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Brane-world black hole entropy from modified dispersion relations

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ABSTRACT

The entropy of the Reissner–Nördstrom black hole is studied within the context of a brane-world scenario. Such a black hole is a solution of the Einstein field equations on the brane, possessing a tidal charge which is a reflection of the extra dimension. We use the modified dispersion relation to obtain the entropy of such brane-world black holes. The resulting entropy differs from that of the standard Bekenstein–Hawking's and contains information on the extra dimension.

1. Introduction

A common feature of all promising candidates for quantum gravity is the existence of a minimal observable length [1–5]. The modified dispersion relation (MDR) is one of the approaches incorporating such a finite resolution of the space–time in the theoretical framework of the standard model. MDR is a common feature in all candidates of quantum gravity. In particular, in the study of loop quantum gravity (LQG) and of models based on non-commutative geometry, there has been strong interest in modifications to the energy–momentum dispersion relation [6–10]. Since black holes are suitable examples of an extreme quantum gravity regime, using MDR to study their thermodynamical behavior and comparing the results with other approaches may further our understanding of their properties and structure.

There are, presumably, other more reliable theories such as string theory and loop quantum gravity with which to study black hole thermodynamics. One may therefore use the results of such studies to impose constraints on the MDR [9–11] which would ultimately result in a better insight into quantum gravity. A study along these lines was performed in a previous work [11]. In this Letter, we use the form of MDR obtained in the above mentioned study where terms proportional to odd powers of energy are not present in our modified dispersion relation [11] and concentrate on the brane-world black hole entropy. The motivation behind the study of brane-world black holes stem from the fact that since the advent of theories with extra non-compact dimensions, great progress has been made in describing some hitherto unexplained problems in particle physics, e.g. the hierarchy problem, without appealing to supersymmetry [12]. In such models, our physical universe is a 3-dimensional brane embedded in a higher dimensional bulk (usually one extra dimension) where the standard matter is confined to the brane except gravity which can propagate into the bulk [12–14] as well as on the brane and the size of the extra dimensions can be much larger than the Planck length scale [12]. The large size of the extra dimension is the key for providing a unification scale of the order of a few TeV [15]. It is therefore plausible to think of the possibility of TeV-sized black holes being produced in the universe. Density perturbations and phase transition in the early universe may lead to such black holes. High energy collision processes in cosmic rays and at future colliders such as LHC can also produce these TeV-sized black holes [16–18]. In such processes, matter on the 3-brane may collapse under gravity to form a black hole. To preserve the general relativity observational predictions, the metric on the brane should be close to the Schwarzschild metric at astrophysical scales [19].

Brane-world black hole solutions have been studied by a number of authors [19–21]. Of particular interest is the solution presented in [19] which is akin to that of the Reissner–Nördstrom solution, but without the electric charge being present. Instead, the Reissner–Nördstrom type correction to the Schwarzschild potential can be thought of as a tidal charge, arising from the projection onto the brane of the free gravitational field effects in the bulk. The study of entropy of brane-world black holes may bring about information on the extra dimensions which, in turn, would provide a deeper insight into the quantum theory of gravity. The entropy of a brane-world black hole was first studied in [22] within the gen-

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eralized uncertainty principle (GUP) formalism and later in [23]. The entropy of a TeV-sized Reissner–Nördstrom type black hole within the context of the ADD brane-world scenario was considered in [15].

In this Letter, we will apply the MDR formalism to obtain the entropy of a Reissner–Nördstrom type black hole obtained in [19]. Such a study would be of interest and complementary to what has been done in [15] in that the results of the two approaches, namely GUP and MDR can be compared and interpreted. This is what we intend to do in what follows.

2. The modified dispersion relation

The modified dispersion relation can be written as [9]

$$(\vec{p})^2 = f(E, m; L_p) \simeq E^2 - \mu^2 + \alpha_1 L_p E^3 + \alpha_2 L_p^2 E^4 + \mathcal{O}\left(L_p^3 E^5\right), \tag{1}$$

where f is the function that gives the exact dispersion relation and L_p is the Planck length. On the right-hand side we have assumed the applicability of a Taylor-series expansion for $E \ll \frac{1}{L_p}$. The coefficients α_i may take different values in different quantum gravity approaches. Note that m is the rest energy of the particle and the mass parameter μ on the right-hand side is directly related to the rest energy, but $\mu \neq m$ if α_i's do not all vanish.

Although MDR is a feature of all quantum gravity scenarios, its functional form depends on the quantum gravity model being used. To incorporate quantum gravitational effects, the Bekenstein–Hawking formalism of black hole thermodynamics needs to be modified. Of course, MDR may provide a perturbation framework for such a modification. On the other hand, loop quantum gravity and string theory give the entropy–area relation of black holes (for $A \gg L_p^2$)

$$S = \frac{A}{4L_p^2} + \rho \ln \frac{A}{L_p^2} + \mathcal{O}\left(\frac{L_p^2}{A}\right), \tag{2}$$

where ρ may have different values in string theory and in loop quantum gravity [9,10,24]. Since string theory and loop quantum gravity are expected to provide a more reliable solution to black hole thermodynamics, these solutions could be considered as a test bed against which other solutions including the ones obtained using MDR [11] should be compared. With that in mind, the entropy of a black hole obtained using Eq. (1) is functionally different from what one obtains using string theory and loop quantum gravity given by Eq. (2). It is then necessary to introduce constraints on the usual form of the MDR to obtain a consistent black hole thermodynamics in both approaches. The result is that terms proportional to odd powers of energy should be ignored in the MDR formula [11]. Consequently, we take the MDR as

$$(\vec{p})^2 = f(E, m; L_p) \simeq E^2 - \mu^2 + \alpha L_p^2 E^4 + \mathcal{O}\left(L_p^4 E^6\right), \tag{3}$$

in what follows. Of course, the black hole thermodynamics obtained via Eq. (3) is now consistent with the result given by Eq. (2).

3. Black holes on the brane

The authors in [19], working in the framework of the Randall–Sundrum scenario, present an exact localized black hole solution which resembles that of a Reissner–Nördstrom solution, but without the electric charge. Instead, the Reissner–Nördstrom type correction to the Schwarzschild potential can be thought of as a tidal charge, arising from the projection of the free gravitational field effects in the bulk onto the brane. These effects are transmitted via the bulk Weyl tensor. The Schwarzschild potential $\phi = -M/(M_p^2 r)$ where M_p is the effective Planck mass on the brane is modified to

$$\phi = -\frac{M}{M_p^2 r} + \frac{Q}{2r^2}, \tag{4}$$

where Q is a tidal charge parameter which may be positive or negative. They showed that an exact black hole solution of the effective field equations on the brane is given by the induced metric

$$ds_4^2 = -f\,dt^2 + f^{-1}\,dr^2 + r^2\,d\Omega^2, \tag{5}$$

where

$$f = 1 - \left(\frac{2M}{M_p^2}\right)\frac{1}{r} + \left(\frac{q}{\widetilde{M}_p^2}\right)\frac{1}{r^2}.$$

Note that $\widetilde{M}_p$ and M_p are fundamental Planck scale in the bulk and effective scale on the brane respectively. Since $q = Q\,\widetilde{M}_p^2$ is a dimensionless tidal charge parameter arising from the projection of gravitational field in the bulk onto the brane, it carries information relating to the extra dimension. For $q > 0$, this metric is analogous to the Reissner–Nördstrom solution with two horizons. Both of these horizons lie inside the Schwarzschild horizon. For $q < 0$, the metric has only one horizon which is larger than the Schwarzschild horizon and is given by

$$r_+ = \frac{M}{M_p^2}\left(1 + \sqrt{1 - \frac{qM_p^4}{M^2\widetilde{M}_p^2}}\right).$$

The negative tidal charge increases the entropy and is therefore considered physically more natural [19]. We will concentrate our attention on $q < 0$ and denote the negative tidal charge by q'. Now the metric on the $4D$ brane can be written as

$$f = 1 - \frac{2M}{r} - \frac{q'}{r^2}, \tag{6}$$

with the roots given by

$$r_\pm = M\left(1 \pm \sqrt{1 + \frac{q'}{M^2}}\right).$$

We note that r_+ is the black hole horizon and r_- is negative and without physical meaning.

4. Entropy of a brane-world black hole in MDR formalism

In this section we derive the entropy of a black hole within the MDR and standard uncertainty principle. Differentiating Eq. (3) and neglecting the rest mass, we write

$$dp = dE\left(1 + \frac{3}{2}\alpha L_p^2 E^2 - \frac{5}{8}\alpha^2 L_p^4 E^4\right). \tag{7}$$

One may then write

$$dE = dp\left(1 - \frac{3}{2}\alpha L_p^2 E^2 + \frac{23}{8}\alpha^2 L_p^4 E^4\right), \tag{8}$$

where we have only considered terms up to the fourth power of the Plank length. Using the standard uncertainty principle, we have

$$dE\,\delta x \geqslant 1 - \frac{3}{2}\alpha\frac{L_p^2}{\delta x^2} + \frac{23}{8}\alpha^2\frac{L_p^4}{\delta x^4}. \tag{9}$$

It is interesting to note that in quantum field theory, the relation between particle localization and its energy is given by $E \geqslant \frac{1}{\delta x}$ where δx is the particle position uncertainty. Certainly, this relation is modified using MDR formula. Assuming $\delta E \sim E$, we have

$$E\delta x \geqslant 1 - \frac{3}{2}\alpha\frac{L_p^2}{\delta x^2} + \frac{23}{8}\alpha^2\frac{L_p^4}{\delta x^4}. \tag{10}$$

When a black hole is absorbing a classical particle of energy E and size R, the minimum increase in the horizon area can be expressed according to the following general relativistic result [25]

$$(\Delta A_d)_{\min} \geqslant \frac{8\pi L_p^{d-2} E R}{(d-3)}. \tag{11}$$

In 4 dimensions, we have $(\Delta A_4)_{\min} \geqslant 8\pi L_p^2 E R$. Note that R can never be smaller than δx. For convenience we may write the horizon shift as

$$(\Delta A_4)_{\min} \simeq \epsilon L_p^2 E \delta x, \tag{12}$$

where ϵ is a parameter to be determined. Particles with a Compton wavelength on the order of the inverse surface gravity are of interest to us [26,27]. Hence we can take δx as

$$\delta x \sim \kappa^{-1} = \frac{2r_+^2}{r_+ - r_-}. \tag{13}$$

For small q' we have

$$\delta x \sim \sqrt{\frac{A}{\pi}}\left(1 - \frac{4\pi q'}{A}\right), \tag{14}$$

where $A = 16\pi M^2$ is the outer horizon area of the black hole. According to information theory [28], the minimum increase of a black hole entropy is simply one bit of information which may be presented by b. To obtain the entropy of the brane-world black hole, we write

$$\frac{dS}{dA} \simeq \frac{(\Delta S_4)_{\min}}{(\Delta A_4)_{\min}} \simeq \frac{b}{\epsilon L_p^2 E \delta x}. \tag{15}$$

Note that L_p is the Planck length on the brane. Assuming that the dimensionless ratio $L_p^2/(\delta x)^2$ is small relative to unity, we can apply a Taylor expansion of the quantity $E\delta x$ in Eq. (10) to find

$$\frac{dS}{dA} \simeq \frac{b}{\epsilon L_p^2}\left(1 + \frac{3}{2}\alpha \frac{L_p^2}{(\delta x)^2} - \frac{5}{8}\alpha^2 \frac{L_p^4}{(\delta x)^4} + \cdots\right). \tag{16}$$

For $q' \ll A$ we have

$$\frac{dS}{dA} \simeq \frac{1}{4L_p^2} + \frac{3}{8}\alpha\pi\left(\frac{1}{A} + \frac{8\pi q'}{A^2} + \frac{48\pi^2 q'^2}{A^3}\right) - \frac{5}{32}\alpha^2 L_p^2 \pi^2 \left(\frac{1}{A} + \frac{8\pi q'}{A^2} + \frac{48\pi^2 q'^2}{A^3}\right)^2 + \cdots. \tag{17}$$

We note that setting $b/\epsilon = 1/4$, the Bekenstein–Hawking area law can be reproduced. Integrating Eq. (17), we find

$$S \simeq \frac{A}{4L_p^2} + \frac{3}{8}\alpha\pi \ln\left(\frac{A}{4L_p^2}\right) - \frac{3}{8}\alpha\pi\left(\frac{8\pi q'}{A} + \frac{24\pi^2 q'^2}{A^2}\right) + \frac{5}{32}\alpha^2 L_p^2 \pi^2 \frac{1}{A} + \frac{5}{32}\alpha^2 L_p^2 \pi^2 \left(\frac{8\pi q'}{A^2} + \frac{160\pi^2 q'^2}{3A^3}\right) + \cdots + C, \tag{18}$$

where C is an integration constant and terms smaller than $\frac{q'^2}{A^2}$ have been neglected. To continue, we ignore C and higher order correction terms in Eq. (18), obtaining

$$S \simeq \frac{A}{4L_p^2} + \frac{3}{8}\alpha\pi \ln\left(\frac{A}{4L_p^2}\right) - \frac{3}{8}\alpha\pi\left(\frac{8\pi q'}{A} + \frac{24\pi^2 q'^2}{A^2}\right) + \frac{5}{32}\alpha^2 L_p^2 \pi^2 \frac{1}{A} + \frac{5}{32}\alpha^2 L_p^2 \pi^2 \left(\frac{8\pi q'}{A^2} + \frac{160\pi^2 q'^2}{3A^3}\right). \tag{19}$$

Of course, it goes without saying that taking into account higher order correction terms would bear no considerable effect on the results as all the properties are restricted to the case of small q'.

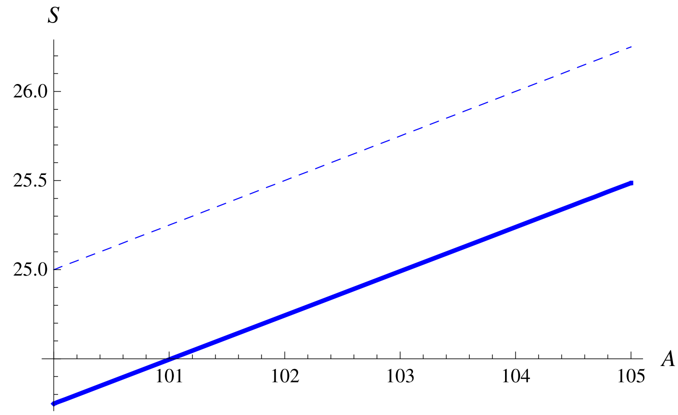

Fig. 1. Entropy of the brane-world black hole plotted as a function of the area for $\alpha = -0.2$, $q' = 0.1$ and $L_p = 1$. The thick line represents the black hole entropy using MDR and the dashed line is that of the Bekenstein–Hawking entropy–area relation.

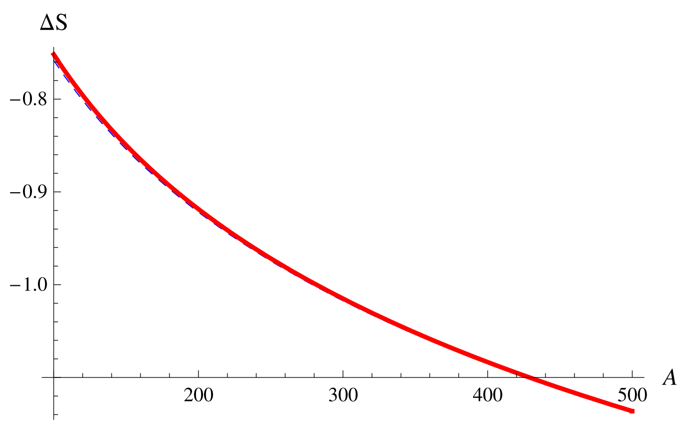

Fig. 2. Variation of $\Delta S = S - S_{BH}$ as a function of the area represented by the thick line, for $\alpha = -0.2$, $q' = 0.1$ and $L_p = 1$. The dashed line represents the logarithmic correction term and is the dominant term.

Fig. 1 shows variation of a black hole entropy as a function of its horizon. One finds that the predicted entropy within our formalism is smaller than that of the standard Bekenstein–Hawking. As in our earlier work [11], we set the parameter α as a negative quantity of order one. There, we compared the results of two approaches, the generalized uncertainty principle and modified dispersion relation within the context of black hole thermodynamics with that of the string theory and loop quantum gravity. Demanding the same results in all approaches and considering string theory and loop quantum gravity as more comprehensive, we put some constraints on the form of GUP and MDR. Also, we found that GUP and MDR are not independent concepts. In fact, they could be equivalent in an ultimate quantum gravity theory. The existence of a positive minimal observable length necessitates a positive value for the model dependent parameter α in the form of GUP. Since we know the relation between the model dependent parameters in GUP and MDR in [11], we set the parameter α as a negative value for MDR in this Letter. It is interesting to note that the existence of a logarithmic term in the entropy–area relation is necessary within the present formalism. It would result in a better insight when dealing and formulating quantum gravity.

The other interesting point is related to the behavior of the variation of $\Delta S = S - S_{BH}$ which is plotted as a function of the area in Fig. 2 as a thick line where S_{BH} is the standard Bekenstein–

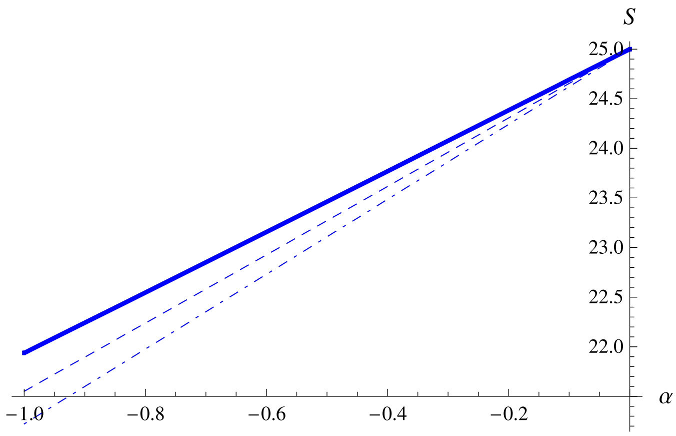

Fig. 3. Relation between the black hole entropy and α for $A = 100$ and $L_p = 1$. The dot-dashed, dashed and thick lines represent $q' = 0$, $q' = 1$ and $q' = 2$ respectively.

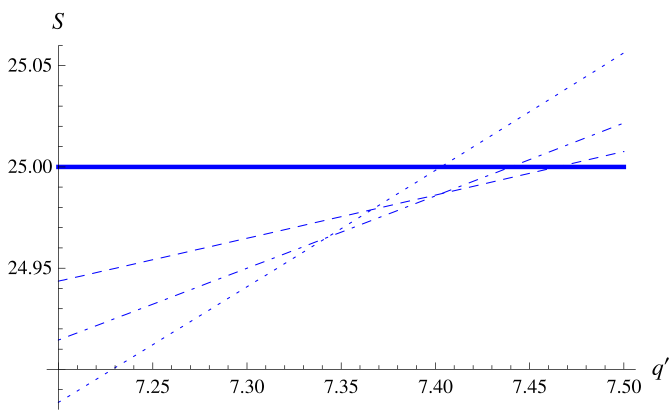

Fig. 4. Variation of entropy as a function of q' (tidal charge) for different values of α for $A = 100$ and $L_p = 1$. The thick, dashed, dot-dashed and dotted lines represent $\alpha = 0$, $\alpha = -0.3$, $\alpha = -0.5$ and $\alpha = -0.8$ respectively.

Hawking entropy. Obviously, the correction terms increase with area, in spite of the decreasing velocity. Comparison of the ΔS curve with that which only contains the logarithmic correction term (the first correction term) in S, represented by a dashed line in Fig. 2, leads one to the conclusion that the logarithmic term dominates the correction terms when q' is small. In other words, the logarithmic term is the dominant correction term.

The impact of parameter α on the entropy of a black hole with negative tidal charge is shown in Fig. 3. It is clear that the entropy decreases with α, regardless of the q' magnitude. However, q' affects the relation between entropy and α. The larger the q' the slower the rate of entropy decrease. In fact, the tidal charge has information about extra dimensions of space–time and large q' refers to strong gravitational field in the bulk. As a result one may find that the gravitational field in the bulk increases the entropy of the black hole on the brane. This effect can also be seen in Fig. 4 where entropy increases as the tidal charge increases, regardless of the value of α. One may find that in any quantum gravity theory with different α, a gravitational field in the bulk would increase the black hole entropy on the brane. It may be due to the free propagation of gravity in the bulk in brane-world scenarios. Since the black hole entropy is closely related to the gravitational field through the area, one may conclude that the brane black hole entropy increases as q' (the tidal charge which is related to bulk gravitational field) increases. It is clear from the definition of r_+ that negative tidal charge will enlarge the brane black hole horizon. Thus it is reasonable for the black hole entropy to be influenced by the effects of the gravitational field in the bulk.

An estimate of the value of α would now be in order. From the LQG point of view, the coefficient of the logarithmic term is $-\frac{1}{2}$. One then finds the value $-\frac{4}{3\pi}$ for α. Certainly the exact value of α is still unknown. Nonetheless, the existence of the logarithmic correction term is demonstrated in the current quantum-corrected black hole entropy. Overall, one can compare the results in our work with the results in [15] which are based on the generalized uncertainty principle formalism to find that MDR and GUP are equivalent and that they yield the same results for the brane-world black hole entropy.

It is worth reiterating at this point that one may also have to impose certain constraints on the general form of GUP according to [11], as was discussed earlier, in order to obtain consistent results. If one tries to investigate the brane-world black hole entropy from a GUP point of view with higher order correction terms, the constraints on the general form of the GUP in [11] must be included. Of course, the results would be functionally consistent with the results of [15], but with different coefficients for similar terms.

5. Conclusions

In this Letter we have studied the MDR corrections to the brane-world black hole entropy. We found a good estimate for the value of α using the LQG approach. The existence of a logarithmic correction term in our approach would be helpful in providing an outlook when studying quantum gravity. The quantum-correction entropy as a function of the horizon area is smaller than the standard Bekenstein–Hawking one. However, what is the effect of pure correction terms on the entropy formula? It is clear that the contribution of the correction terms in the entropy increases with the horizon area in spite of the decreasing velocity. By taking the effects of the correction term only, one finds that it behaves as minus of a logarithmic function. Then the most effective and important correction term in the entropy formula is the logarithmic term and irrespective of the value of the tidal charge, the black hole entropy decreases with α. However, it is clear that a larger charge causes a more slower decrease of entropy with α. Taking into account that the tidal charge is relevant to the gravitational field in the bulk, we may conclude that the gravitational field in the bulk increases the entropy of the black hole on the brane. As Fig. 4 shows, entropy increases with tidal charge in quantum gravity models with different α.

It is necessary to point that all the above properties are relevant to the case of small q' which is considered in this Letter. It is easy to show that the results are also valid when the entropy formula includes higher order correction terms; the contribution of such terms with increasing powers of L_p^2/A become invariably negligible.

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