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On γ_5 in higher-order QCD calculations and the NNLO evolution of the polarized valence distribution

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A R T I C L E   I N F O

A B S T R A C T

We discuss the prescription for the Dirac matrix γ_5 in dimensional regularization used in most second- and third-order QCD calculations of collider cross sections. We provide an alternative implementation of this approach that avoids the use of an explicit form of γ_5 and of its (anti-)commutation relations in the most important case of no more than one γ_5 in each fermion trace. This treatment is checked by computing the third-order corrections to the structure functions F_2 and g_1 in charged-current deep-inelastic scattering with axial-vector couplings to the W-bosons. We derive the so far unknown third-order helicity-difference splitting function $\Delta P_{\rm ns}^{(2){\rm s}}$ that contributes to the next-to-next-to-leading order (NNLO) evolution of the polarized valence quark distribution of the nucleon. This function is negligible at momentum fractions $x \gtrsim 0.3$ but relevant at $x \ll 1$.

Dimensional regularization [1,2], i.e., the analytic continuation of the theory to a non-integer number D of space–time 'dimensions' (see also Ref. [3] for an introduction), is the standard framework for higher-order calculations in gauge field theories including Quantum Chromodynamics (QCD). For some semi-leptonic benchmark observables, e.g., in inclusive deep-inelastic scattering (DIS) and semi-inclusive e^+e^- annihilation (SIA), the use of dimensional regularization requires prescriptions for dealing with the genuinely four-dimensional objects $\varepsilon_{\mu\nu\rho\sigma}^{(4)}$, the totally antisymmetric invariant tensor in four dimensions, and the Dirac matrix $\gamma_5^{(4)} = i\gamma_0\gamma_1\gamma_2\gamma_3 = i/4!\,\varepsilon_{\mu\nu\rho\sigma}^{(4)}\,\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$.

The tensor $\varepsilon_{\mu\nu\rho\sigma}$ enters in the projection of the respective hadronic tensors onto the structure functions F_3 and g_1 in DIS and the fragmentation function F_A in SIA, e.g.,

$$W_{\mu\nu} = \ldots - i\,\varepsilon_{\mu\nu\alpha\beta}\,p^\alpha q^\beta\,\frac{1}{pq}\,F_3(x, Q^2)\;, \tag{1}$$

where the x is the Bjorken scaling variable, $x = Q^2/(2pq)$ with $Q^2 = -q^2$, and where we have suppressed all non-F_3 parts of $W_{\mu\nu}$. It also occurs in the helicity-difference projection of incoming gluons in partonic polarized DIS. The matrix γ_5 enters via the axial-vector coupling of the W and Z bosons to the quarks as well as by the corresponding helicity-difference projection for quarks.

In particular, the issue of γ_5 has attracted a considerable amount of attention. The 'canonical' approach is that of Ref. [2] in which the Dirac algebra, and hence the loop momenta, are split in 4- and $(D-4)$-dimensional sets with

$$\{\gamma_5, \gamma_\mu\} = 0\;, \quad \mu = 0, 1, 2, 3\;,$$
$$[\gamma_5, \gamma_\mu] = 0\;, \quad \text{otherwise}\;, \tag{2}$$

where $\{a, b\}$ and $[a, b]$ denote the standard anti-commutator and commutator, respectively. While Eq. (2) leads to a consistent procedure [4], it has some drawbacks: the occurrence of additional scalar products of $(D-4)$-dimensional loop momenta and an intermediate

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violation of the axial Ward identity. This situation has triggered quite a few of alternative suggestions which we are unable to address in this brief note; the reader is referred to [5–9] and the references therein.

Our focus will be on the scheme developed, on the basis of Ref. [10], in Refs. [11–13] which is closely related to that of Refs. [2,4] but avoids complicating the loop integrals. Consequently this scheme has been employed in almost all higher-order (next-to-next-to-leading order, NNLO, or next-to-next-to-next-to-leading order, N^{3}LO) diagram calculations of splitting and coefficient functions in DIS [13–21] and SIA [22,23], as well as for the determination of the NNLO QCD corrections to the cross section for the production of a pseudoscalar Higgs boson [24–26].

We have been lead to consider this issue by our work on polarized charged-current DIS, in particular the generalization of some of Ref. [27] to the third order, which facilitates the determination of the so far unknown NNLO splitting function $\Delta P_{\rm ns}^{(2)s}$, the longitudinally polarized analogue of $P_{\rm ns}^{(2)s}$ in Ref. [17]. In order to study more cases with more than one γ_5 at the three-loop level, we have redone the calculations of F_2 and F_L of Refs. [15,28,29] and of g_1 of Ref. [19] with axial-vector instead of vector couplings to the gauge bosons. In particular, for the latter case it was useful to employ an algorithm which is equivalent to, but more efficient than, that of Refs. [11–13]. This alternative implementation may be useful for future higher-order calculations in QCD.

In most of the higher-order calculations mentioned above, the prescription of Refs. [11–13], sometimes briefly referred to as the Larin scheme, has been implemented in the form

$$(\gamma_\mu \gamma_5)_L = \frac{1}{6} i \, \varepsilon_{\mu\nu\rho\sigma} \, \gamma^\nu \gamma^\rho \gamma^\sigma \, , \tag{3}$$

i.e., what is continued is the axial-vector matrix, written with a specific order of the two matrices. Alternatively one can use (as, e.g., in Ref. [25])

$$\gamma_{5,L} = \frac{1}{4!} i \, \varepsilon_{\mu\nu\rho\sigma} \, \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \, . \tag{4}$$

Both substitutions, e.g., via Eq. (1), lead to products of two ε-tensors which can be evaluated in terms of the D-dimensional metric tensor $\delta_\alpha^{\,\mu}$ as

$$\varepsilon^{\mu\nu\rho\sigma} \, \varepsilon_{\alpha\beta\kappa\lambda} = \begin{vmatrix} \delta_\alpha^\mu & \delta_\beta^\mu & \delta_\kappa^\mu & \delta_\lambda^\mu \\ \delta_\alpha^\nu & \delta_\beta^\nu & \delta_\kappa^\nu & \delta_\lambda^\nu \\ \delta_\alpha^\rho & \delta_\beta^\rho & \delta_\kappa^\rho & \delta_\lambda^\rho \\ \delta_\alpha^\sigma & \delta_\beta^\sigma & \delta_\kappa^\sigma & \delta_\lambda^\sigma \end{vmatrix} \, . \tag{5}$$

The need to use the D-dimensional metric on the right-hand side has been clearly established, at least for the type of calculations we are considering here, in Ref. [13].

This implies that the traces

$$\text{Tr} \left(\gamma_{\nu_1} \gamma_{\nu_2} \cdots \gamma_{\nu_{2m-1}} \gamma_\mu \gamma_5 \right) \, , \tag{6}$$

evaluated using Eqs. (3) and (4) are not identical at $m \geq 3$, as the additional terms generated by Eq. (4) cancel only at $D = 4$ due to the Schouten identity,

$$\varepsilon^{(4)}_{\nu_3\nu_4\nu_5\nu_6} \delta_{\nu_2}^{\nu_1} + \varepsilon^{(4)}_{\nu_4\nu_5\nu_6\nu_2} \delta_{\nu_3}^{\nu_1} + \varepsilon^{(4)}_{\nu_5\nu_6\nu_2\nu_3} \delta_{\nu_4}^{\nu_1} + \varepsilon^{(4)}_{\nu_6\nu_2\nu_3\nu_4} \delta_{\nu_5}^{\nu_1} + \varepsilon^{(4)}_{\nu_2\nu_3\nu_4\nu_5} \delta_{\nu_6}^{\nu_1} = 0 \, . \tag{7}$$

However, if the above asymmetric non-Hermitian form of the axial-vector matrix is replaced by its symmetric Hermitian counterpart (as done in Ref. [25]),

$$\gamma_\mu \gamma_5 \to \frac{1}{2} \left(\gamma_\mu \gamma_5 - \gamma_5 \gamma_\mu \right) \, , \tag{8}$$

then Eq. (4) leads to exactly the same results as Eq. (3) for the trace (6). The situation is completely analogous if the prescriptions (3) and (4) are applied, for any m and n, to

$$\text{Tr} \left(\gamma_{\nu_1} \gamma_{\nu_2} \cdots \gamma_{\nu_m} \gamma_{\mu_1} \gamma_5 \gamma_{\rho_1} \gamma_{\rho_2} \cdots \gamma_{\rho_n} \gamma_{\mu_2} \gamma_5 \right) : \tag{9}$$

Eq. (3) and Eq. (4) with (8) lead to the same results. Cases with more γ_5 will be addressed below.

Obviously the inconsistent use of Eq. (4) without Eq. (8) leads to wrong results in diagram calculations only in sufficiently complicated cases. For example, re-calculating the third-order corrections for F_3 [16,20] in this manner leads to the same results as Eq. (3) for each individual diagram including its dependence on the gauge parameter. On the other hand, wrong (and unfactorizable, cf. Ref. [13]) results would be obtained for the polarized vector–axial-vector interference structure functions $g_{4,5}$ (using the labeling conventions of Ref. [30]) in which γ_5 occurs not once, as for F_3, but twice.

While the calculation of the Dirac traces is not usually a limiting factor in higher-order calculations, the introduction of additional matrices by Eq. (3) or Eq. (4) has sometimes been considered a drawback of the Larin scheme. For traces with one γ_5, the most important case in QCD calculations (e.g., the only one encountered in Refs. [13–26]), this issue can be avoided by using algorithms which are completely equivalent and do not introduce any additional intermediate matrices.

A procedure equivalent to, but faster than, using Eq. (3) is provided by the following steps:

1. Write the one-γ_5 traces in the form (6) without changing the order of the γ-matrices. This can be viewed as using the cyclicity of the trace, or as reading it from this point, cf. Ref. [6].

2. Evaluate Eq. (6) using

$$\text{Tr}\left(\gamma_{\nu_1}\gamma_{\nu_2}\ldots\gamma_{\nu_{2m-1}}\gamma_\mu\,\gamma_5\right) = -4\,i\,g_{\nu_1\nu_2}\ldots g_{\nu_{2m-5}\nu_{2m-4}}\varepsilon_{\nu_{2m-3}\nu_{2m-2}\nu_{2m-1}\mu}$$
$$\pm \text{ permutations of } \nu_1\ldots\nu_{2m-1}\ . \tag{10}$$

Incidentally, this main step can be programmed in Form [31–33], for an extensive documentation see [34], in a very compact manner for any number of traces with one γ_5, viz

```
repeat;
    id,once,G(m1?,?a,mu?,five) = distrib_(-2,3,G1,G2,?a)*G(mu,five);
    id G2(mu1?,mu2?,mu3?)*G(mu4?,five) = e_(mu1,...,mu4);
endrepeat;
.sort
repeat;
    if ( count(G1,1) );
        id,once,G1(?a) = g_(1,?a);
        Tracen,1;
    endif;
endrepeat;
```

3. For traces with more than one γ_5, use Eq. (3) for all but one (special care is needed for more than two γ_5, see below), then calculate the resulting one-γ_5 trace according to 1. and 2. above.

A corresponding algorithm equivalent to Eq. (4) can be implemented by changing 1. and 2. above to

1. Input all axial-vector matrices in the form (8), then proceed as under 1. above.
2. Evaluate the resulting traces, in which now γ_μ has no special role, as

$$\text{Tr}\left(\gamma_{\nu_1}\gamma_{\nu_2}\ldots\gamma_{\nu_{2m-1}}\gamma_\mu\,\gamma_5\right) = -4\,i\,g_{\nu_1\nu_2}\ldots g_{\nu_{2m-5}\nu_{2m-4}}\varepsilon_{\nu_{2m-3}\nu_{2m-2}\nu_{2m-1}\mu}$$
$$\pm \text{ permutations of } \nu_1\ldots\nu_{2m-1}\mu\ , \tag{11}$$

for which the central two lines of the above Form implementation are changed to the simpler

```
id,once,G(m1?,?a,five) = distrib_(-2,4,G1,G2,?a);
id  G2(mu1?,...,mu4?) = e_(mu1,...,mu4);
```

Eq. (11) has certainly been used elsewhere before; however, we have not seen a clear discussion of the 'implicit-γ_5' relations (10) and (11) to the 'explicit' prescriptions (3) and (4) in the literature.

We now have four equivalent manners to evaluate traces with γ_5 and should briefly address their efficiency: computing Eq. (6) for $m = 7$, i.e., with 14 γ-matrices besides γ_5, requires about 1.2 and 38 seconds, respectively, using (3) and (4) with the internal trace algorithms of Form, but 0.3 and 1.2 seconds with the shown implementations of Eqs. (10) and (11) on a Xeon E5-2667v2 with 3.30 GHz, using one core. The corresponding numbers for $m = 8$ are higher by factors of about 20. This scaling is the same as for the non-γ_5 case, which is, however, faster by almost a factor of 8 than our fastest γ_5 implementation (10). The corresponding execution times for Eq. (9) with $m = n = 5$ (12 γ-matrices besides the two γ_5) are, in the same order, 4.5, 740, 1.3 and 55 seconds; the two faster methods again take longer by about a factor of 20 for $m = n = 6$.

We now move to the application of the above γ_5 scheme in higher-order calculations, focusing on the best known (in general and to us) case of third-order DIS in massless perturbative QCD. This scheme shares the second drawback of the 't Hooft/Veltman scheme (2), the violation of the axial Ward identity. This issue is less serious here than it may be in higher-order calculations in the electroweak theory; it is addressed by 'correcting' the axial current by the renormalization factors Z_5 and Z_A determined to the third order in the strong coupling constant α_s in Ref. [11],

$$Z_A = 1 + a_s^2\,\varepsilon^{-1}\,2\,C_F\beta_0 - a_s^3\left[\varepsilon^{-2}\frac{4}{3}\,C_F\beta_0^2 - \varepsilon^{-1}\frac{2}{9}\,C_F\left(6\,\beta_1 + \beta_0^2 - 42\,C_F\beta_0 + 32\,C_A\beta_0\right)\right], \tag{12}$$

$$Z_5 = 1 - a_s\,4\,C_F + a_s^2\left[22\,C_F^2 - \frac{107}{9}\,C_F C_A + \frac{2}{9}\,C_F n_f\right]$$
$$+ a_s^3\left[C_F^3\left(-\frac{370}{3} + 96\,\zeta_3\right) + C_F^2 C_A\left(\frac{5834}{27} - 160\,\zeta_3\right) + C_F C_A^2\left(-\frac{2147}{27} + 56\,\zeta_3\right)\right.$$
$$\left.+ C_F^2 n_f\left(-\frac{62}{27} - \frac{32}{3}\,\zeta_3\right) + C_A C_F n_f\left(\frac{356}{81} + \frac{32}{3}\,\zeta_3\right) + \frac{52}{81}\,C_F n_f^2\right] \tag{13}$$

for $D = 4 - 2\varepsilon$. These factors are expressed in terms of the renormalized coupling normalized as $a_s = \alpha_s/(4\pi)$, and we have employed the first two coefficients of the β-function of QCD [35–38],

$$\beta_0 = \frac{11}{3}\,C_A - \frac{2}{3}\,n_f\ ,\qquad \beta_1 = \frac{34}{3}\,C_A^2 - \frac{10}{3}\,C_A n_f - \frac{2}{3}\,C_F n_f\ , \tag{14}$$
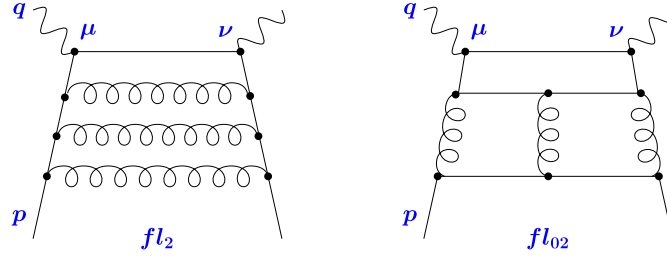
Fig. 1. Typical third-order Feynman diagrams for the two flavour classes contributing to quark-initiated charged-current DIS. Depending on the structure function, the boson lines at the top are replaced by $\varepsilon_{pq\mu\nu}$ or a combination of $g_{\mu\nu}$ and $p_\mu p_\nu$, and the quark lines at the bottom by γ_p or $\gamma_p\gamma_5$ (in Schoonship notation). The vertices with μ and ν represent vector or axial-vector couplings.

to write Eq. (12) in a slightly more compact form.

On top of, or instead of, the multiplication with $Z_5 Z_A$ before performing the mass factorization, a non-trivial factorization-scheme transformation is required in the polarized case for arriving at the splitting and coefficient functions in $\overline{\text{MS}}$ for the helicity-dependent case, see Refs. [19,21,39–42]. At N^{3}LO this transformation is not fully known yet: the pure-singlet quark contribution is missing.

A second yet innocuous effect of using Eqs. (3)–(5) (or any equivalent algorithm) is that all traces, including those of the α_s^0 Born contributions, receive an additional dependence on D. This dependence is factorized and then removed in the projection on the structure functions. E.g., the well-known D-dependence in the projection on the structure function F_3,

$$P_3^{\mu\nu} = -\mathrm{i}\,\frac{1}{(D-3)(D-2)}\,\varepsilon^{\mu\nu\alpha\beta}\,\frac{p_\alpha q_\beta}{p\cdot q}\ ,\tag{15}$$

originates in the basic trace of γ_5 with four other γ-matrices and $\varepsilon_{\mu\nu pq}\,\varepsilon^{\mu\nu pq} \sim (D-2)(D-3)$. This factor is analogous to the $(D-2)^{-1}$ in the F_2 projection that arises from $\gamma_\rho \gamma_\mu \gamma^\rho = (2-D)\gamma_\mu$.

As mentioned above, Eq. (3) has been extensively used in higher-order QCD corrections in cases where only one γ_5 occurs. On the other hand, we are not aware of a corresponding NNLO or N^{3}LO calculation involving two occurrences of γ_5 in either the same or different traces. The former case is more interesting and challenging; a good first example is a third-order calculation of the structure functions F_2 and F_L, for typical forward-Compton diagrams see Fig. 1, with an axial-vector instead of the vector coupling [28,29,43,44] to the gauge boson.

Since we are not looking for a new splitting or coefficient function in this calculation, it is sufficient to keep the full dependence on the Mellin variable N only to two loops and determine the third-order corrections for a few even-integer values of N. At this level the computation is straightforward and virtually automatic; we can use our old diagrams databases and employ our calculation and analysis programs with minor modifications.

The projection on the axial-vector structure functions F_2 and F_L involves a different prefactor. The Born-level trace (p/q are the quark/gauge-boson momenta, $p^2 = 0$) is now

$$\text{Tr}\left(\gamma_p\,\gamma_\mu\gamma_5\,\gamma_{p+q}\,\gamma^\mu\,\gamma_5\right) \sim (D-1)(D-2)(D-3)(D-6)\,pq\ ,\tag{16}$$

i.e., the projections involve an extra factor $[(D-1)(D-3)(6-D)]^{-1}$. Taking this into account, and multiplying the results by $(Z_5 Z_A)^2$ as given by Eqs. (12) and (13) before factorization, we obtain the same splitting functions and quark and gluon coefficient functions for F_2 and F_L as found before,

$$c_{i,q/g}^{(n)\mathrm{aa}}(x) = c_{i,q/g}^{(n)\mathrm{vv}}(x) \quad \text{for} \quad i=2,L \quad \text{at} \quad n=1,2,3\ .\tag{17}$$

This demonstrates that there is no need to resort, as often done, to a fully anti-commuting γ_5 in traces with two γ_5 (which admittedly would lead to the right result here): the scheme considered here can be used also for these cases, at a usually tolerable cost in computing time.

The ultimate γ_5 challenge, in the framework of QCD corrections for structure functions at the lowest order in the electroweak theory, is provided by doing the same for structure function g_1 in polarized DIS. This calculation involves, in addition to that for F_2, an ε-tensor from the projection on g_1, essentially the same as Eq. (15), and a γ_5 or ε-tensor taking the quark or gluon helicity difference. The resulting contractions of four ε-tensors have to be performed with special care as their order matters in the present case, unlike in four dimensions where the results can be shown to be the same by repeated application of the Schouten identity (7), cf. Ref. [45].

In the case at hand, it is correct to pair the ε-tensors from the axial-vector vertices (labeled μ and ν in Fig. 1). This is readily achieved in Form by using the built-in tensor e_- only for these, and to 'protect' the other two ε-tensors by using a different notation until the other contractions and traces have been performed. The fastest implementation is to use Eq. (3) for the axial-vector gauge-boson vertices together with Eq. (10), with e_- suitably renamed in the Form code shown below that equation. A three-fold application of Eq. (3) is also possible, if considerably slower. The symmetric implementations (4) and (11) are yet less efficient; the four-fold application of (4) is prohibitively slow at the third order. Only now all four prescriptions consistently lead to

$$\text{Tr}\left(\gamma_p\gamma_5\,\gamma_\mu\gamma_5\,\gamma_{p+q}\,\gamma_\nu\gamma_5\right)\,\varepsilon^{pq\mu\nu} \sim (D-2)(D-3)^2(D-6)\,(pq)^2\ ,\tag{18}$$

enabling us to verify, by diagram calculations,

$$c_{g_1,q/g}^{(n)\mathrm{aa}}(x) = c_{g_1,q/g}^{(n)\mathrm{vv}}(x) \quad \text{at} \quad n=1,2,3\ .\tag{19}$$

Finally we address the splitting functions $\Delta P_{\rm ns}^{-,\rm v}$ for the polarized quark–antiquark differences

$$\Delta f_{ik}^- = \Delta f_{q_i} - \Delta f_{\bar{q}_i} - \left(\Delta f_{q_k} - \Delta f_{\bar{q}_k}\right) , \tag{20}$$

$$\Delta f^{\rm v} = \sum_{i=1}^{n_f} \left\{\Delta f_{q_i} - \Delta f_{\bar{q}_i}\right\} \tag{21}$$

of helicity-dependent parton distributions, $\Delta f_i = f_i^+ - f_i^-$, where f_i^+ and f_i^- represent the distributions of the parton i with positive and negative helicity, respectively, in a nucleon with positive helicity, and n_f is the number of effectively massless flavours. For general reasons one expects also at NNLO, $n = 2$, a direct relation between the polarized and unpolarized non-singlet cases

$$\Delta P_{\rm ns}^{-(n)} = P_{\rm ns}^{+(n)} , \tag{22}$$

of which the right-hand side was calculated to NNLO in Ref. [17]. On the other hand, the difference

$$\Delta P_{\rm ns}^{\rm s} = \Delta P_{\rm ns}^{\rm v} - \Delta P_{\rm ns}^- \tag{23}$$

can only be determined by a diagram calculation. It is this calculation, via the two-γ_5 polarized vector–axial-vector interference structure function g_5 (cf. Ref. [30]) that lead to our above considerations on γ_5. In particular, $\Delta P_{\rm ns}^{(2)\rm s}$ is obtained from the flavour class fl_{02} in Fig. 1, where the W-bosons are not attached to the external quark line, for the helicity projection $p^\mu \gamma_\mu \gamma_5 \equiv \gamma_p \gamma_5$ and the structure function projection $g_{\mu\nu}$, i.e., with the two γ_5 entering in different traces.

The resulting even-N Mellin-space expression reads

$$\Delta P_{\rm ns}^{(2)\rm s}(N) = 16 n_f\, d^{abc}d_{abc}/n_c \Big(S_{-3}\left(-20\,\eta + 8\,\eta^2\right) + S_{1,-2}\left(8\,\eta - 16\,\eta^2\right)$$
$$+ 32\,\eta\, S_{-2,1} + S_3\left(6\,\eta + 4\,\eta^2\right) + S_{-2}\left(8\,\eta + 20\,\eta^2 + 8\,\eta^3 - 4\,D_0^2\right)$$
$$+ S_1\left(8\,\eta - 14\,\eta^2 - 42\,\eta^3 - 12\,\eta^4 - 2\,D_0^2\right) + D_0^2\Big) \tag{24}$$

in the notation of Ref. [21], i.e., with $D_k = (N + k)^{-1}$, $\eta = D_0 D_1$ and all harmonic sums [46] taken at argument N. The corresponding x-space result, in terms of harmonic polylogarithms [47] at argument x (also suppressed), is given by

$$\Delta P_{\rm ns}^{(2)\rm s}(x) = 16 n_f\, d^{abc}d_{abc}/n_c \Big((1-x)((24 - 20\,\zeta_2)\mathrm{H}_1 - 8\zeta_2\mathrm{H}_{0,-1} - 2\,\mathrm{H}_{1,0,0}$$
$$- 16\,\mathrm{H}_{0,-1,-1,0} + 8\,\mathrm{H}_{0,-1,0,0} + 8\,\mathrm{H}_{0,0,-1,0}) + (1+x)(16\,\mathrm{H}_{-1,0} - 52\zeta_2\mathrm{H}_{-1}$$
$$- 8\zeta_2\mathrm{H}_{0,1} - 40\,\mathrm{H}_{-1,-1,0} + 36\,\mathrm{H}_{-1,0,0} + 32\,\mathrm{H}_{-1,0,1} + 12\,\mathrm{H}_{0,0,0,1} - 4\,\mathrm{H}_{0,1,0,0})$$
$$- x\,(16\,\zeta_3\mathrm{H}_0 + 8\,\mathrm{H}_{0,0} + 36\,\mathrm{H}_{0,0,0} - 8\,\mathrm{H}_{0,0,0,0}) - \mathrm{H}_0(1 + 24\,x) - \mathrm{H}_0(6 - 74\,x)\zeta_2$$
$$- \mathrm{H}_{0,0}(12 + 20\,x)\zeta_2 + \mathrm{H}_{0,1}(10 + 8\,x) + \mathrm{H}_{0,-1,0}(8 + 36\,x) + \mathrm{H}_{0,0,1}(6 - 38\,x)$$
$$- (10 - 8\,x)\zeta_2 + (6 + 88\,x)\zeta_3 + (25 + 15\,x)\zeta_4 \Big) , \tag{25}$$

which can be parametrized, with an accuracy of about 0.1% or better for $10^{-6} \le x \le 0.95$, by

$$\Delta P_{\rm ns}^{(2)\rm s}(x) \cong n_f\,(1-x)\Big(-42.97\,L_0^2 - 29.29\,L_0 + 179.1 + 117.8\,x - 385.5\,x^2 + 75.94\,x^3$$
$$+ x L_0\,(8.818\,L_0 + 460.8) + 2.681\ln(1-x)\Big) + 0.0001\,n_f\,\delta(1-x) , \tag{26}$$

where $L_0 = \ln x$ and the artificial $\delta(1-x)$ contribution can be included to compensate the slightly lesser accuracy at very large x to improve the approximation for high-N moments and large-x convolutions. Eqs. (24) and (25), together with our calculational verification of Eq. (22) from the even moments of g_1 at NNLO, cf. Ref. [27], complete the determination of the third-order helicity-dependent splitting functions of which the main part was performed in Ref. [21].

Eq. (25) can be employed to determine also the odd moments, in particular,

$$\Delta P_{\rm ns}^{(2)\rm s}(N=1) = 8 n_f\, d^{abc}d_{abc}/n_c\,(23 - 12\,\zeta_2 - 16\,\zeta_3) . \tag{27}$$

Together with

$$\Delta P_{\rm ns}^{(1)-}(N=1) = C_F\,(C_A - 2C_F)\,(-13 + 12\,\zeta_2 - 8\,\zeta_3) , \tag{28}$$

$$\Delta P_{\rm ns}^{(2)-}(N=1) = C_F^2\,(C_A - 2C_F)\left(\frac{145}{2} - 62\zeta_2 + 164\,\zeta_3 - 372\,\zeta_4 + 48\,\zeta_2\zeta_3 + 208\,\zeta_5\right)$$
$$+ C_F C_A\,(C_A - 2C_F)\left(\frac{1081}{36} + \frac{245}{3}\zeta_2 - \frac{3214}{9}\zeta_3 + \frac{1058}{3}\zeta_4 - 48\,\zeta_2\zeta_3 - 112\zeta_5\right)$$
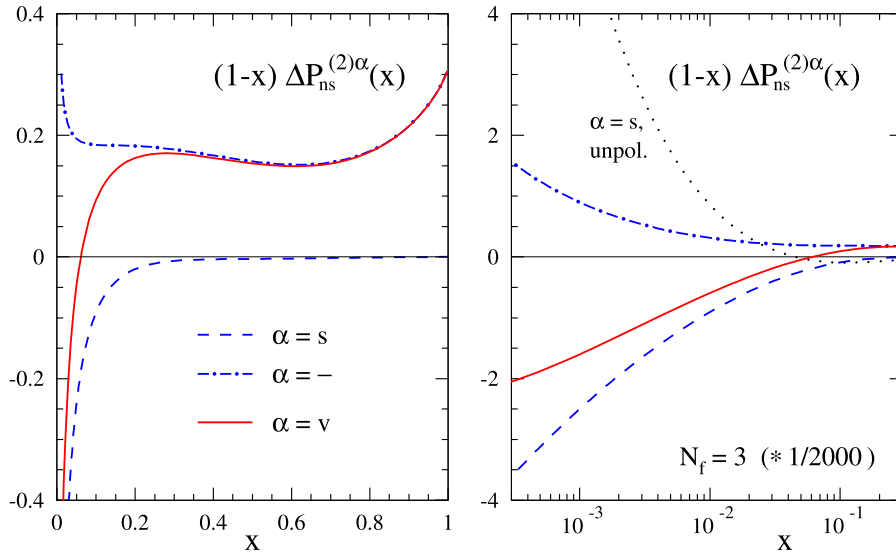$$- C_F n_f\,(C_A - 2C_F)\left(\frac{76}{9} + \frac{44}{3}\zeta_2 - \frac{448}{9}\zeta_3 + \frac{68}{3}\zeta_4\right) \tag{29}$$

Fig. 2. The NNLO splitting functions $\Delta P_{ns}^{(2)-}$ and $\Delta P_{ns}^{(2)v}$ for the polarized quark distributions (20) and (21), together with the previously unknown leading contribution (25) to their difference (23) for three flavours, divided by $2000 \simeq (4\pi)^3$ to compensate for our small expansion parameter $a_s = \alpha_s/(4\pi)$. Also shown, on the right, is the unpolarized counterpart $P_{ns}^{(2)s}(x)$ [17] of Eq. (25).

– note the presence of ζ_2 and the higher weight in the Riemann-ζ function as compared to the 'natural' even moments, cf. Section 3 of Ref. [49] – this leads to the expansion

$$\Delta P_{ns}^{(2)v}(N=1) \cong -0.00810\,\alpha_s^2 - \left(0.04075 - 0.01850\,n_f\right)\alpha_s^3 + \mathcal{O}(\alpha_s^4) \tag{30}$$

in QCD, i.e., for $C_A = n_c = 3$, $C_F = 4/3$ and $d^{abc}d_{abc}/n_c = 5/18$. For the normalization of the latter we had to choose between the earlier convention of Refs. [11,16,17,29] and that of Refs. [20,44] and other more recent articles following Eq. (187) of Ref. [48]. We have done the latter; however, the reader should be aware that the unpolarized counterparts of Eqs. (24)–(25) were presented using the convention $d^{abc}d_{abc}/n_c = 40/9$ in Ref. [17].

Returning to Eq. (30), we note that for $n_f = 4$ Eq. (27) provides almost two thirds of the α_s^3 correction, which is actually larger than the tiny α_s^2 part at normal scales; without it the coefficient of $\alpha_s^3 n_f$ would only amount to 0.00061. At large-x ΔP_{ns}^s is negligible though: it is suppressed by two powers of $(1-x)$ with respect to $\Delta P_{ns}^\pm$, with the leading large-x term the same as for its unpolarized counterpart in Eq. (4.11) of Ref. [17] with $32 n_f d^{abc}d_{abc}/n_c (2\zeta_2 - 3)(1-x)\ln(1-x)$.

The situation is totally different at small x, as shown in Fig. 2: despite an only quadratically logarithmic (negative) small-x enhancement,

$$\Delta P_{ns}^{(2)s}(x) = -16 n_f d^{abc}d_{abc}/n_c \left\{ 6\,\zeta_2\,L_0^2 + (1 + 6\zeta_2)L_0 + \mathcal{O}(1) \right\} \;, \tag{31}$$

its coefficients are such that it overwhelms at $x > 10^{-6}$ the (positive) small-x behaviour of ΔP_{ns}^- which includes terms up to $\ln^4 x$ that are, due to Eq. (22), given by Eq. (4.15) of Ref. [17].

To summarize, we have discussed some subtleties of the (multiple) use of the γ_5 prescription of Refs. [11,12] with a D-dimensionally contracted ε-tensor [13] in higher-order QCD calculations, and provided a procedure that is considerably faster than the algorithm mostly used so far and hence may be useful in some future three- and four-loop calculations in QCD. We have applied our findings to re-derive some third-order results in polarized and unpolarized deep-inelastic scattering, and to calculate the hitherto unknown NNLO splitting function $\Delta P_{ns}^{(2)s}(x)$ which contributes to the evolution of the polarized valence quark distribution, thus completing the determination of the NNLO splitting functions for helicity-dependent parton distributions of hadrons.

A FORM procedure of our alternative implementation of the scheme of Refs. [11–13], as well as FORM and FORTRAN files of our results for $\Delta P_{ns}^{(2)s}$ can be obtained by downloading the source of this article from http://arxiv.org/ or from the authors upon request.

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