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Statistical convergence of double sequences

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Abstract

The idea of statistical convergence was first introduced by Fast (1951) but the rapid developments were started after the papers of Šalát (1980) and Fridy (1985). Now a days it has become one of the most active area of research in the field of summability. In this paper we define and study statistical analogue of convergence and Cauchy for double sequences. We also establish the relation between statistical convergence and strongly Cesàro summable double sequences.

Keywords: Statistical convergence; Statistically Cauchy sequence; Double sequence; Double natural density

1. Introduction

The concept of statistical convergence was first introduced by Fast [4] and also independently by Buck [1] and Schoenberg [10] for real and complex sequences. Further this concept was studied by Šalát [9], Fridy [5], Connor [3] and many others.

Recall that a subset E of the set $\mathbb{N}$ of natural numbers is said to have "natural density" $\delta(E)$ if

$$\delta(E) = \lim_n \frac{1}{n}\left|\{k \leqslant n\colon\ k \in E\}\right|,$$

where the vertical bars denote the cardinality of the enclosed set.

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The number sequence $x = (x_k)$ is said to be *statistically convergent* to the number ℓ if for each $\varepsilon > 0$,

$$\lim_n \frac{1}{n}\left|\left\{k \leqslant n\colon\ |x_k - \ell| \geqslant \varepsilon\right\}\right| = 0;$$

and x is said to be *statistically Cauchy sequence* if for every $\varepsilon > 0$ there exists a number $N = N(\varepsilon)$ such that

$$\lim_n \frac{1}{n}\left|\left\{k \leqslant n\colon\ |x_k - x_N| \geqslant \varepsilon\right\}\right| = 0.$$

In this paper we define these two concepts for double sequences $x = (x_{jk})$ and prove some related results, i.e., two-dimensional analogues supported by some interesting examples.

By the convergence of a double sequence we mean the convergence in Pringsheim's sense [8]. A double sequence $x = (x_{jk})_{j,k=0}^{\infty}$ is said to be *convergent in the Pringsheim's sense* if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{jk} - L| < \varepsilon$ whenever $j, k \geqslant N$. L is called the Pringsheim limit of x.

A double sequence $x = (x_{jk})$ is said to be *Cauchy sequence* if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{pq} - x_{jk}| < \varepsilon$ for all $p \geqslant j \geqslant N$, $q \geqslant k \geqslant N$.

A double sequence x is *bounded* if there exists a positive number M such that $|x_{jk}| < M$ for all j and k, i.e., if

$$\|x\|_{(\infty,2)} = \sup_{j,k} |x_{jk}| < \infty. \tag{1.1}$$

We will denote the set of all bounded double sequences by ℓ_∞^2. Note that in contrast to the case for single sequences, a convergent double sequence need not be bounded.

2. Statistical convergence

Let $K \subseteq \mathbb{N} \times \mathbb{N}$ be a two-dimensional set of positive integers and let $K(n,m)$ be the numbers of (i, j) in K such that $i \leqslant n$ and $j \leqslant m$. Then the two-dimensional analogue of natural density can be defined as follows.

The *lower asymptotic density* of a set $K \subseteq \mathbb{N} \times \mathbb{N}$ is defined as

$$\underline{\delta_2}(K) = \liminf_{n,m} \frac{K(n,m)}{nm}.$$

In case the sequence $(K(n,m)/nm)$ has a limit in Pringsheim's sense then we say that K has a *double natural density* and is defined as

$$\lim_{n,m} \frac{K(n,m)}{nm} = \delta_2(K).$$

For example, let $K = \{(i^2, j^2)\colon\ i, j \in \mathbb{N}\}$. Then

$$\delta_2(K) = \lim_{n,m} \frac{K(n,m)}{nm} \leqslant \lim_{n,m} \frac{\sqrt{n}\sqrt{m}}{nm} = 0,$$

i.e., the set K has double natural density zero, while the set $\{(i, 2j)\colon\ i, j \in \mathbb{N}\}$ has double natural density $1/2$.

Note that, if we set $n = m$, we have a two-dimensional natural density considered by Christopher [2].

We define the statistical analogue for double sequences $x = (x_{jk})$ as follows.

Definition 2.1. A real double sequence $x = (x_{jk})$ is said to be *statistically convergent* to the number ℓ if for each $\varepsilon > 0$, the set

$$\bigl\{(j,k),\ j \leqslant n \text{ and } k \leqslant m\colon\ |x_{jk} - \ell| \geqslant \varepsilon\bigr\}$$

has double natural density zero. In this case we write $st_2\text{-}\lim_{j,k} x_{jk} = \ell$ and we denote the set of all statistically convergent double sequences by st_2.

Remark 1. (a) If x is a convergent double sequence then it is also statistically convergent to the same number. Since there are only a finite number of bounded (unbounded) rows and/or columns,

$$K(n,m) \leqslant s_1 n + s_2 m,$$

where s_1 and s_2 are finite numbers, which we can conclude that x is statistically convergent.

(b) If x is statistically convergent to the number ℓ, then ℓ is determined uniquely.

(c) If x is statistically convergent, then x need not be convergent. Also it is not necessarily bounded. For example, let $x = (x_{jk})$ be defined as

$$x_{jk} = \begin{cases} jk, & \text{if } j \text{ and } k \text{ are squares,} \\ 1, & \text{otherwise.} \end{cases}$$

It is easy to see that $st_2\text{-}\lim x_{jk} = 1$, since the cardinality of the set $\{(j,k)\colon\ |x_{jk} - 1| \geqslant \varepsilon\}$ $\leqslant \sqrt{j}\,\sqrt{k}$ for every $\varepsilon > 0$. But x is neither convergent nor bounded.

We prove some analogues for double sequences. For single sequences such results have been proved by Šalát [9].

Theorem 2.1. *A real double sequence $x = (x_{jk})$ is statistically convergent to a number ℓ if and only if there exists a subset $K = \{(j,k)\} \subseteq \mathbb{N} \times \mathbb{N}$, $j, k = 1, 2, \ldots,$ such that $\delta_2(K) = 1$ and*

$$\lim_{\substack{j,k\to\infty \\ (j,k)\in K}} x_{jk} = \ell.$$

Proof. Let x be statistically convergent to ℓ. Put

$$K_r = \left\{(j,k) \in \mathbb{N} \times \mathbb{N}\colon\ |x_{jk} - \ell| \geqslant \frac{1}{r}\right\}$$

and

$$M_r = \left\{(j,k) \in \mathbb{N} \times \mathbb{N}\colon\ |x_{jk} - \ell| < \frac{1}{r}\right\} \quad (r = 1, 2, \ldots).$$

Then $\delta_2(K_r) = 0$ and

$$(1) \quad M_1 \supset M_2 \supset \cdots \supset M_i \supset M_{i+1} \supset \cdots$$

and

$$(2) \quad \delta_2(M_r) = 1, \quad r = 1, 2, \ldots.$$

Now we have to show that for $(j,k) \in M_r$, (x_{jk}) is convergent to ℓ. Suppose that (x_{jk}) is not convergent to ℓ. Therefore there is $\varepsilon > 0$ such that

$$|x_{jk} - \ell| \geqslant \varepsilon \quad \text{for infinitely many terms.}$$

Let

$$M_\varepsilon = \big\{(j,k)\colon\ |x_{jk} - \ell| < \varepsilon\big\} \quad \text{and} \quad \varepsilon > \frac{1}{r} \quad (r = 1, 2, \ldots).$$

Then

$$(3) \quad \delta_2(M_\varepsilon) = 0,$$

and by (1), $M_r \subset M_\varepsilon$. Hence $\delta_2(M_r) = 0$ which contradicts (2). Therefore (x_{jk}) is convergent to ℓ.

Conversely, suppose that there exists a subset $K = \{(j,k)\} \subseteq \mathbb{N} \times \mathbb{N}$ such that $\delta_2(K) = 1$ and $\lim_{j,k} x_{jk} = \ell$, i.e., there exists $N \in \mathbb{N}$ such that for every $\varepsilon > 0$,

$$|x_{jk} - \ell| < \varepsilon, \quad \forall j, k \geqslant N.$$

Now

$$K_\varepsilon = \big\{(j,k)\colon\ |x_{jk} - \ell| \geqslant \varepsilon\big\} \subseteq \mathbb{N} \times \mathbb{N} - \big\{(j_{N+1}, k_{N+1}), (j_{N+2}, k_{N+2}), \ldots\big\}.$$

Therefore

$$\delta_2(K_\varepsilon) \leqslant 1 - 1 = 0.$$

Hence x is statistically convergent to ℓ. □

Remark 2. If $st\text{-}\lim_{jk} x_{jk} = \ell$, then there exists a sequence $y = (y_{jk})$ such that $\lim_{j,k} y_{jk} = \ell$ and $\delta_2\{(j,k)\colon\ x_{jk} = y_{jk}\} = 1$, i.e.,

$$x_{jk} = y_{jk} \quad \text{for almost all } j, k \text{ (for short a.a. } j, k).$$

Theorem 2.2. *The set $st_2 \cap \ell_\infty^2$ is a closed linear subspace of the normed linear space ℓ_∞^2.*

Proof. Let $x^{(nm)} = (x_{jk}^{(nm)}) \in st_2 \cap \ell_\infty^2$ and $x^{(nm)} \to x \in \ell_\infty^2$. Since $x^{(nm)} \in st_2 \cap \ell_\infty^2$, there exist real numbers a_{nm} such that

$$st_2\text{-}\lim_{j,k} x_{jk}^{(nm)} = a_{nm} \quad (n, m = 1, 2, \ldots).$$

As $x^{(nm)} \to x$, for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|x^{(pq)} - x^{(nm)}| < \varepsilon/3 \tag{2.1}$$

for every $p \geqslant n \geqslant N$, $q \geqslant m \geqslant N$, where $|\cdot|$ denotes the norm in a linear space.

By Theorem 2.1, there exist subsets K_1 and K_2 of $\mathbb{N}\times\mathbb{N}$ with $\delta_2(K_1)=\delta_2(K_2)=1$ and

$$(1)\quad \lim_{j,k}\ {}_{(j,k)\in K_1}\ x_{jk}^{(nm)}=a_{nm},$$

$$(2)\quad \lim_{j,k}\ {}_{(j,k)\in K_2}\ x_{jk}^{(pq)}=a_{pq}.$$

Now the set $K_1\cap K_2$ is infinite since $\delta_2(K_1\cap K_2)=1$.

Choose $(k_1,k_2)\in K_1\cap K_2$. We have from (1) and (2) that

$$\left|x_{k_1,k_2}^{(pq)}-a_{pq}\right|<\varepsilon/3 \tag{2.2}$$

and

$$\left|x_{k_1,k_2}^{(nm)}-a_{nm}\right|<\varepsilon/3. \tag{2.3}$$

Therefore for each $p\geqslant n\geqslant N$ and $q\geqslant m\geqslant N$ we have from (2.1)–(2.3),

$$\begin{aligned}|a_{pq}-a_{nm}| &\leqslant \left|a_{pq}-x_{k_1,k_2}^{pq}\right|+\left|x_{k_1,k_2}^{pq}-x_{k_1,k_2}^{nm}\right|+\left|x_{k_1,k_2}^{nm}-a_{nm}\right| \\ &<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.\end{aligned}$$

That is, the sequence (a_{nm}) is a Cauchy sequence and hence convergent. Let

$$(3)\quad \lim_{n,m} a_{nm}=a.$$

We need to show that x is statistically convergent to a. Since $x^{(nm)}$ is convergent to x, for every $\varepsilon>0$ there is $N_1(\varepsilon)$ such that for $j,k\geqslant N_1(\varepsilon)$,

$$\left|x_{jk}^{(nm)}-x_{jk}\right|<\varepsilon/3.$$

Also from (3) we have for every $\varepsilon>0$ there is $N_2(\varepsilon)$ such that for all $j,k\geqslant N_2(\varepsilon)$,

$$|a_{jk}-a|<\varepsilon/3.$$

Again, since $x^{(nm)}$ is statistically convergent to a_{nm}, there exists a set $K=\{(j,k)\}\subseteq\mathbb{N}\times\mathbb{N}$ such that $\delta_2(K)=1$ and for every $\varepsilon>0$ there is $N_3(\varepsilon)$ such that for all $j,k\geqslant N_3(\varepsilon)$, $(j,k)\in K$,

$$\left|x_{jk}^{(nm)}-a_{nm}\right|<\varepsilon/3.$$

Let $\max\{N_1(\varepsilon),N_2(\varepsilon),N_3(\varepsilon)\}=N_4(\varepsilon)$. Then for a given $\varepsilon>0$ and for all $j,k\geqslant N_4(\varepsilon)$, $(j,k)\in K$,

$$|x_{jk}-a|\leqslant\left|x_{jk}-x_{jk}^{(nm)}\right|+\left|x_{jk}^{(nm)}-a_{jk}\right|+|a_{jk}-a|<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.$$

Therefore x is statistically convergent to a, i.e., $x\in st_2\cap\ell_\infty^2$.

Hence $st_2\cap\ell_\infty^2$ is a closed linear subspace of ℓ_∞^2. $\square$

Theorem 2.3. *The set $st_2\cap\ell_\infty^2$ is nowhere dense in ℓ_∞^2.*

Proof. Since every closed linear subspace of an arbitrary linear normed space S different from S is a nowhere dense set in S (see [7]), by Theorem 2.2 we need only to show that $st_2 \cap \ell_\infty^2 \neq \ell_\infty^2$.

Let the sequence $x = (x_{jk})$ be defined by

$$x_{jk} = \begin{cases} 1, & \text{if } j \text{ and } k \text{ are even}, \\ 0, & \text{otherwise}. \end{cases}$$

It is clear that x is not statistically convergent but x is bounded. Hence $st_2 \cap \ell_\infty^2 \neq \ell_\infty^2$. □

3. Statistically Cauchy sequences

In [5], Fridy has defined the concept of statistically Cauchy single sequences. In this section we define statistically Cauchy double sequences and prove some analogues.

Definition 3.1. A real double sequence $x = (x_{jk})$ is said to be *statistically Cauchy* if for every $\varepsilon > 0$ there exist $N = N(\varepsilon)$ and $M = M(\varepsilon)$ such that for all $j, p \geqslant N$, $k, q \geqslant M$, the set

$$\{(j,k),\ j \leqslant n,\ k \leqslant m\colon |x_{jk} - x_{pq}| \geqslant \varepsilon\}$$

has natural density zero.

Theorem 3.1. *A real double sequence $x = (x_{jk})$ is statistically convergent if and only if x is statistically Cauchy.*

Proof. Let x be statistically convergent to a number ℓ. Then for every $\varepsilon > 0$, the set

$$\{(j,k),\ j \leqslant n,\ k \leqslant m\colon |x_{jk} - \ell| \geqslant \varepsilon\}$$

has natural density zero. Choose two numbers N and M such that $|x_{NM} - \ell| \geqslant \varepsilon$. Now let

$$A_\varepsilon = \{(j,k),\ j \leqslant n,\ k \leqslant m\colon |x_{jk} - x_{NM}| \geqslant \varepsilon\},$$
$$B_\varepsilon = \{(j,k),\ j \leqslant n,\ k \leqslant m\colon |x_{jk} - \ell| \geqslant \varepsilon\},$$
$$C_\varepsilon = \{(j,k),\ j = N \leqslant n,\ k = M \leqslant m\colon |x_{NM} - \ell| \geqslant \varepsilon\}.$$

Then $A_\varepsilon \subseteq B_\varepsilon \cup C_\varepsilon$ and therefore $\delta_2(A_\varepsilon) \leqslant \delta_2(B_\varepsilon) + \delta_2(C_\varepsilon) = 0$. Hence x is statistically Cauchy.

Conversely, let x be statistically Cauchy but not statistically convergent. Then there exist N and M such that the set A_ε has natural density zero. Hence the set

$$E_\varepsilon = \{(j,k),\ j \leqslant n,\ k \leqslant m\colon |x_{jk} - x_{NM}| < \varepsilon\}$$

has natural density 1. In particular, we can write

$$(*) \quad |x_{jk} - x_{NM}| \leqslant 2|x_{jk} - \ell| < \varepsilon$$

if $|x_{jk} - \ell| < \varepsilon/2$. Since x is not statistically convergent, the set B_ε has natural density 1, i.e., the set

$$\{(j,k),\ j \leqslant n,\ k \leqslant m\colon |x_{jk} - \ell| < \varepsilon\}$$

has natural density 0. Therefore by $(*)$, the set

$$\{(j,k),\ j \leqslant n,\ k \leqslant m\colon |x_{jk} - x_{NM}| < \varepsilon\}$$

has natural density 0, i.e., the set A_ε has natural density 1 which is a contradiction. Hence x is statistically convergent. □

From Theorems 2.1 and 3.1 we can state the following for double sequences analogous to the result of Fridy [5].

Theorem 3.2. *The following statements are equivalent*:

(a) *x is statistically convergent to ℓ*;
(b) *x is statistically Cauchy*;
(c) *there exists a subsequence y of x such that*

$$\lim_{jk} y_{jk} = \ell.$$

4. Relation between statistical convergence and strongly Cesàro summable sequences

The following definitions of Cesàro summable double sequences is taken from [6].

Definition 4.1. Let $x = (x_{jk})$ be a double sequence. It is said to be *Cesàro summable* to ℓ if

$$\lim_{n,m} \frac{1}{nm} \sum_{j=1}^{n} \sum_{k=1}^{m} x_{jk} = \ell.$$

We denote the space of all Cesàro summable double sequences by $(C,1,1)$.

Similarly we can define the following as in case of single sequences.

Definition 4.2. Let $x = (x_{jk})$ be a double sequence and p be a positive real number. Then the double sequence x is said to be *strongly p-Cesàro summable* to ℓ if

$$\lim_{n,m} \frac{1}{nm} \sum_{j=1}^{n} \sum_{k=1}^{m} |x_{jk} - \ell|^p = 0.$$

We denote the space of all strongly p-Cesàro summable double sequences by w_p^2.

Remark 3. (i) If $0 < p \leqslant q < \infty$, then $w_q^2 \subseteq w_p^2$ (by Hölder's inequality) and

$$w_p^2 \cap \ell_\infty^2 = w_1^2 \cap \ell_\infty^2 \subseteq (C,1,1) \cap \ell_\infty^2.$$

(ii) If x is convergent but unbounded then x is statistically convergent but x need not be Cesàro nor strongly Cesàro.

Example 1. Let $x = (x_{jk})$ be defined as

$$x_{jk} = \begin{cases} k, & j = 1, \text{ for all } k, \\ j, & k = 1, \text{ for all } j, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\lim_{j,k} x_{jk} = 0$ but

$$\lim_{n,m} \frac{1}{nm} \sum_{j=1}^{n} \sum_{k=1}^{m} x_{jk} = \lim_{n,m} \frac{1}{nm} \frac{1}{2}(m^2 + n^2 + m + n - 2),$$

which does not tend to a finite limit. Hence x is not Cesàro. Also x is not strongly Cesàro but

$$\lim_{n,m} \frac{1}{nm} \left|\left\{(j,k)\colon |x_{jk} - 0| \geqslant \varepsilon\right\}\right| = \lim_{n,m} \frac{m + n - 1}{nm} = 0,$$

i.e., x is statistically convergent to 0.

(iii) If x is a bounded convergent double sequence then it is also $(C, 1, 1)$, w_p^2 and st_2.

The following result is analogue of Theorem 2.1 due to Connor [3].

Theorem 4.1. *Let $x = (x_{jk})$ be a double sequence and p be a positive real number. Then*

(a) *x is statistically convergent to ℓ if it is strongly p-Cesàro summable to ℓ;*
(b) $w_p^2 \cap \ell_\infty^2 = st_2 \cap \ell_\infty^2$.

Proof. (a) Let $K_\varepsilon(p) = \{(j,k),\ j \leqslant n,\ k \leqslant m\colon |x_{jk} - \ell|^p \geqslant \varepsilon\}$. Now since x is strongly p-Cesàro summable to ℓ,

$$\begin{aligned} 0 \leftarrow{}& \frac{1}{nm} \sum_{j=1}^{n} \sum_{k=1}^{m} |x_{jk} - \ell|^p \\ ={}& \frac{1}{nm} \left\{ \sum_{(j,k) \in K_\varepsilon(p)} |x_{jk} - \ell|^p + \sum_{(j,k) \notin K_\varepsilon(p)} |x_{jk} - \ell|^p \right\} \\ \geqslant{}& \frac{1}{nm} \left|\left\{(j,k),\ j \leqslant n,\ k \leqslant m\colon |x_{jk} - \ell|^p \geqslant \varepsilon\right\}\right| \varepsilon. \end{aligned}$$

Hence x is statistically convergent to ℓ.

(b) Let $I_\varepsilon(p) = \{(j,k),\ j \leqslant n,\ k \leqslant m\colon |x_{jk} - \ell| \geqslant (\varepsilon/2)^{1/p}\}$ and $M = \|x\|_{(\infty,2)} + |\ell|$, where $\|x\|_{(\infty,2)}$ is the sup-norm for bounded double sequences $x = (x_{jk})$ given by (1.1).

Since x is a bounded statistically convergent, we can choose $N = N(\varepsilon)$ such that for all $n, m \geqslant N$,

$$\frac{1}{nm} \left| \left\{ (j,k),\ j \leqslant n,\ k \leqslant m\colon |x_{jk} - \ell| \geqslant \left(\frac{\varepsilon}{2}\right)^{1/p} \right\} \right| < \frac{\varepsilon}{2M^p}.$$

Now for all $n, m \geqslant N$ we have

$$\frac{1}{nm}\sum_{j=1}^{n}\sum_{k=1}^{m}|x_{jk}-\ell|^p=\frac{1}{nm}\left\{\sum_{(j,k)\in I_\varepsilon(p)}|x_{jk}-\ell|^p+\sum_{(j,k)\notin I_\varepsilon(p)}|x_{jk}-\ell|^p\right\}$$
$$<\frac{1}{nm}\,nm\,\frac{\varepsilon}{2M^p}M^p+\frac{1}{nm}\,nm\,\frac{\varepsilon}{2}=\varepsilon.$$

Hence x is strongly p-Cesàro summable to ℓ. □

Remark 4. Note that if a bounded sequence x is statistically convergent then it is also $(C,1,1)$ summable but not conversely.

Example 2. Let $x=(x_{jk})$ be defined by

$$x_{jk}=(-1)^j,\quad \forall k;$$

then

$$\lim_{n,m}\frac{1}{nm}\sum_{j=1}^{n}\sum_{k=1}^{m}x_{jk}=0,$$

but obviously x is not statistically convergent.

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