Boundary value problem for a coupled system of nonlinear fractional differential equations

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In this work we discuss a boundary value problem for a coupled differential system of fractional order. The differential operator is taken in the Riemann–Liouville sense and the nonlinear term depends on the fractional derivative of an unknown function. By means of Schauder fixed-point theorem, existence result for the solution is obtained. Our analysis relies on the reduction of the problem considered to the equivalent system of Fredholm integral equations.

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1. Introduction

We consider a boundary value problem of the coupled system

\[
\begin{align*}
D^\alpha u(t) &= f(t, v(t), D^\mu v(t)), & 0 < t < 1, \\
D^\beta v(t) &= g(t, u(t), D^\nu u(t)), & 0 < t < 1, \\
u(0) = u(1) = v(0) = v(1) = 0,
\end{align*}
\]

where \(1 < \alpha, \beta < 2, \mu, \nu > 0, \alpha - \nu \geq 1, \beta - \mu \geq 1, f, g : [0, 1] \times \mathbb{R} \to \mathbb{R} \) are given functions and \(D\) is the standard Riemann–Liouville differentiation.

Fractional differential equation can describe many phenomena in various fields of science and engineering such as control, porous media, electrochemistry, viscoelasticity, electromagnetic, etc. There are a large number of papers dealing with the solvability of nonlinear fractional differential equations. The papers [3–7] considered boundary value problems for fractional differential equations. In [7] the authors investigated the existence and multiplicity of positive solutions for a Dirichlet-type problem of the nonlinear fractional differential equation

\[
D^\alpha_0 u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\
u(0) = u(1) = 0,
\]

where \(1 < \alpha \leq 2\) is a real number, \(f : [0, 1] \times [0, \infty) \to [0, \infty)\) is continuous and \(D^\alpha_0\) is the fractional derivative in the sense of Riemann–Liouville. Because Riemann–Liouville differentiation is not suitable for non-zero boundary values, Zhang, by means of the method of Laplace transforms and fixed-point theorems on a cone, discussed the existence of solutions for nonlinear fractional differential equations with Caputo’s derivative and non-zero boundary values in [5,6] respectively.
study of a coupled differential system of fractional order is also very significant because this kind of system can often occur in applications [9–11]. In [8], Bai established the existence of a positive solution to a singular coupled system of fractional order.

It is worth mentioning that the nonlinear term in the papers [5–8] is independent of the fractional derivative of the unknown function. But the opposite case is more difficult and complicated and this work attempts to deal exactly with this case. The plan of our work is as follows. In the next section, we prepare some material needed to prove our result. The last section is devoted to the existence of solutions for system (1.1).

2. Preliminaries

For completeness, in this section, we recall some definitions and fundamental facts of fractional calculus theory, which can be found in [1,2].

Definition 2.1. The fractional integral of order \( \alpha > 0 \) of a function \( f : (0, \infty) \rightarrow R \) is given by
\[
I_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds,
\]
provided that the integral exists.

Definition 2.2. The fractional derivative of order \( \alpha > 0 \) of a continuous function \( f : (0, \infty) \rightarrow R \) is given by
\[
D_0^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^n \int_0^t \frac{f(s)}{(t-s)^{n-\alpha}} ds,
\]
where \( n = [\alpha] + 1 \) and \( [\alpha] \) denotes the integral part of number \( \alpha \), provided that the right side is pointwise defined on \( (0, \infty) \).

Remark 2.1. The following properties are useful for our discussion: \( D_0^\alpha f(t) = D_0^{\alpha+\beta} f(t), D_0^\alpha D_0^{\beta} f(t) = f(t), \alpha > 0, \beta > 0, f \in L(0, 1); D_0^{\alpha+\beta} f(t) = f(t), 0 < \alpha < 1, f(t) \in C[0, 1] \) and \( D_0^\alpha f(t) \in C(0, 1) \cap L(0, 1); D_0^\alpha : C[0, 1] \rightarrow C[0, 1], \alpha > 0. \)

3. Main results

In this section, we impose growth conditions on \( f \) and \( g \) which allow us to apply the Schauder fixed-point theorem to establish an existence result for solutions for problem (1.1).

Let \( I = [0, 1] \) and \( C(I) \) be the space of all continuous real functions defined on \( I \).

First of all, we present the Green’s function for system (1.1).

Lemma 3.1. Let \( \varphi(t) \in C(I) \) be a given function and \( 1 < \alpha < 2 \); then the unique solution of
\[
\begin{aligned}
D_0^\alpha u(t) &= \varphi(t), \quad 0 < t < 1, \\
\varphi(0) &= \varphi(1) = 0,
\end{aligned}
\]
is \( u(t) = \int_0^1 G_1(t, s) \varphi(s) ds \), where
\[
G_1(t, s) = \begin{cases}
\frac{(t-s)^{\alpha-1} - t(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & s \leq t, \\
\frac{-t(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & t \leq s.
\end{cases}
\]
For the proof of this lemma, we refer the reader to the proof of Lemma 2.3 in [7].

Let
\[
G_2(t, s) = \begin{cases}
\frac{(t-s)^{\beta-1} - t(1-s)^{\beta-1}}{\Gamma(\beta)}, & s \leq t, \\
\frac{-t(1-s)^{\beta-1}}{\Gamma(\beta)}, & t \leq s.
\end{cases}
\]
we call \( (G_1, G_2) \) the Green’s functions of the boundary value problem (1.1).

We define the space \( X = \{ u(t) \mid u(t) \in C(I) \) and \( D_0^\alpha u(t) \in C(I) \} \) endowed with the norm \( \| u \|_X = \max_{t \in I} |u(t)| + \max_{t \in I} |D_0^\alpha u(t)| \).

Lemma 3.2. \( (X, \| \cdot \|_X) \) is a Banach space.
Proof. Let \( \{u_n\}_{n=1}^{\infty} \) be a Cauchy sequence in the space \((X, \| \cdot \|)\); then clearly \( \{u_n\}_{n=1}^{\infty} \) and \( \{D^\nu u_n(t)\}_{n=1}^{\infty} \) are Cauchy sequences in the space \( C(I) \). Therefore, \( \{u_n\}_{n=1}^{\infty} \) and \( \{D^\nu u_n(t)\}_{n=1}^{\infty} \) converge to some \( v \) and \( w \) on \( I \) uniformly and \( v, w \in C(I) \). We need to prove that \( w = D^\nu v \).

Note that
\[
|\Gamma D^\nu u_n(t) - \Gamma w(t)| \leq \frac{1}{\Gamma(v)} \int_0^t (t-s)^{v-1} |D^\nu u_n(s) - w(s)| \, ds
\leq \frac{1}{\Gamma(v + 1)} \max_{t \in I} |D^\nu u_n(t) - w(t)|.
\]

By the convergence of \( \{D^\nu u_n(t)\}_{n=1}^{\infty} \) we have \( \lim_{n \to \infty} \Gamma D^\nu u_n(t) = \Gamma w(t) \) uniformly for \( t \in I \). On the other hand, by Remark 2.1 one has \( \Gamma D^\nu u_n(t) = u_n(t) \). Hence, \( v(t) = \Gamma w(t) \), Remark 2.1 implies that it is equivalent to the relation \( w = D^\nu v \). This completes the proof. \( \square \)

For further purposes we will consider the Banach space \( Y = \{v(t) \mid v(t) \in C(I), D^\mu v(t) \in C(I)\} \) endowed with the norm \( \|v\|_Y = \max_{t \in I} |v(t)| + \max_{t \in I} |D^\mu v(t)| \).

For \( (u, v) \in X \times Y \), let 
\[
\|(u, v)\|_{X \times Y} = \max\{\|u\|_X, \|v\|_Y\}.
\]

Then clearly \((X \times Y, \| \cdot \|_{X \times Y})\) is a Banach space.

Consider the following coupled system of integral equations:
\[
\begin{aligned}
&u(t) = \int_0^1 G_1(t, s)f(s, v(s), D^\mu v(s)) \, ds \\
&v(t) = \int_0^1 G_2(t, s)g(s, u(s), D^\nu u(s)) \, ds.
\end{aligned}
\] (3.1)

Lemma 3.3. Suppose that \( f, g : I \times R \times R \to R \) are continuous. Then \( (u, v) \in X \times Y \) is a solution of (1.1) if and only if \( (u, v) \in X \times Y \) is a solution of system (3.1).

Proof. Let \( (u, v) \in X \times Y \) be a solution of system (1.1). Applying the method used in [7] to prove Lemma 2.3, we can obtain that \( (u, v) \) is a solution of system (3.1). Conversely, let \( (u, v) \in X \times Y \) be a solution of system (3.1). We denote the right-hand side of the first equation in (3.1) by \( w(t) \), i.e., \( w(t) = \Gamma f(t, v(t), D^\nu v(t)) - t^{\alpha - 1} \Gamma f(1, v(1), D^\nu v(1)) \). Using Remark 2.1 and the relation \( D^\nu D^\mu = 0, m = 1, 2, \ldots, N \), where \( N \) is the smallest integer greater than or equal to \( \alpha \) (Remark 2.1 in [7]), we have
\[
D^\mu w(t) = D^\mu [\Gamma f(t, v(t), D^\nu v(t)) - t^{\alpha - 1} \Gamma f(1, v(1), D^\nu v(1))]
= f(t, v(t), D^\nu v(t)),
\]
namely, \( D^\nu u(t) = f(t, v(t), D^\nu v(t)) \). One can verify easily that \( u(0) = D^\mu u(0) = 0 \).

By the same method we can get \( D^\nu v(t) = g(t, u(t), D^\nu u(t)) \), \( v(0) = v(1) = 0 \). Therefore, \( (u, v) \) is a solution of system (1.1). The proof is completed. \( \square \)

Let \( T : X \times Y \to X \times Y \) be the operator defined as
\[
T(u, v)(t) = \left( \int_0^1 G_1(t, s)f(s, v(s), D^\nu v(s)) \, ds, \int_0^1 G_2(t, s)g(s, u(s), D^\nu u(s)) \, ds \right) = (T_1 v(t), T_2 u(t)).
\]

Then by Lemma 3.3, the fixed point of operator \( T \) coincides with the solution of system (1.1).

Now, we give the main result of this work. For convenience, let us define for the following discussion
\[
A = \frac{\Gamma(\alpha - v + 1) + 2 \Gamma(\alpha + 1)}{\Gamma(\alpha + 1) \Gamma(\alpha - v + 1)}, \quad B = \frac{\Gamma(\beta - \mu + 1) + 2 \Gamma(\beta + 1)}{\Gamma(\beta + 1) \Gamma(\beta - \mu + 1)}.
\]

Theorem 3.1. Let \( f, g : I \times R \times R \to R \) be continuous functions. Suppose that one of the following conditions is satisfied.

\( (H_1) \) There exist two nonnegative functions \( a(t), b(t) \in L[0, 1] \) such that \( |f(t, x, y)| \leq a(t) + c_1 |x|^{\rho_1} + c_2 |y|^{\rho_2} \) and \( |g(t, x, y)| \leq b(t) + d_1 |x|^{\rho_1} + d_2 |y|^{\rho_2} \), where \( c_i, d_i \geq 0, 0 < \rho_i, \theta_i < 1 \) for \( i = 1, 2 \).

\( (H_2) \) \( |f(t, x, y)| \leq c_1 |x|^{\rho_1} + c_2 |y|^{\rho_2}, |g(t, x, y)| \leq d_1 |x|^{\rho_1} + d_2 |y|^{\rho_2} \), where \( c_i, d_i > 0, \rho_i, \theta_i > 1 \) for \( i = 1, 2 \).

Then problem (1.1) has a solution.
We shall prove this result by using the Schauder fixed-point theorem. First, let the condition \((H_1)\) be valid. Define
\[
U = \{(u(t), v(t)) \mid (u(t), v(t)) \in X \times Y, \|u(t), v(t)\|_{X \times Y} \leq R, t \in I\},
\]
where
\[
R \geq \max \left\{ \left( \frac{3A_1}{a_1} \right)^{1 \over \alpha-1}, \left( \frac{3A_2}{a_2} \right)^{1 \over \alpha-1}, \left( \frac{3B_1}{b_1} \right)^{1 \over \alpha-1}, \left( \frac{3B_2}{b_2} \right)^{1 \over \alpha-1}, 3k, 3l \right\},
\]
and
\[
k = \max_{t \in I} \int_0^1 |G_1(t, s)a(s)|\,ds + \frac{2}{\Gamma(\alpha - v)} \int_0^1 (1 - s)^{\alpha-v-1}a(s)\,ds,
\]
\[
l = \max_{t \in I} \int_0^1 |G_2(t, s)b(s)|\,ds + \frac{2}{\Gamma(\beta - \mu)} \int_0^1 (1 - s)^{\beta-\mu-1}b(s)\,ds.
\]
Observe that \(U\) is the ball in the Banach space \(X \times Y\).

Now we prove that \(T : U \to U\). For any \((u, v) \in U\), applying Remark 2.1 and the relation \(D^\nu T^\alpha = \frac{r(u)}{r(\alpha-v)} T^\alpha \nu\) (Remark 2.1 in [7]) we have
\[
|T_1 v(t)| = \left| \int_0^1 G_1(t, s)f(s, u(s), D^\nu v(s))\,ds \right|
\]
\[
\leq \int_0^1 |G_1(t, s)a(s)|\,ds + \frac{1}{\Gamma(\alpha - v)} \int_0^1 |G_1(t, s)|\,ds
\]
\[
= \int_0^1 |G_1(t, s)a(s)|\,ds + \frac{1}{\Gamma(\alpha + 1)} \left( \int_0^1 \frac{(1 - s)^{\alpha-1}}{\Gamma(\alpha)} \,ds + \int_0^1 \frac{t(1 - s)^{\alpha-1}}{\Gamma(\alpha)} \,ds \right)
\]
\[
= \int_0^1 |G_1(t, s)a(s)|\,ds + \frac{1}{\Gamma(\alpha + 1)} \left( \int_0^1 (1 - s)^{\alpha-1} \,ds + \frac{1}{\Gamma(\alpha + 1)} \right)
\]
\[
\leq \int_0^1 |G_1(t, s)a(s)|\,ds + \frac{1}{\Gamma(\alpha + 1)}.
\]

\[
|D^\nu T_1 v(t)| = \left| D^\nu f(t, v(t), D^\nu v(t)) - \frac{\Gamma(\alpha)}{\Gamma(\alpha - v)} D^\nu \nu f(1, v(1), D^\nu v(1)) \right|
\]
\[
\leq \frac{1}{\Gamma(\alpha - v)} \left[ \int_0^1 \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} \,ds + \frac{1}{\Gamma(\alpha + 1)} \left( \int_0^1 \frac{(1 - s)^{\alpha-1}}{\Gamma(\alpha)} \,ds \right) \right]
\]
\[
\leq \frac{2}{\Gamma(\alpha - v)} \int_0^1 (1 - s)^{\alpha-1} a(s)\,ds + \frac{1}{\Gamma(\alpha + 1)} \left( \int_0^1 (1 - s)^{\alpha-1} \,ds \right)
\]
\[
= \frac{2}{\Gamma(\alpha - v)} \int_0^1 (1 - s)^{\alpha-1} a(s)\,ds + \frac{1}{\Gamma(\alpha + 1)}.
\]

Hence,
\[
\|T_1 v\|_X \leq k + (c_1 R^\alpha + c_2 R^\nu) A \leq \frac{R}{3} + \frac{R}{3} + \frac{R}{3} = R.
\]

Similarly, one has
\[
\|T_2 u\|_Y \leq l + (d_1 R^\alpha + d_2 R^\nu) B \leq \frac{R}{3} + \frac{R}{3} + \frac{R}{3} = R.
\]

That is, we get \(\|T(u, v)\|_{X \times Y} \leq R\). Notice that \(T_1 v(t), T_2 u(t), D^\nu T_1 v(t)\) and \(D^\nu T_2 u(t)\) are continuous on \(I\). Thus, we have \(T : U \to U\).

Now we are in the position to let \((H_2)\) be satisfied. Choose
\[
0 < R \leq \min \left\{ \left( \frac{1}{2A_1} \right)^{1 \over \alpha-1}, \left( \frac{1}{2A_2} \right)^{1 \over \alpha-1}, \left( \frac{1}{2B_1} \right)^{1 \over \alpha-1}, \left( \frac{1}{2B_2} \right)^{1 \over \alpha-1} \right\}.
\]

Repeating arguments similar to that above we can arrive at
\[
\|T_1 v\|_X \leq (c_1 R^\alpha + c_2 R^\nu) A \leq \frac{R}{2} + \frac{R}{2} = R,
\]

\[
(3.2)
\]

Similarly, one has
\[
\|T_2 u\|_Y \leq l + (d_1 R^\alpha + d_2 R^\nu) B \leq \frac{R}{3} + \frac{R}{3} + \frac{R}{3} = R.
\]

\[
(3.3)
\]

That is, we get \(\|T(u, v)\|_{X \times Y} \leq R\). Notice that \(T_1 v(t), T_2 u(t), D^\nu T_1 v(t)\) and \(D^\nu T_2 u(t)\) are continuous on \(I\). Thus, we have \(T : U \to U\).

Now we are in the position to let \((H_2)\) be satisfied. Choose
\[
0 < R \leq \min \left\{ \left( \frac{1}{2A_1} \right)^{1 \over \alpha-1}, \left( \frac{1}{2A_2} \right)^{1 \over \alpha-1}, \left( \frac{1}{2B_1} \right)^{1 \over \alpha-1}, \left( \frac{1}{2B_2} \right)^{1 \over \alpha-1} \right\}.
\]

Repeating arguments similar to that above we can arrive at
\[
\|T_1 v\|_X \leq (c_1 R^\alpha + c_2 R^\nu) A \leq \frac{R}{2} + \frac{R}{2} = R,
\]

\[
(3.4)
\]
and
\[ \| T_2u \|_V \leq (d_1 R^\alpha + d_2 R^\beta) B \leq R \alpha + R \beta = R. \] (3.5)

Consequently we have \( T : U \to U. \)

In view of the continuity of \( G_1, G_2, f \) and \( g \), it is easy to see that the operator \( T \) is continuous.

In what follows we show that \( T \) is a completely continuous operator. For this we take \( K = \max_{t \in I} |f(t, v(t), D^\alpha v(t))|, L = \max_{t \in I} |g(t, u(t), D^\alpha u(t))| \) for any \( (u, v) \in U \). Let \( t, \tau \in I \) be such that \( t < \tau \); then we have
\[
|T_1v(t) - T_1v(\tau)| = \left| \int_0^t (G_1(t, s) - G_1(\tau, s)) f(s, v(s), D^\alpha v(s)) ds \right|
\leq K \left[ \int_0^t |G_1(t, s) - G_1(\tau, s)| ds + \int_0^\tau |G_1(t, s) - G_1(\tau, s)| ds + \int_\tau^t |G_1(t, s) - G_1(\tau, s)| ds \right]
\leq \frac{K}{\Gamma(\alpha)} \left[ \int_0^t (t - s)^{\alpha - 1} + t^{\alpha - 1} - t^{\alpha - 1} ds \right] + \int_\tau^t \left( t^{\alpha - 1} - (t - s)^{\alpha - 1} \right) ds + \int_\tau^t \left( t^{\alpha - 1} - t^{\alpha - 1} \right) ds
\leq \frac{K}{\Gamma(\alpha + 1)} \left( t^{\alpha - 1} - (t - s)^{\alpha - 1} + t^{\alpha - 1} \right),
\]
\[
|D^\alpha T_1v(t) - D^\alpha T_1v(\tau)| = \left| f^\alpha f(t, v(t), D^\alpha v(t)) = \frac{\Gamma(\alpha)}{\Gamma(\alpha - \nu)} t^{\alpha - \nu - 1} \right| \left| f(1, v(1), D^\nu v(1)) \right|
\leq \frac{1}{\Gamma(\alpha - \nu)} \left[ \int_0^t (t - s)^{\alpha - 1} f(s, v(s), D^\nu v(s)) ds - \int_0^\tau (t - s)^{\alpha - 1} f(s, v(s), D^\nu v(s)) ds \right]
\leq \frac{1}{\Gamma(\alpha - \nu)} \left[ \int_0^t (t - s)^{\alpha - 1} f(s, v(s), D^\nu v(s)) ds - \int_0^\tau (t - s)^{\alpha - 1} f(s, v(s), D^\nu v(s)) ds \right]
\leq \frac{K}{\Gamma(\alpha - \nu)} \left[ \int_0^\tau (t - s)^{\alpha - 1} - (t - s)^{\alpha - 1} ds \right]
\leq \frac{K}{\Gamma(\alpha - \nu)} \left[ \int_0^\tau (t - s)^{\alpha - 1} - (t - s)^{\alpha - 1} ds \right] + \frac{K}{\alpha \Gamma(\alpha - \nu)} \left( t^{\alpha - 1} - t^{\alpha - 1} \right)
\leq \frac{K}{\Gamma(\alpha - \nu)} \left( t^{\alpha - 1} - t^{\alpha - 1} \right).
\]

Similarly,
\[
|T_2u(t) - T_2u(\tau)| \leq \frac{L}{\Gamma(\beta + 1)} \left( t^{\beta - 1} - t^{\beta - 1} + t^{\beta - 1} \right),
\]
\[
|D^\beta T_2u(t) - D^\beta T_2u(\tau)| \leq \frac{L}{\Gamma(\beta + \mu + 1)} \left( t^{\beta - \mu} - t^{\beta - \mu} \right) + \frac{L}{\beta \Gamma(\beta - \mu)} \left( t^{\beta - \mu - 1} - t^{\beta - \mu - 1} \right).
\]

Now, using the fact that the functions \( t^{\alpha}, t^{\beta - 1}, t^{\beta - 1}, t^{\alpha - 1}, t^{\beta - \mu}, t^{\beta - \mu} \) are uniformly continuous on the interval \( I \), we conclude that \( TU \) is an equicontinuous set. Obviously it is uniformly bounded since \( TU \subseteq U \). Thus, \( T \) is completely continuous. The Schauder fixed-point theorem implies the existence of solutions in \( U \) for the problem (1.1) and the theorem is proved. \( \square \)

**Remark 3.1.** For the sake of simplicity, we assume that \( a(t) = b(t) = 0 \) in condition (H2). Otherwise, it should be replaced by a weak but complicated form which can be derived easily from (3.2) and (3.3), and here we omit it.

**Remark 3.2.** If we impose additionally some restriction on \( c_i \) or \( d_i \) in (H1) and (H2), the conclusion of Theorem 3.1 remains true for the nonstrict inequalities \( \rho, \theta_i \leq 1 \) and \( \rho, \theta_i \geq 1 \). For example, we suppose that \( \rho, \theta_i \geq 1 \) in condition (H2); in addition, if \( \rho_i = 1 \), then \( c_i \leq \frac{1}{2a} \) and if \( \theta_i = 1 \), then \( d_i \leq \frac{1}{2a} \). Without loss of generality, let \( \rho_1 = 1 \) and \( \rho_2, \theta_1, \theta_2 > 1 \); then we
may choose

\[
0 < R \leq \min \left\{ \left( \frac{1}{2Ac_2} \right)^{\frac{1}{2}} - \frac{1}{2}, \left( \frac{1}{2Bd_1} \right)^{\frac{1}{2}} - \frac{1}{2}, \left( \frac{1}{2Bd_2} \right)^{\frac{1}{2}} - \frac{1}{2} \right\}.
\]

One can easily obtain the estimates (3.4) and (3.5). Further arguments such as that in Theorem 3.1 yield our desired result.

The following corollary is so simple that we omit its proof.

**Corollary 3.1.** Assume that \( f, g \) are bounded and continuous on \( I \times R \times R \). Then there exists a solution for problem (1.1).

**Example 3.1.** Consider the problem

\[
\begin{align*}
D_t^3 u(t) &= \left( t - \frac{1}{2} \right)^4 \left( (v(t))^{\rho_1} + (D_t^1 v(t))^{\rho_2} \right), \quad 0 < t < 1, \\
D_t^3 v(t) &= \left( t - \frac{1}{2} \right)^4 \left( (u(t))^{\theta_1} + (D_t^1 u(t))^{\theta_2} \right), \quad 0 < t < 1, \\
u(0) = u(1) = v(0) = v(1) = 0.
\end{align*}
\]

(3.6)

where \( 0 < \rho_i, \theta_i < 1 \) or \( \rho_i, \theta_i > 1 \).

Note that \( a(t) = b(t) = 0 \) and \( c_i = d_i = \frac{1}{16} \). By Theorem 3.1 the existence of solutions is obvious for \( 0 < \rho_i, \theta_i < 1 \) or \( \rho_i, \theta_i > 1 \). Consider the nonstrict inequalities \( \rho_i, \theta_i \geq 1 \). With the use of \( \Gamma \left( \frac{1}{2} \right) = \sqrt{\pi} \) and \( \Gamma \left( \frac{3}{4} \right) \approx 1.2254 \), a simple computation shows \( \frac{1}{2A} \approx 0.1817, \frac{1}{2B} \approx 0.2352 \). Since \( c_i < \frac{1}{2A}, d_i < \frac{1}{2B} \), Remark 3.2 implies that problem (3.6) has a solution.

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