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Unbounded critical points for a class of lower semicontinuous functionals

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Abstract

For a general class of lower semicontinuous functionals, we prove existence and multiplicity of critical points, which turn out to be unbounded solutions to the associated Euler equation. We apply a nonsmooth critical point theory developed in [10,12,13] and applied in [8,9,20] to treat the case of continuous functionals.

Keywords: Nonsmooth critical point theory; Lower semicontinuous functionals; Multiplicity of solutions; Quasilinear elliptic equations

1. Introduction

The aim of this paper is to prove existence and multiplicity results of unbounded critical points for a class of lower semicontinuous functionals. Let us consider a bounded open set $\Omega \subset \mathbb{R}^N$ $(N \geqslant 3)$ and define the functional $f : H_0^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$ by

$$f(u) = \int_\Omega j(x, u, \nabla u) - \int_\Omega G(x, u),$$

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where $j(x,s,\xi):\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ is a measurable function with respect to x for all $(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$, and of class C^1 with respect to (s,ξ) for a.e. $x\in\Omega$. We also assume that for almost every x in Ω and every s in $\mathbb{R}$

$$\text{the function } \{\xi\mapsto j(x,s,\xi)\} \text{ is strictly convex.} \tag{1.1}$$

Moreover, we suppose that there exist a constant $\alpha_0>0$ and a positive increasing function $\alpha\in C(\mathbb{R})$ such that the following hypothesis is satisfied for almost every $x\in\Omega$ and for every $(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$:

$$\alpha_0|\xi|^2\leqslant j(x,s,\xi)\leqslant\alpha(|s|)|\xi|^2. \tag{1.2}$$

The functions $j_s(x,s,\xi)$ and $j_\xi(x,s,\xi)$ denote the derivatives of $j(x,s,\xi)$ with respect of the variables s and ξ, respectively. Regarding the function $j_s(x,s,\xi)$, we assume that there exist a positive increasing function $\beta\in C(\mathbb{R})$ and a positive constant R such that the following conditions are satisfied almost everywhere in Ω and for every $\xi\in\mathbb{R}^N$:

$$|j_s(x,s,\xi)|\leqslant\beta(|s|)|\xi|^2 \quad \text{for every } s \text{ in } \mathbb{R}, \tag{1.3}$$

$$j_s(x,s,\xi)s\geqslant 0 \quad \text{for every } s \text{ in } \mathbb{R} \text{ with } |s|\geqslant R. \tag{1.4}$$

Let us notice that, from (1.1) and (1.2), it follows that $j_\xi(x,s,\xi)$ satisfies the following growth condition (see Remark 4.1 for more details):

$$|j_\xi(x,s,\xi)|\leqslant 4\alpha(|s|)|\xi|. \tag{1.5}$$

The function $G(x,s)$ is the primitive with respect to s such that $G(x,0)=0$ of a Carathéodory (i.e. measurable with respect to x and continuous with respect to s) function $g(x,s)$. We will study two different kinds of problems, according to different nonlinearities $g(x,s)$, that have a main common feature. Indeed, in both cases we cannot expect to find critical points in $L^\infty(\Omega)$. In order to be more precise, let us consider a first model example of nonlinearity and suppose that there exists p such that

$$g_1(x,s)=a(x)\operatorname{arctg} s+|s|^{p-2}s, \quad 2<p<\frac{2N}{N-2}, \tag{1.6}$$

where $a(x)\in L^{\frac{2N}{N+2}}(\Omega)$ and $a(x)>0$. Notice that from hypotheses (1.2) and (1.6) it follows that f is lower semicontinuous on $H_0^1(\Omega)$. We will also assume that

$$\lim_{|s|\to\infty}\frac{\alpha(|s|)}{|s|^{p-2}}=0. \tag{1.7}$$

Condition (1.7), together with (1.2), allows f to be unbounded from below, so that we cannot look for a global minimum. Moreover, notice that $g(x,s)$ is odd with respect to s, so that it would be natural to expect, if $j(x,-s,-\xi)=j(x,s,\xi)$, the existence of infinitely many solutions as in the semilinear case (see [1]). Unfortunately, we cannot apply any of the classical results of critical point theory, because our functional f is not of class C^1 on $H_0^1(\Omega)$. Indeed, notice that $\int_\Omega j(x,v,\nabla v)$ is not differentiable. More precisely, since $j_\xi(x,s,\xi)$ and $j_s(x,s,\xi)$ are not supposed to be bounded with respect to s, the terms $j_\xi(x,u,\nabla u)\cdot\nabla v$ and $j_s(x,u,\nabla u)v$ may not be $L^1(\Omega)$ even if $v\in C_0^\infty(\Omega)$. Notice that if $j_s(x,s,\xi)$ and $j_\xi(x,s,\xi)$ were supposed to be bounded with respect to s, f would be Gateaux derivable for every u in $H_0^1(\Omega)$ and along any direction $v\in H_0^1(\Omega)\cap L^\infty(\Omega)$ (see [2,8,9,19,20] for the study of this class of functionals). On the contrary, in our case, for every $u\in H_0^1(\Omega)$, $f'(u)(v)$ does not even exist along directions $v\in H_0^1(\Omega)\cap L^\infty(\Omega)$.

In order to deal with the Euler equation of f let us define the following subspace of $H_0^1(\Omega)$ for a fixed u in $H_0^1(\Omega)$:

$$W_u=\{v\in H_0^1(\Omega): j_\xi(x,u,\nabla u)\cdot\nabla v\in L^1(\Omega) \quad \text{and} \quad j_s(x,u,\nabla u)v\in L^1(\Omega)\}. \tag{1.8}$$

We will see that W_u is dense in $H_0^1(\Omega)$. We give the definition of generalized solution.

Definition 1.1. Let $\Lambda\in H^{-1}(\Omega)$ and assume (1.1), (1.2), (1.3). We say that u is a generalized solution to

$$\begin{cases} -\mathrm{div}\,(j_\xi(x,u,\nabla u))+j_s(x,u,\nabla u)=\Lambda & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases}$$

if $u\in H_0^1(\Omega)$ and it results

$$\begin{cases} j_\xi(x,u,\nabla u)\cdot\nabla u\in L^1(\Omega), \quad j_s(x,u,\nabla u)u\in L^1(\Omega),\\ \int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla v+\int_\Omega j_s(x,u,\nabla u)v=\langle\Lambda,v\rangle \quad \forall v\in W_u. \end{cases}$$

We will prove the following

Theorem 1.2. *Assume conditions* (1.1)–(1.4), (1.6), (1.7). *Moreover, suppose that there exist* $R'>0$ *and* $\delta>0$ *such that*

$$|s|\geqslant R' \Rightarrow pj(x,s,\xi)-j_s(x,s,\xi)s-j_\xi(x,s,\xi)\cdot\xi\geqslant\delta|\xi|^2 \tag{1.9}$$

for a.e. $x\in\Omega$ *and all* $(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$. *Then, if*

$$j(x,-s,-\xi)=j(x,s,\xi),$$

there exists a sequence $\{u_h\} \subset H_0^1(\Omega)$ *of generalized solutions of*

$$\begin{cases} -\mathrm{div}\,(j_\xi(x,u,\nabla u)) + j_s(x,u,\nabla u) = g_1(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{P$_1$}$$

such that $f(u_h) \to +\infty$.

In the nonsymmetric case we consider a different class of nonlinearities $g(x,s)$. A simple model example can be the following:

$$g_2(x,s) = d(x)\mathrm{arctg}\,(s^2) + |s|^{p-2}s, \quad 2 < p < \frac{2N}{N-2}, \tag{1.10}$$

where $d(x) \in L^{\frac{N}{2}}(\Omega)$ and $d(x) > 0$.

We will prove the following

Theorem 1.3. *Assume conditions* (1.1)–(1.4), (1.7), (1.9), (1.10). *Then there exists a nontrivial generalized solution of the problem*

$$\begin{cases} -\mathrm{div}\,(j_\xi(x,u,\nabla u)) + j_s(x,u,\nabla u) = g_2(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{P$_2$}$$

Since the functions $\alpha(|s|)$ and $\beta(|s|)$ in (1.2) and (1.3) are not supposed to be bounded, we are dealing with integrands $j(x,s,\xi)$ which may be unbounded with respect to s. This class of functionals has also been treated in [3–5]. In these papers the existence of a nontrivial solution $u \in L^\infty(\Omega)$ is proved when $g(x,s) = |s|^{p-2}s$. Note that, in this case it is natural to expect solutions in $L^\infty(\Omega)$. In order to prove the existence result, in [4,5], a fundamental step is to prove that every cluster point of a Palais–Smale sequence belongs to $L^\infty(\Omega)$. That is, to prove that u is bounded before knowing that it is a solution. Notice that if u is in $L^\infty(\Omega)$ and $v \in C_0^\infty(\Omega)$ then $j_\xi(x,u,\nabla u) \cdot \nabla v$ and $j_s(x,u,\nabla u)v$ are in $L^1(\Omega)$. Therefore, if $g(x,s) = |s|^{p-2}s$, it would be possible to define a solution as a function $u \in L^\infty(\Omega)$ that satisfies the equation associated to (P_1) (or (P_2)) in the distributional sense. In our case the function $a(x)$ in (1.6) belongs to $L^{2N/(N+2)}(\Omega)$, so that we can only expect to find solutions in $H_0^1(\Omega)$. In the same way, the function $d(x)$ in (1.10) is in $L^{N/2}(\Omega)$ and also in this case the solutions are not expected to be in $L^\infty(\Omega)$. For these reasons, we have given a definition of solution weaker than the distributional one and we have considered the subspace W_u as the space of the admissible test functions. Notice that if $u \in H_0^1(\Omega)$ is a generalized solution of problem (P_1) (resp. (P_2)) and $u \in L^\infty(\Omega)$, then u is a distributional solution of (P_1) (resp. (P_2)).

We want to stress that we have considered here particular nonlinearities (i.e. g_1 and g_2) just to present—in a simple case—the main difficulties we are going to tackle. Indeed, Theorems 1.2 and 1.3 will be proved as consequences of two general results (Theorems 2.1 and 2.3). In order to prove these general results we will use an abstract critical point theory for lower semicontinuous functionals developed in [10,12,13]. So, firstly, we will show that the functional f can be studied by means of this theory (see Theorem 3.11). Then, we will give a definition of a Palais–Smale sequence $\{u_n\}$ suitable to this situation (Definition 6.3), and we will prove that u_n is compact in $H_0^1(\Omega)$ (Theorems 5.1 and 6.9). In order to do this we will follow the arguments of [8,9,19,20] where the case in which $\alpha(s)$ and $\beta(s)$ are bounded is studied. In our case we will have to modify the test functions used in these papers in order to get the compactness result. Indeed, here the main difficulty is to find suitable approximations of u_n that belong to W_{u_n}, in order to choose them as test functions. For this reason a large amount of work (Theorems 4.7–4.10) is devoted to find possible improvements of the class of allowed test functions.

The paper is organized as follows.

In Section 2, we define our general functional, we set the general problem (Problem (P) that we will study and we state the main existence results that we will prove.

In Section 3, we recall (from [10,12,13]) the principal abstract notions and results that we will apply. Moreover, we will study the functional $J : H_0^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$ defined as

$$J(v) = \int_\Omega j(x, v, \nabla v)$$

and we will prove (see Theorem 3.11) that J satisfies a fundamental condition (cf. (3.3)) required in order to apply all the abstract results of Section 3.

In Section 4, we find the conditions under which we can compute the directional derivatives of J (Proposition 4.4). Then, we will prove a fundamental inequality regarding the directional derivatives (Proposition 4.5). Moreover, we will obtain some Brezis–Browder [7] type results (see Theorems 4.7–4.10). These results will be important when determining the class of admissible test functions for Problem (P). In particular, in Theorems 4.9 and 4.10 we study the conditions under which we can give a distributional interpretation to Problem (P).

In Section 5, we obtain a compactness result for J (Theorem 5.1). This theorem will be used to prove that f satisfies our generalized Palais–Smale condition.

In Section 6, we give the proofs of our general results, Theorems 2.1 and 2.3. Then, we will prove Theorems 1.2 and 1.3.

Finally, in Section 7, we prove a summability result (Theorem 7.1) for a generalized solution in dependence of the summability of the function $g(x, s)$.

2. General setting and main results

Let us consider Ω a bounded open set in $\mathbb{R}^N$ $(N \geqslant 3)$. Throughout the paper, we will denote by $\|\cdot\|_p$, $\|\cdot\|_{1,2}$ and $\|\cdot\|_{-1,2}$ the standard norms of the spaces $L^p(\Omega)$, $H_0^1(\Omega)$ and $H^{-1}(\Omega)$, respectively.

Let us define the functional $J : H^1_0(\Omega) \to \mathbb{R} \cup \{+\infty\}$ by

$$J(v) = \int_\Omega j(x, v, \nabla v), \tag{2.1}$$

where $j(x, s, \xi)$ satisfies hypotheses (1.1)–(1.4). We will prove existence and multiplicity results of generalized solutions (see Definition 1.1) of the problem

$$\begin{cases} -\operatorname{div}(j_\xi(x, u, \nabla u)) + j_s(x, u, \nabla u) = g(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{P}$$

In order to do this, we will use variational methods, so that we will study the functional $f : H^1_0(\Omega) \to \mathbb{R} \cup \{+\infty\}$ defined as

$$f(v) = J(v) - \int_\Omega G(x, v),$$

where $G(x, s) = \int_0^s g(x, t)\,\mathrm{d}t$ is the primitive of the function $g(x, s)$ with $G(x, 0) = 0$.

In order to state our multiplicity result let us suppose that $g(x, s)$ satisfies the following conditions. Assume that for every $\varepsilon > 0$ there exists $a_\varepsilon \in L^{2N/(N+2)}(\Omega)$ such that

$$|g(x, s)| \leqslant a_\varepsilon(x) + \varepsilon |s|^{\frac{N+2}{N-2}} \tag{2.2}$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$. Moreover, there exist $p > 2$ and functions $a_0(x)$, $\bar{a}(x) \in L^1(\Omega)$, $b_0(x)$, $\bar{b}(x) \in L^{\frac{2N}{N+2}}(\Omega)$ and $k(x) \in L^\infty(\Omega)$ with $k(x) > 0$ almost everywhere, such that

$$pG(x, s) \leqslant g(x, s)s + a_0(x) + b_0(x)|s|, \tag{2.3}$$

$$G(x, s) \geqslant k(x)|s|^p - \bar{a}(x) - \bar{b}(x)|s| \tag{2.4}$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$ (the constant p is the same as the one in (1.9)).

In this case, we will prove the following

Theorem 2.1. *Assume conditions* (1.1)–(1.4), (1.7), (1.9), (2.2)–(2.4). *Moreover, let*

$$j(x, -s, -\xi) = j(x, s, \xi) \quad \textit{and} \quad g(x, -s) = -g(x, s) \tag{2.5}$$

for a.e. $x \in \Omega$ and every $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$. Then there exists a sequence $\{u_h\} \subset H^1_0(\Omega)$ of generalized solutions of problem (P) with $f(u_h) \to +\infty$.

Remark 2.2. In the classical results of critical point theory different conditions from (2.2)–(2.4) are usually supposed. Indeed, as a growth condition on $g(x, s)$, it is assumed that

$$|g(x,s)| \leqslant a(x) + b|s|^{\sigma-1}, \quad 2 < \sigma < \frac{2N}{N-2}, \quad b \in \mathbb{R}^+, \quad a(x) \in L^{\frac{2N}{N+2}}(\Omega). \tag{2.6}$$

Note that (2.6) implies (2.2). Indeed, suppose that $g(x,s)$ satisfies (2.6), then Young inequality implies that (2.2) is satisfied with $a_\varepsilon(x) = a(x) + C(b,\varepsilon)$. Moreover, as a superlinearity condition, it is usually assumed that there exist $p > 2$ and $R > 0$ with

$$0 < pG(x,s) \leqslant g(x,s)s \quad \text{for every } s \text{ in } \mathbb{R} \text{ with } |s| \geqslant R. \tag{2.7}$$

Note that this condition is stronger than conditions (2.3), (2.4). Indeed, suppose that $g(x,s)$ satisfies (2.7) and notice that this implies that there exists $a_0 \in L^1(\Omega)$ such that

$$pG(x,s) \leqslant g(x,s)s + a_0(x) \quad \text{for every } s \text{ in } \mathbb{R}.$$

Then (2.3) is satisfied with $b_0(x) \equiv 0$. Moreover, from (2.7) we deduce that there exists $\bar{a}(x) \in L^1(\Omega)$ such that

$$G(x,s) \geqslant \frac{1}{R^p} \min\{G(x,R), G(x,-R), 1\}|s|^p - \bar{a}(x)$$

so that also (2.4) is satisfied.

In order to state our existence result in the nonsymmetric case, assume that the function g satisfies the following condition:

$$|g(x,s)| \leqslant a_1(x)|s| + b|s|^{\sigma-1},$$
$$2 < \sigma < \frac{2N}{N-2}, \quad a_1(x) \in L^{\frac{N}{2}}(\Omega), \quad b \in \mathbb{R}^+. \tag{2.8}$$

We will prove the following

Theorem 2.3. *Assume conditions* (1.1)–(1.4), (1.7), (1.9), (2.3), (2.4), (2.8). *Moreover, let*

$$\lim_{s \to 0} \frac{g(x,s)}{s} = 0 \quad \text{a.e. in } \Omega. \tag{2.9}$$

Then there exists a nontrivial generalized solution of problem (P). *In addition, there exist* $\varepsilon > 0$ *such that for every* $\Lambda \in H^{-1}(\Omega)$ *with* $\|\Lambda\|_{-1,2} < \varepsilon$ *the problem*

$$\begin{cases} -\mathrm{div}\,(j_\xi(x,u,\nabla u)) + j_s(x,u,\nabla u) = g(x,u) + \Lambda & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P$_\Lambda$}$$

has at least two generalized solutions u_1, u_2 *with* $f(u_1) \leqslant 0 < f(u_2)$.

Remark 2.4. Notice that, in order to have $g(x,v)v \in L^1(\Omega)$ for every $v \in H_0^1(\Omega)$, the function $a_1(x)$ has to be in $L^{\frac{N}{2}}(\Omega)$. Nevertheless, also in this case we cannot expect to find bounded solution of problem (P). The situation is even worse in problem (P_A), indeed in this case we can only expect to find solutions that belong to $H_0^1(\Omega) \cap \mathrm{dom}(J)$.

Remark 2.5. Notice that condition (2.8) implies (2.2). Indeed, suppose that $g(x,s)$ satisfies (2.8). Then Young inequality implies that, for every $\varepsilon > 0$, we have

$$|g(x,s)| \leqslant \beta(\varepsilon)(a_1(x))^{\frac{N+2}{4}} + \varepsilon|s|^{\frac{N+2}{N-2}} + \gamma(\varepsilon, b),$$

where $\beta(\varepsilon)$ and $\gamma(\varepsilon, b)$ are positive constants depending on ε and b. Now, since we have $a_1(x) \in L^{\frac{N}{2}}(\Omega)$, there holds

$$a_\varepsilon(x) = (\beta(\varepsilon)(a_1(x))^{\frac{N+2}{4}} + \gamma(\varepsilon, b)) \in L^{\frac{2N}{N+2}}(\Omega),$$

which yields (2.2).

3. Abstract results of critical point theory

In this section, we will recall the principal abstract notions and results that we will use in the sequel. We refer the reader to [10,12,13], where this theory is developed. Moreover, we will prove that our functional f satisfies a fundamental property (see condition (3.3) and Theorem 3.11) requested to apply the abstract results.

Let X be a metric space and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. We set

$$\mathrm{dom}(f) = \{u \in X : f(u) < +\infty\} \quad \text{and} \quad \mathrm{epi}(f) = \{(u,\eta) \in X \times \mathbb{R} : f(u) \leqslant \eta\}.$$

The set $\mathrm{epi}(f)$ is endowed with the metric

$$d((u,\eta),(v,\mu)) = (d(u,v)^2 + (\eta - \mu)^2)^{1/2}.$$

Let us define the function $\mathscr{G}_f : \mathrm{epi}(f) \to \mathbb{R}$ by setting

$$\mathscr{G}_f(u,\eta) = \eta. \tag{3.1}$$

Note that $\mathscr{G}_f$ is Lipschitz continuous of constant 1.

From now on we denote with $B(u,\delta)$ the open ball of center u and of radius δ. We recall the definition of the weak slope for a continuous function introduced in [10,12,15,16].

Definition 3.1. Let X be a complete metric space, $g : X \to \mathbb{R}$ a continuous function, and $u \in X$. We denote by $|dg|(u)$ the supremum of the real numbers σ in $[0, \infty)$ such that there exist $\delta > 0$ and a continuous map

$$\mathscr{H} : B(u, \delta) \times [0, \delta] \to X,$$

such that, for every v in $B(u, \delta)$, and for every t in $[0, \delta]$ it results

$$d(\mathscr{H}(v, t), v) \leqslant t,$$

$$g(\mathscr{H}(v, t)) \leqslant g(v) - \sigma t.$$

The extended real number $|dg|(u)$ is called the weak slope of g at u.

According to the previous definition, for every lower semicontinuous function f we can consider the metric space $\mathrm{epi}(f)$ so that the weak slope of $\mathscr{G}_f$ is well defined. Therefore, we can define the weak slope of a lower semicontinuous function f by using $|d\mathscr{G}_f|(u, f(u))$.

More precisely, we have the following

Definition 3.2. For every $u \in \mathrm{dom}(f)$ let

$$|df|(u) = \begin{cases} \dfrac{|d\mathscr{G}_f|(u, f(u))}{\sqrt{1 - |d\mathscr{G}_f|(u, f(u))^2}} & \text{if } |d\mathscr{G}_f|(u, f(u)) < 1, \\ +\infty & \text{if } |d\mathscr{G}_f|(u, f(u)) = 1. \end{cases}$$

The previous notions allow us to give the following

Definition 3.3. Let X be a complete metric space and $f : X \to \mathbb{R} \cup \{+\infty\}$ a lower semicontinuous function. We say that $u \in \mathrm{dom}(f)$ is a (lower) critical point of f if $|df|(u) = 0$. We say that $c \in \mathbb{R}$ is a (lower) critical value of f if there exists a (lower) critical point $u \in \mathrm{dom}(f)$ of f with $f(u) = c$.

Definition 3.4. Let X be a complete metric space, $f : X \to \mathbb{R} \cup \{+\infty\}$ a lower semicontinuous function and let $c \in \mathbb{R}$. We say that f satisfies the Palais–Smale condition at level c ($(PS)_c$ in short), if every sequence $\{u_n\}$ in $\mathrm{dom}(f)$ such that

$$|df|(u_n) \to 0,$$

$$f(u_n) \to c$$

admits a subsequence $\{u_{n_k}\}$ converging in X.

For every $\eta \in \mathbb{R}$, let us define the set

$$f^\eta = \{u \in X : f(u) < \eta\}. \tag{3.2}$$

The next result gives a criterion to obtain an estimate from below of $|df|(u)$ (cf. [12]).

Proposition 3.5. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function defined on the complete metric space X, and let $u \in \mathrm{dom}(f)$. Let us assume that there exist $\delta > 0$, $\eta > f(u)$, $\sigma > 0$ and a continuous function $\mathscr{H} : B(u, \delta) \cap f^\eta \times [0, \delta] \to X$ such that*

$$d(\mathscr{H}(v, t), v) \leqslant t \quad \forall v \in B(u, \delta) \cap f^\eta,$$

$$f(\mathscr{H}(v, t)) \leqslant f(v) - \sigma t \quad \forall v \in B(u, \delta) \cap f^\eta.$$

Then $|df|(u) \geqslant \sigma$.

We will also use the notion of equivariant weak slope (see [9]).

Definition 3.6. Let X be a normed linear space and $f : X \to \mathbb{R} \cup \{+\infty\}$ an even lower semicontinuous function with $f(0) < +\infty$. For every $(0, \eta) \in \mathrm{epi}(f)$ we denote by $|d_{\mathbb{Z}_2}\mathscr{G}_f|(0, \eta)$ the supremum of the numbers σ in $[0, \infty)$ such that there exist $\delta > 0$ and a continuous map

$$\mathscr{H} = (\mathscr{H}_1, \mathscr{H}_2) : (B((0, \eta), \delta) \cap \mathrm{epi}(f)) \times [0, \delta] \to \mathrm{epi}(f)$$

satisfying

$$d(\mathscr{H}((w, \mu), t), (w, \mu)) \leqslant t, \quad \mathscr{H}_2((w, \mu), t) \leqslant \mu - \sigma t,$$

$$\mathscr{H}_1((-w, \mu), t) = -\mathscr{H}_1((w, \mu), t)$$

for every $(w, \mu) \in B((0, \eta), \delta) \cap \mathrm{epi}(f)$ and $t \in [0, \delta]$.

In order to compute $|d\mathscr{G}_f|(u, \eta)$, the next result will be useful (cf. [12]).

Proposition 3.7. *Let X be a normed linear space, $J : X \to \mathbb{R} \cup \{+\infty\}$ a lower semicontinuous functional, $I : X \to \mathbb{R}$ a C^1 functional and let $f = J + I$. Then the following facts hold:*

(a) *for every $(u, \eta) \in \mathrm{epi}(f)$ we have*

$$|d\mathscr{G}_f|(u, \eta) = 1 \Leftrightarrow |d\mathscr{G}_J|(u, \eta - I(u)) = 1,$$

(b) *if J and I are even, for every $\eta \geqslant f(0)$, we have*

$$|d_{\mathbb{Z}_2}\mathscr{G}_f|(0, \eta) = 1 \Leftrightarrow |d_{\mathbb{Z}_2}\mathscr{G}_J|(0, \eta - I(0)) = 1,$$

(c) *if* $u \in \mathrm{dom}(f)$ *and* $I'(u) = 0$, *then*

$$|df|(u) = |dJ|(u).$$

Proof. Assertions (a) and (c) follow by arguing as in [12]. Assertion (b) can be reduced to (a) after observing that, since I is even, it results $I'(0) = 0$. □

In [10,12] variational methods for lower semicontinuous functionals are developed. Moreover, it is shown that the following condition is fundamental in order to apply the abstract theory to the study of lower semicontinuous functions:

$$\forall (u, \eta) \in \mathrm{epi}(f) : f(u) < \eta \Rightarrow |d\mathcal{G}_f|(u, \eta) = 1. \tag{3.3}$$

In the next section, we will prove that the functional f satisfies (3.3).

The next result gives a criterion to verify condition (3.3) (cf. [13, Corollary 2.11]).

Theorem 3.8. *Let* $(u, \eta) \in \mathrm{epi}(f)$ *with* $f(u) < \eta$. *Assume that, for every* $\varrho > 0$, *there exist* $\delta > 0$ *and a continuous map*

$$\mathcal{H} : \{w \in B(u, \delta) : f(w) < \eta + \delta\} \times [0, \delta] \to X$$

satisfying

$$d(\mathcal{H}(w, t), w) \leqslant \varrho t \quad \text{and} \quad f(\mathcal{H}(w, t)) \leqslant (1 - t)f(w) + t(f(u) + \varrho)$$

whenever $w \in B(u, \delta)$, $f(w) < \eta + \delta$ *and* $t \in [0, \delta]$. *Then we have* $|d\mathcal{G}_f|(u, \eta) = 1$. *In addition, if* f *is even,* $u = 0$ *and* $\mathcal{H}(-w, t) = -\mathcal{H}(w, t)$, *then we have* $|d_{\mathbb{Z}_2}\mathcal{G}_f|(0, \eta) = 1$.

Let us now recall from [10] the following

Theorem 3.9. *Let* X *be a Banach space and* $f : X \to \mathbb{R} \cup \{+\infty\}$ *a lower semicontinuous function satisfying* (3.3). *Assume that there exist* $v_0, v_1 \in X$ *and* $r > 0$ *such that* $\|v_1 - v_0\| > r$ *and*

$$\inf\{f(u) : u \in X, \|u - v_0\| = r\} > \max\{f(v_0), f(v_1)\}. \tag{3.4}$$

Let us set

$$\Gamma = \{\gamma : [0, 1] \to \mathrm{dom}(f),\ \gamma \text{ continuous},\ \gamma(0) = v_0 \text{ and } \gamma(1) = v_1\}$$

and assume that

$$c_1 = \inf_{\gamma \in \Gamma} \sup_{[0,1]} f \circ \gamma < +\infty$$

and that f satisfies the Palais–Smale condition at the level c_1. Then, there exists a critical point u_1 of f such that $f(u_1) = c_1$. If, moreover,

$$c_0 = \inf f(\overline{B_r(v_0)}) > -\infty$$

and f satisfies the Palais–Smale condition at the level c_0, then there exists another critical point u_0 of f with $f(u_0) = c_0$.

In the equivariant case we shall apply the following result (see [17]).

Theorem 3.10. *Let X be a Banach space and $f : X \to \mathbb{R} \cup \{+\infty\}$ a lower semicontinuous even function. Let us assume that there exists a strictly increasing sequence (W_h) of finite-dimensional subspaces of X with the following properties:*

(a) *there exist $\rho > 0$, $\gamma > f(0)$ and a subspace $V \subset X$ of finite codimension such that*

$$\forall u \in V : ||u|| = \rho \Rightarrow f(u) \geqslant \gamma,$$

(b) *there exists a sequence (R_h) in (ρ, ∞) such that*

$$\forall u \in W_h : ||u|| \geqslant R_h \Rightarrow f(u) \leqslant f(0),$$

(c) *f satisfies $(PS)_c$ for any $c \geqslant \gamma$ and f satisfies* (3.3),
(d) *$|d_{\mathbb{Z}_2}\mathcal{G}_f|(0, \eta) \neq 0$ for every $\eta > f(0)$.*

Then there exists a sequence $\{u_h\}$ of critical points of f such that $f(u_h) \to +\infty$.

Let us now set $X = H_0^1(\Omega)$ and consider the functional $J : H_0^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$ defined in (2.1). From hypothesis (1.2), we immediately obtain that J is lower semicontinuous. We will now prove that J satisfies (3.3). To this aim, for every $k \geqslant 1$, we define the truncation $T_k : \mathbb{R} \to \mathbb{R}$ at height k, defined as

$$T_k(s) = s \text{ if } |s| \leqslant k, \quad T_k(s) = k\frac{s}{|s|} \text{ if } |s| \geqslant k. \tag{3.5}$$

We will prove the following

Theorem 3.11. *Assume conditions* (1.1), (1.2), (1.4). *Then, for every $(u, \eta) \in \mathrm{epi}(J)$ with $J(u) < \eta$, there holds*

$$|d\mathcal{G}_J|(u, \eta) = 1.$$

Moreover, if $j(x, -s, -\xi) = j(x, s, \xi)$, $\forall \eta > J(0) (= 0)$ it results $|d_{\mathbb{Z}_2}\mathcal{G}_J|(0, \eta) = 1$.

Proof. Let $(u,\eta)\in \mathrm{epi}(J)$ with $J(u)<\eta$ and let $\varrho>0$. Then, there exists $\delta\in(0,1]$, $\delta=\delta(\varrho)$, and $k\geqslant 1$, $k=k(\varrho)$, such that $k\geqslant R$ (where R is as in (1.4)) and

$$||T_k(v)-v||_{1,2}<\varrho \quad \text{for every } v\in B(u,\delta). \tag{3.6}$$

From (1.2) we have

$$j(x,v,\nabla T_k(v))\leqslant \alpha(k)|\nabla v|^2.$$

Then, up to reducing δ, we get the following inequalities:

$$\int_\Omega j(x,v,\nabla T_k(v))<\int_\Omega j(x,u,\nabla T_k(u))+\varrho\leqslant \int_\Omega j(x,u,\nabla u)+\varrho \tag{3.7}$$

for each $v\in B(u,\delta)$. We now prove that, for every $t\in[0,\delta]$ and $v\in B(u,\delta)$, there holds

$$J((1-t)v+tT_k(v))\leqslant(1-t)J(v)+t(J(u)+\varrho). \tag{3.8}$$

From (1.1) and since $j(x,s,\xi)$ is of class C^1 with respect to the variable s, there exists $\theta\in[0,1]$ such that

$$\begin{aligned}
&j(x,(1-t)v+tT_k(v),(1-t)\nabla v+t\nabla T_k(v))-j(x,v,\nabla v)\\
&\quad= j(x,(1-t)v+tT_k(v),(1-t)\nabla v+t\nabla T_k(v))\ -j(x,v,(1-t)\nabla v+t\nabla T_k(v))\\
&\qquad+\ j(x,v,(1-t)\nabla v+t\nabla T_k(v))-j(x,v,\nabla v)\\
&\quad\leqslant tj_s(x,v+\theta t(T_k(v)-v),(1-t)\nabla v+t\nabla T_k(v))(T_k(v)-v)\\
&\qquad+\ t(j(x,v,\nabla T_k(v))-j(x,v,\nabla v)).
\end{aligned}$$

Notice that there holds

$$v(x)\geqslant k \Rightarrow v(x)+\theta t(T_k(v(x))-v(x))\geqslant k\geqslant R,$$

$$v(x)\leqslant -k \Rightarrow v(x)+\theta t(T_k(v(x))-v(x))\leqslant -k\leqslant -R.$$

Then, in light of (1.4) one has

$$j_s(x,v+\theta t(T_k(v)-v),(1-t)\nabla v+t\nabla T_k(v))(T_k(v)-v)\leqslant 0.$$

It follows that

$$j(x,(1-t)v+tT_k(v),(1-t)\nabla v+t\nabla T_k(v))\leqslant(1-t)j(x,v,\nabla v)+tj(x,v,\nabla T_k(v)).$$

Therefore from (3.6) one gets (3.8). In order to apply Theorem 3.8 we define

$$\mathscr{H}:\{v\in B(u,\delta):J(v)<\eta+\delta\}\times[0,\delta]\to H_0^1(\Omega)$$

by setting

$$\mathcal{H}(v,t) = (1-t)v + tT_k(v).$$

Hence, taking into account (3.7) and (3.8), it results

$$d(\mathcal{H}(v,t),v) \leqslant \varrho t \quad \text{and} \quad J(\mathcal{H}(v,t)) \leqslant (1-t)J(v) + t(J(u)+\varrho)$$

for $v \in B(u,\delta)$, $J(v) < \eta + \delta$ and $t \in [0,\delta]$. The first assertion now follows from Theorem 3.8. Finally, since $\mathcal{H}(-v,t) = \mathcal{H}(v,t)$ one also has $|d_{\mathbb{Z}_2}\mathcal{G}_J|(0,\eta) = 1$, whenever $j(x,-s,-\xi) = j(x,s,\xi)$. □

4. The variational setting

This section regards the relations between $|dJ|(u)$ and the directional derivatives of the functional J. Moreover, we will obtain some Brezis–Browder (see [7]) type results.

First of all, we make a few observations.

Remark 4.1. It is readily seen that hypothesis (1.1) and the right inequality of (1.2) imply that there exists a positive increasing function $\bar{\alpha}(|s|)$ such that

$$|j_\xi(x,s,\xi)| \leqslant \bar{\alpha}(|s|)|\xi| \tag{4.1}$$

for a.e. $x \in \Omega$ and every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$. Indeed, from (1.1) one has

$$\forall v \in \mathbb{R}^N : |v| \leqslant 1 \Rightarrow j(x,s,\xi + |\xi|v) \geqslant j(x,s,\xi) + j_\xi(x,s,\xi) \cdot v|\xi|.$$

This, and (1.2) yield

$$j_\xi(x,s,\xi) \cdot v|\xi| \leqslant 4\alpha(|s|)|\xi|^2.$$

From the arbitrariness of v, (4.1) follows. On the other hand, if (4.1) holds we have

$$|j(x,s,\xi)| \leqslant \int_0^1 |j_\xi(x,s,t\xi) \cdot \xi| dt \leqslant \frac{1}{2}\bar{\alpha}(|s|)|\xi|^2.$$

As a consequence, it is not restrictive to suppose that the functions in the right-hand side of (1.2) and (4.1) are the same. Notice that, in particular, there holds $j_\xi(x,s,0) = 0$.

Remark 4.2. It is not restrictive to suppose that the functions $\alpha(s)$ and $\beta(s)$ are both increasing. Indeed, if this is not the case, we can consider the functions

$$A_r(|s|) = \sup_{|s|\leqslant r} \alpha(|s|) \quad \text{and} \quad B_r(|s|) = \sup_{|s|\leqslant r} \beta(|s|),$$

which are increasing.

Remark 4.3. The assumption of strict convexity on the function $\{\xi \to j(x,s,\xi)\}$ implies that, for almost every x in Ω and for every s in $\mathbb{R}$, we have

$$[j_\xi(x,s,\xi) - j_\xi(x,s,\xi^*)] \cdot (\xi - \xi^*) > 0 \tag{4.2}$$

for every ξ, $\xi^* \in \mathbb{R}^N$, with $\xi \neq \xi^*$. Moreover, hypotheses (1.1) and (1.2) imply that,

$$j_\xi(x,s,\xi) \cdot \xi \geqslant \alpha_0 |\xi|^2. \tag{4.3}$$

Indeed, we have

$$0 = j(x,s,0) \geqslant j(x,s,\xi) + j_\xi(x,s,\xi) \cdot (0 - \xi)$$

so that inequality (4.3) follows by virtue of (1.2).

Now, for every $u \in H_0^1(\Omega)$, we define the subspace

$$V_u = \{v \in H_0^1(\Omega) \cap L^\infty(\Omega) : u \in L^\infty(\{x \in \Omega : v(x) \neq 0\})\}. \tag{4.4}$$

As proved in [14], V_u is a vector space dense in $H_0^1(\Omega)$. Since $V_u \subset W_u$, also W_u (see the Introduction) is dense in $H_0^1(\Omega)$. In the following proposition we study the conditions under which we can compute the directional derivatives of J.

Proposition 4.4. *Assume conditions* (1.2), (1.3), (1.5). *Then there exists* $J'(u)(v)$ *for every* $u \in \mathrm{dom}(J)$ *and* $v \in V_u$. *Furthermore, we have*

$$j_s(x,u,\nabla u)v \in L^1(\Omega) \quad \textit{and} \quad j_\xi(x,u,\nabla u) \cdot \nabla v \in L^1(\Omega)$$

and

$$J'(u)(v) = \int_\Omega j_\xi(x,u,\nabla u) \cdot \nabla v + \int_\Omega j_s(x,u,\nabla u)v.$$

Proof. Let $u \in \mathrm{dom}(J)$ and $v \in V_u$. For every $t \in \mathbb{R}$ and a.e. $x \in \Omega$, we set

$$F(x,t) = j(x, u(x) + tv(x), \nabla u(x) + t\nabla v(x)).$$

Since $v \in V_u$ and by using (1.2), it follows that $F(x,t) \in L^1(\Omega)$. Moreover, it results

$$\frac{\partial F}{\partial t}(x,t) = j_s(x, u+tv, \nabla u + t\nabla v)v + j_\xi(x, u+tv, \nabla u + t\nabla v) \cdot \nabla v.$$

From hypotheses (1.3) and (1.5) we get that for every $x \in \Omega$ with $v(x) \neq 0$, it results

$$\left|\frac{\partial F}{\partial t}(x,t)\right| \leqslant ||v||_{\infty}\beta(||u||_{\infty}+||v||_{\infty})(|\nabla u|+|\nabla v|)^2 \\ + \alpha(||u||_{\infty}+||v||_{\infty})(|\nabla u|+|\nabla v|)|\nabla v|.$$

Since the function in the right-hand side of the previous inequality belongs to $L^1(\Omega)$, the assertion follows. □

In the sequel we will often use the cut-off function $H \in C^{\infty}(\mathbb{R})$ given by

$$H(s) = 1 \quad \text{on } [-1,1], \quad H(s) = 0 \text{ outside } [-2,2], \quad |H'(s)| \leqslant 2. \tag{4.5}$$

Now, we can prove a fundamental inequality regarding the weak slope of J.

Proposition 4.5. *Assume conditions* (1.2), (1.3), (1.5). *Then we have*

$$|d(J-w)|(u) \\ \geqslant \sup\left\{\int_{\Omega} j_{\xi}(x,u,\nabla u)\cdot\nabla v + \int_{\Omega} j_s(x,u,\nabla u)v - \langle w,v\rangle : v \in V_u,\ ||v||_{1,2} \leqslant 1\right\}$$

for every $u \in \mathrm{dom}(J)$ *and every* $w \in H^{-1}(\Omega)$.

Proof. If it results $|d(J-w)|(u) = \infty$, or if it holds

$$\sup\left\{\int_{\Omega} j_{\xi}(x,u,\nabla u)\cdot\nabla v + \int_{\Omega} j_s(x,u,\nabla u)v - \langle w,v\rangle : v \in V_u,\ ||v||_{1,2} \leqslant 1\right\} = 0,$$

the inequality holds. Otherwise, let $u \in \mathrm{dom}(J)$ and let $\eta \in \mathbb{R}^+$ be such that $J(u) < \eta$. Moreover, let us consider $\bar{\sigma} > 0$ and $\bar{v} \in V_u$ such that $||\bar{v}||_{1,2} \leqslant 1$ and

$$\int_{\Omega} j_{\xi}(x,u,\nabla u)\cdot\nabla\bar{v} + \int_{\Omega} j_s(x,u,\nabla u)\bar{v} - \langle w,\bar{v}\rangle < -\bar{\sigma}. \tag{4.6}$$

Let us fix $\varepsilon > 0$ and let us prove that there exists $k_0 \geqslant 1$ such that

$$\left|\left|H\left(\frac{u}{k_0}\right)\bar{v}\right|\right|_{1,2} < 1 + \varepsilon \tag{4.7}$$

and

$$\int_{\Omega} j_s(x,u,\nabla u)H\left(\frac{u}{k_0}\right)\bar{v} \\ + \int_{\Omega} j_{\xi}(x,u,\nabla u)\cdot\nabla\left(H\left(\frac{u}{k_0}\right)\bar{v}\right) - \left\langle w, H\left(\frac{u}{k_0}\right)\bar{v}\right\rangle < -\bar{\sigma}. \tag{4.8}$$

Let us set $v_k = H(u/k)\bar{v}$, where $H(s)$ is defined as in (4.5). Since $\bar{v} \in V_u$ we deduce that $v_k \in V_u$ for every $k \geqslant 1$ and v_k converges to $\bar{v}$ in $H_0^1(\Omega)$. This, together with the fact that $\|\bar{v}\|_{1,2} \leqslant 1$, implies (4.7). Moreover, Proposition 4.4 implies that we can consider $J'(u)(v_k)$. In addition, as k goes to infinity, we have

$$j_s(x, u(x), \nabla u(x)) v_k(x) \to j_s(x, u(x), \nabla u(x)) \bar{v}(x) \quad \text{for a.e. } x \in \Omega,$$

$$j_\xi(x, u(x), \nabla u(x)) \cdot \nabla v_k(x) \to j_\xi(x, u(x), \nabla u(x)) \cdot \nabla \bar{v}(x) \quad \text{for a.e. } x \in \Omega.$$

Moreover, we get

$$\left| j_s(x, u, \nabla u) H\left(\frac{u}{k}\right) \bar{v} \right| \leqslant |j_s(x, u, \nabla u)\bar{v}|,$$

$$|j_\xi(x, u, \nabla u) \cdot \nabla v_k| \leqslant |j_\xi(x, u, \nabla u)| \, |\nabla \bar{v}| + 2|\bar{v}| \, |j_\xi(x, u, \nabla u) \cdot \nabla u|.$$

Since $\bar{v} \in V_u$ and by using (1.3) and (1.5), we can apply Lebesgue Dominated Convergence Theorem to obtain

$$\lim_{k \to \infty} \int_\Omega j_s(x, u, \nabla u) v_k = \int_\Omega j_s(x, u, \nabla u) \bar{v},$$

$$\lim_{k \to \infty} \int_\Omega j_\xi(x, u, \nabla u) \cdot \nabla v_k = \int_\Omega j_\xi(x, u, \nabla u) \cdot \nabla \bar{v},$$

which, together with (4.6), implies (4.8). Since we want to apply Proposition 3.5, let us consider J^η as defined in (3.2). Let us now show that there exists $\delta_1 > 0$ such that

$$\left\| H\left(\frac{z}{k_0}\right) \bar{v} \right\| \leqslant 1 + \varepsilon, \tag{4.9}$$

as well as

$$\int_\Omega j_\xi(x, z, \nabla z) \cdot \nabla \left(H\left(\frac{z}{k_0}\right) \bar{v} \right) + \int_\Omega j_s(x, z, \nabla z) H\left(\frac{z}{k_0}\right) \bar{v} - \left\langle w, H\left(\frac{z}{k_0}\right) \bar{v} \right\rangle < -\bar{\sigma} \tag{4.10}$$

for every $z \in B(u, \delta_1) \cap J^\eta$. Indeed, take $u_n \in J^\eta$ such that $u_n \to u$ in $H_0^1(\Omega)$ and set

$$v_n = H\left(\frac{u_n}{k_0}\right) \bar{v}.$$

We have that $v_n \to H(u/k_0)\bar{v}$ in $H_0^1(\Omega)$, so that (4.9) follows from (4.7). Moreover, note that $v_n \in V_{u_n}$, so that from Proposition 4.4 we deduce that we can consider

$J'(u_n)(v_n)$. From (1.3) and (1.5) it follows

$$|j_s(x,u_n,\nabla u_n)v_n| \leqslant \beta(2k_0)\|\bar{v}\|_\infty |\nabla u_n|^2,$$

$$|j_\xi(x,u_n,\nabla u_n)\cdot \nabla v_n| \leqslant \alpha(2k_0)|\nabla u_n|\left[\frac{2}{k_0}\|\bar{v}\|_\infty |\nabla u_n| + |\nabla \bar{v}|\right].$$

Then, we obtain

$$\lim_{n\to\infty}\int_\Omega j_s(x,u_n,\nabla u_n)v_n = \int_\Omega j_s(x,u,\nabla u)H\left(\frac{u}{k_0}\right)\bar{v},$$
$$\lim_{n\to\infty}\int_\Omega j_\xi(x,u_n,\nabla u_n)\cdot\nabla v_n = \int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla\left[H\left(\frac{u}{k_0}\right)\bar{v}\right],$$

which, together with (4.8), implies (4.10). Now, observe that (4.10) is equivalent to say that $J'(z)\big(H\big(\frac{z}{k}\big)\bar{v}\big) - \langle w, H\big(\frac{z}{k}\big)\bar{v}\rangle < -\bar{\sigma}$. Thus, there exists $\delta<\delta_1$ with

$$J\left(z+\frac{t}{1+\varepsilon}H\left(\frac{z}{k_0}\right)\bar{v}\right) - J(z) - \left\langle w, \frac{t}{1+\varepsilon}H\left(\frac{z}{k}\right)\bar{v}\right\rangle \leqslant -\frac{\bar{\sigma}}{1+\varepsilon}t \tag{4.11}$$

for every $t\in[0,\delta]$ and $z\in B(u,\delta)\cap J^\eta$. Finally, let us define the continuous function $\mathscr{H}: B(u,\delta)\cap J^\eta\times[0,\delta]\to H_0^1(\Omega)$ given by

$$\mathscr{H}(z,t) = z+\frac{t}{1+\varepsilon}H\left(\frac{z}{k_0}\right)\bar{v}.$$

From (4.9) and (4.11) we deduce that $\mathscr{H}$ satisfies all the hypotheses of Proposition 3.5. Then, $|d(J-w)|(u)>\frac{\bar{\sigma}}{1+\varepsilon}$, and the conclusion follows from the arbitrariness of ε. □

The next lemma will be useful in proving two Brezis–Browder type results for J.

Lemma 4.6. *Assume conditions* (1.1)–(1.4) *and let* $u\in\mathrm{dom}(J)$. *Then*

$$\int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla u + \int_\Omega j_s(x,u,\nabla u)u \leqslant |dJ|(u)\|u\|_{1,2}. \tag{4.12}$$

In particular, if $|dJ|(u)<\infty$, *there holds*

$$j_\xi(x,u,\nabla u)\cdot\nabla u\in L^1(\Omega) \quad \textit{and} \quad j_s(x,u,\nabla u)u\in L^1(\Omega).$$

Proof. First, notice that if u is such that $|dJ|(u)=\infty$, or

$$\int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla u + \int_\Omega j_s(x,u,\nabla u)u \leqslant 0$$

then the conclusion holds. Otherwise, let $k \geqslant 1$, $u \in \mathrm{dom}(J)$ with $|dJ|(u) < \infty$, and $\sigma > 0$ be such that

$$\int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla T_k(u) + \int_\Omega j_s(x,u,\nabla u)T_k(u) > \sigma \|T_k(u)\|_{1,2},$$

where $T_k(s)$ is defined in (3.5). We will prove that $|dJ|(u) \geqslant \sigma$. Fixed $\varepsilon > 0$, we first want to show that there exists $\delta_1 > 0$ such that

$$\|T_k(w)\|_{1,2} \leqslant (1+\varepsilon)\|T_k(u)\|_{1,2}, \tag{4.13}$$

$$\int_\Omega j_\xi(x,w,\nabla w)\cdot\nabla T_k(w) + \int_\Omega j_s(x,w,\nabla w)T_k(w) > \sigma \|T_k(u)\|_{1,2} \tag{4.14}$$

for every $w \in H_0^1(\Omega)$ with $\|w-u\|_{1,2} < \delta_1$. Indeed, take $w_n \in H_0^1(\Omega)$ such that $w_n \to u$ in $H_0^1(\Omega)$. Then, (4.13) follows directly. Moreover, notice that from (1.3) and (1.4) there holds

$$j_s(x,w_n(x),\nabla w_n(x))w_n(x) \geqslant -R\beta(R)|\nabla w_n(x)|^2.$$

Since $w_n \to u$ in $H_0^1(\Omega)$, from (4.3) and by applying Fatou Lemma we get

$$\liminf_{n\to\infty}\left[\int_\Omega j_\xi(x,w_n,\nabla w_n)\cdot\nabla T_k(w_n) + \int_\Omega j_s(x,w_n,\nabla w_n)T_k(w_n)\right]$$
$$\geqslant \int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla T_k(u) + \int_\Omega j_s(x,u,\nabla u)T_k(u) > \sigma\|T_k(u)\|_{1,2},$$

which yields (4.14). Consider now the continuous map $\mathscr{H} : B(u,\delta_1)\times[0,\delta_1] \to H_0^1(\Omega)$ defined as

$$\mathscr{H}(w,t) = w - \frac{t}{\|T_k(u)\|_{1,2}(1+\varepsilon)}T_k(w).$$

From (4.13) and (4.14) we deduce that there exists $\delta < \delta_1$ such that

$$d(\mathscr{H}(w,t),w) \leqslant t,$$

$$J(\mathscr{H}(w,t)) - J(w) \leqslant -\frac{\sigma}{1+\varepsilon}$$

for every $t \in [0,\delta]$ and $w \in H_0^1(\Omega)$ with $\|w-u\|_{1,2} < \delta$ and $J(w) < J(u)+\delta$. Then, the arbitrariness of ε yields $|dJ|(u) \geqslant \sigma$. Therefore, for every $k \geqslant 1$ we get

$$\int_\Omega j_s(x,u,\nabla u)T_k(u) + \int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla T_k(u) \leqslant |dJ|(u)\|T_k(u)\|_{1,2}.$$

Taking the limit as $k \to \infty$, the Monotone Convergence Theorem yields (4.12). □

Notice that a generalized solution u (see Definition 1.1) is not, in general, a distributional solution. This, because a test function $v \in W_u$ may not belong to C_0^∞. Thus, it is natural to study the conditions under which it is possible to enlarge the class of admissible test functions. This kind of argument was introduced in [7].

More precisely, suppose we have a function $u \in H_0^1(\Omega)$ such that

$$\int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla z + \int_\Omega j_s(x,u,\nabla u)z = \langle w, z\rangle \quad \forall z \in V_u, \tag{4.15}$$

where V_u is defined in (4.4) and $w \in H^{-1}(\Omega)$. A natural question is whether or not we can take as test function $v \in H_0^1(\Omega)\cap L^\infty(\Omega)$. The next result gives an answer to this question.

Theorem 4.7. *Assume that conditions* (1.1)–(1.3) *hold. Let* $w \in H^{-1}(\Omega)$ and $u \in H_0^1(\Omega)$ *that satisfies* (4.15). *Moreover, suppose that* $j_\xi(x,u,\nabla u)\cdot\nabla u \in L^1(\Omega)$ *and there exist* $v \in H_0^1(\Omega)\cap L^\infty(\Omega)$ *and* $\eta \in L^1(\Omega)$ *such that*

$$j_s(x,u,\nabla u)v + j_\xi(x,u,\nabla u)\cdot\nabla v \geqslant \eta. \tag{4.16}$$

Then $j_\xi(x,u,\nabla u)\cdot\nabla v + j_s(x,u,\nabla u)v \in L^1(\Omega)$ *and*

$$\int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla v + \int_\Omega j_s(x,u,\nabla u)v = \langle w, v\rangle.$$

Proof. Since $v \in H_0^1(\Omega)\cap L^\infty(\Omega)$, then $H(\frac{u}{k})v \in V_u$. From (4.15) we have

$$\int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla\left[H\left(\frac{u}{k}\right)v\right] + \int_\Omega j_s(x,u,\nabla u)H\left(\frac{u}{k}\right)v = \left\langle w, H\left(\frac{u}{k}\right)v\right\rangle \tag{4.17}$$

for every $k \geqslant 1$. Note that

$$\int_\Omega \left| j_\xi(x,u,\nabla u)\cdot\nabla u H'\left(\frac{u}{k}\right)\frac{v}{k}\right| \leqslant \frac{2}{k}\|v\|_\infty \int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla u.$$

Since $j_\xi(x,u,\nabla u)\cdot\nabla u \in L^1(\Omega)$, the Lebesgue Dominated Convergence Theorem yields

$$\lim_{k\to\infty}\int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla u H'\left(\frac{u}{k}\right)\frac{v}{k} = 0,$$
$$\lim_{k\to\infty}\left\langle w, H\left(\frac{u}{k}\right)v\right\rangle = \langle w, v\rangle.$$

As far as concerns the remaining terms in (4.17), notice that from (4.16) it follows

$$[j_s(x,u,\nabla u)v + j_\xi(x,u,\nabla u)\cdot\nabla v]H\left(\frac{u}{k}\right) \geqslant H\left(\frac{u}{k}\right)\eta \geqslant -\eta^- \in L^1(\Omega).$$

Thus, we can apply Fatou Lemma and obtain

$$\int_\Omega j_s(x,u,\nabla u)v + j_\xi(x,u,\nabla u)\cdot\nabla v \leqslant \langle w, v\rangle.$$

The previous inequality and (4.16) imply that

$$j_s(x,u,\nabla u)v + j_\xi(x,u,\nabla u)\cdot\nabla v \in L^1(\Omega). \tag{4.18}$$

Now, notice that

$$\left|[j_s(x,u,\nabla u)v + j_\xi(x,u,\nabla u)\cdot\nabla v]H\left(\frac{u}{k}\right)\right| \leqslant |j_s(x,u,\nabla u)v + j_\xi(x,u,\nabla u)\cdot\nabla v|.$$

From (4.18) we deduce that we can use Lebesgue Dominated Convergence Theorem to pass to the limit in (4.17) and obtain the conclusion. □

In the next result, we find the conditions under which we can use $v\in H^1_0(\Omega)$ in (4.15). Moreover, we prove, under suitable hypotheses, that if u satisfies (4.15) then u is a generalized solution (see Definition 1.1) of the corresponding problem.

Theorem 4.8. *Assume that conditions* (1.1)–(1.4) *hold. Let* $w\in H^{-1}(\Omega)$, *and let* $u\in H^1_0(\Omega)$ *be such that* (4.15) *is satisfied. Moreover, suppose that* $j_\xi(x,u,\nabla u)\cdot\nabla u\in L^1(\Omega)$, *and that there exist* $v\in H^1_0(\Omega)$ *and* $\eta\in L^1(\Omega)$ *such that*

$$j_s(x,u,\nabla u)v \geqslant \eta \quad \text{and} \quad j_\xi(x,u,\nabla u)\cdot\nabla v \geqslant \eta. \tag{4.19}$$

Then $j_s(x,u,\nabla u)v\in L^1(\Omega)$, $j_\xi(x,u,\nabla u)\cdot\nabla v\in L^1(\Omega)$ *and*

$$\int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla v + \int_\Omega j_s(x,u,\nabla u)v = \langle w, v\rangle. \tag{4.20}$$

In particular, it results $j_s(x,u,\nabla u)u$, $j_s(x,u,\nabla u)\in L^1(\Omega)$ *and*

$$\int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla u + \int_\Omega j_s(x,u,\nabla u)u = \langle w, u\rangle.$$

Moreover, u is a generalized solution of the problem

$$\begin{cases} -\mathrm{div}\,(j_\xi(x,u,\nabla u)) + j_s(x,u,\nabla u) = w & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega. \end{cases} \tag{4.21}$$

Proof. Let $k\geqslant 1$ be fixed. For every $v\in H^1_0(\Omega)$ we have that $T_k(v)\in H^1_0(\Omega)\cap L^\infty(\Omega)$ and $-v^-\leqslant T_k(v)\leqslant v^+$. Then, from (4.19), we get

$$j_s(x,u,\nabla u)T_k(v) \geqslant -\eta^- \in L^1(\Omega). \tag{4.22}$$

Moreover,

$$j_\xi(x,u,\nabla u)\cdot\nabla T_k(v)\geqslant -[j_\xi(x,u,\nabla u)\cdot\nabla T_k(v)]^-\geqslant -\eta^-\in L^1(\Omega). \qquad (4.23)$$

Then, we can apply Theorem 4.7 and obtain

$$\int_\Omega j_s(x,u,\nabla u)T_k(v)+\int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla T_k(v)=\langle w,T_k(v)\rangle \qquad (4.24)$$

for every $k\geqslant 1$. By using again (4.22) and (4.23) and by arguing as in Theorem 4.7 we obtain

$$j_s(x,u,\nabla u)v\in L^1(\Omega)\quad\text{and}\quad j_\xi(x,u,\nabla u)\cdot\nabla v\in L^1(\Omega).$$

Thus, we can use Lebesgue Dominated Convergence Theorem to pass to the limit in (4.24) and get (4.20). In particular, by (1.3), (1.4) and (4.3) we can choose $v=u$. Finally, since

$$j_s(x,u,\nabla u)=j_s(x,u,\nabla u)\chi_{\{|u|<1\}}+j_s(x,u,\nabla u)\chi_{\{|u|\geqslant 1\}}$$

and

$$|j_s(x,u,\nabla u)\chi_{\{|u|\geqslant 1\}}|\leqslant|j_s(x,u,\nabla u)u|$$

by (1.3) it results also $j_s(x,u,\nabla u)\in L^1(\Omega)$. Finally, notice that if $v\in W_u$ we can take $\eta=j_\xi(x,u,\nabla u)\cdot\nabla v$ and $\eta=j_s(x,u,\nabla u)v$, so that (4.20) is satisfied. Thus, u is a generalized solution to Problem (4.21). □

We point out that the previous result implies that if $u\in H_0^1(\Omega)$ satisfies (4.15) and $j_\xi(x,u,\nabla u)\cdot\nabla u\in L^1(\Omega)$, it results that $j_s(x,u,\nabla u)\in L^1(\Omega)$, then $j_s(x,u,\nabla u)v\in L^1(\Omega)$ for every $v\in C_0^\infty(\Omega)$. Instead, the term which has not a distributional interpretation in (4.15) is $j_\xi(x,u,\nabla u)$. In the next result we show that if we multiply $j_\xi(x,u,\nabla u)$ by a suitable sequence of C_c^1 functions, we obtain, passing to the limit, a distributional interpretation of (4.15).

Theorem 4.9. *Assume conditions* (1.1)–(1.4). *Let $w\in H^{-1}(\Omega)$ and $u\in H_0^1(\Omega)$ be such that* (4.15) *is satisfied. Let (ϑ_h) be a sequence in $C_c^1(\mathbb{R})$ with*

$$\sup_{h\geqslant 1}\|\vartheta_h\|_\infty<\infty,\quad \sup_{h\geqslant 1}\|\vartheta_h'\|_\infty<\infty,$$
$$\lim_{h\to\infty}\vartheta_h(s)=1,\quad \lim_{h\to\infty}\vartheta_h'(s)=0.$$

If $j_\xi(x,u,\nabla u)\cdot\nabla u\in L^1(\Omega)$, the sequence

$$\operatorname{div}[\vartheta_h(u)j_\xi(x,u,\nabla u)]$$

is strongly convergent in $W^{-1,q}(\Omega)$ *for every* $1<q<\frac{N}{N-1}$, *and*

$$\lim_{h\to\infty}\{-\mathrm{div}\,[\vartheta_h(u)j_\xi(x,u,\nabla u)]\}+j_s(x,u,\nabla u)=w\quad \textit{in } W^{-1,q}(\Omega).$$

Proof. Let $w=-\mathrm{div}\,F$ with $F\in L^2(\Omega,\mathbb{R}^N)$ and $v\in C_c^\infty(\Omega)$. Then $\vartheta_h(u)v\in V_u$ and we can take v as test function in (4.15). It results

$$\begin{aligned}\int_\Omega j_\xi(x,u,\nabla u)\vartheta_h(u)\cdot\nabla v = &-\int_\Omega j_\xi(x,u,\nabla u)\vartheta_h'(u)\cdot\nabla uv-\int_\Omega j_s(x,u,\nabla u)\vartheta_h(u)v\\ &+\int_\Omega F\vartheta_h{}'(u)\nabla uv+\int_\Omega F\vartheta_h(u)\nabla v.\end{aligned}$$

Then u is a solution of the following equation:

$$-\mathrm{div}\,[\vartheta_h(u)j_\xi(x,u,\nabla u)]=\xi_h\quad\text{in } \mathscr{D}'(\Omega),$$

where

$$\xi_h=-[\vartheta_h'(u)(j_\xi(x,u,\nabla u)-F)\cdot\nabla u+\vartheta_h(u)j_s(x,u,\nabla u)]-\mathrm{div}\,(\vartheta_h(u)F).$$

Now, notice that

$$\vartheta_h(u)F\to F\quad\text{strongly in } L^2(\Omega).$$

Then, $\mathrm{div}\,(\vartheta_h(u)F)$ is a convergent sequence in $H^{-1}(\Omega)$. Since the embedding of $H^{-1}(\Omega)$ in $W^{-1,q}(\Omega)$ is continuous, we get the desired convergence. Moreover, Theorem 4.8 implies that $j_s(x,u,\nabla u)\in L^1(\Omega)$. Then, the remaining terms in ξ_h converge strongly in $L^1(\Omega)$. Thus, we get the conclusion by observing that the embedding of $L^1(\Omega)$ in $W^{-1,q}(\Omega)$ is continuous. □

Consider the case $j(x,s,\xi)=a(x,s)|\xi|^2$ with $a(x,s)$ measurable with respect to x, continuous with respect to s and such that hypotheses (1.1)–(1.4), (1.7) are satisfied. The next result proves that, in particular, if there exists $u\in H_0^1(\Omega)$ that satisfies (4.15) and if $a(x,u)|\nabla u|^2\in L^1(\Omega)$, then u satisfies (4.15) in the sense of distribution.

Theorem 4.10. *Assume conditions* (1.1)–(1.4), (1.7). *Let* $w\in H^{-1}(\Omega)$ *and* $u\in H_0^1(\Omega)$ *that satisfies* (4.15). *Moreover, suppose that* $j_\xi(x,u,\nabla u)\cdot\nabla u\in L^1(\Omega)$ *and that*

$$j(x,s,\xi)=\widehat{j}(x,s,|\xi|). \tag{4.25}$$

Then $j_\xi(x,u,\nabla u)\in L^1(\Omega)$ *and u is a distributional solution to*

$$\begin{cases}-\mathrm{div}\,(j_\xi(x,u,\nabla u))+j_s(x,u,\nabla u)=w & \textit{in } \Omega,\\ u=0 & \textit{on } \partial\Omega.\end{cases}$$

Proof. It is readily seen that, in view of (1.1) and (4.25), it results

$$|\xi|\,|j_\xi(x,s,\xi)| \leqslant \sqrt{2} j_\xi(x,s,\xi)\cdot\xi$$

for a.e. $x\in\Omega$, every $s\in\mathbb{R}$ and $\xi\in\mathbb{R}^N$. Then

$$j_\xi(x,u,\nabla u)\chi_{\{|\nabla u|>1\}}\in L^1(\Omega).$$

Moreover, we take into account (1.7), and we observe that (1.5) implies that there exists a positive constant C such that

$$|\xi|\leqslant 1 \Rightarrow |j_\xi(x,s,\xi)|\leqslant 4\alpha(|s|)\leqslant C(|s|^{p-2}+1),$$

which by Sobolev embedding implies also that $j_\xi(x,u,\nabla u)\chi_{\{|\nabla u|\leqslant 1\}}\in L^1(\Omega)$. Then $j_\xi(x,u,\nabla u)\in L^1(\Omega)$. Moreover, from (1.3) and (1.4) we have

$$j_s(x,u,\nabla u)u \geqslant j_s(x,u,\nabla u)u\chi_{\{x\,:\,|u(x)|<R\}}\in L^1(\Omega).$$

Then Theorem 4.8 implies that $j_s(x,u,\nabla u)u\in L^1(\Omega)$. Finally, again Theorem 4.8 yields the conclusion. □

5. A compactness result for J

In this section, we will prove the following compactness result for J. We will follow an argument similar to the one used in [9] and in [20].

Theorem 5.1. *Assume conditions* (1.1)–(1.4). *Let* $\{u_n\}\subset H^1_0(\Omega)$ *be a bounded sequence with* $j_\xi(x,u_n,\nabla u_n)\cdot\nabla u_n\in L^1(\Omega)$ *and let* $\{w_n\}\subset H^{-1}(\Omega)$ *be such that*

$$\forall v\in V_{u_n}: \int_\Omega j_s(x,u_n,\nabla u_n)v + j_\xi(x,u_n,\nabla u_n)\cdot\nabla v = \langle w_n, v\rangle. \tag{5.1}$$

If w_n *is strongly convergent in* $H^{-1}(\Omega)$, *then, up to a subsequence,* u_n *is strongly convergent in* $H^1_0(\Omega)$.

Proof. Let w be the limit of $\{w_n\}$ and let $L>0$ be such that

$$\|u_n\|_{1,2}\leqslant L \quad \text{for every } n\geqslant 1. \tag{5.2}$$

From (5.2) we deduce that there exists $u\in H^1_0(\Omega)$ such that, up to a subsequence,

$$u_n \rightharpoonup u \quad \text{weakly in } H^1_0(\Omega). \tag{5.3}$$

Step 1: Let us first prove that u is such that

$$\int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla\psi + \int_\Omega j_s(x,u,\nabla u)\psi = \langle w,\psi\rangle \quad \forall\psi\in V_u. \tag{5.4}$$

First of all, from Rellich Compact Embedding Theorem, up to a subsequence,

$$\begin{cases} u_n \to u & \text{in } L^q(\Omega) \quad \forall q \in [1, 2N/(N-2)), \\ u_n(x) \to u(x) & \text{for a.e. } x \in \Omega. \end{cases} \tag{5.5}$$

We now want to prove that, up to a subsequence,

$$\nabla u_n(x) \to \nabla u(x) \quad \text{for a.e. } x \in \Omega. \tag{5.6}$$

Let $h \geqslant 1$. For every $v \in C_c^\infty(\Omega)$ we have that $H(\frac{u_n}{h}) v \in V_{u_n}$ (where H is again the function defined in (4.5)), then

$$\begin{aligned} &\int_\Omega H\Big(\frac{u_n}{h}\Big) j_\xi(x, u_n, \nabla u_n) \cdot \nabla v \\ &\quad = -\int_\Omega \Big[H\Big(\frac{u_n}{h}\Big) j_s(x, u_n, \nabla u_n) + H'\Big(\frac{u_n}{h}\Big) j_\xi(x, u_n, \nabla u_n) \cdot \frac{\nabla u_n}{h} \Big] v \\ &\qquad + \Big\langle w_n, H\Big(\frac{u_n}{h}\Big) v \Big\rangle. \end{aligned}$$

Let $w_n = -\text{div}\,(F_n)$, with (F_n) strongly convergent in $L^2(\Omega, \mathbb{R}^N)$. Then it follows that:

$$\begin{aligned} &\int_\Omega H\Big(\frac{u_n}{h}\Big) j_\xi(x, u_n, \nabla u_n) \cdot \nabla v \\ &\quad = \int_\Omega \Big[H'\Big(\frac{u_n}{h}\Big)(F_n - j_\xi(x, u_n, \nabla u_n)) \cdot \frac{\nabla u_n}{h} - H\Big(\frac{u_n}{h}\Big) j_s(x, u_n, \nabla u_n) \Big] v \\ &\qquad + \int_\Omega H\Big(\frac{u_n}{h}\Big) F_n \cdot \nabla v. \end{aligned}$$

Since the square bracket is bounded in $L^1(\Omega)$ and $\big(H(\frac{u_n}{h}) F_n\big)$ is strongly convergent in $L^2(\Omega, \mathbb{R}^N)$ we can apply [11, Theorem 5] with

$$b_n(x, \xi) = H\Big(\frac{u_n(x)}{h}\Big) j_\xi(x, u_n(x), \xi) \quad \text{and} \quad E = E_h = \{x \in \Omega : |u(x)| \leqslant h\}$$

and deduce (5.6) by the arbitrariness of $h \geqslant 1$. Notice that, by virtue of Theorem 4.8, for every n we have

$$\int_\Omega j_\xi(x, u_n, \nabla u_n) \cdot \nabla u_n + \int_\Omega j_s(x, u_n, \nabla u_n) u_n = \langle w_n, u_n \rangle.$$

Then, in view of (1.4), one has

$$\sup_{n \geqslant 1} \int_\Omega j_\xi(x, u_n, \nabla u_n) \cdot \nabla u_n < \infty. \tag{5.7}$$

Let now $k \geqslant 1$, $\varphi \in C_c^\infty(\Omega)$, $\varphi \geqslant 0$ and consider

$$v = \varphi e^{-M_k(u_n+R)^+} H\Big(\frac{u_n}{k}\Big), \quad \text{where } M_k = \frac{\beta(2k)}{\alpha_0}. \tag{5.8}$$

Note that $v \in V_{u_n}$ and

$$\nabla v = \nabla \varphi e^{-M_k(u_n+R)^+} H\Big(\frac{u_n}{k}\Big) - M_k \varphi e^{-M_k(u_n+R)^+} \nabla (u_n + R)^+ H\Big(\frac{u_n}{k}\Big) + \varphi e^{-M_k(u_n+R)^+} H'\Big(\frac{u_n}{k}\Big)\frac{\nabla u_n}{k}.$$

Taking v as test function in (5.1), we obtain

$$\begin{aligned}
&\int_\Omega j_\xi(x, u_n, \nabla u_n) \cdot e^{-M_k(u_n+R)^+} H\Big(\frac{u_n}{k}\Big)\nabla\varphi \\
&\quad + \int_\Omega [j_s(x, u_n, \nabla u_n) - M_k j_\xi(x, u_n, \nabla u_n) \cdot \nabla (u_n + R)^+]\varphi e^{-M_k(u_n+R)^+} H\Big(\frac{u_n}{k}\Big) \\
&\quad = \int_\Omega j_\xi(x, u_n, \nabla u_n) \cdot \varphi e^{-M_k(u_n+R)^+} H'\Big(\frac{u_n}{k}\Big)\frac{\nabla u_n}{k} \\
&\qquad + \Big\langle w_n, \varphi e^{-M_k(u_n+R)^+} H\Big(\frac{u_n}{k}\Big)\Big\rangle.
\end{aligned} \tag{5.9}$$

Observe that

$$[j_s(x, u_n, \nabla u_n) - M_k j_\xi(x, u_n, \nabla u_n) \cdot \nabla (u_n + R)^+]\varphi e^{-M_k(u_n+R)^+} H\Big(\frac{u_n}{k}\Big) \leqslant 0.$$

Indeed, the assertion follows from (1.4), for almost every x such that $u_n(x) \leqslant -R$ while, for almost every x in $\{x : -R \leqslant u_n(x) \leqslant 2k\}$ from (1.5), (4.3) and (5.8) we get

$$[j_s(x, u_n, \nabla u_n) - M_k j_\xi(x, u_n, \nabla u_n) \cdot \nabla (u_n + R)^+] \leqslant (\beta(2k) - \alpha_0 M_k)|\nabla u_n|^2 \leqslant 0.$$

Moreover, from (1.5), (5.2), (5.5) and (5.6) we have

$$\int_\Omega j_\xi(x, u_n, \nabla u_n) \cdot e^{-M_k(u_n+R)^+} H\Big(\frac{u_n}{k}\Big)\nabla\varphi \to \int_\Omega j_\xi(x, u, \nabla u) \cdot e^{-M_k(u+R)^+} H\Big(\frac{u}{k}\Big)\nabla\varphi,$$

$$\Big\langle w_n, \varphi e^{-M_k(u_n+R)^+} H\Big(\frac{u_n}{k}\Big)\Big\rangle \to \Big\langle w, \varphi e^{-M_k(u+R)^+} H\Big(\frac{u}{k}\Big)\Big\rangle$$

as $n \to \infty$. Moreover, we take into account (5.7) and deduce that there exists a positive constant C such that

$$\left| \int_\Omega j_\xi(x, u_n, \nabla u_n) \cdot \varphi e^{-M_k(u_n+R)^+} H'\Big(\frac{u_n}{k}\Big)\frac{\nabla u_n}{k} \right| \leqslant \frac{C}{k}.$$

We take the superior limit in (5.9) and we apply Fatou Lemma to obtain

$$\int_\Omega j_\xi(x,u,\nabla u)\cdot e^{-M_k(u+R)^+}H\Big(\frac{u}{k}\Big)\nabla\varphi+\int_\Omega j_s(x,u,\nabla u)\varphi e^{-M_k(u+R)^+}H\Big(\frac{u}{k}\Big)$$

$$-\ M_k\int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla u^+\varphi e^{-M_k(u+R)^+}H\Big(\frac{u}{k}\Big)$$

$$\geqslant-\frac{C}{k}+\Big\langle w,\varphi e^{-M_k(u+R)^+}H\Big(\frac{u}{k}\Big)\Big\rangle \tag{5.10}$$

for every $\varphi\in C_c^\infty(\Omega)$ with $\varphi\geqslant 0$. Then, the previous inequality holds for every $\varphi\in H_0^1\cap L^\infty(\Omega)$ with $\varphi\geqslant 0$. We now choose in (5.10) the admissible test function

$$\varphi=e^{M_k(u+R)^+}\psi,\quad \psi\in V_u,\ \ \psi\geqslant 0.$$

It results

$$\int_\Omega j_\xi(x,u,\nabla u)\cdot H\Big(\frac{u}{k}\Big)\nabla\psi+\int_\Omega j_s(x,u,\nabla u)H\Big(\frac{u}{k}\Big)\psi$$

$$\geqslant-\frac{C}{k}+\Big\langle w,H\Big(\frac{u}{k}\Big)\psi\Big\rangle. \tag{5.11}$$

Notice that

$$\Big|j_\xi(x,u,\nabla u)\cdot H\Big(\frac{u}{k}\Big)\nabla\psi\Big|\leqslant|j_\xi(x,u,\nabla u)|\,|\nabla\psi|,$$

$$\Big|j_s(x,u,\nabla u)H\Big(\frac{u}{k}\Big)\psi\Big|\leqslant|j_s(x,u,\nabla u)\psi|.$$

Since $\psi\in V_u$ and from (1.3) and (1.5) we deduce that we can pass to the limit in (5.11) as $k\to\infty$, and we obtain

$$\int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla\psi+\int_\Omega j_s(x,u,\nabla u)\psi\geqslant\langle w,\psi\rangle\quad\forall\psi\in V_u,\psi\geqslant 0.$$

In order to show the opposite inequality, we can take $v=\varphi e^{-M_k(u_n-R)^-}H\big(\frac{u_n}{k}\big)$ as test function in (5.1) and we can repeat the same argument as before. Thus, (5.4) follows.

Step 2: In this step we will prove that $u_n\to u$ strongly in $H_0^1(\Omega)$. From (4.3), (5.7) and Fatou Lemma, we have

$$0\leqslant\int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla u\leqslant\liminf_n\int_\Omega j_\xi(x,u_n,\nabla u_n)\cdot\nabla u_n<\infty$$

so that $j_\xi(x,u,\nabla u)\cdot\nabla u\in L^1(\Omega)$. Therefore, by Theorem 4.8 we deduce

$$\int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla u+\int_\Omega j_s(x,u,\nabla u)u=\langle w,u\rangle. \tag{5.12}$$

In order to prove that u_n converges to u strongly in $H_0^1(\Omega)$ we follow the argument of [20, Theorem 3.2] and we consider the function $\zeta:\mathbb{R}\to\mathbb{R}$ defined by

$$\zeta(s)=\begin{cases} Ms & \text{if } 0<s<R, \quad M=\frac{\beta(R)}{\alpha_0},\\ MR & \text{if } s\geqslant R,\\ -Ms & \text{if } -R<s<0,\\ MR & \text{if } s\leqslant -R.\end{cases} \tag{5.13}$$

We have that $v_n=u_n e^{\zeta(u_n)}$ belongs to $H_0^1(\Omega)$, and conditions (1.3)–(1.5) imply that hypotheses of Theorem 4.8 are satisfied. Then, we can use v_n as test function in (5.1). It results

$$\int_\Omega j_\xi(x,u_n,\nabla u_n)\cdot\nabla u_n e^{\zeta(u_n)} = \langle w_n, v_n\rangle - \int_\Omega [j_s(x,u_n,\nabla u_n)+j_\xi(x,u_n,\nabla u_n)\cdot\nabla u_n\zeta'(u_n)]v_n.$$

Note that v_n converges to $ue^{\zeta(u)}$ weakly in $H_0^1(\Omega)$ and almost everywhere in Ω. Moreover, conditions (1.3), (1.4) and (5.13) allow us to apply Fatou Lemma and get that

$$\limsup_h \int_\Omega j_\xi(x,u_n,\nabla u_n)\cdot\nabla u_n e^{\zeta(u_n)} \leqslant \langle w, ue^{\zeta(u)}\rangle - \int_\Omega [j_s(x,u,\nabla u)+j_\xi(x,u,\nabla u)\cdot\nabla u\zeta'(u)]ue^{\zeta(u)}. \tag{5.14}$$

On the other hand (5.12) and (5.13) imply that

$$\begin{cases} j_\xi(x,u,\nabla u)\cdot\nabla[ue^{\zeta(u)}]+j_s(x,u,\nabla u)ue^{\zeta(u)}\in L^1(\Omega),\\ j_\xi(x,u,\nabla u)\cdot\nabla[ue^{\zeta(u)}]\in L^1(\Omega).\end{cases} \tag{5.15}$$

Therefore, from Theorem 4.8, there holds

$$\int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla[ue^{\zeta(u)}]+\int_\Omega j_s(x,u,\nabla u)ue^{\zeta(u)} = \langle w, ue^{\zeta(u)}\rangle. \tag{5.16}$$

Thus, (5.14) and (5.16) imply that

$$\begin{aligned}\int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla u e^{\zeta(u)} &\leqslant \liminf_{n\to\infty}\int_\Omega j_\xi(x,u_n,\nabla u_n)\cdot\nabla u_n e^{\zeta(u_n)}\\ &\leqslant \limsup_{n\to\infty}\int_\Omega j_\xi(x,u_n,\nabla u_n)\cdot\nabla u_n e^{\zeta(u_n)}\\ &\leqslant \int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla u e^{\zeta(u)}.\end{aligned}$$

Then (4.3) implies that $u_n\to u$ strongly in $H_0^1(\Omega)$. $\square$

6. Proofs of Theorems 2.1 and 2.3

In this section we give the definition of a Concrete Palais–Smale sequence, we study the relation between a Palais–Smale sequence and a Concrete Palais–Smale sequence, and we prove that f satisfies the $(PS)_c$ for every $c \in \mathbb{R}$. Finally, we conclude by giving the proofs of Theorems 2.1 and 2.3.

Let us consider the functional $I : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$I(v) = -\int_\Omega G(x, v) - \langle \Lambda, v \rangle,$$

where $\Lambda \in H^{-1}(\Omega)$, $G(x,s) = \int_0^s g(x,t)dt$ and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying assumption (2.2). Then (1.2) implies that the functional $f : H_0^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$ defined by $f(v) = J(v) + I(v)$ is lower semicontinuous.

In order to apply the abstract theory, it is crucial the following

Theorem 6.1. *Assume conditions* (1.1), (1.2), (1.4),(2.2). *Then, for every* $(u, \eta) \in \mathrm{epi}(f)$ *with* $f(u) < \eta$, *it results*

$$|d\mathscr{G}_f|(u, \eta) = 1.$$

Moreover, if $j(x, -s, -\xi) = j(x, s, \xi)$, $g(x, -s) = -g(x, s)$ *and* $\Lambda = 0$, *for every* $\eta > f(0)$ *one has* $|d_{\mathbb{Z}_2}\mathscr{G}_f|(0, \eta) = 1$.

Proof. Since G is of class C^1, Theorem 3.11 and Proposition 3.7 imply the result. □

Furthermore, since G a C^1 functional, as a consequence of Proposition 4.5 one has the following

Proposition 6.2. *Assume conditions* (1.2), (1.3), (1.5), (2.2) *and consider* $u \in \mathrm{dom}(f)$ *with* $|df|(u) < \infty$. *Then there exists* $w \in H^{-1}(\Omega)$ *such that* $\|w\|_{-1,2} \leqslant |df|(u)$ *and*

$$\forall v \in V_u : \int_\Omega j_\xi(x, u, \nabla u) \cdot \nabla v + \int_\Omega j_s(x, u, \nabla u)v - \int_\Omega g(x, u)v - \langle \Lambda, v \rangle = \langle w, v \rangle.$$

Proof. Given $u \in \mathrm{dom}(f)$ with $|df|(u) < \infty$, let

$$\widehat{J}(v) = J(v) - \int_\Omega g(x, u)v - \langle \Lambda, v \rangle,$$

$$\widehat{I}(v) = I(v) + \int_\Omega g(x, u)v + \langle \Lambda, v \rangle.$$

Then, since $\widehat{I}$ is of class C^1 with $\widehat{I}'(u) = 0$, by (c) of Proposition 3.7 we get $|df|(u) = |d\widehat{J}|(u)$. By Proposition 4.5, there exists $w \in H^{-1}(\Omega)$ with $\|w\|_{-1,2} \leqslant |df|(u)$ and

$$\forall v \in V_u : \int_\Omega j_\xi(x,u,\nabla u) \cdot \nabla v + \int_\Omega j_s(x,u,\nabla u)v - \int_\Omega g(x,u)v - \langle \Lambda, v \rangle = \langle w, v \rangle$$

and the assertion is proved. □

We can now give the definition of the Concrete Palais–Smale condition.

Definition 6.3. Let $c \in \mathbb{R}$. We say that $\{u_n\}$ is a Concrete Palais–Smale sequence for f at level c ($(CPS)_c$-sequence for short) if there exists $w_n \in H^{-1}(\Omega)$ with $w_n \to 0$ such that $j_\xi(x,u_n,\nabla u_n) \cdot \nabla u_n \in L^1(\Omega)$ for every $n \geqslant 1$, and

$$f(u_n) \to c, \tag{6.1}$$

$$\int_\Omega j_\xi(x,u_n,\nabla u_n) \cdot \nabla v + \int_\Omega j_s(x,u_n,\nabla u_n)v - \int_\Omega g(x,u_n)v - \langle \Lambda, v \rangle = \langle w_n, v \rangle, \quad \forall v \in V_{u_n}. \tag{6.2}$$

We say that f satisfies the Concrete Palais–Smale condition at level c ($(CPS)_c$ for short) if every $(CPS)_c$-sequence for f admits a strongly convergent subsequence in $H_0^1(\Omega)$.

Proposition 6.4. *Assume conditions* (1.2)–(1.5), (2.2). *If* $u \in \mathrm{dom}(f)$ *satisfies* $|df|(u) = 0$, *then* u *is a generalized solution to*

$$\begin{cases} -\mathrm{div}\,(j_\xi(x,u,\nabla u)) + j_s(x,u,\nabla u) = g(x,u) + \Lambda & \textit{in } \Omega, \\ u = 0 & \textit{on } \partial\Omega. \end{cases}$$

Proof. It is sufficient to combine Lemma 4.6, Proposition 6.2, and Theorem 4.8. □

The following result concerns the relation between the $(\mathrm{PS})_c$ condition and the $(\mathrm{CPS})_c$ condition.

Proposition 6.5. *Assume conditions* (1.2)–(1.5), (2.2). *Then if* f *satisfies the* $(CPS)_c$ *condition, it satisfies the* $(PS)_c$ *condition.*

Proof. Let $\{u_n\} \subset \mathrm{dom}(f)$ that satisfies the Definition 3.4. From Lemma 4.6 and Proposition 6.2 we get that u_n satisfies the conditions in Definition 6.3. Thus, there exists a subsequence, which converges in $H_0^1(\Omega)$. □

We now want to prove that f satisfies the $(\mathrm{CPS})_c$ condition at every level c. In order to do this, let us consider a $(\mathrm{CPS})_c$-sequence $\{u_n\}\in\mathrm{dom}(f)$.

From Theorem 5.1 we deduce the following

Proposition 6.6. *Assume that conditions* (1.1)–(1.4), (2.2) *are satisfied. Let* $\{u_n\}$ *be a* $(CPS)_c$*-sequence for* f*, bounded in* $H_0^1(\Omega)$*. Then* $\{u_n\}$ *admits a strongly convergent subsequence in* $H_0^1(\Omega)$.

Proof. Let $\{u_n\}\subset\mathrm{dom}(f)$ be a concrete Palais–Smale sequence for f at level c. Taking into account that, as known, by (2.2) the map $\{u\mapsto g(x,u)\}$ is compact from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$, it suffices to apply Theorem 5.1 to see that $\{u_n\}$ is strongly compact in $H_0^1(\Omega)$. □

Proposition 6.7. *Assume conditions* (1.1)–(1.4), (1.9), (2.2),(2.3). *Then every* $(CPS)_c$*-sequence* $\{u_n\}$ *for* f *is bounded in* $H_0^1(\Omega)$.

Proof. Condition (1.4) and (4.3) allow us to apply Theorem 4.8 to deduce that we may choose $v=u_n$ as test functions in (6.2). Taking into account conditions (1.9), (2.2), (2.7), (6.1), the boundedness of $\{u_n\}$ in $H_0^1(\Omega)$ follows by arguing as in [20, Lemma 4.3]. □

Remark 6.8. Notice that we use condition (1.9) only in Proposition 6.7.

We can now state the following

Theorem 6.9. *Assume conditions* (1.1)–(1.4), (1.9), (2.2), (2.3). *Then the functional* f *satisfies the* $(PS)_c$ *condition at every level* $c\in\mathbb{R}$.

Proof. Let $\{u_n\}\subset\mathrm{dom}(f)$ be a concrete Palais–Smale sequence for f at level c. From Proposition 6.7 it follows that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. By Proposition 6.6 f satisfies the Concrete Palais–Smale condition. Finally Proposition 6.5 implies that f satisfies the $(\mathrm{PS})_c$ condition. □

We are now able to prove Theorem 2.1.

Proof of Theorem 2.1. We will prove Theorem 2.1 as a consequence of Theorem 3.10. First, note that (1.2) and (2.2) imply that f is lower semicontinuous. Moreover, from (2.5) we deduce that f is an even functional, and from Theorem 3.11 we deduce that (3.3) and condition (d) of Theorem 3.10 are satisfied. Hypotheses (2.4) implies that condition (b) of Theorem 3.10 is verified (see the subsequent proof of Theorem 2.3). Let now (λ_h,φ_h) be the sequence of solutions of $-\Delta u=\lambda u$ with homogeneous Dirichlet boundary conditions. Moreover, let us consider $V^+=\overline{\mathrm{span}}\{\varphi_h\in H_0^1(\Omega):h\geqslant h_0\}$ and note that V^+ has finite codimension. In order to prove (a) of Theorem 3.10 it is enough to show that there exist $h_0,\gamma>0$ such that for all $u\in V^+$ with $||\nabla u||_2=1$ there holds $f(u)\geqslant\gamma$. First, note that condition (2.2) implies that, for every $\varepsilon>0$, we find $a_\varepsilon^{(1)}\in C_c^\infty(\Omega)$ and $a_\varepsilon^{(2)}\in L^{2N/(N+2)}(\Omega)$ with $||a_\varepsilon^{(2)}||_{2N/(N+2)}\leqslant\varepsilon$ and

$$|g(x,s)|\leqslant a_\varepsilon^{(1)}(x)+a_\varepsilon^{(2)}(x)+\varepsilon|s|^{\frac{N+2}{N-2}}.$$

Now, let $u\in V^+$ and notice that there exist two positive constants c_1,c_2 such that

$$\begin{aligned}
f(u)&\geqslant \alpha_0||\nabla u||_2^2-\int_\Omega G(x,u)\\
&\geqslant \alpha_0||\nabla u||_2^2-\int_\Omega\left((a_\varepsilon^{(1)}+a_\varepsilon^{(2)})|u|+\frac{N-2}{2N}\varepsilon|u|^{\frac{2N}{N-2}}\right)\\
&\geqslant \alpha_0||\nabla u||_2^2-||a_\varepsilon^{(1)}||_2||u||_2-c_1||a_\varepsilon^{(2)}||_{\frac{2N}{N+2}}||\nabla u||_2-\varepsilon c_2||\nabla u||_2^{\frac{2N}{N-2}}\\
&\geqslant \alpha_0||\nabla u||_2^2-||a_\varepsilon^{(1)}||_2||u||_2-c_1\varepsilon||\nabla u||_2-\varepsilon c_2||\nabla u||_2^{\frac{2N}{N-2}}.
\end{aligned}$$

Then if h_0 is sufficiently large, since $\lambda_h\to+\infty$, for all $u\in V^+$, $||\nabla u||_2=1$ implies $||a_\varepsilon^{(1)}||_2||u||_2\leqslant\alpha_0/2$. Thus, for $\varepsilon>0$ small enough, $||\nabla u||_2=1$ implies $f(u)\geqslant\gamma$ for some $\gamma>0$. Then also (a) of Theorem 3.10 is satisfied. Theorem 6.9 implies that f satisfies $(PS)_c$ condition at every level c, so that we get the existence of a sequence of critical points $\{u_h\}\subset H_0^1(\Omega)$ with $f(u_h)\to+\infty$. Proposition 6.4 yields the assertion. □

Let us conclude this section by proving Theorem 2.3.

Proof of Theorem 2.3. We will prove Theorem 2.3 as a consequence of Theorem 3.9. In order to do this, let us notice that, from (1.2) and (2.8), f is lower semicontinuous on $H_0^1(\Omega)$. Moreover, Theorem 3.11 implies that condition (3.3) is satisfied. From Theorem 6.9 we deduce that f satisfies $(PS)_c$ condition at every level c. It is left to show that f satisfies the geometrical assumptions of Theorem 3.9.

Let us first consider the case in which $\Lambda=0$. Notice that conditions (1.2), (2.8) and (2.9) imply that there exist $\gamma>0$ and $r>0$ such that for $||u||_{1,2}=r$ there holds $f(u)\geqslant\gamma$. Conditions (1.2) and (2.4) imply that there holds

$$f(v)\leqslant\int_\Omega\alpha(|v|)|\nabla v|^2-\int_\Omega k(x)|v|^p+||\bar{a}||_1+C_0||\bar{b}||_{\frac{2N}{N+2}}||v||_{1,2}. \tag{6.3}$$

Now, let us consider a finite-dimensional subspace W of $H_0^1(\Omega)$ such that $W\subset L^\infty(\Omega)$. Condition (1.7) implies that, for every $\varepsilon>0$, there exists $R>r$, $w\in W$, with $||w||_\infty>R$ and a positive constant C_ε such that

$$\int_\Omega\alpha(|w|)|\nabla w|^2\leqslant\varepsilon C_W||w||_{1,2}^p+C_\varepsilon||w||_{1,2}^2, \tag{6.4}$$

where C_W is a positive constant depending on W. Then, by suitably choosing ε, (6.3) and (6.4) yield condition (3.4) for a suitable $v_1 \in H_0^1(\Omega)$ and for $v_0 = 0$. Thus, we can apply Theorem 3.9 and deduce the existence of a nontrivial critical point u of f. From Proposition 6.4, u is a generalized solution of Problem (P).

Now, let us consider the case in which $\Lambda \neq 0$. Let φ_1 be the first eigenfunction of the Laplace operator with homogeneous Dirichlet boundary conditions and set $v_0 = t_0\varphi_1$ for $t_0 > 0$. Then, if t_0 sufficiently small, thanks to (1.2) and (2.8), we get $f(v_0) < 0$. As before, (1.2), (2.8) and (2.9) imply that there exist $\varepsilon > 0$, $r = r(\varepsilon) > 0$ and $\gamma > 0$ such that, for every $\Lambda \in H^{-1}(\Omega)$ with $||\Lambda||_{-1} < \varepsilon$, there holds

$$f(u) \geqslant \gamma \quad \text{for every } u \text{ with } ||u - v_0||_{1,2} = r.$$

Moreover, we use condition (1.2), (1.7) and (2.4) and we argue as before to deduce the existence of $v_1 \in H_0^1(\Omega)$ with $||v_1 - v_0|| > r$ and $f(v_1) < 0$. Condition (3.4) is thus fulfilled. Then, we can apply Theorem 3.9 getting the existence of two distinct nontrivial critical points of f. Finally, Proposition 6.4 yields the conclusion. □

Remark 6.10. Notice that Theorems 1.2 and 1.3 are an easy consequence of Theorems 2.1 and 2.3, respectively. Indeed, consider for example $g_1(x,s) = a(x)\mathrm{arctg}\, s + |s|^{p-2}s$. In order to prove Theorem 1.2, it is left to show that $g_1(x,s)$ satisfies conditions (2.2), (2.3) and (2.4). First, notice that Young inequality implies that, for every $\varepsilon > 0$, there exists a positive constant $\beta(\varepsilon)$ such that (2.2) holds with $a_\varepsilon(x) = \beta(\varepsilon) + a(x)$. Moreover, (2.3) is satisfied with $a_0(x) = 0$ and $b_0(x) = \pi/2(p-1)$. Finally, (2.4) is verified with $k(x) = 1/p$, $\bar{a}(x) = 0$ and $\bar{b}(x) = (\pi/2 + C)a(x)$ where $C \in \mathbb{R}^+$ is sufficiently large. Theorem 1.3 can be obtained as a consequence of Theorem 2.3 in a similar fashion.

7. Summability results

In this section, we suppose that $g(x,s)$ satisfies the following growth condition

$$|g(x,s)| \leqslant a(x) + b|s|^{\frac{N+2}{N-2}}, \quad a(x) \in L^r(\Omega), \quad b \in \mathbb{R}^+. \tag{7.1}$$

Note that (2.2) implies (7.1). Let us set $2^* = 2N/(N-2)$. We prove the following:

Theorem 7.1. *Assume conditions* (1.1)–(1.4), (7.1). *Let* $u \in H_0^1(\Omega)$ *be a generalized solution of problem* (P). *Then the following conclusions hold*:

(a) *if* $r \in (2N/(N+2), N/2)$, *then* u *belongs to* $L^{r^{**}}(\Omega)$, *where* $r^{**} = Nr/(N-2r)$,
(b) *if* $r > N/2$, *then* u *belongs to* $L^\infty(\Omega)$.

Theorem 7.1 will be proved as a consequence of the following

Lemma 7.2. *Let us assume that conditions* (1.1)–(1.4) *are satisfied. Let* $u \in H_0^1(\Omega)$ *be a generalized solution of the problem*

$$\begin{cases} -\mathrm{div}\,(j_\xi(x,u,\nabla u)) + j_s(x,u,\nabla u) + c(x)u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{7.2}$$

Then the following conclusions hold:

(i) *if* $c \in L^{\frac{N}{2}}(\Omega)$ *and* $f \in L^r(\Omega)$, *with* $r \in (2N/(N+2), N/2)$, *then u belongs to* $L^{r^{**}}(\Omega)$, *where* $r^{**} = Nr/(N-2r)$,

(ii) *if* $c \in L^t(\Omega)$ *with* $t > N/2$ *and* $f \in L^q(\Omega)$, *with* $q > N/2$, *then u belongs to* $L^\infty(\Omega)$.

Proof. Let us first prove conclusion (i). For every $k > R$ (where R is defined in (1.4)), let us define the function $\eta_k(s) : \mathbb{R} \to \mathbb{R}$ such that $\eta_k \in C^1$, η_k is odd and

$$\eta_k(s) = \begin{cases} 0 & \text{if } 0 < s < R, \\ (s-R)^{2\gamma+1} & \text{if } R < s < k, \\ b_k s + c_k & \text{if } s > k, \end{cases} \tag{7.3}$$

where b_k and c_k are constant such that η_k is C^1. Since u is a generalized solution of (7.2), $v = \eta_k(u)$ belongs to W_u. Then we can take it as test function, moreover, $j_s(x,u,\nabla u)\eta_k(u) \geqslant 0$. Then from (1.4) and (4.3) we get

$$\alpha_0 \int_\Omega \eta_k'(u)|\nabla u|^2 \leqslant \int_\Omega f(x)\eta_k(u) - \int_\Omega c(x)u\eta_k(u). \tag{7.4}$$

Now, let us consider the odd function $\psi_k(s) : \mathbb{R} \to \mathbb{R}$ defined by

$$\psi_k(s) = \int_0^s \sqrt{\eta_k'(t)}\, dt. \tag{7.5}$$

The following properties of the functions ψ_k and η_k can be deduced from (7.3) and (7.5) by easy calculations

$$[\psi_k{}'(s)]^2 = \eta_k'(s), \tag{7.6}$$

$$0 \leqslant \eta_k(s)(s-R) \leqslant C_0 \psi_k(s)^2, \tag{7.7}$$

$$|\eta_k(s)| \leqslant C_0 |\psi_k(s)|^{\frac{2\gamma+1}{\gamma+1}}, \tag{7.8}$$

where C_0 is a positive constant. Notice that for every $\varepsilon > 0$ there exist $c_1(x) \in L^{\frac{N}{2}}(\Omega)$, with $\|c_1\|_{\frac{N}{2}} \leqslant \varepsilon$ and $c_2 \in L^\infty(\Omega)$ such that $c(x) = c_1(x) + c_2(x)$. From (7.4), (7.6), (7.7) and Hölder inequality, we deduce

$$\alpha_0 \int_\Omega |\nabla(\psi_k(u))|^2 \leqslant C_0 ||c_1(x)||_{\frac{N}{2}} \left[\int_\Omega |\psi_k(u)|^{2^*}\right]^{\frac{2}{2^*}} + \int_\Omega |f(x) - Rc_1(x) - c_2(x)u|\, |\eta_k(u)|.$$

We fix $\varepsilon = (\alpha_0 \mathscr{S})/(2C_0)$, where $\mathscr{S}$ is the Sobolev constant. We obtain

$$\left[\int_\Omega |\psi_k(u)|^{2^*}\right]^{\frac{2}{2^*}} \leqslant C \int_\Omega |f(x) - Rc_1(x) - c_2(x)u|\, |\eta_k(u)|. \tag{7.9}$$

Now, let us define the function

$$h(x) = |f(x) - Rc_1(x) - c_2(x)u(x)| \tag{7.10}$$

and note that $h(x)$ belongs to $L^t(\Omega)$ with

$$t = \min\{r, 2^*\}. \tag{7.11}$$

Let us consider first the case in which $t = r$, then from (7.8) and (7.9), we get

$$\left[\int_\Omega |\psi_k(u)|^{2^*}\right]^{\frac{2}{2^*}} \leqslant C||h||_r \left[\int_\Omega |\psi_k(u)|^{r'\frac{2\gamma+1}{\gamma+1}}\right]^{\frac{1}{r'}}.$$

Since $2N/(N+2) < r < N/2$ we can define $\gamma \in \mathbb{R}^+$ by

$$\gamma = \frac{r(N+2) - 2N}{2(N-2r)} \Rightarrow 2^*(\gamma+1) = r'(2\gamma+1) = r^{**}. \tag{7.12}$$

Moreover, since $r < N/2$ we have that $2/2^* > 1/r'$, then

$$\left[\int_\Omega |\psi_k(u)|^{2^*}\right]^{\frac{2}{2^*} - \frac{1}{r'}} \leqslant C||h||_r. \tag{7.13}$$

Notice that $|\psi_k(u)| \to C(\gamma)|u - R|^{\gamma+1}\chi_{\{x\,:\,|u(x)|>R\}}$ almost everywhere in Ω. Then Fatou Lemma implies that $|u - R|^{\gamma+1}\chi_{\{x\,:\,|u(x)|>R\}}$ belongs to $L^{2^*}(\Omega)$. Thus, u belongs to $L^{2^*(\gamma+1)}(\Omega) = L^{r^{**}}(\Omega)$ and the conclusion follows. Consider now the case in which $t = 2^*$ and note that this implies that $N > 6$. In this case we get

$$\left[\int_\Omega |\psi_k(u)|^{2^*}\right]^{\frac{2}{2^*}} \leqslant C||h||_{2^*} \left[\int_\Omega s|\psi_k(u)|^{(2^*)'\frac{2\gamma+1}{\gamma+1}}\right]^{\frac{1}{(2^*)'}}.$$

Since $N > 6$ it results $2/2^* > 1/(2^*)'$. Moreover, we can choose γ such that

$$2^*(\gamma+1) = (2^*)'(2\gamma+1).$$

Thus, we follow the same argument as in the previous case and we deduce that u belongs to $L^{s_1}(\Omega)$ where

$$s_1 = \frac{2^* N}{N - 2\,2^*}.$$

If it still holds $s_1 < r$ we can repeat the same argument to gain more summability on u. In this way for every $s \in [2^*, r)$ we can define the increasing sequence

$$s_0 = 2^*, \quad s_{n+1} = \frac{N s_n}{N - 2 s_n}$$

and we deduce that there exists $\bar{n}$ such that $s_{\bar{n}-1} < r$ and $s_{\bar{n}} \geqslant r$. At this step from (7.11) we get that $t = r$ and then $u \in L^{r^{**}}(\Omega)$, that is the maximal summability we can achieve.

Now, let us prove conclusion (ii). First, note that since $f \in L^q(\Omega)$, with $q > N/2$, f belongs to $L^r(\Omega)$ for every $r > (2N)/(N+2)$. Then, conclusion (i) implies that $u \in L^\sigma(\Omega)$ for every $\sigma > 1$. Now, take $\delta > 0$ such that $t - \delta > N/2$, since $u \in L^{\frac{t}{\delta}}(\Omega)$ it results

$$\int_\Omega |c(x)u(x)|^{t-\delta} \leqslant ||c(x)||_t^{t-\delta} \left[\int_\Omega |u(x)|^{\frac{t}{\delta}} \right]^{\frac{\delta}{t}} < \infty\,.$$

Then, the function $d(x) = f(x) - c(x)u(x)$ belongs to $L^r(\Omega)$ with $r = \min\{q, t - \delta\} > N/2$. Let us take $k > R$ (R is defined in (1.4)) and consider the function $v = G_k(u) = u - T_k(u)$ (where $T_k(s)$ is defined in (3.5)). Since u is a generalized solution of (7.2) we can take v as test function. From (1.4) and (4.3) it results

$$\alpha_0 \int_\Omega |\nabla G_k(u)|^2 \leqslant \int_\Omega |d(x)|\,|G_k(u)|.$$

The conclusion follows from Theorem 4.2 of [21]. □

Remark 7.3. In classical results of this type (see e.g. [18] or [6]) it is usually considered as test function $v = |u|^{2\gamma} u$. Note that this type of function cannot be used here for it does not belong to the space W_u. Moreover, the classical truncation T_u seems not to be useful because of the presence of $c(x)u$. Then, we have chosen a suitable truncation of u in order to manage also the term $c(x)u$.

Now we are able to prove Theorem 7.1.

Proof of Theorem 7.1. Theorem 7.1 will be proved as a consequence of Lemma 7.2. So, consider u a generalized solution of Problem (P), we have to prove that u is a generalized solution of Problem (7.2) for suitable $f(x)$ and $c(x)$. This is shown in Theorem 2.2.5 of [9], then we will give here a sketch of the proof of [9] just for clearness. We set

$$g_0(x,s) = \min\{\max\{g(x,s), -a(x)\}, a(x)\},$$

$$g_1(x,s) = g(x,s) - g_0(x,s).$$

It follows that $g(x,s) = g_0(x,s) + g_1(x,s)$ and $|g_0(x,s)| \leqslant a(x)$ so that we can set $f(x) = g_0(x, u(x))$. Moreover, we define

$$c(x) = \begin{cases} -\dfrac{g_1(x,u(x))}{u(x)} & \text{if } u(x) \neq 0, \\ 0 & \text{if } u(x) = 0. \end{cases}$$

Then $|c(x)| \leqslant b|u(x)|^{\frac{4}{N-2}}$, so that $c(x) \in L^{\frac{N}{2}}(\Omega)$. Lemma 7.2 implies that conclusion (a) holds. Now, if $r > N/2$ we have that $f(x) \in L^r(\Omega)$ with $r > N/2$. Moreover, conclusion (a) implies that $u \in L^t(\Omega)$ for every $t < \infty$, so that $c(x) \in L^t(\Omega)$ with $t > N/2$. Then Lemma 7.2 implies that $u \in L^\infty(\Omega)$. □

Remark 7.4. When dealing with quasilinear equations (i.e. $j(x,s,\xi) = a(x,s)\xi \cdot \xi$), a standard technique, to prove summability results, is to reduce the problem to the linear one and to apply the classical result (see e.g. [21]). Note that here this is not possible due to the general form of j.

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