analyzing not only stochastic processes driven by fractional Brownian motion, but also nonrandom fractional phenomena in physics, nonrandom fractional optimal control, see [1,7,13,22,26,34,40] and references therein. One of the emerging branches of this study is the theory of abstract partial differential equations that involve fractional derivatives in time (including fractional diffusion equations), for short, we call fractional evolution equations. Let us point out that a strong motivation for investigating such equations comes from physics. For example, as stated in [15], fractional diffusion equations describe anomalous diffusion on fractals (physical objects of fractional dimension, like some amorphous semiconductors or strongly porous materials; see [1,33] and references therein). In normal diffusion (described by, such as the heat equation) the mean square displacement of a diffusive particle behaves like const $\cdot\, t$ for $t \to \infty$. A typical behavior for anomalous diffusion is const $\cdot\, t^{\alpha}$ for some $0 < \alpha < 1$. Also, as indicated in [11,19,28,32], this class of equations can provide a nice instrument for the description of memory and hereditary properties of various materials and processes. What we want to emphasize is that this is the main advantage of fractional models in comparison with classical integer-order models, in which such effects are in fact neglected. At present, much interest has developed regarding the class of equations (see, e.g., [15,25,30,36,39]). In particular, in [15] S.D. Eidelman and A.N. Kochubei considered the Cauchy problem of an evolution equation with the fractional derivative with respect to the time variable and a uniformly elliptic operator with variable coefficients acting in the spatial variables, where a fundamental solution of the Cauchy problem was constructed and investigated. We mention that much of the previous research on the fractional evolution equations was done provided that the operator in the linear part is the infinitesimal generator of a strongly continuous operator semigroup, an analytic semigroup, or a compact semigroup, or a Hille–Yosida operator, much less is known about the fractional evolution equations with almost sectorial operators.

To explain the results better we need to introduce some terminology. We set $I=(0,T)$ for some $T>0$ and use the following notation for $\beta \geqslant 0$, $g_\beta(t)=\begin{cases}\frac{1}{\Gamma(\beta)}t^{\beta-1}, & t>0,\\ 0, & t\leqslant 0,\end{cases}$ and $g_0(t)=0$, where $\Gamma(\beta)$ is the Gamma function.

Definition 1.2. Let $f\in L^1(I;X)$ and $\alpha \geqslant 0$. Then the expression

$$J_t^\alpha f(t) := (g_\alpha * f)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)\,ds, \quad t>0,\ \alpha>0,$$

with $J_t^0 f(t)=f(t)$, is called Riemann–Liouville integral of order α of f.

Definition 1.3. Let $f(t)\in C^{m-1}(I;X)$, $g_{m-\alpha}*f\in W^{m,1}(I,X)$ ($m\in\mathbb{N}$, $0\leqslant m-1<\alpha<m$). The regularized Caputo fractional derivative of order α of f is defined by

$$ {}_cD_t^\alpha f(t) = D_t^m J_t^{m-\alpha}\left(f(t)-\sum_{i=0}^{m-1} f^{(i)}(0)g_{i+1}(t)\right), \tag{1.5}$$

where $D_t^m := \frac{d^m}{dt^m}$.

In this work, motivated by the above consideration, we are interested in studying the Cauchy problem for the linear evolution equation

$$\begin{cases} {}_cD_t^\alpha u(t) + Au(t) = f(t), & t>0,\\ u(0)=u_0, \end{cases} \tag{LCP}$$

as well as the Cauchy problem for the corresponding semilinear fractional evolution equation

$$\begin{cases} {}_cD_t^\alpha u(t) + Au(t) = f\big(t,u(t)\big), & t>0,\\ u(0)=u_0 \end{cases} \tag{SLCP}$$

in X, where ${}_cD_t^\alpha$, $0<\alpha<1$, is the regularized Caputo fractional derivative of order α and A is an almost sectorial operator, that is, $A\in\Theta_\omega^\gamma(X)$ ($-1<\gamma<0$, $0<\omega<\pi/2$). The main purpose is to study the existence and uniqueness of mild solutions and classical solutions of Cauchy problems (LCP) and (SLCP). To do this, we construct two operator families based on the generalized Mittag-Leffler-type functions and the resolvent operators associated with A, present deep analysis on basic properties for these families including the study of the compactness, and prove that, under natural assumptions, reasonable concepts of solutions can be given to problems (LCP) and (SLCP), which in turn is used to find solutions to the Cauchy problems.

Remark 1.4. We make no assumption on the density of the domain of A.

Remark 1.5. (i) M.M. Dzhrbashyan and A.B. Nersessyan in [14] (see also [34]) showed that the solution of the Cauchy problem

$$\begin{cases} {}_cD_t^{\alpha}u(t)+\lambda u(t)=0, & t>0,\\ u(0)=1, & 0<\alpha<1, \end{cases}$$

has the form $u(t)=E_{\alpha}(-\lambda t^{\alpha})$, where E_{α} is the known Mittag-Leffler function. This result issues a warning to us that no matter how smooth the data u_0 is, it is inappropriate to define the mild solution of problem (LCP) as follows

$$u(t)=T(t)u_0+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}T(t-s)f(s)\,ds,$$

where $T(t)$ is the semigroup generated by A (see Remark 1.3), though this fashion was used in some situations of previous research (see, e.g., [20]).

(ii) Let us point out that in the treatment of problems (LCP) and (SLCP), one of the difficult points is to give reasonable concept of solutions (see also the work of E. Hernandez et al. [18]). Another is that even though the operator A generates a semigroup $T(t)$ in X, it will not be continuous at $t=0$ for nonsmooth initial data u_0.

(iii) It is worth mentioning that if it is the case when A is a matrix (or even bounded linear operators) then A.A. Kilbas et al. [23, Section 7.4] obtained an explicit representation of mild solution to problem (LCP).

Let us now give a short summary of this paper, which is organized in a way close to that given by A.N. Carvalho et al. [6]. In Section 2 we give brief overview of the construction of functional calculus about almost sectorial operators, state some results about the analytic semigroups of growth order $1+\gamma$, describe the necessity to use the regularized fractional derivative (1.5), and summarize some properties on Caputo fractional derivative and two special functions. In Section 3, we construct a pair of families of operators and present a deep analysis on the properties for these families. Based on the families of operators defined in Section 3, a reasonable concept of solution is given in Section 4 to problems (LCP), which in turn is used to analyze the existence of mild solutions and classical solutions to the Cauchy problem. The corresponding semilinear problem (SLCP) is studied in Section 5. We first investigate the existence of mild solutions, and then the existence of classical solutions. Finally, based mainly on [6,38], we present three examples in Section 6 to illustrate our results.

Remark 1.6. Let us note that results in this paper can be easily extended to the case of (general) sectorial operators.

2. Preliminaries

We first introduce some special functions and classes of functions which will be used in the following, for more details, we refer to [29,38]. Let $-1<\gamma<0$, and let S_{μ}^0 with $0<\mu<\pi$ be the open sector $\{z\in\mathbb{C}\setminus\{0\};\ |\arg z|<\mu\}$ and S_{μ} be its closure, that is $S_{\mu}:=\{z\in\mathbb{C}\setminus\{0\};\ |\arg z|\leqslant\mu\}\cup\{0\}$. Set

$$\mathcal{F}_0^{\gamma}\big(S_{\mu}^0\big)=\bigcup_{s<0}\Psi_s^{\gamma}\big(S_{\mu}^0\big)\cup\Psi_0\big(S_{\mu}^0\big),$$

$$\mathcal{F}\big(S_{\mu}^0\big)=\big\{f\in\mathcal{H}\big(S_{\mu}^0\big);\ \text{there } k,n\in\mathbb{N} \text{ such that } f\psi_n^k\in\mathcal{F}_0\big(S_{\mu}^0\big)\big\},$$

where

$$\mathcal{H}\big(S_{\mu}^0\big)=\big\{f:S_{\mu}^0\mapsto\mathbb{C};\ f \text{ is holomorphic}\big\},$$

$$\mathcal{H}^{\infty}\big(S_{\mu}^0\big)=\big\{f\in\mathcal{H}\big(S_{\mu}^0\big);\ f \text{ is bounded}\big\},$$

$$\varphi_0(z)=\frac{1}{1+z},\qquad \psi_n(z):=\frac{z}{(1+z)^n},\quad z\in\mathbb{C}\backslash\{-1\},\ n\in\mathbb{N}\cup\{0\},$$
$$\Psi_0\big(S_\mu^0\big)=\left\{f\in\mathcal{H}\big(S_\mu^0\big);\ \sup_{z\in S_\mu^0}\left|\frac{f(z)}{\varphi_0(z)}\right|<\infty\right\},$$

and for each $s<0$,

$$\Psi_s^\gamma\big(S_\mu^0\big):=\Big\{f\in\mathcal{H}\big(S_\mu^0\big);\ \sup_{z\in S_\mu^0}\big|\psi_n^s(z)f(z)\big|<\infty\Big\},$$

where n is the smallest integer such that $n\geqslant 2$ and $\gamma+1<-(n-1)s$.

Observe that the classes of functions introduced above satisfy the inclusions

$$\mathcal{F}_0^\gamma\big(S_\mu^0\big)\subset\mathcal{H}^\infty\big(S_\mu^0\big)\subset\mathcal{F}\big(S_\mu^0\big)\subset\mathcal{H}\big(S_\mu^0\big).$$

Moreover, taking $k,n\in\mathbb{N}\cup\{0\}$ with $n>k$, one easily sees that $\psi_n^k\in\mathcal{F}_0^\gamma(S_\mu^0)$.

Assume that $A\in\Theta_\omega^\gamma(X)$ with $-1<\gamma<0$ and $0<\omega<\pi/2$. Following F. Periago and B. Straub [38] (see also A. McIntosh [31] and M. Cowling et al. [8]), a closed linear operator $f\to f(A)$ can be constructed for every $f\in\mathcal{F}(S_\mu^0)$ via an extended functional calculus. In the following we give a short overview to this construction.

For $f\in\mathcal{F}_0^\gamma(S_\mu^0)$, via the Dunford–Riesz integral, the operator $f(A)$ is defined by

$$f(A)=\frac{1}{2\pi i}\int_{\Gamma_\theta}f(z)R(z;A)\,dz,\tag{2.1}$$

where the integral contour $\Gamma_\theta:=\{\mathbb{R}_+e^{i\theta}\}\cup\{\mathbb{R}_+e^{-i\theta}\}$, is oriented counter-clockwise and $\omega<\theta<\mu<\pi$. It follows that the integral is absolutely convergent and defines a bounded linear operator on X, and its value does not depend on the choice of θ.

Notice in particular that for $k,n\in\mathbb{N}\cup\{0\}$ with $n>k$,

$$\psi_n^k(A)=A^k(A+1)^{-n}$$

and the operator $\psi_n^k(A)$ is injective. Notice also that if $f\in\mathcal{F}(S_\mu^0)$, then there exist $k,n\in\mathbb{N}$ such that $f\psi_n^k\in\mathcal{F}_0^\gamma(S_\mu^0)$. Hence, for $f\in\mathcal{F}(S_\mu^0)$, one can define a closed linear operator, still denoted by $f(A)$,

$$D\big(f(A)\big)=\big\{x\in X;\ \big(f\psi_n^k\big)(A)x\in D\big(A^{(n-1)k}\big)\big\},$$
$$f(A)=\big(\psi_n^k(A)\big)^{-1}\big(f\psi_n^k\big)(A),$$

and the definition of $f(A)$ does not depend on the choice of k and n. We emphasize that $f(A)$ is indeed an extension of the original one and the triple $(\mathcal{F}_0^\gamma(S_\mu^0),\mathcal{F}(S_\mu^0),f(A))$ is called an **abstract functional calculus** on X (see [29]).

With respect to this construction we collect some basic properties. For more details, we refer to [38].

Proposition 2.1. *The following assertions hold.*

(i) $\alpha f(A)+\beta g(A)=(\alpha f+\beta g)(A)$, $(fg)(A)=f(A)g(A)$ *for all* $f,g\in\mathcal{F}_0^\gamma(S_\mu^0)$, $\alpha,\beta\in\mathbb{C}$;
(ii) $f(A)g(A)\subset(fg)(A)$ *for all* $f,g\in\mathcal{F}(S_\mu^0)$; *and*
(iii) $f(A)g(A)=(fg)(A)$, *provided that* $g(A)$ *is bounded or* $D((fg)(A))\subset D(g(A))$.

Since for each $\beta \in \mathbb{C}$, $z^\beta \in \mathcal{F}(S^0_\mu)$ $(z \in \mathbb{C} \setminus (-\infty, 0],\ 0 < \mu < \pi)$, one can define, via the triple $(\mathcal{F}^\gamma_0(S^0_\mu), \mathcal{F}(S^0_\mu), f(A))$, the complex powers of A which are closed by $A^\beta = z^\beta(A)$ $(\beta \in \mathbb{C})$. However, in difference to the case of sectorial operators, having $0 \in \rho(A)$ does not imply that the complex powers $A^{-\beta}$ with $Re\,\beta > 0$, are bounded. The operator $A^{-\beta}$ belongs to $\mathscr{L}(X)$ whenever $v\,Re\,\beta > 1+\gamma$. So, in this situation, the linear space $X^\beta := D(A^\beta)$, $\beta > 1+\gamma$, endowed with the graph norm $\|x\|_\beta = \|A^\beta x\|$ $(x \in X^\beta)$, is a Banach space.

Next, we turn our attention to the semigroup associated with A. Since given $t \in S^0_{\frac{\pi}{2}-\omega}$, $e^{-tz} \in \mathcal{H}^\infty(S^0_\mu)$ satisfies the conditions (a) and (b) of [38, Lemma 2.13], the family

$$T(t) = e^{-tz}(A) = \frac{1}{2\pi i}\int_{\Gamma_\theta} e^{-tz}R(z;A)\,dz, \quad t \in S^0_{\frac{\pi}{2}-\omega}, \tag{2.2}$$

here $\omega < \theta < \mu < \frac{\pi}{2} - |\arg t|$, forms an analytic semigroup of growth order $1+\gamma$. For more properties on $T(t)$, please see the following proposition.

Proposition 2.2. *(See [38, Theorem 3.9].) Let $A \in \Theta^\gamma_\omega(X)$ with $-1 < \gamma < 0$ and $0 < \omega < \pi/2$. Then the following properties remain true.*

(i) *$T(t)$ is analytic in $S^0_{\frac{\pi}{2}-\omega}$ and $\frac{d^n}{dt^n}T(t) = (-A)^n T(t)$ $(t \in S^0_{\frac{\pi}{2}-\omega})$;*
(ii) *The functional equation $T(s+t) = T(s)T(t)$ for all $s, t \in S^0_{\frac{\pi}{2}-\omega}$ holds;*
(iii) *There is a constant $C_0 = C_0(\gamma) > 0$ such that $\|T(t)\| \leqslant C_0 t^{-\gamma-1}$ $(t > 0)$;*
(iv) *The range $R(T(t))$ of $T(t)$, $t \in S^0_{\frac{\pi}{2}-\omega}$, is contained in $D(A^\infty)$. Particularly, $R(T(t)) \subset D(A^\beta)$ for all $\beta \in \mathbb{C}$ with $Re\,\beta > 0$,*

$$A^\beta T(t)x = \frac{1}{2\pi i}\int_{\Gamma_\theta} z^\beta e^{-tz}R(z;A)x\,dz, \quad \textit{for all } x \in X,$$

and hence there exists a constant $C' = C'(\gamma, \beta) > 0$ such that

$$\left\|A^\beta T(t)\right\| \leqslant C' t^{-\gamma - Re\,\beta - 1}, \quad \textit{for all } t > 0;$$

(v) *If $\beta > 1+\gamma$, then $D(A^\beta) \subset \Sigma_T = \{x \in X;\ \lim_{t\to 0;\, t>0} T(t)x = x\}$.*

Remark 2.1. We note that the condition (ii) of the proposition does not satisfy for $t = 0$ or $s = 0$.

Recall that semigroups of growth $1+\gamma$ were investigated earlier in [10,41].

The relation between the resolvent operators of A and the semigroup $T(t)$ is characterized by

Proposition 2.3. *(See [38, Theorem 3.13].) Let $A \in \Theta^\gamma_\omega(X)$ with $-1 < \gamma < 0$ and $0 < \omega < \pi/2$. Then for every $\lambda \in \mathbb{C}$ with $Re\,\lambda > 0$, one has $R(\lambda, -A) = \int_0^\infty e^{-\lambda t}T(t)\,dt$.*

Below we briefly state the necessary notions and facts on fractional calculus. Let us begin with the following definition.

Definition 2.1. Let $f(t) \in L^1(I, X)$, $g_{m-\alpha} * f \in W^{m,1}(I, X)$ $(m \in \mathbb{N},\ 0 \leqslant m-1 < \alpha < m)$. The Riemann–Liouville fractional derivative of order α of f is defined by ${}_RD^\alpha_t f(t) := D^m_t(g_{m-\alpha} * f)(t) = D^m_t J^{m-\alpha}_t f(t)$, where $D^m_t := \frac{d^m}{dt^m}$.

Assume that $0 < \alpha < 1$. We mention that the Caputo definition for the fractional derivative incorporates the initial values of the function and of its integer derivatives of lower order and the relevant property that the derivative of a constant is zero is preserved. Moreover, the setting in (LCP) or (SLCP) determines the necessity to use the regularized fractional derivative (1.5). In particular, if, for example, one considers instead of (1.5) the Riemann–Liouville fractional derivative, but without subtracting $t^{-\alpha}u(0)$, then the appropriate initial data will be the limit value, as $t \to 0$, of the fractional integral of a solution of the order $1-\alpha$, not the limit value of the solution itself. On the other hand, note that for a smooth enough function $u(t)$, the Caputo fractional derivative ${}_cD_t^\alpha u$ can be written as

$$ {}_cD_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u'(s)\, ds. $$

In the physical literature the expression on the right is used as the basic object for formulating fractional diffusion equations (cf., e.g., [15]).

We summarize some properties on Riemann–Liouville integral and Caputo fractional derivative as follows (cf., e.g., [34,39,40]):

Proposition 2.4. *Let $\alpha, \beta > 0$. The following properties hold.*

(i) $J_t^\alpha J_t^\beta f = J_t^{\alpha+\beta} f$ *for all* $f \in L^1(I; X)$;
(ii) $J_t^\alpha (f * g) = J_t^\alpha f * g$ *for all* $g, f \in L^p(I; X)$ $(1 \leqslant p < +\infty)$;
(iii) *The Caputo fractional derivative* ${}_cD_t^\alpha$ *is a left inverse of* J_t^α:

$$ {}_cD_t^\alpha J_t^\alpha f = f, \quad \text{for all } f \in L^1(I; X), $$

but in general not a right inverse, in fact, for all $f(t) \in C^{m-1}(I; X)$ *with* $g_{m-\alpha} * f \in W^{m,1}(I, X)$ $(m \in \mathbb{N}$, $0 \leqslant m-1 < \alpha < m)$*, one has*

$$ J_t^\alpha {}_cD_t^\alpha f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0) g_{k+1}(t). $$

At the end of this section, we present some properties of two special functions. Denote by $E_{\alpha,\beta}$ the generalized Mittag-Leffler special function (cf., e.g., [29,34,39]) defined by

$$ E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} = \frac{1}{2\pi i} \int_{\Upsilon} \frac{\lambda^{\alpha-\beta} e^{\lambda}}{\lambda^\alpha - z}\, d\lambda, \quad \alpha, \beta > 0,\ z \in \mathbb{C}, $$

where Υ is a contour which starts and ends at $-\infty$ and encircles the disc $|\lambda| \leqslant |z|^{1/\alpha}$ counterclockwise. If $0 < \alpha < 1$, $\beta > 0$, then the asymptotic expansion of $E_{\alpha,\beta}$ as $z \to \infty$ is given by

$$ E_{\alpha,\beta}(z) = \begin{cases} \frac{1}{\alpha} z^{(1-\beta)/\alpha} \exp(z^{1/\alpha}) + \varepsilon_{\alpha,\beta}(z), & |\arg z| \leqslant \frac{1}{2}\alpha\pi, \\ \varepsilon_{\alpha,\beta}(z), & |\arg(-z)| < (1 - \frac{1}{2}\alpha)\pi, \end{cases} \tag{2.3} $$

where

$$ \varepsilon_{\alpha,\beta}(z) = -\sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(\beta - \alpha n)} + O\big(|z|^{-N}\big), \quad \text{as } z \to \infty. $$

For short, set

$$E_\alpha(z) := E_{\alpha,1}(z), \qquad e_\alpha(z) := E_{\alpha,\alpha}(z).$$

Then we have

$${}_cD_t^\alpha E\big(\omega t^\alpha\big) = \omega E\big(\omega t^\alpha\big), \qquad J_t^{1-\alpha}\big(t^{\alpha-1}e_\alpha\big(\omega t^\alpha\big)\big) = E_\alpha\big(\omega t^\alpha\big).$$

Consider also the function of Wright-type

$$\Psi_\alpha(z) := \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(-\alpha n + 1 - \alpha)} = \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{(-z)^n}{(n-1)!}\Gamma(n\alpha)\sin(n\pi\alpha), \quad z \in \mathbb{C},$$

with $0 < \alpha < 1$. For $-1 < r < \infty$, $\lambda > 0$, the following results hold.

(W_1) $\Psi_\alpha(t) \geqslant 0$, $t > 0$;
(W_2) $\int_0^\infty \frac{\alpha}{t^{\alpha+1}}\Psi_\alpha(\frac{1}{t^\alpha})e^{-\lambda t}\,dt = e^{-\lambda^\alpha}$;
(W_3) $\int_0^\infty \Psi_\alpha(t)t^r\,dt = \frac{\Gamma(1+r)}{\Gamma(1+\alpha r)}$;
(W_4) $\int_0^\infty \Psi_\alpha(t)e^{-zt}\,dt = E_\alpha(-z)$, $z \in \mathbb{C}$;
(W_5) $\int_0^\infty \alpha t\Psi_\alpha(t)e^{-zt}\,dt = e_\alpha(-z)$, $z \in \mathbb{C}$.

3. Properties of the operators $\mathcal{S}_\alpha(t)$ and $\mathcal{P}_\alpha(t)$

Throughout this section we let A be an operator in the class $\Theta_\omega^\gamma(X)$ and $-1 < \gamma < 0$, $0 < \omega < \pi/2$. In the sequel, we will define two families of operators based on the generalized Mittag-Leffler-type functions and the resolvent operators associated with A. They will be two families of linear and bounded operators. In order to check the properties of the families, we will need a third object, namely the semigroup associated with A. We stress that these families will be used very frequently throughout the rest of this paper. Below the letter C will denote various positive constants.

Define operator families $\{\mathcal{S}_\alpha(t)\}|_{t\in S^0_{\frac{\pi}{2}-\omega}}$, $\{\mathcal{P}_\alpha(t)\}|_{t\in S^0_{\frac{\pi}{2}-\omega}}$ by

$$\mathcal{S}_\alpha(t) := E_\alpha\big(-zt^\alpha\big)(A) = \frac{1}{2\pi i}\int_{\Gamma_\theta} E_\alpha\big(-zt^\alpha\big)R(z;A)\,dz,$$

$$\mathcal{P}_\alpha(t) := e_\alpha\big(-zt^\alpha\big)(A) = \frac{1}{2\pi i}\int_{\Gamma_\theta} e_\alpha\big(-zt^\alpha\big)R(z;A)\,dz,$$

where the integral contour $\Gamma_\theta := \{\mathbb{R}_+e^{i\theta}\} \cup \{\mathbb{R}_+e^{-i\theta}\}$ is oriented counter-clockwise and $\omega < \theta < \mu < \frac{\pi}{2} - |\arg t|$.

We need some basic properties of these families which are used further in this paper.

Theorem 3.1. *For each fixed $t \in S^0_{\frac{\pi}{2}-\omega}$, $\mathcal{S}_\alpha(t)$ and $\mathcal{P}_\alpha(t)$ are linear and bounded operators on X. Moreover, there exist constants $C_s = C(\alpha,\gamma) > 0$, $C_p = C(\alpha,\gamma) > 0$ such that for all $t > 0$,*

$$\big\|\mathcal{S}_\alpha(t)\big\| \leqslant C_s t^{-\alpha(1+\gamma)}, \qquad \big\|\mathcal{P}_\alpha(t)\big\| \leqslant C_p t^{-\alpha(1+\gamma)}. \tag{3.1}$$

Proof. Note, from the asymptotic expansion of $E_{\alpha,\beta}$ that for each fixed $t \in S^0_{\frac{\pi}{2}-\omega}$, $E_\alpha(-zt^\alpha)$, $e_\alpha(-zt^\alpha) \in \mathcal{F}_0^\gamma(S^0_\mu)$. Therefore, by (2.1), the operator families $\{\mathcal{S}_\alpha(t)\}|_{t\in S^0_{\frac{\pi}{2}-\omega}}$, $\{\mathcal{P}_\alpha(t)\}|_{t\in S^0_{\frac{\pi}{2}-\omega}}$ are well defined, and for each $t \in S^0_{\frac{\pi}{2}-\omega}$, $\mathcal{S}_\alpha(t)$ and $\mathcal{P}_\alpha(t)$ are linear bounded operators on X. So, to prove the theorem, it is sufficient to prove that the estimates in (3.1) hold.

Let $T(t)$, $t \in S^0_{\frac{\pi}{2}-\omega}$, be the semigroup defined by (2.2). Then by (W_4) and the Fubini Theorem, we get

$$\begin{aligned}\mathcal{S}_\alpha(t)x &= \frac{1}{2\pi i}\int_{\Gamma_\theta} E_\alpha\big(-zt^\alpha\big)R(z;A)x\,dz\\ &= \frac{1}{2\pi i}\int_0^\infty \Psi_\alpha(\lambda)\int_{\Gamma_\theta} e^{-\lambda z t^\alpha}R(z;A)x\,dz\,d\lambda\\ &= \int_0^\infty \Psi_\alpha(s)T\big(st^\alpha\big)x\,ds, \quad t\in S^0_{\frac{\pi}{2}-\omega},\ x\in X. \end{aligned} \tag{3.2}$$

A similar argument shows that

$$\mathcal{P}_\alpha(t)x = \int_0^\infty \alpha s\Psi_\alpha(s)T\big(st^\alpha\big)x\,ds, \quad t\in S^0_{\frac{\pi}{2}-\omega},\ x\in X. \tag{3.3}$$

Hence, by (3.2), (3.3), Proposition 2.2(iii), (W_1) and (W_3), we have

$$\begin{aligned}\big\|\mathcal{S}_\alpha(t)x\big\| &\leqslant C_0\int_0^\infty \Psi_\alpha(s)s^{-(1+\gamma)}t^{-\alpha(1+\gamma)}\|x\|\,ds\\ &\leqslant C_0\frac{\Gamma(-\gamma)}{\Gamma(1-\alpha(1+\gamma))}t^{-\alpha(1+\gamma)}\|x\|, \quad t>0,\ x\in X,\\ \big\|\mathcal{P}_\alpha(t)x\big\| &\leqslant \alpha C_0\int_0^\infty \Psi_\alpha(s)s^{-\gamma}t^{-\alpha(1+\gamma)}\|x\|\,ds\\ &\leqslant \alpha C_0\frac{\Gamma(1-\gamma)}{\Gamma(1-\alpha\gamma)}t^{-\alpha(1+\gamma)}\|x\|, \quad t>0,\ x\in X.\end{aligned}$$

Therefore the estimates in (3.1) hold. This completes the proof. □

From now on, we will frequently use the representations (3.2) and (3.3) for operators $\mathcal{S}_\alpha(t)$ and $\mathcal{P}_\alpha(t)$, respectively.

Theorem 3.2. *For $t>0$, $\mathcal{S}_\alpha(t)$ and $\mathcal{P}_\alpha(t)$ are continuous in the uniform operator topology. Moreover, for every $r>0$, the continuity is uniform on $[r,\infty)$.*

Proof. Let $\epsilon>0$ be given. For every $r>0$, it follows from (W_3) that we may choose $\delta_1, \delta_2 > 0$ such that

$$\frac{2C_0}{r^{\alpha(1+\gamma)}}\int_0^{\delta_1}\Psi_\alpha(s)s^{-(1+\gamma)}\,ds\leqslant\frac{\epsilon}{3},\qquad \frac{2C_0}{r^{\alpha(1+\gamma)}}\int_{\delta_2}^{\infty}\Psi_\alpha(s)s^{-(1+\gamma)}\,ds\leqslant\frac{\epsilon}{3}. \tag{3.4}$$

Then we deduce, by Proposition 2.2(i), that there exists a positive constant δ such that

$$\int_{\delta_1}^{\delta_2}\Psi_\alpha(s)\left\|T\left(t_1^\alpha s\right)-T\left(t_2^\alpha s\right)\right\|ds\leqslant\frac{\epsilon}{3}, \tag{3.5}$$

for $t_1,t_2\geqslant r$ and $|t_1-t_2|<\delta$.

On the other hand, using (3.4), (3.5) and Theorem 3.1, we get

$$\begin{aligned}
&\left\|\mathcal{S}_\alpha(t_1)x-\mathcal{S}_\alpha(t_2)x\right\|\\
&\quad\leqslant\int_0^{\delta_1}\Psi_\alpha(s)\left(\left\|T\left(t_1^\alpha s\right)\right\|+\left\|T\left(t_2^\alpha s\right)\right\|\right)\|x\|\,ds+\int_{\delta_1}^{\delta_2}\Psi_\alpha(s)\left\|T\left(t_1^\alpha s\right)-T\left(t_2^\alpha s\right)\right\|\|x\|\,ds\\
&\qquad+\int_{\delta_2}^{\infty}\Psi_\alpha(s)\left(\left\|T\left(t_1^\alpha s\right)\right\|+\left\|T\left(t_2^\alpha s\right)\right\|\right)\|x\|\,ds\\
&\quad\leqslant\frac{2C_0}{r^{\alpha(1+\gamma)}}\int_0^{\delta_1}\Psi_\alpha(s)s^{-(1+\gamma)}\|x\|\,ds+\int_{\delta_1}^{\delta_2}\Psi_\alpha(s)\left\|T\left(t_1^\alpha s\right)-T\left(t_2^\alpha s\right)\right\|\|x\|\,ds\\
&\qquad+\frac{2C_0}{r^{\alpha(1+\gamma)}}\int_{\delta_2}^{\infty}\Psi_\alpha(s)s^{-(1+\gamma)}\|x\|\,ds\\
&\quad\leqslant\epsilon\|x\|,\quad\text{for any } x\in X,
\end{aligned}$$

that is,

$$\left\|\mathcal{S}_\alpha(t_1)-\mathcal{S}_\alpha(t_2)\right\|\leqslant\epsilon,$$

which implies that $\mathcal{S}_\alpha(t)$ is uniformly continuous on $[r,\infty)$ in the uniform operator topology and hence, by the arbitrariness of $r>0$, $\mathcal{S}_\alpha(t)$ is continuous in the uniform operator topology for $t>0$. A similar argument enables us to give the characterization of continuity on $\mathcal{P}_\alpha(t)$. This completes the proof. □

Theorem 3.3. *Let $0<\beta<1-\gamma$. Then*

(i) *the range $R(\mathcal{P}_\alpha(t))$ of $\mathcal{P}_\alpha(t)$ for $t>0$, is contained in $D(A^\beta)$;*
(ii) *$\mathcal{S}_\alpha'(t)x=-t^{\alpha-1}A\mathcal{P}_\alpha(t)x$ $(x\in X)$, and $\mathcal{S}_\alpha'(t)x$ for $x\in D(A)$ is locally integrable on $(0,\infty)$;*
(iii) *for all $x\in D(A)$ and $t>0$, $\|A\mathcal{S}_\alpha(t)x\|\leqslant Ct^{-\alpha(1+\gamma)}\|Ax\|$, here C is a constant depending on γ,α.*

Proof. It follows from Proposition 2.2(iv) that for all $x\in X$, $t>0$, $T(t)x\in D(A^\beta)$ with $\beta>0$. Therefore, in view of (3.3), Proposition 2.2(iv) and (W_3), we have

$$\begin{aligned}\left\|A^{\beta}\mathcal{P}_\alpha(t)x\right\| &\leqslant \int_0^\infty \alpha s\Psi_\alpha(s)\left\|A^{\beta}T\left(t^\alpha s\right)\right\|\|x\|\,ds\\ &\leqslant \alpha C' t^{-\alpha(\gamma+\beta+1)}\int_0^\infty \Psi_\alpha(s)s^{-(\beta+\gamma)}\,ds\,\|x\|\\ &\leqslant \alpha C'\frac{\Gamma(1-\beta-\gamma)}{\Gamma(1-\alpha(\beta+\gamma+1))}t^{-\alpha(1+\beta+\gamma)}\|x\|,\end{aligned}$$

which implies that the assertion (i) holds.

From (i), it is easy to see that for all $x\in X$, $\mathcal{S}'_\alpha(t)x=-t^{\alpha-1}A\mathcal{P}_\alpha(t)x$. Moreover, for every $x\in D(A)$, one has by Proposition 2.2(iv),

$$\left\|t^{\alpha-1}A\mathcal{P}_\alpha(t)x\right\|\leqslant t^{\alpha-1}\int_0^\infty \alpha s\Psi_\alpha(s)\left\|T\left(t^\alpha s\right)\right\|\|Ax\|\,ds\leqslant \alpha C_0\frac{\Gamma(1-\gamma)}{\Gamma(1-\alpha\gamma)}t^{-\alpha\gamma-1}\|Ax\|.$$

Since $-\alpha\gamma-1>-1$, this shows that $\mathcal{S}'_\alpha(t)x$ for each $x\in D(A)$ is locally integrable on $(0,\infty)$, that is, (ii) is true.

Moreover, Proposition 2.2(iv) and (3.2) imply that

$$\begin{aligned}\left\|A\mathcal{S}_\alpha(t)x\right\| &\leqslant C_0 t^{-\alpha(1+\gamma)}\int_0^\infty \Psi_\alpha(s)s^{-1-\gamma}\,ds\|Ax\|\\ &\leqslant C_0\frac{\Gamma(-\gamma)}{\Gamma(1-\alpha(1+\gamma))}t^{-\alpha(1+\gamma)}\|Ax\|,\quad x\in D(A).\end{aligned}$$

This means that (iii) holds, and completes the proof. □

Remark 3.1. Particularly, from the proof of Theorem 3.3(i) we can conclude that

$$\left\|A\mathcal{P}_\alpha(t)\right\|\leqslant Ct^{-\alpha(2+\gamma)},$$

where C is a constant depending on γ,α. Moreover, using a similar argument with that in Theorem 3.2, we have that $A\mathcal{P}_\alpha(t)$ for $t>0$ is continuous in the uniform operator topology.

Theorem 3.4. *The following properties hold.*

(i) *Let $\beta>1+\gamma$. For all $x\in D(A^\beta)$, $\lim_{t\to 0;t>0}\mathcal{S}_\alpha(t)x=x$;*
(ii) *For all $x\in D(A)$, $(\mathcal{S}_\alpha(t)-I)x=\int_0^t -s^{\alpha-1}A\mathcal{P}_\alpha(s)x\,ds$;*
(iii) *For all $x\in D(A)$, $t>0$, $D_t^\alpha\mathcal{S}_\alpha(t)x=-A\mathcal{S}_\alpha(t)x$;*
(iv) *For all $t>0$, $\mathcal{S}_\alpha(t)=J_t^{1-\alpha}(t^{\alpha-1}\mathcal{P}_\alpha(t))$.*

Proof. For any $x\in X$, note by (3.2) and (W_3) that

$$\mathcal{S}_\alpha(t)x-x=\int_0^\infty \Psi_\alpha(s)\left(T\left(t^\alpha s\right)x-x\right)ds.$$

On the other hand, by Theorem 2.2(v) it follows that $D(A^\beta) \subset \Sigma_T$ in view of $\beta > 1 + \gamma$. Therefore, we deduce, using Proposition 2.2(iii), that for any $x \in D(A^\beta)$, there exists a function $\eta(s) \in L^1(0, +\infty)$ depending on $\Psi_\alpha(s)$ such that

$$\left\| \Psi_\alpha(s)\bigl(T\bigl(t^\alpha s\bigr)x - x\bigr)\right| \leqslant \eta(s).$$

Hence, by means of the Lebesgue dominated convergence theorem we obtain

$$\mathcal{S}_\alpha(t)x - x \to 0, \quad \text{as } t \to 0,$$

that is, the assertion (i) remains true.

From (i) and Theorem 3.3(ii) we get for all $x \in D(A)$,

$$\bigl(\mathcal{S}_\alpha(t) - I\bigr)x = \lim_{s\to 0}\bigl(\mathcal{S}_\alpha(t)x - \mathcal{S}_\alpha(s)x\bigr) = \int_0^t -\lambda^{\alpha-1} A\mathcal{P}_\alpha(\lambda)x\,d\lambda,$$

which implies that the assertion (ii) holds.

To prove (iii), first it is easy to see that $\frac{1}{\varphi_0} \in \mathcal{F}(S_\mu^0)$ and the operator $\varphi_0(A)$ is injective. Taking $x \in D(A)$, by Proposition 2.1(iii) one has

$$\mathcal{S}_\alpha(t)x = E_\alpha\bigl(-zt^\alpha\bigr)(A)x = \bigl(E_\alpha\bigl(-zt^\alpha\bigr)\varphi_0\bigr)(A)\left(\frac{1}{\varphi_0}\right)(A)x.$$

Moreover, by (2.3), we have $\sup_{z\to\infty} |zt^\alpha E_\alpha(-zt^\alpha)| < \infty$, which implies that

$$\left|zE_\alpha\bigl(-zt^\alpha\bigr)(1+z)^{-1}\right| \leqslant C|z|^{-1}t^{-\alpha}, \quad \text{as } z \to \infty,$$

where C is a constant which is independent of t. Consequently,

$$-zE_\alpha\bigl(-zt^\alpha\bigr)(1+z)^{-1} \in \mathcal{F}_0^\gamma\bigl(S_\mu^0\bigr). \tag{3.6}$$

Notice also that

$${}_cD_t^\alpha E_\alpha\bigl(-zt^\alpha\bigr)(1+z)^{-1}R(z;A) = (-z)E_\alpha\bigl(-zt^\alpha\bigr)(1+z)^{-1}R(z;A).$$

Combining Proposition 2.1(ii) and (3.6), we get

$$\begin{aligned}
{}_cD_t^\alpha\bigl(\bigl(E_\alpha\bigl(-zt^\alpha\bigr)\bigl(1+z^\beta\bigr)^{-1}\bigr)(A)\bigr) &= \frac{1}{2\pi i}\int_{\Gamma_\theta} (-z)E_\alpha\bigl(-zt^\alpha\bigr)(1+z)^{-1}R(z;A)\,dz \\
&= (-z)(A)\bigl(E_\alpha\bigl(-zt^\alpha\bigr)(1+z)^{-1}\bigr)(A) \\
&= -A\bigl(E_\alpha\bigl(-zt^\alpha\bigr)(1+z)^{-1}\bigr)(A).
\end{aligned}$$

Hence, we obtain

$$\begin{aligned} D_t^\alpha \mathcal{S}_\alpha(t)x &= -A\big(E_\alpha\big(-zt^\alpha\big)(1+z)^{-1}\big)(A)(1+z)(A)x \\ &= -A\big(E_\alpha\big(-zt^\alpha\big)\big)(A)x \\ &= -A\mathcal{S}_\alpha(t)x. \end{aligned}$$

This proves (iii).

For (iv), by a similar argument with (iii), one can prove that $t^{\alpha-1}e_\alpha(-zt^\alpha)$ belongs to $\mathcal{F}_0^\gamma(S_\mu^0)$ for $t>0$ and hence

$$J_t^\alpha\big(t^{\alpha-1}\mathcal{P}_\alpha(t)\big) = J_t^\alpha\big(t^{\alpha-1}e_\alpha\big(-zt^\alpha\big)(A)\big) = \big(E_\alpha\big(-zt^\alpha\big)\big)(A) = \mathcal{S}_\alpha(t),$$

in view of $J_t^\alpha(t^{\alpha-1}e_\alpha(-zt^\alpha)) = E_\alpha(-zt^\alpha)$. This completes the proof. □

Before proceeding with our theory further, we present the following result.

Lemma 3.1. *If $R(\lambda, -A)$ is compact for every $\lambda>0$, then $T(t)$ is compact for every $t>0$.*

Proof. Note first that as a consequence of Theorem 3.13 in [38], for every $\lambda\in\mathbb{C}$ with $Re\,\lambda>0$, $R(\lambda; -A) = \int_0^\infty e^{-\lambda s}T(s)\,ds$ defines a bounded linear operator on X. Therefore, we obtain

$$\lambda R(\lambda; -A)T(t) - T(t) = \lambda\int_0^\infty e^{-\lambda s}\big(T(t+s)-T(t)\big)\,ds. \tag{3.7}$$

Let $\epsilon>0$ be given. For every $\lambda>0$ and $t>0$, it follows from Theorem 3.2 that there exists a $\nu>0$ such that $\sup_{s\in[0,\nu]}\|T(s+t)-T(t)\| \leqslant \frac{\epsilon}{2}$. So

$$\lambda\int_0^\nu e^{-s\lambda}\big\|T(t+s)-T(t)\big\|\,ds \leqslant \frac{\epsilon}{2}. \tag{3.8}$$

On the other hand, by Theorem 2.2(iii), we get

$$\begin{aligned} \lambda\left\|\int_\nu^\infty e^{-s\lambda}\big(T(s+t)-T(t)\big)\,ds\right\| &\leqslant \lambda C\int_\nu^\infty e^{-s\lambda}\big((t+s)^{-1-\gamma}+t^{-\gamma-1}\big)\,ds \\ &\leqslant 2Ct^{-\gamma-1}e^{-\lambda\nu}, \end{aligned}$$

which implies that there exists a $\lambda_0>0$ large enough such that

$$\lambda\left\|\int_\nu^\infty e^{-s\lambda}\big(T(s+t)-T(t)\big)\,ds\right\| \leqslant \frac{\epsilon}{2}, \quad \lambda\geqslant\lambda_0. \tag{3.9}$$

Thus, for all $\lambda\geqslant\lambda_0$, using (3.7), (3.8) and (3.9) we deduce that

$$\big\|\lambda R(\lambda; -A)T(t)-T(t)\big\| \leqslant \lambda\int_0^\nu e^{-s\lambda}\big\|T(t+s)-T(t)\big\|\,ds$$

$$+\lambda\int_{\nu}^{\infty} e^{-s\lambda}\big\|T(s+t)-T(t)\big\|\,ds$$
$$\leqslant \epsilon.$$

It follows from the arbitrariness of $\nu > 0$ that

$$\lim_{\lambda\to\infty}\big\|\lambda R(\lambda;-A)T(t)-T(t)\big\| = 0.$$

Since $\lambda R(\lambda;-A)T(t)$ is compact for every $\lambda > 0$ and $t > 0$, $T(t)$ is compact for every $t > 0$. □

With the help of this lemma we now show the following result.

Theorem 3.5. *If $R(\lambda, -A)$ is compact for every $\lambda > 0$, then $\mathcal{S}_\alpha(t)$, $\mathcal{P}_\alpha(t)$ are compact for every $t > 0$.*

Proof. Let $\epsilon > 0$ be arbitrary. Put

$$\varsigma_\epsilon(t) := \int_{\epsilon}^{\infty}\Psi_\alpha(s)T\big(st^\alpha-\epsilon t^\alpha\big)\,ds, \qquad \zeta_\epsilon(t) := \int_{\epsilon}^{\infty}\Psi_\alpha(s)T\big(st^\alpha\big)\,ds.$$

Then, $\zeta_\epsilon(t) = T(\epsilon t^\alpha)\varsigma_\epsilon(t)$, and it is easy to prove that for every $t > 0$, $\varsigma_\epsilon(t)$ is a bounded linear operator on X. Therefore, by the compactness of $T(t)$, $t > 0$, we see that $\zeta_\epsilon(t)$ is compact for every $t > 0$.

On the other hand, note that

$$\big\|\zeta_\epsilon(t)-\mathcal{S}_\alpha(t)\big\| \leqslant \left\|\int_0^{\epsilon}\Psi_\alpha(s)T\big(st^\alpha\big)\,ds\right\| \leqslant C_0 t^{-\alpha(1+\gamma)}\int_0^{\epsilon}\Psi_\alpha(s)s^{-1-\gamma}\,ds.$$

Hence, it follows from the compactness of $\zeta_\epsilon(t)$, $t > 0$, that $\mathcal{S}_\alpha(t)$ is compact for every $t > 0$. By a similar technique we can conclude that $\mathcal{P}_\alpha(t)$ is compact for every $t > 0$. The proof is completed. □

4. Linear problems

Let $A \in \Theta_\omega^\gamma(X)$ with $-1 < \gamma < 0$ and $0 < \omega < \pi/2$. We discuss the existence and uniqueness of mild solution and classical solutions for the inhomogeneous linear abstract Cauchy problem

$$\begin{cases} {}_cD_t^\alpha u(t) + Au(t) = f(t), & 0 < t \leqslant T, \\ u(0) = u_0, \end{cases} \tag{LCP}$$

where ${}_cD_t^\alpha$, $0 < \alpha < 1$, is the Caputo fractional derivative of order α, and u_0 is given belonging to a subset of X.

Assumption. Assume that $u(\cdot) : [0, T] \to X$ is a function such that

(H^*) $u \in C([0,T];X)$, $g_{1-\alpha} * u \in C^1((0,T];X)$, $u(t) \in D(A)$ for $t \in (0,T]$, $Au \in L^1((0,T);X)$, and u satisfies (LCP).

Then, by Definitions 1.2 and 1.3, one can rewrite (LCP) as

$$u(t) = u_0 - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Au(s)\,ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds, \tag{4.1}$$

for $t \in [0, T]$.

Before presenting the definition of mild solution of problem (LCP), we first prove the following lemma.

Lemma 4.1. *If $u : [0, T] \to X$ is a function satisfying Assumption (H^*), then $u(t)$ satisfies the following integral equation*

$$u(t) = \mathcal{S}_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1} \mathcal{P}_\alpha(t-s) f(s)\,ds, \quad t \in (0, T].$$

Proof. Note that the Laplace transform of an abstract function $f \in L^1(\mathbb{R}^+, X)$ is defined by $\widehat{f}(\lambda) := \int_0^\infty e^{-\lambda t} f(t)\,dt$ $(\lambda > 0)$. Applying the Laplace transform to (4.1) we get $\widehat{u}(\lambda) = \frac{u_0}{\lambda} - \frac{1}{\lambda^\alpha} A\widehat{u}(\lambda) + \frac{\widehat{f}(\lambda)}{\lambda^\alpha}$, that is,

$$\widehat{u}(\lambda) = \lambda^{\alpha-1}\left(\lambda^\alpha + A\right)^{-1} u_0 + \left(\lambda^\alpha + A\right)^{-1} \widehat{f}(\lambda).$$

On the other hand, using Proposition 2.3 and (W_2) we deduce that

$$\begin{aligned}
&\lambda^{\alpha-1}\left(\lambda^\alpha + A\right)^{-1} u_0 + \left(\lambda^\alpha + A\right)^{-1} \widehat{f}(\lambda) \\
&\quad = \lambda^{\alpha-1} \int_0^\infty e^{-\lambda^\alpha t} T(t) u_0\,dt + \int_0^\infty e^{-\lambda^\alpha t} T(t) \widehat{f}(\lambda)\,dt \\
&\quad = \int_0^\infty \frac{d}{d\lambda} e^{-(\lambda t)^\alpha} T\left(t^\alpha\right) u_0\,dt + \int_0^\infty \int_0^\infty \alpha t^{\alpha-1} e^{-(\lambda t)^\alpha t} T\left(t^\alpha\right) f(s) e^{-s\lambda}\,ds\,dt \\
&\quad = \int_0^\infty \int_0^\infty \frac{\alpha t}{\tau^\alpha} \Psi_\alpha\left(\frac{1}{\tau^\alpha}\right) e^{-\lambda t \tau} T\left(t^\alpha\right) d\tau\,dt \\
&\qquad + \int_0^\infty \int_0^\infty \int_0^\infty \frac{\alpha}{\tau^{2\alpha}} t^{\alpha-1} \Psi\left(\frac{1}{\tau^\alpha}\right) e^{-\lambda t} T\left(\frac{t^\alpha}{\tau^\alpha}\right) f(s) e^{-s\lambda}\,d\tau\,ds\,dt \\
&\quad = \int_0^\infty \int_0^\infty \frac{\alpha}{\tau^{\alpha+1}} \Psi_\alpha\left(\frac{1}{\tau^\alpha}\right) e^{-\lambda t} T\left(\frac{t^\alpha}{\tau^\alpha}\right) d\tau\,dt \\
&\qquad + \int_0^\infty \int_0^\infty \int_0^\infty \alpha \tau t^{\alpha-1} \Psi(\tau) T\left(t^\alpha \tau\right) f(s) e^{-(s+t)\lambda}\,d\tau\,ds\,dt
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty e^{-\lambda t} \int_0^\infty \Psi_\alpha(\tau) T\big(t^\alpha \tau\big)\, d\tau\, dt \\
&\quad + \int_0^\infty e^{-t\lambda} \int_0^t (t-s)^{\alpha-1} f(s) \left(\int_0^\infty \alpha\tau\Psi(\tau) T\big((t-s)^\alpha \tau\big)\, d\tau \right) ds\, dt \\
&= \int_0^\infty e^{-\lambda t} \mathcal{S}_\alpha(t)\, dt + \int_0^\infty e^{-\lambda t} \int_0^t (t-s)^\alpha \mathcal{P}_\alpha(t-s) f(s)\, ds\, dt \\
&= \int_0^\infty e^{-\lambda t} \left(\mathcal{S}_\alpha(t) u_0 + \int_0^t (t-s)^{\alpha-1} \mathcal{P}_\alpha(t-s) f(s)\, ds \right) dt.
\end{aligned}
$$

This implies that

$$
\widehat{u}(\lambda) = \int_0^\infty e^{-\lambda t} \left(\mathcal{S}_\alpha(t) u_0 + \int_0^t (t-s)^{\alpha-1} \mathcal{P}_\alpha(t-s) f(s)\, ds \right) dt.
$$

Now using the uniqueness of the Laplace transform (cf. [43, Theorem 1.1.6]), we deduce that the assertion of the lemma holds. This completes the proof. □

Motivated by Lemma 4.1, we adopt the following concept of mild solution to problem (LCP).

Definition 4.1. By a mild solution of problem (LCP), we mean a function $u \in C((0,T];X)$ satisfying

$$
u(t) = \mathcal{S}_\alpha(t) u_0 + \int_0^t (t-s)^{\alpha-1} \mathcal{P}_\alpha(t-s) f(s)\, ds, \quad t \in (0,T].
$$

Remark 4.1. It is to be noted that:

(a) Unlike the case of strongly continuous operator semigroups, we do not require the mild solution of problem (LCP) to be continuous at $t=0$. Moreover, in general, since the operator $\mathcal{S}_\alpha(t)$ is singular at $t=0$, solutions to problem (LCP) are assumed to have the same kind of singularity at $t=0$ as the operator $\mathcal{S}_\alpha(t)$. This is the case, for instance, if $f \equiv 0$ so that we have that $u(t) = \mathcal{S}_\alpha(t) u_0$, which presents a discontinuity at the initial time;
(b) When $u_0 \in D(A^\beta)$, $\beta > 1+\gamma$, it follows from Theorem 3.4(i) that the mild solution is continuous at $t=0$.

For $f \in L^1((0,T);X)$, the initial problem (LCP) has a unique mild solution for every $u_0 \in X$. We will now be interested in imposing further condition on f and u_0 so that the mild solution will become a classical solution. To this end we first introduce the following definition.

Definition 4.2. By a classical solution to problem (LCP), we mean a function $u(t) \in C([0,T];X)$ with ${}_cD_t^\alpha u(t) \in C((0,T];X)$, which, for all $t \in (0,T]$, takes values in $D(A)$ and satisfies (LCP).

We are now ready to state our main result in this section.

Theorem 4.1. *Let $A \in \Theta_\omega^\gamma(X)$ with $0 < \omega < \frac{\pi}{2}$. Suppose that $f(t) \in D(A)$ for all $0 < t \leqslant T$, $Af(t) \in L^\infty((0,T);X)$, and $f(t)$ is Hölder continuous with an exponent $\theta' > \alpha(1+\gamma)$, that is,*

$$\|f(t) - f(s)\| \leqslant K|t-s|^{\theta'}, \quad \text{for all } 0 < t, s \leqslant T.$$

Then, for every $u_0 \in D(A)$, there exists a classical solution to problem (LCP) *and this solution is unique.*

Proof. For $u_0 \in D(A)$, let $u(t) = \mathcal{S}_\alpha(t)u_0$ $(t > 0)$. Then it follows from Theorem 3.4(i), (iii) that $u(t)$ is a classical solution of the following problem

$$\begin{cases} {}_cD_t^\alpha u(t) + Au(t) = 0, \quad 0 < t \leqslant T, \\ u(0) = u_0. \end{cases} \tag{4.2}$$

Moreover, from Lemma 4.1, it is easy to see that $u(t)$ is the only solution to problem (4.2). Put

$$w(t) = \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)f(s)\,ds, \quad 0 < t \leqslant T.$$

Then from the assumptions on f and Theorem 3.1 we obtain

$$\begin{aligned} \|Aw(t)\| &\leqslant \int_0^t (t-s)^{\alpha-1}\|\mathcal{P}_\alpha(t-s)\|\|Af(t)\|_{L^\infty((0,T);X)}\,ds \\ &\leqslant C_p\|Af(t)\|_{L^\infty((0,T);X)}\frac{1}{-\alpha\gamma}t^{-\gamma\alpha}, \end{aligned}$$

which implies that $w(t) \in D(A)$ for all $0 < t \leqslant T$.

Next, we show ${}_cD_t^\alpha w(t) \in C((0,T];X)$. Since $w(0) = 0$ and hence

$${}_cD_t^\alpha w(t) = D_t^1 J_t^{1-\alpha} w(t) = D_t^1\big(\big(J_t^{1-\alpha}\mathcal{Q}_\alpha\big) * f\big) = D_t^1(\mathcal{S}_\alpha * f), \tag{4.3}$$

in view of Proposition 2.4 and Theorem 3.4(iv), where $\mathcal{Q}_\alpha(t) := t^{\alpha-1}\mathcal{P}_\alpha(t)$, it remains to prove $v(t) := (\mathcal{S}_\alpha * f)(t) \in C^1((0,T];X)$. Let $h > 0$ and $h \leqslant T - t$. Then it is easy to verify the identity

$$\frac{v(t+h) - v(t)}{h} = \int_0^t \frac{\mathcal{S}_\alpha(t+h-s) - \mathcal{S}_\alpha(t-s)}{h} f(s)\,ds + \frac{1}{h}\int_t^{t+h} \mathcal{S}_\alpha(t+h-s)f(s)\,ds.$$

Again by the assumptions on f and Theorem 3.1, we have, for $t > 0$ fixed,

$$\|(t-s)^{\alpha-1}A\mathcal{P}_\alpha(t-s)f(s)\| \leqslant C_p(t-s)^{-\alpha\gamma-1}\|Af(s)\| \in L^1\big((0,T);X\big),$$

for all $s \in [0,t)$. Therefore, using Theorem 3.3(ii) and the Dominated Convergence Theorem we get

$$\begin{aligned} \lim_{h\to 0}\int_0^t \frac{\mathcal{S}_\alpha(t+h-s) - \mathcal{S}_\alpha(t-s)}{h} f(s)\,ds &= \int_0^t (t-s)^{\alpha-1}(-A)\mathcal{P}_\alpha(t-s)f(s)\,ds \\ &= -Aw(t). \end{aligned} \tag{4.4}$$

Furthermore, note that

$$\frac{1}{h}\int_t^{t+h}\mathcal{S}_\alpha(t+h-s)f(s)\,ds=\frac{1}{h}\int_0^h\mathcal{S}_\alpha(s)f(t+h-s)\,ds$$
$$=\frac{1}{h}\int_0^h\mathcal{S}_\alpha(s)\big(f(t+h-s)-f(t-s)\big)\,ds$$
$$+\frac{1}{h}\int_0^h\mathcal{S}_\alpha(s)\big(f(t-s)-f(t)\big)\,ds+\frac{1}{h}\int_0^h\mathcal{S}_\alpha(s)f(t)\,ds.$$

From Theorem 3.1 and the Hölder continuity on f we have

$$\frac{1}{h}\left\|\int_0^h\mathcal{S}_\alpha(s)\big(f(t+h-s)-f(t-s)\big)\,ds\right\|\leqslant\frac{C_sKh^{\theta'-\alpha(1+\gamma)}}{1-\alpha(1+\gamma)},$$
$$\frac{1}{h}\left\|\int_0^h\mathcal{S}_\alpha(s)\big(f(t-s)-f(t)\big)\,ds\right\|\leqslant\frac{C_sKh^{\theta'-\alpha(1+\gamma)}}{1+\theta-\alpha(1+\gamma)}.$$

Also, since $f(t)\in D(A)$ $(0<t\leqslant T)$, $\lim_{h\to 0}\frac{1}{h}\int_0^h\mathcal{S}_\alpha(s)f(t)\,ds=f(t)$ in view of Theorem 3.4(i). Hence,

$$\frac{1}{h}\int_t^{t+h}\mathcal{S}_\alpha(t+h-s)f(s)\,ds\to f(t)\quad\text{as } h\to 0^+. \tag{4.5}$$

Combining (4.4) and (4.5) we deduce that v is differentiable from the right at t and $v'_+(t)=f(t)-Aw(t)$ $(t\in(0,T])$. By a similar argument with the above, one has that v is differentiable from the left at t and $v'_-(t)=f(t)-Aw(t)$ $(t\in(0,T])$. Next, we prove $Aw(t)\in C((0,T];X)$. To the end, let $Aw(t)=I_1(t)+I_2(t)$, where

$$I_1(t):=\int_0^t(t-s)^{\alpha-1}A\mathcal{P}_\alpha(t-s)\big(f(s)-f(t)\big)\,ds,$$
$$I_2(t):=\int_0^t A(t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)f(t)\,ds.$$

By Theorem 3.4(ii), we obtain $I_2(t)=-(\mathcal{S}_\alpha(t)-I)f(t)$. So, by the assumption of f and Theorem 3.2, we see that $I_2(t)$ is continuous for $0<t\leqslant T$. To prove the same conclusion for $I_1(t)$, we let $0<h\leqslant T-t$ and write

$$I_1(t+h)-I_1(t)$$
$$=\int_0^t\big((t+h-s)^{\alpha-1}A\mathcal{P}_\alpha(t+h-s)-(t-s)^{\alpha-1}A\mathcal{P}_\alpha(t-s)\big)\big(f(s)-f(t)\big)\,ds$$

$$
\begin{aligned}
&+\int_0^t (t+h-s)^{\alpha-1} A\mathcal{P}_\alpha(t+h-s)\big(f(t)-f(t+h)\big)\,ds\\
&+\int_t^{t+h} (t+h-s)^{\alpha-1} A\mathcal{P}_\alpha(t+h-s)\big(f(s)-f(t+h)\big)\,ds\\
:=\ & h_1(t)+h_2(t)+h_3(t).
\end{aligned}
$$

For $h_1(t)$, on the one hand, it follows from Theorem 3.2 that

$$
\begin{aligned}
&\lim_{h\to 0}(t+h-s)^{\alpha-1} A\mathcal{P}_\alpha(t+h-s)\big(f(s)-f(t)\big)\\
&\quad=(t-s)^{\alpha-1} A\mathcal{P}_\alpha(t-s)\big(f(s)-f(t)\big).
\end{aligned}
$$

On the other hand, for $t\in(0,T]$ fixed, by Remark 3.1 and the assumption on f, we get

$$
\begin{aligned}
&\big\|(t+h-s)^{\alpha-1} A\mathcal{P}_\alpha(t+h-s)\big(f(s)-f(t)\big)\big\|\\
&\quad\leqslant C'_p K(t+h-s)^{-\alpha(1+\gamma)-1}(t-s)^{\theta'}\\
&\quad\leqslant C'_p K(t-s)^{(\theta'-\alpha-\alpha\gamma)-1}\in L^1\big((0,t);X\big).
\end{aligned}
$$

Thus, by means of the Dominated Convergence Theorem one has

$$
\begin{aligned}
&\lim_{h\to 0}\int_0^t (t+h-s)^{\alpha-1} A\mathcal{P}_\alpha(t+h-s)\big(f(s)-f(t)\big)\,ds\\
&\quad=\int_0^t (t-s)^{\alpha-1} A\mathcal{P}_\alpha(t-s)\big(f(s)-f(t)\big)\,ds,
\end{aligned}
$$

which implies that $h_1(t)\to 0$ as $h\to 0^+$.

For $h_2(t)$, using Theorem 3.3(i) and Remark 3.1, we obtain

$$
\begin{aligned}
&\int_0^t (t+h-s)^{\alpha-1}\big\|A\mathcal{P}_\alpha(t+h-s)\big\|_{\mathcal{L}[X]}\big\|f(t)-f(t+h)\big\|\,ds\\
&\quad\leqslant\int_0^t C'_p K(t+h-s)^{-\alpha(1+\gamma)-1}h^{\theta'}\,ds\\
&\quad=\frac{C'_p K h^{\theta'}}{\alpha(1+\gamma)}\big(h^{-\alpha(1+\gamma)}-(h+t)^{-\alpha(1+\gamma)}\big).
\end{aligned}
$$

This yields $h_2(t)\to 0$ as $h\to 0^+$.

Moreover, $h_3(t) \to 0$ as $h \to 0^+$ by the following estimate

$$\left\| \int_t^{t+h} (t+h-s)^{\alpha-1} \mathcal{P}_\alpha(t+h-s)\big(Af(s) - Af(t+h)\big)\, ds \right\| \leqslant \frac{2C_p}{-\alpha\gamma} \left\| Af(s) \right\|_{L^\infty(0,T;X)} h^{-\alpha\gamma}$$

in view of $Af(s) \in L^\infty((0,T);X)$ and Theorem 3.2.

The same reasoning gives $I_1(t-h) - I_1(h) \to 0$ as $h \to 0^+$. Consequently, $Aw \in C((0,T];X)$, which implies that $v' \in C((0,T];X)$, provided that f is continuous on $(0,T]$. Thus, by (4.3) we have ${}_cD_t^\alpha w \in C((0,T];X)$. Hence, we prove that $u + w$ is a classical solution to problem (LCP), and Lemma 4.1 implies that it is unique. This completes the proof. □

5. Nonlinear problems

In this section we apply the theory developed in the previous sections to the nonlinear fractional abstract Cauchy problem

$$\begin{cases} {}_cD_t^\alpha u(t) + Au(t) = f\big(t, u(t)\big), & t > 0, \\ u(0) = u_0, \end{cases} \tag{SLCP}$$

where $A \in \Theta_\omega^\gamma(X)$ with $0 < \omega < \frac{\pi}{2}$, and ${}_cD_t^\alpha$, $0 < \alpha < 1$, is the Caputo fractional derivative of order α.

Definition 5.1. By a mild solution to problem (SLCP), we mean a function $u \in C((0,T];X)$ satisfying $u(t) = \mathcal{S}_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s) f(s,u(s))\, ds$ $(t \in (0,T])$.

Theorem 5.1. *Let $A \in \Theta_\omega^\gamma(X)$ with $-1 < \gamma < -\frac{1}{2}$ and $0 < \omega < \frac{\pi}{2}$. Suppose that the nonlinear mapping $f : (0,T] \times X \to X$ is continuous with respect to t and there exist constants $M, N > 0$ such that*

$$\left\| f(t,x) - f(t,y) \right\| \leqslant M\big(1 + \|x\|^{\nu-1} + \|y\|^{\nu-1}\big)\|x - y\|,$$
$$\left\| f(t,x) \right\| \leqslant N\big(1 + \|x\|^\nu\big),$$

for all $t \in (0,T]$ and for each $x, y \in X$, where ν is a constant in $[1, -\frac{\gamma}{1+\gamma})$. Then, for every $u_0 \in X$, there exists a $T_0 > 0$ such that the problem (SLCP) *has a unique mild solution defined on* $(0, T_0]$.

Proof. For fixed $r > 0$, we introduce the metric space

$$F_r(T,u_0) = \big\{u \in C\big((0,T];X\big);\ \rho_T\big(u, \mathcal{S}_\alpha(t)u_0\big) \leqslant r\big\},$$
$$\rho_T(u_1,u_2) = \sup_{t \in (0,T]} \left\| u_1(t) - u_2(t) \right\|.$$

It is not difficult to see that, with this metric, $F_r(T,u_0)$ is a complete metric space. Take $L := T^{\alpha(1+\gamma)} r + C_s\|u_0\|$. Then for any $u \in F_r(T,u_0)$, we have

$$\left\| s^{\alpha(1+\gamma)} u(s) \right\| \leqslant s^{\alpha(1+\gamma)} \left\| u - \mathcal{S}_\alpha(t)u_0 \right\| + s^{\alpha(1+\gamma)} \left\| \mathcal{S}_\alpha(t)u_0 \right\| \leqslant L.$$

Choose $0 < T_0 \leqslant T$ such that

$$C_p N \frac{T_0^{-\alpha\gamma}}{-\alpha\gamma} + C_p N L^\nu T_0^{-\alpha(\nu(1+\gamma)+\gamma)} \beta\big(-\gamma\alpha, 1-\nu\alpha(1+\gamma)\big) \leqslant r, \tag{5.1}$$

$$M C_p \frac{T_0^{-\alpha\gamma}}{-\alpha\gamma} + 2L^{\rho-1} T_0^{-\alpha(\gamma+(1+\gamma)(\nu-1))} \beta\big(-\alpha\gamma, 1-\alpha(1+\gamma)(\nu-1)\big) \leqslant \frac{1}{2}, \tag{5.2}$$

where $\beta(\eta_1, \eta_2)$ with $\eta_i > 0$, $i = 1, 2$, denotes the Beta function. Assume that $u_0 \in X$. Consider the mapping Γ^α given by

$$\big(\Gamma^\alpha u\big)(t) = \mathcal{S}_\alpha(t) u_0 + \int_0^t (t-s)^{\alpha-1} \mathcal{P}_\alpha(t-s) f\big(s, u(s)\big)\, ds, \quad u \in F_r(T_0, u_0).$$

By the assumptions on f, Theorems 3.1 and 3.2, we see that $(\Gamma^\alpha u)(t) \in C((0,T]; X)$ and

$$\begin{aligned}
&\big\|\big(\Gamma^\alpha u\big)(t) - \mathcal{S}_\alpha(t) u_0\big\| \\
&\quad\leqslant C_p N \int_0^t (t-s)^{-\alpha\gamma-1} \big(1 + \|u(s)\|^\nu\big)\, ds \\
&\quad\leqslant C_p N \frac{T_0^{-\alpha\gamma}}{-\alpha\gamma} + \int_0^t C_p N L^\nu (t-s)^{-\alpha\gamma-1} s^{-\nu\alpha(1+\gamma)}\, ds \\
&\quad\leqslant C_p N \frac{T_0^{-\alpha\gamma}}{-\alpha\gamma} + C_p N L^\nu T_0^{-\alpha(\nu(1+\gamma)+\gamma)} \beta\big(-\gamma\alpha, 1-\nu\alpha(1+\gamma)\big) \\
&\quad\leqslant r,
\end{aligned}$$

in view of (5.1). So, Γ^α maps $F_r(T_0, u_0)$ into itself. Next, for any $u, v \in F_r(T_0, u_0)$, by the assumptions on f and Theorem 3.1 we have

$$\begin{aligned}
&\big\|\big(\Gamma^\alpha u\big)(t) - \big(\Gamma^\alpha v\big)(t)\big\| \\
&\quad\leqslant C_p M \int_0^t (t-s)^{-\alpha\gamma-1} \big(1 + \|u(s)\|^{\rho-1} + \|v(s)\|^{\rho-1}\big) \|u(s) - v(s)\|\, ds \\
&\quad\leqslant C_p M \rho_t(u, v) \int_0^t (t-s)^{-\alpha\gamma-1} \big(1 + 2L^{\nu-1} s^{-\alpha(\nu-1)(1+\gamma)}\big)\, ds \\
&\quad\leqslant 2L^{\rho-1} T_0^{-\alpha(\gamma+(1+\gamma)(\nu-1))} \beta\big(-\alpha\gamma, 1-\alpha(1+\gamma)(\nu-1)\big) \rho_{T_0}(u, v) \\
&\qquad + M C_p \frac{T_0^{-\alpha\gamma}}{-\alpha\gamma} \rho_{T_0}(u, v).
\end{aligned}$$

This yields that Γ^α is a contraction on $F_r(T_0, u_0)$ due to (5.2). So Γ_α has a unique fixed point $u \in F_r(T_0, u_0)$ by the Banach Fixed Point Theorem, which is a mild solution to problem (SLCP) on $(0, T_0]$. The proof is completed. □

By a similar argument as in the proof of Theorem 5.1 we have

Corollary 5.1. *Let $A \in \Theta_\omega^\gamma(X)$ with $-1 < \gamma < -\frac{2}{3}$ and $0 < \omega < \frac{\pi}{2}$. Suppose that $f : (0, T] \times X^\beta \to X$ ($\beta \in (1+\gamma, -1-2\gamma)$) is continuous with respect to t and there exist constants $M, N > 0$ such that*

$$\|f(t,x) - f(t,y)\| \leqslant M\big(1 + \|x\|_\beta^{\nu-1} + \|y\|_\beta^{\nu-1}\big)\|x-y\|_\beta,$$
$$\|f(t,x)\| \leqslant N\big(1 + \|x\|_\beta^\nu\big),$$

for all $t \in (0, T]$ and for each $x, y \in X^\beta$, where ν is a constant in $[1, -\frac{\gamma+\beta}{1+\gamma})$. Then, for every $u_0 \in X^\beta$, there exists a $T_0 > 0$ such that the problem (SLCP) *has a unique mild solution $u \in C((0, T_0]; X^\beta)$.*

Remark 5.1. If $A \in \Theta_\omega^\gamma(X)$ with $-1 < \gamma < 0$ and $0 < \omega < \frac{\pi}{2}$, then we can derive the local existence and uniqueness of mild solutions to problem (SLCP), under the conditions:

(1) $u_0 \in X^\beta$ with $\beta > 1 + \gamma$;
(2) the nonlinear mapping $f : [0, T] \times X \to X$ is continuous with respect to t and there exists a continuous function $L_f(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|f(t,x) - f(t,y)\| \leqslant L_f(r)\|x-y\|,$$

for all $0 \leqslant t \leqslant T$ and for each $x, y \in X$ satisfying $\|x\|, \|y\| \leqslant r$.

Indeed, for $r > \frac{C_p T_0^{-\alpha\gamma}}{-\alpha\gamma} \sup_{t\in[0,T]} \|f(t,u_0)\|$ fixed, we may choose $0 < T_0 \leqslant T$ such that

$$\sup_{t\in[0,T_0]} \big\|\big(\mathcal{S}_\alpha(t) - I\big)u_0\big\| + \frac{C_p T_0^{-\alpha\gamma}}{-\alpha\gamma}\Big(L_f(r)r + \sup_{t\in[0,T_0]}\big\|f(t,u_0)\big\|\Big) < r \tag{5.3}$$

in view of Theorem 3.4(i). Assume that the map Γ^α is defined the same as in Theorem 5.1 and the space $F_r(T_0, u_0)$ is replaced by the following Banach space:

$$F_r'(T_0,u_0) = \Big\{u \in C\big([0,T_0]; X\big);\ u(0) = u_0 \text{ and } \sup_{t\in[0,T_0]}\|u - u_0\| \leqslant r\Big\}.$$

Then, it is easy to verify, thanks to the assumptions on f and (5.3), that Γ^α maps $F_r'(T_0, u_0)$ into itself and is a contraction on $F_r'(T_0, u_0)$, which implies that the problem (SLCP) has a unique mild solution defined on $[0, T_0]$.

Since $1 > 1 + \gamma$ $(-1 < \gamma < -\frac{1}{2})$, $X^1 = D(A)$ is a Banach space endowed with the graph norm $\|x\|_{X^1} = \|Ax\|$ $(x \in X^1)$. The following is the existence of X^1-smooth solutions.

Theorem 5.2. *Let $A \in \Theta_\omega^\gamma(X)$ with $-1 < \gamma < -\frac{1}{2}$, $0 < \omega < \frac{\pi}{2}$ and $u_0 \in X^1$. Let there exist a continuous function $M_f(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ and a constant $N_f > 0$ such that the mapping $f : (0, T] \times X^1 \to X^1$ satisfies*

$$\|f(t,x) - f(t,y)\|_{X^1} \leqslant M_f(r)\|x-y\|_{X^1},$$
$$\big\|f\big(t, \mathcal{S}_\alpha(t)u_0\big)\big\|_{X^1} \leqslant N_f\big(1 + t^{-\alpha(1+\gamma)}\|u_0\|_{X^1}\big),$$

for all $0 < t \leqslant T$ and for each $x, y \in X^1$ satisfying $\sup_{t\in(0,T]}\|x(t) - \mathcal{S}_\alpha(t)u_0\|_{X^1} \leqslant r$, $\sup_{t\in(0,T]}\|y(t) - \mathcal{S}_\alpha(t)u_0\|_{X^1} \leqslant r$. Then there is a $T_0 > 0$ such that the problem (SLCP) *has a unique mild solution defined on $(0, T_0]$.*

Proof. For $u_0 \in X^1$ and $r > 0$, set

$$F_r''(T,u_0) = \Big\{u \in C\big((0,T];X^1\big);\ \sup_{t\in(0,T]} \big\|u - \mathcal{S}_\alpha(t)u_0\big\|_{X^1} \leqslant r\Big\}.$$

For any $u \in F_r''(T,u_0)$, by the assumptions on f and Theorem 3.1 we have

$$\begin{aligned}
&\big\|\big(\Gamma^\alpha u\big)(t) - \mathcal{S}_\alpha(t)u_0\big\|_{X^1} \\
&\quad \leqslant \int_0^t (t-s)^{\alpha-1}\big\|\mathcal{P}_\alpha(t-s)\big\|\big\|f\big(s,u(s)\big) - f\big(s,\mathcal{S}_\alpha(t)u_0\big)\big\|_{X^1}\,ds \\
&\qquad + \int_0^t (t-s)^{\alpha-1}\big\|\mathcal{P}_\alpha(t-s)\big\|\big\|f\big(s,\mathcal{S}_\alpha(t)u_0\big)\big\|_{X^1}\,ds \\
&\quad \leqslant C_p \int_0^t (t-s)^{-\alpha\gamma-1}\big(M_f(r)r + N_f + N_f s^{-\alpha(1+\gamma)}\|u_0\|\big)\,ds \\
&\quad \leqslant C_p\big(M_f(r)r + N_f\big)\frac{T^{-\alpha\gamma}}{-\alpha\gamma} + C_p N_f T^{-\alpha(1+2\gamma)}\beta\big(-\gamma\alpha, 1-\alpha(1+\gamma)\big)\|u_0\|.
\end{aligned}$$

Using this result and an analogous idea as in Theorem 5.1, we obtain the conclusion of the theorem. Here we omit the details. □

Next, we will derive mild solutions under the condition of compactness on the resolvent of A.

Theorem 5.3. *Let $A \in \Theta_\omega^\gamma(X)$ with $-1 < \gamma < 0$ and $0 < \omega < \frac{\pi}{2}$. Let*

(H_1) $R(\lambda,-A)$ be compact for every $\lambda > 0$;
(H_2) $f : [0,T] \times X \to X$ be a Carathéodory function and for any $r > 0$, there exists a function $m_r(t) \in L^p((0,T);\mathbb{R}^+)$ with $p > -\frac{1}{\alpha\gamma}$ such that

$$\big\|f(t,x)\big\| \leqslant m_r(t), \quad \text{and} \quad \liminf_{r\to+\infty} \frac{\|m_r(t)\|_{L^p(0,T)}}{r} = \sigma < \infty$$

for a.e. $t \in [0,T]$ and all $x \in X$ satisfying $\|x\| \leqslant r$.

Then for every $u_0 \in D(A^\beta)$ with $\beta > 1+\gamma$, the problem (SLCP) *has at least a mild solution, provided that*

$$C_p\sigma\left(\frac{T^{1-(1+\alpha\gamma)q}}{1-(1+\alpha\gamma)q}\right)^{\frac{1}{q}} < 1, \tag{5.4}$$

where $q = p/(p-1)$.

Proof. Assume that $u_0 \in D(A^\beta)$. On $C([0,T];X)$ define the map

$$\big(\Gamma^\alpha u\big)(t) = \mathcal{S}_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)f\big(s,u(s)\big)\,ds.$$

From our assumptions it is easy to see that Γ_μ is well defined and maps $C([0,T];X)$ into itself. Put

$$\Omega_r = \{u \in C([0,T];X);\ \|u\| \leqslant r,\ \text{for all } 0 \leqslant t \leqslant T\},$$

for $r > 0$ as selected below. We seek for solutions in Ω_r. We claim that there exists an integer $r > 0$ such that Γ^α maps Ω_r into Ω_r. In fact, if this is not the case, then for each $r > 0$, there would exist $u^r \in \Omega_r$ and $t^r \in [0,T]$ such that $\|(\Gamma^\alpha u^r)(t^r)\| > r$. On the other hand, by (H_2) and Theorem 3.1 we get

$$\begin{aligned} r &< \|(\Gamma^\alpha u^r)(t^r)\| \\ &\leqslant \|\mathcal{S}_\alpha(t^r)u_0\| + \int_0^{t^r} \|(t^r - s)^{\alpha-1}\mathcal{P}_\alpha(t^r - s) f(s, u(s))\|\,ds \\ &\leqslant \sup_{t\in[0,T]}\|\mathcal{S}_\alpha(t)u_0\| + \int_0^{t^r} C_p(t^r - s)^{-1-\alpha\gamma} m_r(s)\,ds \\ &\leqslant \sup_{t\in[0,T]}\|\mathcal{S}_\alpha(t)u_0\| + C_p\left(\int_0^{t^r} s^{-(1+\alpha\gamma)q}\,ds\right)^{\frac{1}{q}}\left(\int_0^{t^r} m_r^p(s)\,ds\right)^{\frac{1}{p}} \\ &\leqslant \sup_{t\in[0,T]}\|\mathcal{S}_\alpha(t)u_0\| + C_p\|m_r\|_{L^p(0,T)}\left(\frac{T^{1-(1+\alpha\gamma)q}}{1-(1+\alpha\gamma)q}\right)^{\frac{1}{q}}, \end{aligned}$$

where $q = p/(p-1)$. Dividing both sides by r and taking the lower limit as $r \to \infty$, one has $1 \leqslant C_p\sigma(\frac{T^{1-(1+\alpha\gamma)q}}{1-(1+\alpha\gamma)q})^{1/q}$, which contradicts (5.4). Hence for some positive integer r, $\Gamma^\alpha(\Omega_r) \subset \Omega_r$.

The rest of the proof is divided into three steps.

Step 1. Γ^α is continuous on Ω_r.

Take $\{u_n\}_{n=1}^\infty \subset \Omega_r$ with $u_n \to u$ in $C([0,T];X)$. Then by the continuity of f with respect to the second argument we deduce that

$$f(s, u_n(s)) \to f(s, u(s)) \quad \text{for a.e. } s \in [0,T].$$

Moreover, observe from (H_2) and Theorem 3.1 that for a fixed $0 < t \leqslant T$,

$$(t-s)^{\alpha-1}\|\mathcal{P}_\alpha(t-s) f(s, u_n(s))\| \leqslant C_p(t-s)^{-1-\alpha\gamma} m_r(s) \in L^1(0,t).$$

Thus, by means of the Lebesgue dominated convergence theorem we obtain

$$\int_0^t (t-s)^{\alpha-1}\|\mathcal{P}_\alpha(t-s)\| \cdot \|f(s, u_n(s)) - f(s, u(s))\|\,ds \to 0,$$

which means that $\lim_{n\to\infty}\|\Gamma^\alpha u_n - \Gamma^\alpha u\|_\infty = 0$. So Γ^α is continuous on Ω_r.

Step 2. $P := \{(\Gamma^\alpha u)(\cdot);\ \cdot \in [0,T],\ u \in \Omega_r\}$ is equicontinuous.

For $0 < t_1 < t_2 \leqslant T$ and $\delta > 0$ small enough, we have

$$\|(\Gamma^\alpha u)(t_1) - (\Gamma^\alpha u)(t_2)\| \leqslant I_1 + I_2 + I_3 + I_4 + I_5,$$

where

$$I_1 = \left\|\mathcal{S}_\alpha(t_1)u_0 - \mathcal{S}_\alpha(t_2)u_0\right\|,$$

$$I_2 = \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1}\left\|\mathcal{P}_\alpha(t_2 - s) f\big(s, u(s)\big)\right\| ds,$$

$$I_3 = \int_0^{t_1-\delta} (t_1 - s)^{\alpha-1}\left\|\mathcal{P}_\alpha(t_2 - s) - \mathcal{P}_\alpha(t_1 - s)\right\|\left\|f\big(s, u(s)\big)\right\| ds,$$

$$I_4 = \int_{t_1-\delta}^{t_1} (t_1 - s)^{\alpha-1}\left\|\mathcal{P}_\alpha(t_2 - s) - \mathcal{P}_\alpha(t_1 - s)\right\|\left\|f\big(s, u(s)\big)\right\| ds,$$

$$I_5 = \int_0^{t_1}\left|(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\right| \cdot \left\|\mathcal{P}_\alpha(t_2 - s)\right\|\left\|f\big(s, u(s)\big)\right\| ds.$$

From Theorem 3.2 and Theorem 3.4(i) it is easy to see that $I_1 \to 0$ when $t_1 \to t_2$. Moreover, using (H_2) and Theorem 3.1 we get

$$I_2 \leqslant C_p\left(\frac{(t_2 - t_1)^{1-(1+\alpha\gamma)q}}{1 - (1+\alpha\gamma)q}\right)^{\frac{1}{q}} \|m_r\|_{L^p(0,T)},$$

$$I_3 \leqslant \sup_{s\in[0,t_1-\delta]}\left\|\mathcal{P}_\alpha(t_2 - s) - \mathcal{P}_\alpha(t_1 - s)\right\|\left(\int_0^{t_1-\delta} (t_1 - s)^{q\alpha-q} q\, ds\right)^{\frac{1}{q}} \|m_r\|_{L^p(0,T)}$$

$$\leqslant \sup_{s\in[0,t_1-\delta]}\left\|\mathcal{P}_\alpha(t_2 - s) - \mathcal{P}_\alpha(t_1 - s)\right\|\left(\frac{t_1^{1+q(\alpha-1)} - \delta^{1+q(\alpha-1)}}{1 + q(\alpha - 1)}\right)\|m_r\|_{L^p(0,T)},$$

$$I_4 \leqslant C_p \int_{t_1-\delta}^{t_1} (t_1 - s)^{\alpha-1} \cdot 2(t_1 - s)^{-\alpha(\gamma+1)} m_r(s)\, ds$$

$$\leqslant C_p\left(\int_0^{t_1}\left((t_1 - s)^{-q(\gamma\alpha+1)} - (t_2 - s)^{-q(\alpha\gamma+1)}\right) ds\right)^{\frac{1}{q}} \|m_r\|_{L^p(0,T)}$$

$$\leqslant C_p\left(\int_0^{t_1}(t_1 - s)^{-q(\alpha\gamma+1)} - (t_2 - s)^{-q(\gamma\alpha+1)}\, ds\right)^{\frac{1}{q}} \|m_r\|_{L^p(0,T)}$$

$$= C_p\left(\frac{(t_2 - t_1)^{1-(1+\alpha\gamma)q}}{1 - (1+\alpha\gamma)q} + \frac{t_1^{1-(1+\alpha\gamma)q} - t_2^{1-(1+\alpha\gamma)q}}{1 - (1+\alpha\gamma)q}\right)^{\frac{1}{q}} \|m_r\|_{L^p(0,T)}.$$

It follows from Theorem 3.2 that I_i $(i = 2, 3, 4, 5)$ tends to zero independent of $u \in \Omega_r$ as $t_2 - t_1 \to 0$, $\delta \to 0$. Hence, we can conclude that $\|(\Gamma^\alpha u)(t_1) - (\Gamma^\alpha u)(t_2)\| \to 0$ as $t_2 - t_1 \to 0$, and the limit is independent of $u \in \Omega_r$. For the case when $0 = t_1 < t_2 \leqslant T$, since

$$\int_0^{t_2} (t_2-s)^{\alpha-1} \left\| P(t_2-s) f\big(s,u(s)\big) \right\| ds \leqslant C_p \left(\frac{t_2^{1-q(\alpha\gamma+1)}}{1-q(\alpha\gamma+1)} \right)^{\frac{1}{q}} \|m_r\|_{L^p(0,T)},$$

in view of (H_2) and Theorem 3.1, $\|(\Gamma^\alpha u)(t_2)\|$ can be made small when t_2 is small independently of $u \in \Omega_r$. Thus, the assertion in Step 2 holds.

Step 3. For each $t \in [0,T]$, $\{(\Gamma^\alpha u)(t);\ u \in \Omega_r\}$ is precompact in X.

For the case when $t=0$, it is not difficult to see that $\{(\Gamma^\alpha u)(0);\ u \in \Omega_r\} = \{u_0\colon\ u \in \Omega_r\}$ is compact. Let $t \in (0,T]$ be fixed and $\epsilon, \delta > 0$. For $u \in \Omega_r$, define the map $\Gamma^\alpha_{\epsilon,\delta}$ by

$$\big(\Gamma^\alpha_{\epsilon,\delta} u\big)(t) = \mathcal{S}_\alpha(t) u_0 + \int_0^{t-\epsilon} \int_\delta^\infty \alpha\tau(t-s)^{\alpha-1} \Psi_\alpha(\tau) T\big((t-s)^\alpha \tau\big) f\big(s,u(s)\big)\, d\tau\, ds.$$

Since A has compact resolvent, $\{T(t)\}_{t>0}$ is compact in view of Theorem 3.5. Thus, for each $t \in (0,T]$, $\{\Gamma^\alpha_{\epsilon,\delta} u)(t);\ u \in \Omega_r,\ \delta > 0,\ 0 < \epsilon < t\}$ is precompact in X. On the other hand, by (H_2) and Theorem 3.1, a direct calculation yields

$$\begin{aligned}
&\big\| \big(\Gamma^\alpha u\big)(t) - \big(\Gamma^\alpha_{\epsilon,\delta} u\big)(t) \big\| \\
&\quad \leqslant \left\| \int_0^t \int_0^\delta \alpha\tau(t-s)^{\alpha-1} \Psi_\alpha(\tau) T\big((t-s)^\alpha \tau\big) f\big(s,u(s)\big)\, d\tau\, ds \right\| \\
&\qquad + \left\| \int_{t-\epsilon}^t \int_\delta^\infty \alpha\tau(t-s)^{\alpha-1} \Psi_\alpha(\tau) T\big((t-s)^\alpha \tau\big) f\big(s,u(s)\big)\, d\tau\, ds \right\| \\
&\quad \leqslant \int_0^t C_p (t-s)^{-1-\alpha\gamma} m_r(s) ds \int_0^\delta \tau^{-\gamma} \Psi_\alpha(\tau)\, d\tau \\
&\qquad + \int_{t-\epsilon}^t C_p (t-s)^{-1-\alpha\gamma} m_r(s)\, ds \int_\delta^\infty \tau^{-\gamma} \Psi_\alpha(\tau)\, d\tau \\
&\quad \leqslant C_p \left(\frac{T^{1-(1+\alpha\gamma)q}}{1-(1+\alpha\gamma)q} \right)^{\frac{1}{q}} \|m_r\|_{L^p(0,T)} \int_0^\delta \tau^{-\gamma} \Psi_\alpha(\tau)\, d\tau \\
&\qquad + C_p \left(\frac{\epsilon^{1-(1+\alpha\gamma)q}}{1-(1+\alpha\gamma)q} \right)^{\frac{1}{q}} \|m_r\|_{L^p(0,T)} \frac{\Gamma(1-\gamma)}{\Gamma(1-\gamma\alpha)}.
\end{aligned}$$

Using the total boundedness we have that for each $t \in (0,T]$, $\{(\Gamma^\alpha u)(t);\ u \in \Omega_r\}$ is precompact in X. Therefore, for each $t \in [0,T]$, $\{(\Gamma^\alpha u)(t);\ u \in \Omega_r\}$ is precompact in X.

Finally, by Steps 1–3 and the Arzelà–Ascoli theorem, Γ^α is a compact operator. So, by Schauder's second fixed point theorem, Γ^α has a fixed point, which gives a mild solution. This completes the proof. □

Theorem 5.4. *Let $A \in \Theta^\gamma_\omega(X)$ with $0 < \omega < \frac{\pi}{2}$ and $-1 < \gamma < -\frac{1}{2}$. Suppose that there exists a continuous function $M'_f(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ and a constant $\kappa > \alpha(1+\gamma)$ such that the mapping $f : [0,T] \times X \to X$ satisfies*

$$\big\| f(t,x) - f(s,y) \big\| \leqslant M'_f(r)\big(|t-s|^{\kappa} + \|x-y\|\big),$$

for all $0 \leqslant t \leqslant T$ *and* $x, y \in X$ *satisfying* $\|x\|, \|y\| \leqslant r$. *In addition, let the assumptions of Theorem* 5.2 *be satisfied and u be a mild solution corresponding to* u_0, *defined on* $[0, T_0]$. *Then u is the unique classical solution to problem* (SLCP) *on* $[0, T_0]$, *provided that* $u_0 \in D(A)$ *with* $Au_0 \in D(A^{\beta})$, $\beta > (1+\gamma)$.

Proof. In order to prove that u is a classical solution, by Theorem 4.1 and the condition on f, we only have to verify that u is Hölder continuous with an exponent $\varsigma > \alpha(1+\gamma)$ on $(0, T_0]$. For fixed $t \in (0, T_0]$, take $0 < h < 1$ such that $h + t \leqslant T_0$. We estimate the difference

$$\begin{aligned}
&\big\|u(t+h) - u(t)\big\| \\
&\quad\leqslant \big\|\mathcal{S}_\alpha(t+h)u_0 - \mathcal{S}_\alpha(t)u_0\big\| + \left\| \int_0^h (t+h-s)^{\alpha-1}\mathcal{P}(t+h-s) f\big(s, u(s)\big)\, ds \right\| \\
&\qquad + \left\| \int_0^t (t-s)^{\alpha-1}\mathcal{P}(t-s)\big[f\big(s+h, u(s+h)\big) - f\big(s, u(s)\big)\big]\, ds \right\| \\
&\quad= I_1 + I_2 + I_3.
\end{aligned}$$

By Theorem 3.1, Theorem 3.3(ii) and the assumptions on f we obtain

$$\begin{aligned}
I_1 &= \left\| \int_0^t -s^{\alpha-1} A\mathcal{P}_\alpha(s)u_0\, ds \right\| \leqslant \frac{C_p}{-\alpha\gamma}\big((t+h)^{-\alpha\gamma} - t^{-\alpha\gamma}\big), \\
I_3 &\leqslant M'C_p \int_0^t (t-s)^{-\alpha\gamma-1}\big(|h|^{\kappa} + \big\|u(s+h) - u(s)\big\|\big)\, ds \\
&\leqslant \frac{M'C_p}{-\alpha\gamma} T_0^{-\alpha\gamma} h^{\kappa} + M'C_p \int_0^t (t-s)^{-\alpha\gamma-1}\big\|u(s+h) - u(s)\big\|\, ds.
\end{aligned}$$

Put $N_2 := \sup_{t\in(0,T_0)} \|f(t, u(t))\|$. Then, it follows from Theorem 3.1 that

$$I_2 \leqslant C_p \int_0^h (t+h-s)^{-\alpha\gamma-1}\big\| f\big(s, u(s)\big)\big\|\, ds \leqslant \frac{C_p N_2}{-\alpha\gamma}\big((t+h)^{-\alpha\gamma} - t^{-\alpha\gamma}\big).$$

Collecting these estimates and using the inequality $(t+h)^{-\alpha\gamma} - t^{-\alpha\gamma} \leqslant h^{-\alpha\gamma}$ $(0 < -\alpha\gamma < 1)$ we have

$$\begin{aligned}
\big\|u(t+h) - u(t)\big\| &\leqslant \frac{C_p N_2 + C_p}{-\alpha\gamma}\big((t+h)^{-\alpha\gamma} - t^{-\alpha\gamma}\big) + \frac{M'_p}{-\alpha\gamma} T_0^{-\alpha\gamma} h^{\kappa} \\
&\quad + M'C_p \int_0^t (t-s)^{-\alpha\gamma-1}\big\|u(s+h) - u(s)\big\|\, ds
\end{aligned}$$

$$\leqslant \frac{C_p N_2 + C_p + M' C_p}{-\alpha\gamma} h^{\varsigma} + M' C_p \int_0^t (t-s)^{-\alpha\gamma-1} \left\| u(s+h) - u(s) \right\| ds,$$

where $\varsigma = \min\{\kappa, -\alpha\gamma\} > \alpha(\gamma+1)$. Now, it follows from the Gronwall inequality that u is Hölder continuous on $(0, T_0]$. This completes the proof of the theorem. □

6. Applications

In this section, we present three examples (Examples 6.1–6.3) motivated from physics, which do not aim at generality but indicate how our theorems can be applied to concrete problems. Examples 6.1 and 6.2 are inspired directly from the work of A.N. Carvalho et al. [6], and they describe anomalous diffusion on fractals (physical objects of fractional dimension, like some amorphous semiconductors or strongly porous materials; see [1,33] and references therein). Example 6.1 is the limit problem of certain fractional diffusion equations in complex systems on domains of "dumb-bell with a thin handle" (see, e.g., [1,33]). Example 6.2 displays anomalous dynamical behavior of anomalous transport processes (see, e.g., [1,33]). Example 6.3 is a modified fractional Schrödinger equation with fractional Laplacians whose physical background is statistical physics and fractional quantum mechanics (see, e.g., [22,39]). We refer the reader to M. Kirane et al. [24] and references therein for more research results related to fractional Laplacians.

Example 6.1. Consider the system of fractional partial differential equations in the form

$$\begin{cases} {}_cD_t^\alpha w - \Delta w + w = f(w), \quad x \in \Omega,\ t > 0, \\ \dfrac{\partial w}{\partial n} = 0, \quad x \in \partial\Omega, \\ {}_cD_t^\alpha v - \dfrac{1}{g}(g v_x)_x + v = f(v), \quad x \in (0,1), \\ v(0) = w(P_0), \quad v(1) = w(P_1), \\ w(x,0) = w_0(x), \quad x \in \Omega, \quad v(x,0) = v_0(x), \quad x \in (0,1), \end{cases} \tag{6.1}$$

where $\Omega = D_1 \cup D_2$ and D_1 and D_2 are mutually disjoint bounded domains in $\mathbb{R}^N$ $(N \geqslant 2)$ with smooth boundaries, joined by the line segment Q_0, and ${}_cD_t^\alpha$, $0 < \alpha < 1$, is the regularized Caputo fractional derivative of order α, that is,

$$\left({}_cD_t^\alpha u\right)(t,x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{\partial}{\partial t} \int_0^t (t-s)^{-\alpha} u(s,x)\, ds - t^{-\alpha} u(0,x) \right). \tag{6.2}$$

When $\alpha = 1$, we regard (6.1) as the limit problem of (1.1) as $\varepsilon \to 0$, which is described more detail in Example 1.1. Here, our objective is to show that system (6.1) is well posed in $V_0^p := L^p(\Omega) \oplus L_g^p(0,1)$ $(1 \leqslant p < \infty)$.

Let the operators $A_0 : D(A_0) \subset V_0^p \mapsto V_0^p$ be defined by

$$D(A_0) = \big\{(w,v) \in V_0^p;\ w \in D(\Delta_\Omega),\ v \in L_g^p(0,1),$$
$$w(P_0) = v(0),\ w(P_1) = v(1)\big\},$$
$$A_0(w,v) = \left(-\Delta w + w, -\frac{1}{g}\big(g v'\big)' + v\right), \quad (w,v) \in V_0^p,$$

where Δ_Ω is the Laplace operator with homogeneous Neumann boundary conditions in $L^p(\Omega)$ and

$$D(\Delta_\Omega)=\left\{u\in W^{2,p}(\Omega);\ \frac{\partial u}{\partial n}\bigg|_{\partial\Omega}=0\right\}.$$

From Example 1.1, if $p>\frac{N}{2}$, then $A_0\in\Theta_\mu^{-\gamma'}(V_0^p)$ for some $\gamma'\in(0,1-\frac{N}{2p})$ and $\mu\in(0,\frac{\pi}{2})$. Therefore, system (6.1) can be seen as an abstract evolution equation in the form

$$\begin{cases} {}_cD_t^\alpha u+A_0u=f(u), & t>0,\\ u(0)=u_0=(w_0,v_0)\in V_0^p. \end{cases} \tag{6.3}$$

We assume that the nonlinearity $f:\mathbb{R}\to\mathbb{R}$ is globally Lipschitz continuous. It can define a Nemytskiĭ operator from V_0^p into itself by $f(w,v)=(f_\Omega(w),f_I(v))$ with $f_\Omega(w)(x)=f(w(x))$, $x\in\Omega$, and $f_I(v)(x)=f(v(x))$, $x\in(0,1)$, such that

$$\|f(u)-f(u')\|_{V_0^p}\leqslant L''(r)\|u-u'\|_{V_0^p},$$

for all $u,u'\in V_0^p$ satisfying $\|u\|_{V_0^p},\|u'\|_{V_0^p}\leqslant r$. Hence, from Remark 5.1, (6.3) (that is, (6.1)) has a unique mild solution provided that $u_0\in D(A_0^\beta)$ with $\beta>1-\gamma'$ (in particular, $u_0\in D(A_0)$).

Example 6.2. Let Ω be a bounded domain in $\mathbb{R}^N$ $(N\geqslant 1)$ with boundary $\partial\Omega$ of class C^4. Consider the fractional initial–boundary value problem

$$\begin{cases} ({}_cD_t^\alpha u)(t,x)-\Delta u(t,x)=f(u(t,x)), & t>0,\ x\in\Omega,\\ u|_{\partial\Omega}=0,\\ u(0,x)=u_0(x), & x\in\Omega, \end{cases} \tag{6.4}$$

in the space $C^l(\overline{\Omega})$ $(0<l<1)$, where Δ stands for the Laplacian with respect to the spatial variable and ${}_cD_t^\alpha$, representing the regularized Caputo fractional derivative of order α $(0<\alpha<1)$, is given by (6.2). Set

$$\widetilde{A}=-\Delta,\qquad D(\widetilde{A})=\{u\in C^{2+l}(\overline{\Omega});\ u=0\ \text{on}\ \partial\Omega\}.$$

It follows from Example 1.2 that there exist $\nu,\varepsilon>0$ such that $\widetilde{A}+\nu\in\Theta_{\frac{\pi}{2}-\varepsilon}^{\frac{l}{2}-1}(C^l(\overline{\Omega}))$. Then, problem (6.4) can be written abstractly as

$$\begin{cases} {}_cD_t^\alpha u(t)+\widetilde{A}u(t)=f(u), & t>0,\\ u(0)=u_0. \end{cases}$$

With respect to the nonlinearity f, we assume that $f:\mathbb{R}\to\mathbb{R}$ is continuously differentiable and satisfies the condition

$$|f(x)-f(y)|\leqslant\frac{k(r)}{r}|x-y|,\quad |x|,|y|\leqslant r, \tag{6.5}$$

for any $r>0$. It defines a Nemytskiĭ operator from $C^l(\overline{\Omega})$ into $C^l(\overline{\Omega})$ by $f(u)(x)=f(u(x))$ with

$$\|f(u)-f(v)\|_{C^l(\overline{\Omega})}\leqslant k(r)\|u-v\|_{C^l(\overline{\Omega})},\qquad \|v\|_{C^l(\overline{\Omega})},\|u\|_{C^l(\overline{\Omega})}\leqslant r.$$

Noting $\frac{l}{2} - 1 \in (-1, -\frac{1}{2})$, we then obtain (i) according to Remark 5.1, (6.4) has a unique mild solution for each $u_0 \in D(\widetilde{A}^\beta)$ with $\beta > \frac{l}{2}$. Moreover, (ii) if f', f'' are continuously differentiable functions satisfying the condition (6.5), then one finds that the Nemytskiĭ operator satisfies the assumptions of Theorem 5.2 and Theorem 5.4, which implies that for each $u_0 \in D(\widetilde{A})$ with $\widetilde{A}u_0 \in D(\widetilde{A}^\beta)$ $(\beta > \frac{l}{2})$, the corresponding mild solution to (6.4) is also a unique classical solution.

Example 6.3. Consider the following fractional Cauchy problem

$$\begin{cases} ({}_cD_t^\alpha u)(t,x) + (-i\Delta + \sigma)^{\frac{1}{2}} u(t,x) = f\big(u(t,x)\big), & t > 0,\ x \in \mathbb{R}^2, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^2, \end{cases} \tag{6.6}$$

in $L^3(\mathbb{R}^2)$, where $\sigma > 0$ is a suitable constant, $i\Delta$ is the Schrödinger operator and ${}_cD_t^\alpha$ $(0 < \alpha < 1)$ is given by (6.2). Let

$$\widehat{A} = (-i\Delta + \sigma)^{\frac{1}{2}}, \qquad D(\widehat{A}) = W^{1,3}\big(\mathbb{R}^2\big) \quad \text{(a Sobolev space)}.$$

Then $i\Delta$ generates a β-times integrated semigroup $S^\beta(t)$ with $\beta = \frac{5}{12}$ on $L^3(\mathbb{R}^2)$ such that $\|S^\beta(t)\|_{\mathscr{L}(L^3(\mathbb{R}^2))} \leqslant \widehat{M} t^\beta$ for all $t \geqslant 0$ and some constants $\widehat{M} > 0$ (see [35]). Therefore, by virtue of [43, Theorem 1.3.5(P.15), Definition 1.3.1 for $C = I$ (P.12)], we deduce that the operator $i\Delta + \sigma$ belongs to $\Theta_{\frac{\pi}{2}}^{\beta-1}(L^3(\mathbb{R}^2))$, which denotes the family of all linear closed operators $A : D(A) \subset L^3(\mathbb{R}^2) \to L^3(\mathbb{R}^2)$ satisfying $\sigma(A) \subset S_{\frac{\pi}{2}} = \{z \in \mathbb{C} \setminus \{0\};\ |\arg z| \leqslant \frac{\pi}{2}\} \cup \{0\}$, and for every $\frac{\pi}{2} < \mu < \pi$ there exists a constant C_μ such that $\|R(z; A)\| \leqslant C_\mu |z|^{\beta-1}$ for all $z \in \mathbb{C} \setminus S_\mu$. Thus, it follows from [38, Proposition 3.6] that $\widehat{A} \in \Theta_\omega^{-1+2\beta}(L^3(\mathbb{R}^2))$ for some $0 < \omega < \frac{\pi}{2}$. Moreover, the system (6.6) can be rewritten as follows:

$$\begin{cases} {}_cD_t^\alpha u + \widehat{A}u = f(u), & t > 0 \\ u(0,x) = u_0 \in L^3\big(\mathbb{R}^2\big). \end{cases}$$

Assume that $f : \mathbb{C} \to \mathbb{C}$ is globally Lipschitz continuous. Then we have a Nemytskiĭ operator from $L^3(\mathbb{R}^2)$ to itself given by $f(u)(x) = f(u(x))$, and $\|f(u) - f(v)\|_{L^3(\mathbb{R}^2)} \leqslant \widehat{L}(r)\|u - v\|_{L^3(\mathbb{R}^2)}$ for a constant $\widehat{L}(r)$ and all $u, v \in L^3(\mathbb{R}^2)$ such that $\|u\|_{L^3(\mathbb{R}^2)} \leqslant r$ and $\|v\|_{L^3(\mathbb{R}^2)} \leqslant r$. Consequently, it follows from Remark 5.1 that (6.6) has a unique mild solution provided $u_0 \in D(\widehat{A})^\tau$ with $\tau > \frac{5}{6}$.

Acknowledgment

The authors would like to thank the referee very much for his/her valuable suggestions and comments.

References

[1] Vo V. Anh, N.N. Leonenko, Spectral analysis of fractional kinetic equations with random data, J. Stat. Phys. 104 (2001) 1349–1387.
[2] J.M. Arrieta, Rates of eigenvalues on a dumbbell domain. Simple eigenvalue case, Trans. Amer. Math. Soc. 347 (9) (1995) 3503–3531.
[3] J.M. Arrieta, A. Carvalho, G. Lozada-Cruz, Dynamics in dumbbell domains I. Continuity of the set of equilibria, J. Differential Equations 231 (2006) 551–597.
[4] J.M. Arrieta, A. Carvalho, G. Lozada-Cruz, Dynamics in dumbbell domains II. The limiting problem, J. Differential Equations 247 (2009) 174–202.
[5] J.M. Arrieta, A. Carvalho, G. Lozada-Cruz, Dynamics in dumbbell domains III. Continuity of attractors, J. Differential Equations 247 (2009) 225–259.
[6] A.N. Carvalho, T. Dlotko, M.J.D. Nescimento, Non-autonomous semilinear evolution equations with almost sectorial operators, J. Evol. Equ. 8 (2008) 631–659.

[7] D. del-Castillo-Negrete, B.A. Carreras, V. Lynch, Front dynamics in reaction–diffusion systems with Levy flights: A fractional diffusion approach, Phys. Rev. Lett. 91 (1) (2003) 018302.
[8] M. Cowling, I. Doust, A. McIntosh, A. Yagi, Banach space operators with a bounded H^∞ calculus, J. Aust. Math. Soc. Sect. A 60 (1996) 51–89.
[9] E.N. Dancer, D. Daners, Domain perturbation of elliptic equations subject to Robin boundary conditions, J. Differential Equations 74 (1997) 86–132.
[10] R. deLaubenfels, Existence Families, Functional Calculi and Evolution Equations, Springer-Verlag, Berlin, 1994.
[11] K. Diethelm, A.D. Freed, On the solution of nonlinear fractional order differential equations used in the modeling of viscoplasticity, in: F. Keil, W. Mackens, H. Voss, J. Werther (Eds.), Scientific Computing in Chemical Engineering II-Computational Fluid Dynamics, Reaction Engineering and Molecular Properties, Springer-Verlag, Heidelberg, 1999, pp. 217–224.
[12] A. Ducrot, P. Magal, K. Prevost, Integrated semigroups and parabolic equations. Part I: linear perturbation of almost sectorial operators, J. Evol. Equ. 10 (2010) 263–291.
[13] T.E. Duncan, Y. Hu, B. Pasik-Duncan, Stochastic calculus for fractional Brownian motion, I. Theory, SIAM J. Control Optim. 38 (2000) 582–612.
[14] M.M. Dzhrbashyan, A.B. Nersessyan, Fractional derivatives and Cauchy problem for differential equations of fractional order, Izv. AN Arm. SSR. Mat. 3 (1968) 3–29 (in Russian).
[15] Samuil D. Eidelman, Anatoly N. Kochubei, Cauchy problem for fractional diffusion equations, J. Differential Equations 199 (2) (2004) 211–255.
[16] R.R. Gadyl'shin, On the eigenvalues of a dumb-bell with a thin handle, Izv. Math. 69 (2) (2005) 265–329.
[17] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Springer, Berlin, 1981.
[18] E. Hernandez, D. O'Regan, E. Balachandran, On recent developments in the theory of abstract differential equations with fractional derivatives, Nonlinear Anal. 73 (2010) 3462–3471.
[19] H. Hilfer, Applications of Fractional Calculus in Physics, World Scientific Publ. Co., Singapore, 2000.
[20] O.K. Jaradat, A. Al-Omari, S. Momani, Existence of the mild solution for fractional semilinear initial value problems, Nonlinear Anal. 69 (2008) 3153–3159.
[21] S. Jimbo, The singularly perturbed domain and the characterization for the eigenfunctions with Neumann boundary conditions, J. Differential Equations 77 (1989) 322–350.
[22] Y. Hu, G. Kallianpur, Schröodinger equations with fractional Laplacians, Appl. Math. Optim. 42 (2000) 281–290.
[23] A.A. Kilbas, Hari M. Srivastava, J. Juan Trujillo, Theory and Applications of Fractional Differential Equations, North-Holland Math. Stud., vol. 204, Elsevier Science B.V., Amsterdam, 2006.
[24] M. Kirane, Y. Laskri, N.-e. Tatar, Critical exponents of Fujita type for certain evolution equations and systems with spatio-temporal fractional derivatives, J. Math. Anal. Appl. 312 (2005) 488–501.
[25] M. Kirane, S.A. Malik, The profile of blowing-up solutions to a nonlinear system of fractional differential equations, Nonlinear Anal. 73 (2010) 3723–3736.
[26] M. Kirane, N.-e. Tatar, Exponential growth for a fractionally damped wave equation, Z. Anal. Anwend. 22 (2003) 167–177.
[27] A. Lunardi, Analytic Semigroups and Optimal Regularity in Parabolic Problems, Birkhäuser, Basel, 1995.
[28] F. Mainardi, Fractional calculus: Some basic problems in continuum and statistical mechanics, in: A. Carpinteri, F. Mainardi (Eds.), Fractals and Fractional Calculus in Continuum Mechanics, Springer-Verlag, Wien, 1997, pp. 291–348.
[29] H. Markus, The Functional Calculus for Sectorial Operators, Oper. Theory Adv. Appl., vol. 69, Birkhäuser-Verlag, Basel, 2006.
[30] B. Maslowski, D. Nualart, Evolution equations driven by a fractional Brownian motion, J. Funct. Anal. 202 (2003) 277–305.
[31] A. McIntosh, Operators which have an H^∞ functional calculus, in: Proc. Centre Math. Anal. Austral. Nat. Univ., vol. 14, 1986, pp. 210–231.
[32] F. Metzler, W. Schick, H.G. Kilian, T.F. Nonnenmacher, Relaxation in filled polymers: A fractional calculus approach, J. Chem. Phys. 103 (1995) 7180–7186.
[33] R. Metzler, J. Klafter, The random walk's guide to anomalous diffusion: a fractional dynamics approach, Phys. Rep. 339 (2000) 1–77.
[34] K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Differential Equations, John Wiley, New York, 1993.
[35] J.M.A.M. van Neerven, B. Straub, On the existence and growth of mild solutions of the abstract Cauchy problem for operators with polynomially bounded resolvent, Houston J. Math. 24 (1998) 137–171.
[36] D. Nualart, P. Vuillermot, Variational solutions for partial differential equations driven by fractional noise, J. Funct. Anal. 232 (2006) 390–454.
[37] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, New York, 1983.
[38] F. Periago, B. Straub, A functional calculus for almost sectorial operators and applications to abstract evolution equations, J. Evol. Equ. 2 (2002) 41–68.
[39] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.
[40] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives, Theory and Applications, Gordon and Breach, Yverdon, 1993.
[41] S.P. Toropova, Relation of semigroups with singularity to integrated semigroups, Russian Math. 47 (2003) 66–74.
[42] W. von Wahl, Gebrochene Potenzen eines elliptischen Operators und parabolische Differentialgleichungen in Räumen hölderstetiger Funktionen, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. 11 (1972) 231–258.
[43] T.J. Xiao, J. Liang, The Cauchy Problem for Higher Order Abstract Differential Equations, Lecture Notes in Math., vol. 1701, Springer, Berlin, New York, 1998.