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A Lagrangian representation of tangles

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Abstract

We construct a functor from the category of oriented tangles in $\mathbb{R}^3$ to the category of Hermitian modules and Lagrangian relations over $\mathbb{Z}[t, t^{-1}]$. This functor extends the Burau representations of the braid groups and its generalization to string links due to Le Dimet.

Keywords: Braids; Tangles; String links; Burau representation; Lagrangian relations

1. Introduction

The aim of this paper is to generalize the classical Burau representation of braid groups to tangles. The Burau representation is a homomorphism from the group of braids on n strands to the group of $(n \times n)$-matrices over the ring $\Lambda = \mathbb{Z}[t, t^{-1}]$, where n is a positive integer. This representation has been extensively studied by various authors since the foundational work of Burau [2]. In the last 15 years, new important representations of braid groups came to light, specifically those associated with the Jones knot polynomial, R-matrices, and ribbon categories. These latter representations do extend to tangles, so it is natural to ask whether the Burau representation has a similar property.

An extension of the Burau representation to a certain class of tangles was first pointed out by Le Dimet [5]. He considered so-called 'string links', which are tangles whose all components are intervals going from the bottom to the top but not necessarily monotonically. The string links on n strands form a

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monoid with respect to the usual composition of tangles. Le Dimet's work yields a homomorphism of this monoid into the group of $(n \times n)$-matrices over the quotient field of Λ. For braids, this gives the Burau representation. The construction of Le Dimet also applies to colored string links, giving a generalization of the Gassner representation of the pure braid group. These representations of Le Dimet were studied by Kirk et al. [4] (see also [6,9]).

To extend the Burau representation to arbitrary oriented tangles, we first observe that oriented tangles do not form a group or a monoid but rather a category **Tangles** whose objects are finite sequences of ± 1. An extension of the Burau representation to **Tangles** should be a functor from **Tangles** to some algebraically defined category. We show that the relevant algebraic category is the one of Hermitian Λ-modules and Lagrangian relations. Our principal result is a construction of a functor from **Tangles** to this category. For braids and string links, our constructions are equivalent to those of Burau and Le Dimet.

The appearance of Lagrangian relations rather than homomorphisms is parallel to the following well-known observations concerning cobordisms. Generally speaking, a cobordism (W, M_-, M_+) does not induce a homomorphism from the homology (with any coefficients) of the bottom base M_- to the homology of the top base M_+. However, the kernel of the inclusion homomorphism $H_*(M_-) \oplus H_*(M_+) \to H_*(W)$ can be viewed as a morphism from $H_*(M_-)$ to $H_*(M_+)$ determined by W. This kernel is Lagrangian with respect to the usual intersection form in homology. These observations suggest a definition of a Lagrangian category over any integral domain with involution. Applying these ideas to the infinite cyclic covering of the tangle exterior, we obtain our functor from the category of tangles to the category of Lagrangian relations over Λ. Parallel constructions involving 2-fold coverings are studied in [7].

Note that recently, a most interesting representation of braid groups due to R. Lawrence was shown to be faithful by S. Bigelow and D. Krammer. We do not know whether this representation extends to tangles.

The organization of the paper is as follows. In Section 2, we introduce the category $\mathbf{Lagr}_{\Lambda}$ of Lagrangian relations over the ring Λ. In Section 3, we define our functor **Tangles** $\to$ $\mathbf{Lagr}_{\Lambda}$. Section 4 deals with the proof of three technical lemmas stated in the previous section. In Section 5, we discuss the case of braids and string links. Finally, Section 6 outlines a multivariable generalization of the theory as well as a high-dimensional version.

2. Category of Lagrangian relations

Fix throughout this section an integral domain Λ (i.e., a commutative ring with unit and without zero-divisors) with ring involution $\Lambda \to \Lambda$, $\lambda \mapsto \tilde{\lambda}$.

2.1. Hermitian modules

A *skew-hermitian form* on a Λ-module H is a form $\omega\colon H \times H \to \Lambda$ such that for all $x, x', y \in H$ and all $\lambda, \lambda' \in \Lambda$,

(i) $\omega(\lambda x + \lambda' x', y) = \lambda\omega(x, y) + \lambda'\omega(x', y)$,
(ii) $\omega(x, y) = -\widetilde{\omega(y, x)}$.

Such a form is called *non-degenerate* when it satisfies:

(iii) If $\omega(x, y) = 0$ for all $y \in H$, then $x = 0$.

A *Hermitian Λ-module* is a finitely generated Λ-module H endowed with a non-degenerate skew-hermitian form ω. The same module H with the opposite form $-\omega$ will be denoted by $-H$. Note that a Hermitian Λ-module is always torsion-free.

For a submodule $A \subset H$, denote by $\mathrm{Ann}(A)$ the annihilator of A with respect to ω, that is, the module $\{x \in H \mid \omega(x, a) = 0$ for all $a \in A\}$. We say that A is *isotropic* if $A \subset \mathrm{Ann}(A)$, and *Lagrangian* if $A = \mathrm{Ann}(A)$.

Given a submodule A of H, set

$$\overline{A} = \{x \in H \mid \lambda x \in A \text{ for a non-zero } \lambda \in \Lambda\}.$$

Clearly $A \subset \overline{A}$ and $\overline{\mathrm{Ann}(A)} = \mathrm{Ann}(A) = \mathrm{Ann}(\overline{A})$. Note that for any Lagrangian $A \subset H$, we have $A = \overline{A}$.

Lemma 2.1. *For any submodule A of a Hermitian Λ-module H,*

$$\mathrm{Ann}(\mathrm{Ann}(A)) = \overline{A}.$$

Proof. Let $Q = Q(\Lambda)$ denote the field of fractions of Λ. Given a Λ-module F, denote by F_Q the vector space $F \otimes_\Lambda Q$. Note that the kernel of the natural homomorphism $F \to F_Q$ is the Λ-torsion $\mathrm{Tors}_\Lambda F \subset F$.

The form ω uniquely extends to a skew-hermitian form $H_Q \times H_Q \to Q$. Given a linear subspace V of H_Q, let $\mathrm{Ann}_Q(V)$ be the annihilator of V with respect to the latter form. Observe that $\mathrm{Ann}_Q(\mathrm{Ann}_Q(V)) = V$. Indeed, one inclusion is trivial and the other one follows from dimension count, since $\dim(\mathrm{Ann}_Q(V)) = \dim(H_Q) - \dim(V)$.

The inclusion $A \hookrightarrow H$ induces an inclusion $A_Q \hookrightarrow H_Q$. Since H is torsion-free, $H \subset H_Q$ (and $A \subset A_Q$). Clearly, $\overline{A} = A_Q \cap H$ and $\mathrm{Ann}(A)_Q = \mathrm{Ann}_Q(A_Q)$. Replacing in the latter formula A with $\mathrm{Ann}(A)$, we obtain

$$\mathrm{Ann}(\mathrm{Ann}(A))_Q = \mathrm{Ann}_Q(\mathrm{Ann}(A)_Q) = \mathrm{Ann}_Q(\mathrm{Ann}_Q(A_Q)) = A_Q.$$

Therefore

$$\overline{A} = A_Q \cap H = \mathrm{Ann}(\mathrm{Ann}(A))_Q \cap H = \overline{\mathrm{Ann}(\mathrm{Ann}(A))} = \mathrm{Ann}(\mathrm{Ann}(A)),$$

and the lemma is proved. □

Lemma 2.2. *For any submodules $A, B \subset H$,*

$$\mathrm{Ann}(A + B) = \mathrm{Ann}(A) \cap \mathrm{Ann}(B)\,, \quad \mathrm{Ann}(A \cap B) = \overline{\mathrm{Ann}(A) + \mathrm{Ann}(B)}.$$

Proof. The first equality is obvious, and implies

$$\begin{aligned}\mathrm{Ann}(\mathrm{Ann}(A) + \mathrm{Ann}(B)) &= \mathrm{Ann}(\mathrm{Ann}(A)) \cap \mathrm{Ann}(\mathrm{Ann}(B))\\ &= \overline{A} \cap \overline{B} = \overline{A \cap B}.\end{aligned}$$

Therefore

$$\mathrm{Ann}(A \cap B) = \mathrm{Ann}(\overline{A \cap B}) = \mathrm{Ann}(\mathrm{Ann}(\mathrm{Ann}(A) + \mathrm{Ann}(B))),$$

which is equal to $\overline{\mathrm{Ann}(A) + \mathrm{Ann}(B)}$ by Lemma 2.1. □

Lemma 2.3. *For any submodules $A \subset B \subset H$, we have $\overline{B}/A = \overline{B/A} \subset H/A$.*

Proof. Consider the canonical projection $\pi\colon H \to H/A$. Clearly,

$$\pi(\overline{B}) = \{\xi \in H/A \mid \lambda\xi \in B/A \text{ for a non-zero } \lambda \in \Lambda\} = \overline{B/A}.$$

Also $\ker(\pi|_{\overline{B}}) = \ker(\pi) \cap \overline{B} = A \cap \overline{B} = A$. Hence $\overline{B}/A = \overline{B/A}$. □

2.2. Lagrangian contractions

The results above in hand, we can develop the theory of Lagrangian contractions and Lagrangian relations over Λ by mimicking the well-known theory over $\mathbb{R}$ (see, for instance, [10, Section IV.3]).

Let (H, ω) be a Hermitian Λ-module as above. Let A be an isotropic submodule of H such that $A = \overline{A}$. Denote by $H|A$ the quotient module $\mathrm{Ann}(A)/A$ with the skew-hermitian form

$$(x \bmod A, y \bmod A) = \omega(x, y).$$

For a submodule $L \subset H$, set

$$L|A = ((L + A) \cap \mathrm{Ann}(A))/A \subset H|A.$$

We say that $L|A$ is obtained from L by *contraction along* A.

Lemma 2.4. *$H|A$ is a Hermitian Λ-module. If L is a Lagrangian submodule of H, then $\overline{L|A}$ is a Lagrangian submodule of $H|A$.*

Proof. To check that the form on $H|A$ is non-degenerate, pick $x \in \mathrm{Ann}(A)$ such that $\omega(x, y) = 0$ for all $y \in \mathrm{Ann}(A)$. Then, $x \in \mathrm{Ann}(\mathrm{Ann}(A)) = \overline{A} = A$ so that $x \bmod A = 0$.

To prove the second claim of the lemma, set $B = (L + A) \cap \mathrm{Ann}(A) \subset H$. We claim that $\mathrm{Ann}(B) = \overline{B}$. Since both A and L are isotropic, it is easy to check that $B \subset \mathrm{Ann}(B)$ and therefore $\overline{B} \subset \mathrm{Ann}(B)$. Let us verify the opposite inclusion. Lemmas 2.1 and 2.2 imply that

$$\begin{aligned}\mathrm{Ann}(B) &= \mathrm{Ann}((L + A) \cap \mathrm{Ann}(A)) = \overline{\mathrm{Ann}(L + A) + \mathrm{Ann}(\mathrm{Ann}(A))}\\ &\subset \overline{\mathrm{Ann}(L) + \overline{A}} = \overline{L + A}.\end{aligned}$$

Since $A \subset B$, we have $\mathrm{Ann}(B) \subset \mathrm{Ann}(A)$ and therefore

$$\mathrm{Ann}(B) \subset \overline{L + A} \cap \mathrm{Ann}(A) = \overline{(L + A) \cap \mathrm{Ann}(A)} = \overline{B}.$$

Thus $\mathrm{Ann}(B) = \overline{B}$. This implies that $\mathrm{Ann}(B/A) = \overline{B}/A$, which is equal to $\overline{B/A}$ by Lemma 2.3. So $\overline{B/A}$ is Lagrangian. □

2.3. Category of Lagrangian relations

Let H_1, H_2 be Hermitian Λ-modules. A *Lagrangian relation* between H_1 and H_2 is a Lagrangian submodule of $(-H_1) \oplus H_2$ (the latter is a Hermitian Λ-module in the obvious way). For a Lagrangian relation $N \subset (-H_1) \oplus H_2$, we shall use the notation $N: H_1 \Rightarrow H_2$.

For a Hermitian Λ-module H, the submodule of $H \oplus H$

$$\mathrm{diag}_H = \{h \oplus h \in (-H) \oplus H \mid h \in H\}$$

is clearly a Lagrangian relation $H \Rightarrow H$. It is called the *diagonal Lagrangian relation*. Given two Lagrangian relations $N_1: H_1 \Rightarrow H_2$ and $N_2: H_2 \Rightarrow H_3$, their composition is defined by $N_2 \circ N_1 = \overline{N_2 N_1}: H_1 \Rightarrow H_3$, where $N_2 N_1$ denotes the following submodule of $(-H_1) \oplus H_3$:

$$N_2 N_1 = \{h_1 \oplus h_3 \mid h_1 \oplus h_2 \in N_1 \text{ and } h_2 \oplus h_3 \in N_2 \text{ for a certain } h_2 \in H_2\}.$$

Lemma 2.5. *The composition of two Lagrangian relations is a Lagrangian relation.*

Proof. Given two Lagrangian relations $N_1: H_1 \Rightarrow H_2$ and $N_2: H_2 \Rightarrow H_3$, consider the Hermitian Λ-module $H = (-H_1) \oplus H_2 \oplus (-H_2) \oplus H_3$ and its isotropic submodule

$$A = 0 \oplus \mathrm{diag}_{H_2} \oplus 0 = \{0 \oplus h \oplus h \oplus 0 \mid h \in H_2\}.$$

Note that $\overline{A} = A$. It follows from the non-degeneracy of H_2 that $\mathrm{Ann}(A) = (-H_1) \oplus \mathrm{diag}_{H_2} \oplus H_3$. Therefore $H|A = (-H_1) \oplus H_3$. Observe that $N_2 N_1 = (N_1 \oplus N_2)|A$. Lemma 2.4 implies that $N_2 \circ N_1 = \overline{N_2 N_1}$ is a Lagrangian submodule of $(-H_1) \oplus H_3$. □

Lemma 2.6. *For any submodules $N_1 \subset H_1 \oplus H_2$ and $N_2 \subset H_2 \oplus H_3$, we have $\overline{N_2 N_1} = \overline{\overline{N}_2 \overline{N}_1}$.*

Proof. Consider an element $h_1 \oplus h_3$ of $\overline{N}_2 \overline{N}_1$. By definition, $h_1 \oplus h_2 \in \overline{N_1}$ and $h_2 \oplus h_3 \in \overline{N_2}$ for some $h_2 \in H_2$, so $\lambda_1(h_1 \oplus h_2) \in N_1$ and $\lambda_2(h_2 \oplus h_3) \in N_2$ for some $\lambda_1, \lambda_2 \neq 0$. Then $\lambda_1 \lambda_2 (h_1 \oplus h_3) \in N_2 N_1$, so $h_1 \oplus h_3 \in \overline{N_2 N_1}$. Hence, $\overline{N}_2 \overline{N}_1 \subset \overline{N_2 N_1}$. Taking the closure on both sides, we get $\overline{\overline{N}_2 \overline{N}_1} \subset \overline{N_2 N_1}$.The opposite inclusion is obvious. □

Theorem 2.7. *Hermitian Λ-modules, as objects, and Lagrangian relations, as morphisms, form a category.*

Proof. The composition law is well-defined by Lemma 2.5; let us check that it is associative. Consider Lagrangian relations $N_1: H_1 \Rightarrow H_2$, $N_2: H_2 \Rightarrow H_3$, and $N_3: H_3 \Rightarrow H_4$. By Lemma 2.6,

$$N_3 \circ (N_2 \circ N_1) = \overline{N_3 \overline{N_2 N_1}} = \overline{\overline{N}_3 \overline{N_2 N_1}} = \overline{N_3 (N_2 N_1)}.$$

Similarly, $(N_3 \circ N_2) \circ N_1 = \overline{(N_3 N_2) N_1}$. It follows from the definitions that $N_3(N_2 N_1) = (N_3 N_2) N_1$; this implies the associativity. The role of the identity morphisms is played by the diagonal Lagrangian relations. Indeed, for any Lagrangian relation $N: H_1 \Rightarrow H_2$,

$$\mathrm{diag}_{H_2} \circ N = \overline{\mathrm{diag}_{H_2} N} = \overline{N},$$

which is equal to N since N is Lagrangian. Similarly, $N \circ \mathrm{diag}_{H_1} = N$. □

We shall call this category the *category of Lagrangian relations over Λ*. It will be denoted by $\mathbf{Lagr}_\Lambda$.

2.4. Lagrangian relations from unitary isomorphisms

By the *graph* of a homomorphism $f: A \to B$ of abelian groups, we mean the set

$$\Gamma_f = \{a \oplus f(a) | a \in A\} \subset A \oplus B.$$

Let H_1, H_2 be Hermitian Λ-modules. Consider the Hermitian Q-modules $H_1 \otimes Q$ and $H_2 \otimes Q$, where $Q = Q(\Lambda)$ is the field of fractions of Λ and $\otimes = \otimes_\Lambda$. For a unitary Q-isomorphism $\varphi: H_1 \otimes Q \to H_2 \otimes Q$, we define its *restricted graph* Γ^0_φ by

$$\Gamma^0_\varphi = \Gamma_\varphi \cap (H_1 \oplus H_2) = \{h \oplus \varphi(h) | h \in H_1, \varphi(h) \in H_2\} \subset H_1 \oplus H_2.$$

If φ is induced by a unitary Λ-isomorphism $f: H_1 \to H_2$, then clearly $\Gamma^0_\varphi = \Gamma_f$.

Lemma 2.8. *Given any unitary isomorphism $\varphi: H_1 \otimes Q \to H_2 \otimes Q$, the restricted graph Γ^0_φ is a Lagrangian relation $H_1 \Rightarrow H_2$.*

Proof. Denote by ω_1 (resp. ω_2, ω) the skew-hermitian form on H_1 (resp. H_2, $(-H_1) \oplus H_2$), and pick $h, h' \in H_1$ such that $\varphi(h), \varphi(h') \in H_2$. Then,

$$\omega(h \oplus \varphi(h), h' \oplus \varphi(h')) = -\omega_1(h, h') + \omega_2(\varphi(h), \varphi(h')) = 0.$$

Therefore, Γ^0_φ is isotropic. To check that it is Lagrangian, consider an element $x = x_1 \oplus x_2$ of $\mathrm{Ann}(\Gamma^0_\varphi) \subset (-H_1) \oplus H_2$. For all h in H_1 such that $\varphi(h) \in H_2$,

$$\begin{aligned} 0 &= \omega(x, h \oplus \varphi(h)) = -\omega_1(x_1, h) + \omega_2(x_2, \varphi(h)) \\ &= -\omega_2(\varphi(x_1), \varphi(h)) + \omega_2(x_2, \varphi(h)) = \omega_2(x_2 - \varphi(x_1), \varphi(h)). \end{aligned}$$

Since φ is an isomorphism, we have $H_2 \subset \overline{\{\varphi(h) | h \in H_1, \varphi(h) \in H_2\}}$. Therefore, $\omega_2(x_2 - \varphi(x_1), h_2) = 0$ for all $h_2 \in H_2$. Since ω_2 is non-degenerate, it follows that $x_2 = \varphi(x_1)$ so $x = x_1 \oplus \varphi(x_1) \in \Gamma^0_\varphi$ and the lemma is proved. □

Therefore, Lagrangian relations can be understood as a generalization of unitary isomorphisms. More precisely, let $\mathbf{U}_\Lambda$ be the category of Hermitian Λ-modules and unitary Λ-isomorphisms. Also, let $\mathbf{U}^\mathbf{0}_\Lambda$ be the category of Hermitian Λ-modules, where the morphisms between H_1 and H_2 are the unitary Q-isomorphisms between $H_1 \otimes Q$ and $H_2 \otimes Q$.

Theorem 2.9. *The maps $f \mapsto f \otimes \mathrm{id}_Q$, $\varphi \mapsto \Gamma^0_\varphi$ and $f \mapsto \Gamma_f$ define embeddings of categories* $\mathbf{U}_\Lambda \subset \mathbf{U}^\mathbf{0}_\Lambda \subset \mathbf{Lagr}_\Lambda$ and $\mathbf{U}_\Lambda \subset \mathbf{Lagr}_\Lambda$ *which fit in the commutative diagram*

$$\begin{array}{ccc} \mathbf{U}_\Lambda & \hookrightarrow & \mathbf{U}^\mathbf{0}_\Lambda \\ & \searrow & \downarrow \\ & & \mathbf{Lagr}_\Lambda. \end{array}$$

Proof. The first embedding being clear, we check the second one. By Lemma 2.8, Γ^0_φ is a Lagrangian relation. Also, note that $\Gamma^0_\varphi \otimes Q = \Gamma_\varphi$. Therefore, given two unitary Q-isomorphisms φ_1 and φ_2,

$$\begin{aligned}\Gamma^0_{\varphi_2\circ\varphi_1} &= \Gamma_{\varphi_2\circ\varphi_1} \cap (H_1 \oplus H_3) = \Gamma_{\varphi_2}\Gamma_{\varphi_1} \cap (H_1 \oplus H_3)\\ &= (\Gamma^0_{\varphi_2} \otimes Q)(\Gamma^0_{\varphi_1} \otimes Q) \cap (H_1 \oplus H_3) = (\Gamma^0_{\varphi_2}\Gamma^0_{\varphi_1} \otimes Q) \cap (H_1 \oplus H_3)\\ &= \overline{\Gamma^0_{\varphi_2}\Gamma^0_{\varphi_1}} = \Gamma^0_{\varphi_2} \circ \Gamma^0_{\varphi_1}.\end{aligned}$$

It is clear that a Q-isomorphism φ is entirely determined by its restricted graph Γ^0_φ. Finally, the graph Γ_f of a unitary Λ-isomorphism f is equal to the restricted graph of the induced unitary Q-isomorphism $f \otimes \mathrm{id}_Q$. Therefore, the diagram commutes. The theorem follows. □

3. The Lagrangian representation

3.1. The category of oriented tangles

Let D^2 be the closed unit disk in $\mathbb{R}^2$. Given a positive integer n, denote by x_i the point $((2i-n-1)/n, 0)$ in D^2, for $i = 1, \dots, n$. Let ε and ε' be sequences of ± 1 of respective length n and n'. An $(\varepsilon, \varepsilon')$-*tangle* is the pair consisting of the cylinder $D^2 \times [0, 1]$ and its oriented piecewise linear 1-submanifold τ whose oriented boundary $\partial\tau$ is $\sum_{j=1}^{n'} \varepsilon'_j (x'_j, 1) - \sum_{i=1}^{n} \varepsilon_i (x_i, 0)$. Note that for such a tangle to exist, we must have $\sum_i \varepsilon_i = \sum_j \varepsilon'_j$.

Two $(\varepsilon, \varepsilon')$-tangles $(D^2 \times [0, 1], \tau_1)$ and $(D^2 \times [0, 1], \tau_2)$ are *isotopic* if there exists an auto-homeomorphism h of $D^2 \times [0, 1]$, keeping $D^2 \times \{0, 1\}$ fixed, such that $h(\tau_1) = \tau_2$ and $h|_{\tau_1}: \tau_1 \simeq \tau_2$ is orientation-preserving. We shall denote by $T(\varepsilon, \varepsilon')$ the set of isotopy classes of $(\varepsilon, \varepsilon')$-tangles, and by id_ε the isotopy class of the trivial $(\varepsilon, \varepsilon)$-tangle $(D^2, \{x_1, \dots, x_n\}) \times [0, 1]$.

Given an $(\varepsilon, \varepsilon')$-tangle τ_1 and an $(\varepsilon', \varepsilon'')$-tangle τ_2, their *composition* is the $(\varepsilon, \varepsilon'')$-tangle $\tau_2 \circ \tau_1$ obtained by gluing the two cylinders along the disk corresponding to ε' and shrinking the length of the resulting cylinder by a factor 2 (see Fig. 1). Clearly, the composition of tangles induces a composition

$$T(\varepsilon, \varepsilon') \times T(\varepsilon', \varepsilon'') \longrightarrow T(\varepsilon, \varepsilon'')$$

on the isotopy classes of tangles.

The *category of oriented tangles* **Tangles** is defined as follows: the objects are the finite sequences ε of ± 1, and the morphisms are given by $\mathrm{Hom}(\varepsilon, \varepsilon') = T(\varepsilon, \varepsilon')$. The composition is clearly associative, and the trivial tangle id_ε plays the role of the identity endomorphism of ε. The aim of this section is to construct a functor **Tangles** $\to$ $\mathbf{Lagr}_\Lambda$.

3.2. Objects

Denote by $\mathcal{N}(\{x_1, \dots, x_n\})$ an open tubular neighborhood of $\{x_1, \dots, x_n\}$ in $D^2 \subset \mathbb{R}^2$, and by S^2 the 2-sphere $\mathbb{R}^2 \cup \{\infty\}$. Given a sequence $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ of ± 1, let ℓ_ε be the sum $\sum_{i=1}^{n} \varepsilon_i$. We shall denote

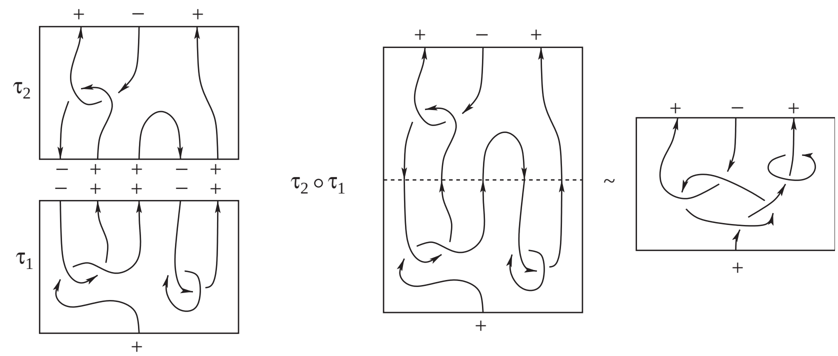

Fig. 1. A tangle composition.

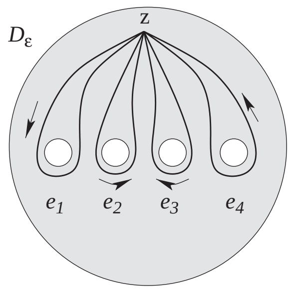

Fig. 2. The space D_ε for $\varepsilon = (+1, +1, -1, +1)$.

by D_ε the compact surface

$$D_\varepsilon = \begin{cases} D^2 \backslash \mathcal{N}(\{x_1, \dots, x_n\}) & \text{if } \ell_\varepsilon \neq 0, \\ S^2 \backslash \mathcal{N}(\{x_1, \dots, x_n\}) & \text{if } \ell_\varepsilon = 0, \end{cases}$$

endowed with the counterclockwise orientation, a base point z, and the generating family $\{e_1, \dots, e_n\}$ of $\pi_1(D_\varepsilon, z)$, where e_i is a simple loop turning once around x_i counterclockwise if $\varepsilon_i = +1$, clockwise if $\varepsilon_i = -1$ (see Fig. 2). The same space with the clockwise orientation will be denoted by $-D_\varepsilon$.

The natural epimorphism $\pi_1(D_\varepsilon) \to \mathbb{Z}$, $e_i \mapsto 1$ gives an infinite cyclic covering $\widehat{D}_\varepsilon \to D_\varepsilon$. Choosing a generator t of the group of the covering transformations endows the homology $H_1(\widehat{D}_\varepsilon)$ with a structure of module over $\Lambda = \mathbb{Z}[t, t^{-1}]$. If $\ell_\varepsilon \neq 0$, then D_ε retracts by deformation on the wedge of n circles representing $e_1, \dots, e_n$, and one easily checks that $H_1(\widehat{D}_\varepsilon)$ is a free Λ-module with basis $v_1 = \hat{e}_1 - \hat{e}_2, \dots, v_{n-1} = \hat{e}_{n-1} - \hat{e}_n$, where $\hat{e}_i$ is the path in $\widehat{D}_\varepsilon$ lifting e_i starting at some fixed lift $\hat{z} \in \widehat{D}_\varepsilon$ of z. If $\ell_\varepsilon = 0$, then $H_1(\widehat{D}_\varepsilon) = \bigoplus_i \Lambda v_i / \Lambda \hat{\gamma}$, where $\hat{\gamma}$ is a lift of $\gamma = e_1^{\varepsilon_1} \cdots e_n^{\varepsilon_n}$ to $\widehat{D}_\varepsilon$. Note that in any case, $H_1(\widehat{D}_\varepsilon)$ is a free Λ-module.

Let $\langle \ , \ \rangle \colon H_1(\widehat{D}_\varepsilon) \times H_1(\widehat{D}_\varepsilon) \to \mathbb{Z}$ be the ($\mathbb{Z}$-bilinear, skew-symmetric) intersection form induced by the orientation of D_ε lifted to $\widehat{D}_\varepsilon$. Consider the pairing $\omega_\varepsilon \colon H_1(\widehat{D}_\varepsilon) \times H_1(\widehat{D}_\varepsilon) \to \Lambda$ given by

$$\omega_\varepsilon(x, y) = \sum_k \langle t^k x, y \rangle t^{-k}.$$

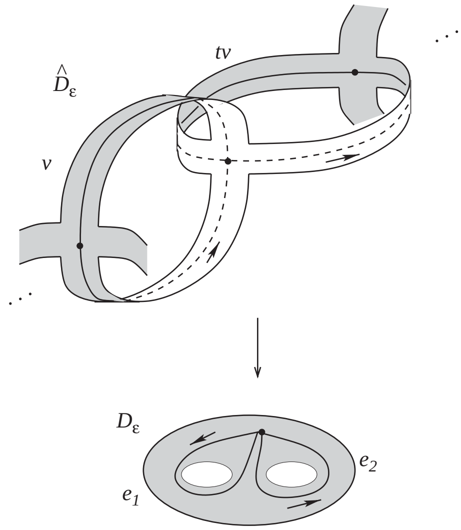

Fig. 3. Computation of ω_ε for $\varepsilon = (+1, +1)$.

Note that this form is well-defined since, for any given $x, y \in H_1(\widehat{D}_\varepsilon)$, the intersection $\langle t^k x, y\rangle$ vanishes for all but a finite number of integers k. The multiplication by t being an isometry with respect to the intersection form, it is easy to check that ω_ε is skew-hermitian with respect to the involution $\Lambda \to \Lambda$ induced by $t \mapsto t^{-1}$.

Example 3.1. Consider ε of length 2. If $\varepsilon_1 + \varepsilon_2 = 0$, then $\widehat{D}_\varepsilon$ is contractible so $H_1(\widehat{D}_\varepsilon) = 0$. If $\varepsilon_1 + \varepsilon_2 \neq 0$, then $H_1(\widehat{D}_\varepsilon) = \Lambda v$ with $v = \hat{e}_1 - \hat{e}_2$, and $\omega_\varepsilon(v, v) = \frac{\varepsilon_1+\varepsilon_2}{2}(t - t^{-1})$, cf. Fig. 3.

We shall give a proof of the following result in Section 4.

Lemma 3.2. *For any ε, the form $\omega_\varepsilon: H_1(\widehat{D}_\varepsilon) \times H_1(\widehat{D}_\varepsilon) \to \Lambda$ is non-degenerate.*

3.3. Morphisms

Given an $(\varepsilon, \varepsilon')$-tangle $\tau \subset D^2 \times [0, 1]$, denote by $\mathcal{N}(\tau)$ an open tubular neighborhood of τ and by X_τ its exterior

$$X_\tau = \begin{cases} (D^2 \times [0,1]) \backslash \mathcal{N}(\tau) & \text{if } \ell_\varepsilon \neq 0, \\ (S^2 \times [0,1]) \backslash \mathcal{N}(\tau) & \text{if } \ell_\varepsilon = 0. \end{cases}$$

Note that $\ell_\varepsilon = \ell_{\varepsilon'}$. We shall orient X_τ so that the induced orientation on ∂X_τ extends the orientation on $(-D_\varepsilon) \sqcup D_{\varepsilon'}$. If $\ell_\varepsilon \neq 0$, then the exact sequence of the pair $(D^2 \times [0, 1], X_\tau)$ and the excision isomorphism give

$$H_1(X_\tau) = H_2(D^2 \times [0,1], X_\tau) = H_2(\overline{\mathcal{N}(\tau)}, \overline{\mathcal{N}(\tau)} \cap X_\tau),$$
$$= \bigoplus_{j=1}^{\mu} H_2(\overline{\mathcal{N}(\tau_j)}, \overline{\mathcal{N}(\tau_j)} \cap X_\tau),$$

where $\tau_1, \ldots, \tau_\mu$ are the connected components of τ. Since $(\overline{\mathcal{N}(\tau_j)}, \overline{\mathcal{N}(\tau_j)} \cap X_\tau)$ is homeomorphic to $(\tau_j \times D^2, \tau_j \times S^1)$, we have $H_2(\overline{\mathcal{N}(\tau_j)}, \overline{\mathcal{N}(\tau_j)} \cap X_\tau) = \mathbb{Z} m_j$, where m_j is a meridian of τ_j oriented so that its linking number with τ_j is 1. Hence, $H_1(X_\tau) = \bigoplus_{j=1}^{\mu} \mathbb{Z} m_j$. If $\ell_\varepsilon = 0$, then $H_1(X_\tau) = \bigoplus_{j=1}^{\mu} \mathbb{Z} m_j / \sum_{i=1}^{n} \varepsilon_i e_i$.

The composition of the Hurewicz homomorphism and the homomorphism $H_1(X_\tau) \to \mathbb{Z}, m_j \mapsto 1$ gives an epimorphism $\pi_1(X_\tau) \to \mathbb{Z}$ which extends the previously defined homomorphisms $\pi_1(D_\varepsilon) \to \mathbb{Z}$ and $\pi_1(D_{\varepsilon'}) \to \mathbb{Z}$. As before, it determines an infinite cyclic covering $\widehat{X}_\tau \to X_\tau$, so the homology of $\widehat{X}_\tau$ is endowed with a natural structure of module over $\Lambda = \mathbb{Z}[t, t^{-1}]$.

Let $i_\tau \colon H_1(\widehat{D}_\varepsilon) \to H_1(\widehat{X}_\tau)$ and $i'_\tau \colon H_1(\widehat{D}_{\varepsilon'}) \to H_1(\widehat{X}_\tau)$ be the homomorphisms induced by the obvious inclusion $\widehat{D}_\varepsilon \sqcup \widehat{D}_{\varepsilon'} \subset \widehat{X}_\tau$. Denote by j_τ the homomorphism $H_1(\widehat{D}_\varepsilon) \oplus H_1(\widehat{D}_{\varepsilon'}) \to H_1(\widehat{X}_\tau)$ given by $j_\tau(x, x') = i'_\tau(x') - i_\tau(x)$. Finally, set

$$N(\tau) = \overline{\ker(j_\tau)} \subset H_1(\widehat{D}_\varepsilon) \oplus H_1(\widehat{D}_{\varepsilon'}).$$

Note that if τ and τ' are two isotopic $(\varepsilon, \varepsilon')$-tangles, then $N(\tau) = N(\tau')$.

Lemma 3.3. *$N(\tau)$ is a Lagrangian submodule of $(-H_1(\widehat{D}_\varepsilon)) \oplus H_1(\widehat{D}_{\varepsilon'})$.*

Lemma 3.4. *If $\tau_1 \in T(\varepsilon, \varepsilon')$ and $\tau_2 \in T(\varepsilon', \varepsilon'')$, then $N(\tau_2 \circ \tau_1) = N(\tau_2) \circ N(\tau_1)$.*

We postpone the proof of these lemmas to the next section, and summarize our results in the following theorem.

Theorem 3.5. *Given a sequence ε of ± 1, denote by $\mathfrak{F}(\varepsilon)$ the Hermitian Λ-module $(H_1(\widehat{D}_\varepsilon), \omega_\varepsilon)$. For $\tau \in T(\varepsilon, \varepsilon')$, let $\mathfrak{F}(\tau)$ be the Lagrangian relation $N(\tau)\colon H_1(\widehat{D}_\varepsilon) \Rightarrow H_1(\widehat{D}_{\varepsilon'})$. Then, $\mathfrak{F}$ is a functor* **Tangles** $\to$ $\mathbf{Lagr}_\Lambda$.

The usual notions of cobordism and *I*-equivalence for links generalize to tangles in the obvious way. (The surface in $D^2 \times [0,1] \times [0,1]$ interpolating between two tangles $\tau_1, \tau_2 \subset D^2 \times [0,1]$ should be standard on $D^2 \times \{0,1\} \times [0,1]$ and homeomorphic to $\tau_1 \times [0,1]$.) It is easy to see (cf. [4, Theorem 5.1 and the proof of Proposition 5.3]) that the Lagrangian relation $N(\tau)$ is an *I*-equivalence invariant of τ.

The usual computation of the Alexander module of a link L from a diagram of L extends to our setting. This gives a computation of $H_1(\widehat{D}_\varepsilon) \oplus H_1(\widehat{D}_{\varepsilon'}) \xrightarrow{j_\tau} H_1(\widehat{X}_\tau)$ (cf. [4, Proposition 4.4]). Hence, it is possible to compute $N(\tau)$ from a diagram of τ.

Finally, given an $(\varepsilon, \varepsilon)$-tangle τ, one can construct an oriented link $\hat{\tau} \subset S^3$ by 'closing' τ in the obvious way. Although we shall not discuss it here, note that the Lagrangian submodule $N(\tau)$ is closely related to the Alexander polynomial of $\hat{\tau}$.

3.4. Freeness of $N(\tau)$

As pointed out in Section 3.2, the functor $\mathfrak{F}$: **Tangles** $\to$ $\mathbf{Lagr}_{\Lambda}$ maps the objects to free modules over the ring $\Lambda = \mathbb{Z}[t, t^{-1}]$. What about the morphisms? Given an oriented tangle τ, is the Λ-module $N(\tau)$ free? The following theorem answers this question.

Theorem 3.6. *Given any tangle $\tau \in T(\varepsilon, \varepsilon')$, the Λ-module $N(\tau)$ is free. Its rank is given by*

$$\mathrm{rk}_{\Lambda} N(\tau) = \begin{cases} 0 & \text{if } n = n' = 0, \\ \dfrac{n+n'}{2} - 1 & \text{if } \ell_{\varepsilon} \neq 0 \text{ or } nn' = 0 \text{ and } (n, n') \neq (0,0), \\ \dfrac{n+n'}{2} - 2 & \text{if } \ell_{\varepsilon} = 0 \text{ and } nn' > 0, \end{cases}$$

where n and n' denote the length of ε and ε'.

In order to prove this result, we shall need several notions of homological algebra, that we recall now. Let Λ be a commutative ring with unit. The *projective dimension* $\mathrm{pd}(A)$ of a Λ-module A is the minimum integer n (if it exists) such that there is a projective resolution of length n of A, that is, an exact sequence

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to A \to 0,$$

where all the P_i's are projective modules. It is a well-known fact that if $0 \to K_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to A \to 0$ is any resolution of A with $\mathrm{pd}(A) \leqslant n$ and all the P_i's projective, then K_n is projective as well (see, for instance, [11, Lemma 4.1.6]). The *global dimension* of a ring Λ is the (possibly infinite) number $\sup\{\mathrm{pd}(A) \mid A \text{ is a } \Lambda\text{-module}\}$. For example, the global dimension of Λ is zero if Λ is a field, and at most one if Λ is a principal ideal domain.

Lemma 3.7. *Let $\Lambda = \mathbb{Z}[t, t^{-1}]$. Consider an exact sequence of Λ-modules*

$$0 \longrightarrow K \longrightarrow P \longrightarrow F,$$

where P and F are free Λ-modules. Then K is free.

Proof. Note that the ring Λ has global dimension 2 (see e.g. [11, Theorem 4.3.7]). We shall also need the fact that all projective Λ-modules are free [8, Chapter 3.3]. Let A be the image of the homomorphism $P \to F$. We claim that the projective dimension of A is at most 1. Indeed, since the global dimension of Λ is at most two, there is a projective resolution $0 \to P_2 \xrightarrow{\partial} P_1 \to P_0 \to A \to 0$ of A. Splicing this resolution with the exact sequence $0 \to A \hookrightarrow F \to F/A \to 0$, we get a resolution of F/A

$$0 \to P_1/\partial P_2 \to P_0 \to F \to F/A \to 0,$$

where P_0 and F are projective. Since the global dimension of Λ is 2, we have $\mathrm{pd}(F/A) \leqslant 2$. Hence, $P_1/\partial P_2$ is projective as well. Therefore, the resolution of A

$$0 \to P_1/\partial P_2 \to P_0 \to A \to 0$$

is projective, so $\mathrm{pd}(A) \leqslant 1$. Now, the exact sequence $0 \to K \to P \to A \to 0$ together with the fact that P is free and $\mathrm{pd}(A) \leqslant 1$, implies that K is projective. Therefore, it is free. $\square$

Proof of Theorem 3.6. Consider the exact sequence

$$0 \to N(\tau) \hookrightarrow H_1(\widehat{D}_\varepsilon) \oplus H_1(\widehat{D}_{\varepsilon'}) \to (H_1(\widehat{D}_\varepsilon) \oplus H_1(\widehat{D}_{\varepsilon'}))/N(\tau) \to 0.$$

Clearly, the latter module is finitely generated and torsion free. Since Λ is a noetherian ring, such a module embeds in a free Λ-module F, giving an exact sequence

$$0 \to N(\tau) \hookrightarrow H_1(\widehat{D}_\varepsilon) \oplus H_1(\widehat{D}_{\varepsilon'}) \to F.$$

By Lemma 3.7, $N(\tau)$ is free. Since $N(\tau)$ is a Lagrangian submodule of $H_1(\widehat{D}_\varepsilon) \oplus H_1(\widehat{D}_{\varepsilon'})$, we have $\mathrm{rk}_\Lambda N(\tau) = \frac{1}{2}\mathrm{rk}_\Lambda(H_1(\widehat{D}_\varepsilon) \oplus H_1(\widehat{D}_{\varepsilon'}))$. If ε has length n, we know from Section 3.2 that

$$\mathrm{rk}_\Lambda H_1(\widehat{D}_\varepsilon) = \begin{cases} 0 & \text{if } n = 0, \\ n-1 & \text{if } \ell_\varepsilon \neq 0, \\ n-2 & \text{if } \ell_\varepsilon = 0 \text{ and } n > 0. \end{cases}$$

The result follows. □

4. Proof of the lemmas

The proof of Lemmas 3.2 and 3.3 rely on the *Blanchfield duality theorem*. We recall this fundamental result referring for a proof and further details to [3, Appendix E].

Let M be a piecewise linear compact connected oriented m-dimensional manifold possibly with boundary. Consider an epimorphism of $\pi_1(M)$ onto a finitely generated free abelian group G. It induces a G-covering $\widehat{M} \to M$, so the homology modules of $\widehat{M}$ are modules over $\Lambda = \mathbb{Z}G$. For any integer q, let $\langle\ ,\ \rangle\colon H_q(\widehat{M}) \times H_{m-q}(\widehat{M}, \partial\widehat{M}) \to \mathbb{Z}$ be the $\mathbb{Z}$-bilinear intersection form induced by the orientation of M lifted to $\widehat{M}$. The *Blanchfield pairing* is the form $S\colon H_q(\widehat{M}) \times H_{m-q}(\widehat{M}, \partial\widehat{M}) \to \Lambda$ given by

$$S(x, y) = \sum_{g \in G} \langle gx, y\rangle g^{-1}.$$

Note that S is Λ-sesquilinear with respect to the involution of Λ given by $\sum_{g\in G} n_g g \mapsto \sum_{g \in G} n_g g^{-1}$. The form S induces a Λ-sesquilinear form

$$H_q(\widehat{M})/\mathrm{Tors}_\Lambda H_q(\widehat{M}) \times H_{m-q}(\widehat{M}, \partial\widehat{M})/\mathrm{Tors}_\Lambda H_{m-q}(\widehat{M}, \partial\widehat{M}) \longrightarrow \Lambda.$$

Theorem 4.1 (*Blanchfield*). *The latter form is non-degenerate*.

Let us now prove the lemmas stated in the previous section.

Proof of Lemma 3.2. Consider the Blanchfield pairing

$$S_\varepsilon\colon H_1(\widehat{D}_\varepsilon) \times H_1(\widehat{D}_\varepsilon, \partial\widehat{D}_\varepsilon) \longrightarrow \Lambda.$$

It follows from the definitions that $\omega_\varepsilon(x, y) = S_\varepsilon(x, j_\varepsilon(y))$, where $j_\varepsilon\colon H_1(\widehat{D}_\varepsilon) \to H_1(\widehat{D}_\varepsilon, \partial\widehat{D}_\varepsilon)$ is the inclusion homomorphism. Note that $\partial\widehat{D}_\varepsilon$ consists of a finite number of copies of $\mathbb{R}$, so $H_1(\partial\widehat{D}_\varepsilon) = 0$ and j_ε is injective. Pick $y \in H_1(\widehat{D}_\varepsilon)$ and assume that for all $x \in H_1(\widehat{D}_\varepsilon)$, $0 = \omega_\varepsilon(x, y) = S_\varepsilon(x, j_\varepsilon(y))$. By the Blanchfield duality theorem, $j_\varepsilon(y) \in \mathrm{Tors}_\Lambda(H_1(\widehat{D}_\varepsilon, \partial\widehat{D}_\varepsilon))$, so $0 = \lambda j_\varepsilon(y) = j_\varepsilon(\lambda y)$ for some $\lambda \in \Lambda$, $\lambda \neq 0$. Since j_ε is injective, $\lambda y = 0$. As $H_1(\widehat{D}_\varepsilon)$ is torsion-free, $y = 0$, so ω_ε is non-degenerate. □

Proof of Lemma 3.3. Let $H_1(\widehat{D}_\varepsilon) \oplus H_1(\widehat{D}_{\varepsilon'}) \stackrel{i}{\to} H_1(\partial\widehat{X}_\tau)$ be the inclusion homomorphism, and denote by

$$H_2(\widehat{X}_\tau, \partial\widehat{X}_\tau) \stackrel{\partial}{\longrightarrow} H_1(\partial\widehat{X}_\tau) \stackrel{j}{\longrightarrow} H_1(\widehat{X}_\tau)$$

the homomorphisms appearing in the exact sequence of the pair $(\widehat{X}_\tau, \partial\widehat{X}_\tau)$. Also, denote by ω the pairing $(-\omega_\varepsilon) \oplus \omega_{\varepsilon'}$ on $(-H_1(\widehat{D}_\varepsilon)) \oplus H_1(\widehat{D}_{\varepsilon'})$ and by

$$S_{\partial X}\colon H_1(\partial\widehat{X}_\tau) \times H_1(\partial\widehat{X}_\tau) \to \Lambda, \qquad S_X\colon H_1(\widehat{X}_\tau) \times H_2(\widehat{X}_\tau, \partial\widehat{X}_\tau) \to \Lambda$$

the Blanchfield pairings. Clearly, $N(\tau) = ((-1)\mathrm{id} \oplus \mathrm{id}')(\overline{L})$, where $L = \ker(j \circ i)$ and id (resp. id$'$) is the identity endomorphism of $H_1(\widehat{D}_\varepsilon)$ (resp. $H_1(\widehat{D}_{\varepsilon'})$). Then, $\mathrm{Ann}(N(\tau)) = ((-1)\mathrm{id} \oplus \mathrm{id}')\mathrm{Ann}(L)$ and we just need to check that $\mathrm{Ann}(L) = \overline{L}$.

First, we check that $K = \ker(j) = \mathrm{Im}(\partial)$ satisfies $\mathrm{Ann}_{\partial X}(K) = \overline{K}$, where $\mathrm{Ann}_{\partial X}$ denotes the annihilator with respect to the form $S_{\partial X}$. Observe that for any $x \in H_1(\partial\widehat{X}_\tau)$ and $Y \in H_2(\widehat{X}_\tau, \partial\widehat{X}_\tau)$, we have $S_{\partial X}(x, \partial(Y)) = S_X(j(x), Y)$. Therefore

$$\begin{aligned}\mathrm{Ann}_{\partial X}(K) &= \{x \in H_1(\partial\widehat{X}_\tau) \mid S_{\partial X}(x, K) = 0\}\\ &= \{x \in H_1(\partial\widehat{X}_\tau) \mid S_X(j(x), H_2(\widehat{X}_\tau, \partial\widehat{X}_\tau)) = 0\}.\end{aligned}$$

By the Blanchfield duality, the latter set is just $j^{-1}(\mathrm{Tors}_\Lambda(H_1(\widehat{X}_\tau))) = \overline{K}$.

Clearly, $i(L) \subset K$. The exact sequence of the pair $(\partial\widehat{X}_\tau, \widehat{D}_\varepsilon \sqcup \widehat{D}_{\varepsilon'})$ gives

$$H_1(\widehat{D}_\varepsilon) \oplus H_1(\widehat{D}_{\varepsilon'}) \stackrel{i}{\longrightarrow} H_1(\partial\widehat{X}_\tau) \longrightarrow T,$$

where T is a torsion Λ-module. This implies that $K \subset \overline{i(L)}$ and therefore $\overline{i(L)} = \overline{K}$. Since the forms ω and $S_{\partial X}$ are compatible under i,

$$\begin{aligned}\mathrm{Ann}(L) &= i^{-1}(\mathrm{Ann}_{\partial X}(i(L))) = i^{-1}(\mathrm{Ann}_{\partial X}(\overline{i(L)})) = i^{-1}(\mathrm{Ann}_{\partial X}(\overline{K}))\\ &= i^{-1}(\mathrm{Ann}_{\partial X}(K)) = i^{-1}(\overline{K}) = \overline{L},\end{aligned}$$

and the lemma is proved. □

Proof of Lemma 3.4. Denote by τ the composition $\tau_2 \circ \tau_1$. Note that it is sufficient to check the equality $\ker(j_\tau) = \ker(j_{\tau_2})\ker(j_{\tau_1})$. Indeed, Lemma 2.6 then implies

$$\begin{aligned}N(\tau) &= \overline{\ker(j_\tau)} = \overline{\ker(j_{\tau_2})\ker(j_{\tau_1})}\\ &= \overline{\overline{\ker(j_{\tau_2})}\,\overline{\ker(j_{\tau_1})}} = \overline{N(\tau_2)N(\tau_1)} = N(\tau_2) \circ N(\tau_1).\end{aligned}$$

Since $X_\tau = X_{\tau_1} \cup X_{\tau_2}$ and $X_{\tau_1} \cap X_{\tau_2} = D_{\varepsilon'}$, we get the following Mayer–Vietoris exact sequence of Λ-modules:

$$H_1(\widehat{D}_{\varepsilon'}) \stackrel{\alpha}{\to} H_1(\widehat{X}_{\tau_1}) \oplus H_1(\widehat{X}_{\tau_2}) \stackrel{\beta}{\to} H_1(\widehat{X}_\tau) \to H_0(\widehat{D}_{\varepsilon'}) \stackrel{\alpha_0}{\to} H_0(\widehat{X}_{\tau_1}) \oplus H_0(\widehat{X}_{\tau_2}).$$

The homomorphism α_0 is clearly injective, so β is onto and we get a short exact sequence which fits in the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_{\varepsilon'} & \xrightarrow{i} & H_\varepsilon \oplus H_{\varepsilon'} \oplus H_{\varepsilon''} & \xrightarrow{\pi} & H_\varepsilon \oplus H_{\varepsilon''} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle j_\tau} & & \\ 0 & \longrightarrow & \ker(\beta) & \longrightarrow & H_1(\widehat{X}_{\tau_1}) \oplus H_1(\widehat{X}_{\tau_2}) & \xrightarrow{\beta} & H_1(\widehat{X}_\tau) & \longrightarrow & 0\,, \end{array}$$

where $H_\bullet$ denotes $H_1(\widehat{D}_\bullet)$, i is the natural inclusion, π the canonical projection, and $\varphi(x, x', x'') = (j_{\tau_1}(x, x'), j_{\tau_2}(x', x''))$. Clearly,

$$\pi(\ker(\varphi)) = \{x \oplus x'' \mid \varphi(x, x', x'') = 0 \text{ for some } x' \in H_{\varepsilon'}\} = \ker(j_{\tau_2}) \ker(j_{\tau_1}).$$

Therefore, we just need to check that $\pi(\ker(\varphi)) = \ker(j_\tau)$, which is an easy diagram chasing exercise using the surjectivity of $\alpha: H_{\varepsilon'} \to \ker(\beta)$. □

5. Examples

5.1. Braids

An $(\varepsilon, \varepsilon')$-tangle $\tau = \tau_1 \cup \cdots \cup \tau_n \subset D^2 \times [0, 1]$ is called an *oriented braid* if every component τ_j of τ is strictly increasing or strictly decreasing with respect to the projection to [0, 1]. Note that for such an oriented braid to exist, we must have $\sharp\{i \mid \varepsilon_i = 1\} = \sharp\{j \mid \varepsilon'_j = 1\}$ and $\sharp\{i \mid \varepsilon_i = -1\} = \sharp\{j \mid \varepsilon'_j = -1\}$. The finite sequences of ± 1, as objects, and the isotopy classes of oriented braids, as morphisms, form a subcategory **Braids** of the category of oriented tangles. We shall now investigate the restriction of the functor $\mathfrak{F}$ to this subcategory.

Consider an oriented braid $\beta = \beta_1 \cup \cdots \cup \beta_n \subset D^2 \times [0, 1]$. Clearly, there exists an isotopy $H_\beta: D^2 \times [0, 1] \to D^2 \times [0, 1]$ with $H_\beta(x, t) = (x, t)$ for $(x, t) \in (D^2 \times \{0\}) \cup (\partial D^2 \times [0, 1])$, such that $t \mapsto H_\beta(x_i, t)$ is a homeomorphism of [0, 1] onto the arc β_i for $i = 1, \ldots, n$. Let $h_\beta: D_\varepsilon \to D_{\varepsilon'}$ be the homeomorphism given by $x \mapsto H_\beta(x, 1)$, and by the identity on $S^2 \backslash D^2$ if $\varepsilon_1 + \cdots + \varepsilon_n = 0$. It is a standard result that the isotopy class (rel ∂D^2) of h_β only depends on the isotopy class of β. Consider the lift $\hat{h}_\beta: \widehat{D}_\varepsilon \to \widehat{D}_{\varepsilon'}$ of h_β fixing $\partial \widehat{D}^2$ pointwise, and denote by f_β the induced unitary isomorphism $(\hat{h}_\beta)_*: H_1(\widehat{D}_\varepsilon) \to H_1(\widehat{D}_{\varepsilon'})$.

The isotopy H_β provides a deformation retraction of X_β to $D_{\varepsilon'}$: let us identify $H_1(\widehat{X}_\beta)$ and $H_1(\widehat{D}_{\varepsilon'})$ via this deformation. Clearly, the homomorphism $j_\beta: H_1(\widehat{D}_\varepsilon) \oplus H_1(\widehat{D}_{\varepsilon'}) \to H_1(\widehat{X}_\beta)$ is given by $j_\beta(x, y) = y - f_\beta(x)$. Therefore,

$$N(\beta) = \overline{\ker(j_\beta)} = \ker(j_\beta) = \{x \oplus f_\beta(x) \mid x \in H_1(\widehat{D}_\varepsilon)\} = \Gamma_{f_\beta},$$

the graph of the unitary isomorphism f_β. We have proved:

Proposition 5.1. *The restriction of* $\mathfrak{F}$ *to the subcategory of oriented braids gives a functor* **Braids** $\to$ $\mathbf{U}_\Lambda$.

Consider an $(\varepsilon, \varepsilon')$-tangle $\tau = \tau_1 \cup \cdots \cup \tau_n \subset D^2 \times [0, 1]$ such that every component τ_i of τ is strictly increasing with respect to the projection to [0, 1]. Here, $\varepsilon = \varepsilon' = (1, \ldots, 1)$. We will simply call τ a *braid*,

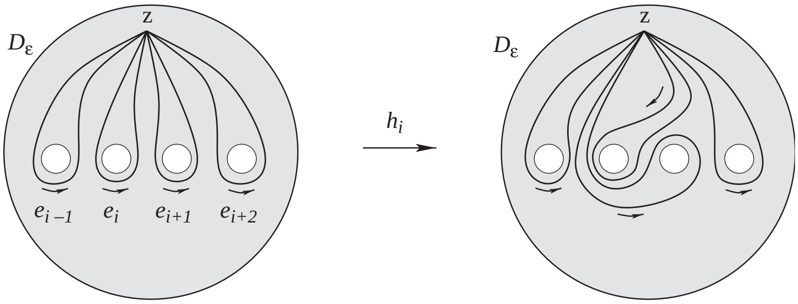

Fig. 4. The action of h_i on the loops $e_{i-1}, \dots, e_{i+2}$.

or an *n-strand braid*. As usual, we will denote by B_n the group of isotopy classes of n-strand braids, and by $\sigma_1, \dots, \sigma_{n-1}$ its standard set of generators (see Fig. 5). Recall that the *Burau representation* $B_n \to \mathrm{GL}_n(\Lambda)$ maps the generator σ_i to the matrix

$$I_{i-1} \oplus \begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix} \oplus I_{n-i-1},$$

where I_k denotes the identity $(k \times k)$-matrix. This representation is reducible: it splits into the direct sum of an $(n-1)$-dimensional representation ρ and the trivial one-dimensional representation (see e.g. [1]). Using the Artin presentation of B_n, one easily checks that the map $\sigma_i \mapsto \rho(\sigma_i)^{\mathrm{T}}$, where T denotes the transposition, also defines a representation $\rho^{\mathrm{T}}\colon B_n \to \mathrm{GL}_{n-1}(\Lambda)$.

Proposition 5.2. *The restriction of the functor $\mathfrak{F}$ to B_n gives a linear anti-representation $B_n \to \mathrm{GL}_{n-1}(\Lambda)$ which is the dual of ρ^{T}.*

Proof. Consider two braids $\alpha, \beta \in B_n$. By Proposition 5.1, $N(\alpha)$ (resp. $N(\beta)$, $N(\alpha\beta)$) is the graph of a unitary automorphism f_α (resp. f_β, $f_{\alpha\beta}$) of $H_1(\widehat{D}_\varepsilon)$. Note that the product $\alpha\beta \in B_n$ represents the composition $\beta \circ \alpha$ in the category of tangles. Clearly, $f_{\alpha\beta} = f_\beta \circ f_\alpha$. Therefore, $\mathfrak{F}$ restricted toB_n is an anti-representation. In order to check that it corresponds to the dual of ρ^{T}, we just need to verify that these anti-representations coincide on the generators σ_i of B_n.

Denote by f_i the unitary isomorphism corresponding to σ_i. We shall now compute the matrix of f_i with respect to the basis $v_1, \dots, v_{n-1}$ of $H_1(\widehat{D}_\varepsilon)$. Consider the homeomorphism h_i of D_ε associated with σ_i. As shown in Fig. 4, its action on the loops e_j is given by

$$h_i(e_j) = \begin{cases} e_i e_{i+1} e_i^{-1} & \text{if } j = i, \\ e_i & \text{if } j = i+1, \\ e_j & \text{else.} \end{cases}$$

Therefore, the lift $\hat{h}_i$ of h_i satisfies

$$\hat{h}_i(\hat{e}_j) = \begin{cases} \hat{e}_i - t(\hat{e}_i - \hat{e}_{i+1}) & \text{if } j = i, \\ \hat{e}_i & \text{if } j = i+1, \\ \hat{e}_j & \text{else,} \end{cases}$$

and the matrix of $f_i = (\hat{h}_i)_*$ with respect to the basis $v_j = \hat{e}_j - \hat{e}_{j+1}$ is

$$M_{f_1} = \begin{pmatrix} -t & 1 \\ 0 & 1 \end{pmatrix} \oplus I_{n-3}, \quad M_{f_{n-1}} = I_{n-3} \oplus \begin{pmatrix} 1 & 0 \\ t & -t \end{pmatrix},$$

$$M_{f_i} = I_{i-2} \oplus \begin{pmatrix} 1 & 0 & 0 \\ t & -t & 1 \\ 0 & 0 & 1 \end{pmatrix} \oplus I_{n-i-2} \quad \text{for } 2 \leqslant i \leqslant n-2.$$

This is exactly $\rho(\sigma_i)$ (see, for instance, [1, p. 121]). □

5.2. String links

An $(\varepsilon, \varepsilon')$-tangle $\tau = \tau_1 \cup \cdots \cup \tau_n \subset D^2 \times [0, 1]$ is called an *oriented string link* if every component τ_j of τ joins $D^2 \times \{0\}$ and $D^2 \times \{1\}$. Oriented string links clearly form a category **Strings** which satisfies

Braids $\subset$ **Strings** $\subset$ **Tangles**,

where all the inclusions denote embeddings of categories.

Proposition 5.3. *The restriction of* $\mathfrak{F}$ *to the subcategory of oriented string links gives a functor* **Strings** $\to$ $\mathbf{U}^{\mathbf{0}}_{\mathbf{\Lambda}}$.

Proof. Since τ is an oriented string link, the inclusions $D_\varepsilon \subset X_\tau$ and $D_{\varepsilon'} \subset X_\tau$ induce isomorphisms in integral homology. Therefore, the induced homomorphisms $H_1(\widehat{D}_\varepsilon; Q) \overset{i_\tau}{\to} H_1(\widehat{X}_\tau; Q)$ and $H_1(\widehat{D}_{\varepsilon'}; Q) \overset{i'_\tau}{\to} H_1(\widehat{X}_\tau; Q)$ are isomorphisms (see e.g. [4, Proposition 2.3]). Since $Q = Q(\Lambda)$ is a flat Λ-module, $\ker(j_\tau) \otimes Q$ is the kernel of

$$H_1(\widehat{D}_\varepsilon; Q) \oplus H_1(\widehat{D}_{\varepsilon'}; Q) \overset{i'_\tau - i_\tau}{\longrightarrow} H_1(\widehat{X}_\tau; Q).$$

Hence,

$$\mathfrak{F}(\tau) = \overline{\ker(j_\tau)} = (\ker(j_\tau) \otimes Q) \cap (H_1(\widehat{D}_\varepsilon) \oplus H_1(\widehat{D}_{\varepsilon'})) = \Gamma^0_\varphi,$$

the restricted graph of the unitary Q-isomorphism $\varphi = (i'_\tau)^{-1} \circ i_\tau$. □

If all the components of an oriented string link τ are oriented from bottom to top, we will simply speak of τ as a *string link*. By Proposition 5.3, the restriction of $\mathfrak{F}$ to the category of string links gives a functor to the category $\mathbf{U}^{\mathbf{0}}_{\mathbf{\Lambda}}$. This functor is due to Le Dimet [5] and was studied further in [4].

5.3. Elementary tangles

Every tangle $\tau \in T(\varepsilon, \varepsilon')$ can be expressed as a composition of the *elementary tangles* given in Fig. 5, where the orientation of the strands is determined by the signs ε and ε'. We shall now compute explicitly the functor $\mathfrak{F}$ on these tangles, assuming that $\ell_\varepsilon \neq 0$.

Let us start with the tangle $u \in T(\varepsilon, \varepsilon')$. Here, $H_1(\widehat{D}_\varepsilon) = \bigoplus_{i=1}^{n-3} \Lambda v_i$ and $H_1(\widehat{D}_{\varepsilon'}) = \bigoplus_{i=1}^{n-1} \Lambda v'_i$ where $v_i = \hat{e}_i - \hat{e}_{i+1}$ and $v'_i = \hat{e}'_i - \hat{e}'_{i+1}$. Moreover, X_u is homeomorphic to the exterior of the trivial $(\varepsilon'', \varepsilon'')$-tangle,

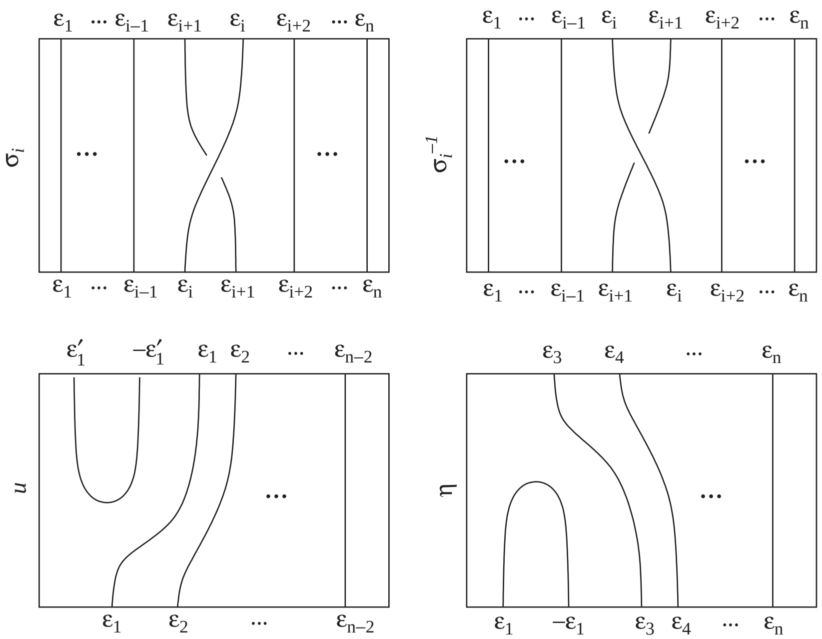

Fig. 5. The elementary tangles.

where $\varepsilon'' = (-\varepsilon'_1, \varepsilon_1, \ldots, \varepsilon_{n-2}) = (\varepsilon'_2, \ldots, \varepsilon'_n)$. Hence, $H_1(\widehat{X}_u) = \bigoplus_{i=1}^{n-2} \Lambda v''_i$ with $v''_i = \hat{e}''_i - \hat{e}''_{i+1}$ and the homomorphism $j_u \colon H_1(\widehat{D}_\varepsilon) \oplus H_1(\widehat{D}_{\varepsilon'}) \to H_1(\widehat{X}_u)$ is given by $j_u(v_i) = -v''_{i+1}$ for $i = 1, \ldots, n-3$, $j_u(v'_1) = 0$ and $j_u(v'_i) = v''_{i-1}$ for $i = 2, \ldots, n-1$. Therefore,

$$N(u) = \overline{\ker(j_u)} = \ker(j_u) = \Lambda v'_1 \oplus \bigoplus_{i=1}^{n-3} \Lambda(v_i \oplus v'_{i+2}).$$

Similarly, we easily compute

$$N(\eta) = \Lambda v_1 \oplus \bigoplus_{i=1}^{n-3} \Lambda(v_{i+2} \oplus v'_i).$$

Now, consider the oriented braid $\sigma_i \in T(\varepsilon, \varepsilon')$ given in Fig. 5. Then, $N(\sigma_i)$ is equal to the graph Γ_{f_i} of a unitary isomorphism $f_i \colon H_1(\widehat{D}_\varepsilon) \to H_1(\widehat{D}_{\varepsilon'})$. As in the proof of Proposition 5.2, we can compute the matrix M_{f_i} of f_i with respect to the bases $v_1, \ldots, v_{n-1}$ of $H_1(\widehat{D}_\varepsilon)$ and $v'_1, \ldots, v'_{n-1}$ of $H_1(\widehat{D}_{\varepsilon'})$:

$$M_{f_1} = \begin{pmatrix} -t^{\varepsilon_2} & 1 \\ 0 & 1 \end{pmatrix} \oplus I_{n-3}, \quad M_{f_{n-1}} = I_{n-3} \oplus \begin{pmatrix} 1 & 0 \\ t^{\varepsilon_n} & -t^{\varepsilon_n}, \end{pmatrix},$$

$$M_{f_i} = I_{i-2} \oplus \begin{pmatrix} 1 & 0 & 0 \\ t^{\varepsilon_{i+1}} & -t^{\varepsilon_{i+1}} & 1 \\ 0 & 0 & 1 \end{pmatrix} \oplus I_{n-i-2} \quad \text{for } 2 \leqslant i \leqslant n-2.$$

Finally, consider the tangle σ_i^{-1} given in Fig. 5. Since it is an oriented braid, $N(\sigma_i^{-1})$ is equal to the graph of a unitary isomorphism $g_i\colon H_1(\widehat{D}_{\varepsilon'}) \to H_1(\widehat{D}_\varepsilon)$. Furthermore, we have

$$\mathrm{diag}_{H_1(\widehat{D}_\varepsilon)} = N(\mathrm{id}_\varepsilon) = N(\sigma_i^{-1} \circ \sigma_i) = N(\sigma_i^{-1}) \circ N(\sigma_i) = \Gamma_{g_i} \circ \Gamma_{f_i} = \Gamma_{g_i \circ f_i}.$$

Therefore, $g_i \circ f_i$ is the identity endomorphism of $H_1(\widehat{D}_\varepsilon)$, so the matrix of g_i with respect to the basis given above is equal to $M_{g_i} = M_{f_i}^{-1}$.

With these elementary tangles, we can sketch an alternative proof of Lemma 3.3 which does not make use of the Blanchfield duality. Indeed, any tangle $\tau \in T(\varepsilon, \varepsilon')$ can be written as a composition of σ_i, σ_i^{-1}, u and η. By Lemmas 2.5 and 3.4, we just need to check that $N(\sigma_i)$, $N(\sigma_i^{-1})$, $N(u)$ and $N(\eta)$ are Lagrangian. For $N(\sigma_i)$ and $N(\sigma_i^{-1})$, this follows from Proposition 5.1, Lemma 3.2 and Lemma 2.8. For $N(u)$ and $N(\eta)$, it can be verified by a direct computation of ω_ε.

Using the results above, it is possible to compute $N(\tau_2 \circ \tau_1)$ from $N(\tau_1)$ for any elementary tangle τ_2 and a tangle τ_1. This leads to a recursive computation of $N(\tau)$ for $(\varepsilon, \varepsilon')$-tangles with no closed components and at least one strand joining D_ε with $D_{\varepsilon'}$.

6. Generalizations

6.1. The category of m-colored tangles

Fix throughout this section a positive integer m. An *m-colored tangle* is an oriented tangle τ together with a map c assigning to each component τ_j of τ a *color* $c(j) \in \{1, \dots, m\}$. The composition of two m-colored tangles is defined if and only if it is compatible with the coloring of each component. Finally, we say that an m-colored tangle is an *oriented m-colored braid* (resp. an *oriented m-colored string link*) if the underlying tangle is a braid (resp. a string link).

More formally, m-colored tangles can be understood as morphisms of a category in the following way. Consider two maps $\varphi\colon \{1, \dots, n\} \to \{\pm 1, \dots, \pm m\}$ and $\varphi'\colon \{1, \dots, n'\} \to \{\pm 1, \dots, \pm m\}$, where n and n' are non-negative integers. We will say that an m-colored tangle (τ, c) is a (φ, φ')-*tangle* if the following conditions hold:

- τ is an $(\varepsilon, \varepsilon')$-tangle, where $\varepsilon = \varphi/|\varphi|$ and $\varepsilon' = \varphi'/|\varphi'|$;
- if $x_i \in D^2 \times \{0\}$ (resp. $x_i' \in D^2 \times \{1\}$) is an endpoint of a component τ_j of τ, then $|\varphi(i)| = c(j)$ (resp. $|\varphi'(i)| = c(j)$).

Two (φ, φ')-tangles are isotopic if they are isotopic as $(\varepsilon, \varepsilon')$-tangles under an isotopy that respects the color of each component. We denote by $T(\varphi, \varphi')$ the set of isotopy classes of (φ, φ')-tangles. The composition of oriented tangles induces a composition $T(\varphi, \varphi') \times T(\varphi', \varphi'') \to T(\varphi, \varphi'')$ for any φ, φ' and φ''.

This allows us to define the *category of m-colored tangles* $\mathbf{Tangles_m}$. Its objects are the maps $\varphi\colon \{1, \dots, n\} \to \{\pm 1, \dots, \pm m\}$ with $n \geqslant 0$, and its morphisms are given by $\mathrm{Hom}(\varphi, \varphi') = T(\varphi, \varphi')$. Clearly, oriented m-colored braids and oriented m-colored string links form categories $\mathbf{Braids_m}$ and $\mathbf{Strings_m}$ such that

$$\mathbf{Braids_m} \subset \mathbf{Strings_m} \subset \mathbf{Tangles_m}.$$

6.2. The multivariable Lagrangian representation

We now define a functor $\mathfrak{F}_m\colon \mathbf{Tangles_m} \to \mathbf{Lagr_{\Lambda_m}}$, where Λ_m denotes the ring $\mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$. This construction generalizes the functor of Theorem 3.5, which corresponds to the case $m = 1$. It also extends the works of Gassner for pure braids and Le Dimet for pure string links.

Consider an object of $\mathbf{Tangles_m}$, that is, a map $\varphi\colon \{1, \dots, n\} \to \{\pm 1, \dots, \pm m\}$ with $n \geqslant 0$. Set $\ell_\varphi = (\ell_\varphi^{(1)}, \dots, \ell_\varphi^{(m)}) \in \mathbb{Z}^m$, where $\ell_\varphi^{(j)} = \sum_{\{i \mid \varphi(i) = \pm j\}} sign(\varphi(i))$ for $j = 1, \dots, m$. Using the notation of Section 3.2, we define

$$D_\varphi = \begin{cases} D^2 \backslash \mathcal{N}(\{x_1, \dots, x_n\}) & \text{if } \ell_\varphi \neq (0, \dots, 0), \\ S^2 \backslash \mathcal{N}(\{x_1, \dots, x_n\}) & \text{if } \ell_\varphi = (0, \dots, 0). \end{cases}$$

As in the case of oriented tangles, we endow D_φ with the counterclockwise orientation, a base point z, and generators $e_1, \dots, e_n$ of $\pi_1(D_\varphi, z)$. Consider the homomorphism from $\pi_1(D_\varphi)$ to the free abelian group $G \cong \mathbb{Z}^m$ with basis $t_1, \dots, t_m$ given by $e_i \mapsto t_{|\varphi(i)|}$. It defines a regular G-covering $\widehat{D}_\varphi \to D_\varphi$, so the homology $H_1(\widehat{D}_\varphi)$ is a module over $\mathbb{Z}G = \Lambda_m$. Finally, let $\omega_\varphi\colon H_1(\widehat{D}_\varphi) \times H_1(\widehat{D}_\varphi) \to \Lambda_m$ be the skew-hermitian pairing given by

$$\omega_\varphi(x, y) = \sum_{g \in G} \langle gx, y \rangle g^{-1},$$

where $\langle\ ,\ \rangle\colon H_1(\widehat{D}_\varphi) \times H_1(\widehat{D}_\varphi) \to \mathbb{Z}$ is the intersection form induced by the orientation of D_φ lifted to $\widehat{D}_\varphi$.

Consider now a (φ, φ')-tangle (τ, c). Note that $\ell_\varphi = \ell_{\varphi'}$. Let X_τ be the compact manifold

$$X_\tau = \begin{cases} (D^2 \times [0,1]) \backslash \mathcal{N}(\tau) & \text{if } \ell_\varphi \neq (0, \dots, 0), \\ (S^2 \times [0,1]) \backslash \mathcal{N}(\tau) & \text{if } \ell_\varphi = (0, \dots, 0), \end{cases}$$

oriented so that the induced orientation on ∂X_τ extends the orientation on $(-D_\varphi) \sqcup D_{\varphi'}$. We know from Section 3.3 that $H_1(X_\tau) = \bigoplus_{j=1}^{\mu} \mathbb{Z} m_j$ if $\ell_\varphi \neq (0, \dots, 0)$, and $H_1(X_\tau) = \bigoplus_{j=1}^{\mu} \mathbb{Z} m_j / \sum_{i=1}^{n} \mathrm{sign}(\varphi(i)) e_i$ otherwise. Hence, the coloring of τ defines a homomorphism $H_1(X_\tau) \to G$, $m_j \mapsto t_{c(j)}$ which induces a homomorphism $\pi_1(X_\tau) \to G$ extending the homomorphisms $\pi_1(D_\varphi) \to G$ and $\pi_1(D_{\varphi'}) \to G$. It gives a G-covering $\widehat{X}_\tau \to X_\tau$.

Consider the inclusion homomorphisms $i_\tau\colon H_1(\widehat{D}_\varphi) \to H_1(\widehat{X}_\tau)$ and $i'_\tau\colon H_1(\widehat{D}_{\varphi'}) \to H_1(\widehat{X}_\tau)$. Denote by j_τ the homomorphism $H_1(\widehat{D}_\varphi) \oplus H_1(\widehat{D}_{\varphi'}) \to H_1(\widehat{X}_\tau)$ given by $j_\tau(x, x') = i'_\tau(x') - i_\tau(x)$. Set

$$\mathfrak{F}_m(\tau) = \overline{\ker(j_\tau)} \subset H_1(\widehat{D}_\varphi) \oplus H_1(\widehat{D}_{\varphi'}).$$

Theorem 6.1. *Let $\mathfrak{F}_m$ assign to each map $\varphi\colon \{1, \dots, n\} \to \{\pm 1, \dots, \pm m\}$ the pair $(H_1(\widehat{D}_\varphi), \omega_\varphi)$ and to each $\tau \in T(\varphi, \varphi')$ the submodule $\mathfrak{F}_m(\tau)$ of $H_1(\widehat{D}_\varphi) \oplus H_1(\widehat{D}_{\varphi'})$. Then, $\mathfrak{F}_m$ is a functor*

$\mathbf{Tangles_m} \to \mathbf{Lagr}_{\Lambda_\mathbf{m}}$ *which fits in the diagram*

$$\begin{array}{ccccc} \mathbf{Braids_m} & \hookrightarrow & \mathbf{Strings_m} & \hookrightarrow & \mathbf{Tangles_m} \\ \downarrow & & \downarrow & & \downarrow \mathfrak{F}_m \\ \mathbf{U}_{\Lambda_\mathbf{m}} & \hookrightarrow & \mathbf{U}^\mathbf{0}_{\Lambda_\mathbf{m}} & \hookrightarrow & \mathbf{Lagr}_{\Lambda_\mathbf{m}}, \end{array}$$

where the horizontal arrows denote embeddings of categories.

Proof. Lemmas 3.2, 3.3, 3.4, Proposition 5.1, Proposition 5.3 and their proofs extend to our setting with obvious changes. The only 'topological' facts required are the following:

(i) $H_1(\partial\widehat{D}_\varphi)=0$,
(ii) the Λ_m-module $H_1(\widehat{D}_\varphi)$ is torsion-free,
(iii) $H_1(\partial\widehat{X}_\tau,\widehat{D}_\varphi\sqcup\widehat{D}_{\varphi'})$ is a torsion Λ_m-module.

The definition of $\widehat{D}_\varphi$ easily implies that $\partial\widehat{D}_\varphi$ consists of copies of $\mathbb{R}$, so the first claim is checked. Since D_φ has the homotopy type of a 1-dimensional CW-complex Y_φ, the Λ_m-module $H_1(\widehat{D}_\varphi)=H_1(\widehat{Y}_\varphi)=Z_1(\widehat{Y}_\varphi)$ is a submodule of the free Λ_m-module $C_1(\widehat{Y}_\varphi)$. Therefore, $H_1(\widehat{D}_\varphi)$ is torsion-free. Finally, the third claim follows easily from the definitions and the excision theorem. $\square$

6.3. High-dimensional Lagrangian representations

The Lagrangian representation of Theorem 3.5 can be generalized in another direction by considering high-dimensional manifolds. We conclude the paper with a brief sketch of this construction.

Fix throughout this section an integer $n\geqslant 1$. In the sequel, all the manifolds are assumed piecewise linear, compact and oriented. Consider a homology $2n$-sphere D. To this manifold, we associate a category $\mathscr{C}_D$ as follows. Its objects are codimension-2 submanifolds M of D such that $H_n(M)=0$. The morphisms between $M\subset D$ and $M'\subset D$ are given by properly embedded codimension-2 submanifolds T of $D\times[0,1]$ such that the oriented boundary ∂T of T satisfies $\partial T\cap(D\times\{0\})=-M$ and $\partial T\cap(D\times\{1\})=M'$, where $-M$ denotes M with the opposite orientation. The composition is defined in the obvious way.

If D_M is the complement of an open tubular neighborhood of M in D, we easily check that $H_1(D_M)\cong H_0(M)$. Therefore, the epimorphism $H_0(M)\to\mathbb{Z}$ which sends every generator to 1 determines a $\mathbb{Z}$-covering $\widehat{D}_M\to D_M$. The lift of the orientation of D_M to $\widehat{D}_M$ defines a $\mathbb{Z}$-bilinear intersection form on $H_n(\widehat{D}_M)$. This gives a Λ-sesquilinear form on $H_n(\widehat{D}_M)$, which in turn induces a Λ-sesquilinear form ω_M on $BH_n(\widehat{D}_M)$, where $BH=H/\mathrm{Tors}_\Lambda H$ for a Λ-module H. (Note that ω_M is skew-hermitian if n is odd, and Hermitian if n is even.) Using the fact that $H_n(M)=0$, the proof of Lemma 3.2 can be applied to this setting, showing that ω_M is non-degenerate. Let $\mathfrak{F}_D(M)$ denote the Λ-module $BH_n(\widehat{D}_M)$ endowed with the non-degenerate Λ-sesquilinear form ω_M.

Given a codimension-2 submanifold T of $D\times[0,1]$, denote by X_T the complement of an open tubular neighborhood of T in $D\times[0,1]$. Since $H_1(X_T)\cong H_0(T)$, we have a $\mathbb{Z}$-covering $\widehat{X}_T\to X_T$ given by the homomorphism $H_0(T)\to\mathbb{Z}$ which sends every generator to 1. There are obvious inclusions $\widehat{D}_M\subset\widehat{X}_T$ and $\widehat{D}_{M'}\subset\widehat{X}_T$ which induce homomorphisms i and i' in n-dimensional homology. Let $j\colon H_n(\widehat{D}_M)\oplus H_n(\widehat{D}_{M'})\to H_n(\widehat{X}_T)$ be the homomorphism given by $j(x,x')=i'(x')-i(x)$. It induces a homomorphism

$$BH_n(\widehat{D}_M) \oplus BH_n(\widehat{D}_{M'}) \xrightarrow{j_T} BH_n(\widehat{X}_T).$$

Set $\mathfrak{F}_D(T) = \ker(j_T)$. The proof of Lemma 3.3 can be applied to check that $\mathfrak{F}_D(T)$ is a Lagrangian submodule of $(-BH_n(\widehat{D}_M)) \oplus BH_n(\widehat{D}_{M'})$. Lemma 3.4 can also be adapted to our setting to show that $\mathfrak{F}_D(T_2 \circ T_1) = \mathfrak{F}_D(T_2) \circ \mathfrak{F}_D(T_1)$. Therefore, $\mathfrak{F}_D$ is a functor from $\mathscr{C}_D$ to the Lagrangian category $\mathbf{Lagr}_\Lambda$ amended as follows: the non-degenerate form is Hermitian if n is even, skew-hermitian if n is odd.

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