Asymptotic behavior of solutions for a class of predator–prey reaction–diffusion systems with time delays

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Received 12 July 2005
Available online 12 June 2006
Submitted by C.V. Pao

Abstract

The aim of this paper is to investigate the asymptotic behavior of solutions for a class of three-species predator–prey reaction–diffusion systems with time delays under homogeneous Neumann boundary condition. Some simple and easily verifiable conditions are given to the rate constants of the reaction functions to ensure the convergence of the time-dependent solution to a constant steady-state solution. The conditions for the convergence are independent of diffusion coefficients and time delays, and the conclusions are directly applicable to the corresponding parabolic-ordinary differential system and to the corresponding system without time delays.

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Keywords: Reaction–diffusion system; Predator–prey model; Time delay; Asymptotic behavior; Upper and lower solutions
1. Introduction

Differential equations with time delays are traditionally formulated in the framework of ordinary differential systems. In recent years considerable attention has been given to parabolic systems with time delays, especially in relation to reaction–diffusion systems where the reaction functions depend on the unknown functions with time delays (see [1,2,4,5,8–24]). In this paper we investigate the asymptotic behavior of solutions for a class of three-species predator–prey reaction–diffusion systems with time delays, especially in relation to reaction–diffusion systems where the reaction functions depend on the unknown functions with time delays (see [1,2,4,5,8–24]). In this paper we investigate the asymptotic behavior of solutions for a class of three-species predator–prey reaction–diffusion systems with time delays in a bounded domain \( \Omega \) in \( \mathbb{R}^n \) under Neumann boundary condition. The system under the consideration is given in the form

\[
\begin{align*}
\frac{\partial u_1}{\partial t} &= d_1 \Delta u_1 + u_1(a_1 - b_{11} u_1 - b_{12} u_2(x,t - \tau_2)), \\
\frac{\partial u_2}{\partial t} &= d_2 \Delta u_2 + u_2(-a_2 + b_{21} u_1(x,t - \tau_1) - b_{22} u_2 - b_{23} u_3(x,t - \tau_3)), \\
\frac{\partial u_3}{\partial t} &= d_3 \Delta u_3 + u_3(-a_3 + b_{32} u_2(x,t - \tau_2) - b_{33} u_3), \quad (x,t) \in \Omega \times (0, \infty), \\
\frac{\partial u_i}{\partial t} &= \frac{\partial u_i}{\partial \nu} = \frac{\partial u_i}{\partial \nu} = 0, \quad (x,t) \in \partial \Omega \times (0, \infty), \\
u_i(x,t) &= \eta_i(x,t) \geq 0, \quad (x,t) \in \Omega \times [-\tau_i, 0] (i = 1, 2, 3),
\end{align*}
\]

where \( \Delta \) is the Laplace operator, \( \partial u_i/\partial \nu \) denotes the outward normal derivative of \( u_i \) on the boundary \( \partial \Omega \) of \( \Omega \), the constants \( a_i, b_{ij}, d_i \) and \( \tau_i \) satisfy \( a_i > 0, b_{ij} > 0, d_i \geq 0 \) and \( \tau_i > 0 \) for all \( i, j = 1, 2, 3 \), and the initial function \( \eta_i(x,t) \) is Hölder continuous on \( \overline{\Omega} \times [\tau_i, 0] \) (cf. [14]). The asymptotic behavior of the solution of (1.1) has been investigated in [2,10], and various sufficient conditions for the convergence of the solution to a constant steady-state solution are obtained. Clearly, a constant steady-state solution \( (c_1, c_2, c_3) \) of (1.1) is governed by

\[
\begin{align*}
c_1(a_1 - b_{11} c_1 - b_{12} c_2) &= 0, \\
_c_2(-a_2 + b_{21} c_1 - b_{22} c_2 - b_{23} c_3) &= 0, \\
_c_3(-a_3 + b_{32} c_2 - b_{33} c_3) &= 0.
\end{align*}
\]

It is obvious that the above system possesses the trivial nonnegative solution \((0,0,0)\) and the semitrivial nonnegative solution \((a_2/b_{12}, 0, 0)\). If \(a_1 b_{21} > a_2 b_{11}\), then it has also the semitrivial nonnegative solution \((\hat{c}_1, \hat{c}_2, 0)\) where

\[
\begin{align*}
\hat{c}_1 &= \frac{a_1 b_{22} + a_2 b_{12}}{b_{11} b_{22} + b_{12} b_{21}}, \quad \hat{c}_2 = \frac{a_1 b_{21} - a_2 b_{11}}{b_{11} b_{22} + b_{12} b_{21}},
\end{align*}
\]

and if

\[
a_1 b_{21} b_{32} > a_2 b_{11} b_{32} + a_3 (b_{11} b_{22} + b_{12} b_{21}),
\]
then it has the positive solution \((\tilde{c}_1, \tilde{c}_2, \tilde{c}_3)\) where

\[
\begin{align*}
\tilde{c}_1 &= \frac{a_1(b_{22}b_{33} + b_{23}b_{32}) + a_2b_{12}b_{33} - a_3b_{12}b_{23}}{b_{11}(b_{22}b_{33} + b_{23}b_{32}) + b_{12}b_{21}b_{33}}, \\
\tilde{c}_2 &= \frac{a_1b_{21}b_{33} + a_3b_{11}b_{23} - a_2b_{11}b_{33}}{b_{11}(b_{22}b_{33} + b_{23}b_{32}) + b_{12}b_{21}b_{33}}, \\
\tilde{c}_3 &= \frac{a_1b_{21}b_{32} - a_2b_{11}b_{32} - a_3(b_{11}b_{22} + b_{12}b_{21})}{b_{11}(b_{22}b_{33} + b_{23}b_{32}) + b_{12}b_{21}b_{33}}.
\end{align*}
\]

For a better clarity of the further presentation, let us summarize the main results of \([2,10]\) in the following theorem.

**Theorem A.** Let \((u_1(x, t), u_2(x, t), u_3(x, t))\) be the unique nonnegative solution of (1.1). Then the following results hold:

(a) If \(\eta_1(x, 0) \equiv 0\), then \((u_1(x, t), u_2(x, t), u_3(x, t))\) converges uniformly to \((0, 0, 0)\) as \(t \to \infty\).

(b) If \(\eta_2(x, 0) \equiv 0\) and \(\eta_1(x, 0) \not\equiv 0\), then \((u_1(x, t), u_2(x, t), u_3(x, t))\) converges uniformly to \((a_1/b_{11}, 1, 0)\) as \(t \to \infty\).

(c) If \(\eta_3(x, 0) \equiv 0\), \(\eta_1(x, 0) \not\equiv 0\), \(\eta_2(x, 0) \not\equiv 0\) and \(a_1b_{21} \leq a_2b_{11}\), then \((u_1(x, t), u_2(x, t), u_3(x, t))\) converges uniformly to \((a_1/b_{11}, 0, 0)\) as \(t \to \infty\).

(d) If \(\eta_3(x, 0) \equiv 0\), \(\eta_1(x, 0) \not\equiv 0\), \(\eta_2(x, 0) \not\equiv 0\) and

\[
a_1b_{21} > a_2b_{11}, \quad b_{11}b_{22} > b_{12}b_{21},
\]

then \((u_1(x, t), u_2(x, t), u_3(x, t))\) converges uniformly to \((\hat{c}_1, \hat{c}_2, 0)\) as \(t \to \infty\) provided the initial functions \(\eta_1(x, t)\) and \(\eta_2(x, t)\) satisfy

\[
\hat{c}_i - \alpha_i \leq \eta_i(x, t) \leq \hat{c}_i + \beta_i \quad \text{on } \overline{\Omega} \times [-\tau_i, 0], \quad i = 1, 2,
\]

where the positive constants \(\alpha_i\) and \(\beta_i\) satisfy

\[
0 < \alpha_i < \hat{c}_i \quad (i = 1, 2), \quad b_{12}\alpha_2 < b_{11}\beta_1, \quad b_{12}\beta_2 < b_{11}\alpha_1, \quad b_{21}\beta_1 < b_{22}\beta_2, \quad b_{21}\alpha_1 < b_{22}\alpha_2. \quad (1.8)
\]

(e) If \(\eta_i(x, 0) \not\equiv 0\) \( (i = 1, 2, 3)\) and \(a_1b_{21} \leq a_2b_{11}\), then \((u_1(x, t), u_2(x, t), u_3(x, t))\) converges uniformly to \((a_1/b_{11}, 0, 0)\) as \(t \to \infty\).

(f) If \(\eta_i(x, 0) \not\equiv 0\) \( (i = 1, 2, 3)\), condition (1.4) holds and

\[
b_{11}b_{22}b_{33} > b_{12}b_{21}b_{33} + b_{11}b_{23}b_{32}, \quad (1.9)
\]

then \((u_1(x, t), u_2(x, t), u_3(x, t))\) converges uniformly to \((\tilde{c}_1, \tilde{c}_2, \tilde{c}_3)\) as \(t \to \infty\) provided the initial functions \(\eta_1(x, t), \eta_2(x, t)\) and \(\eta_3(x, t)\) satisfy

\[
\tilde{c}_i - \alpha_i \leq \eta_i(x, t) \leq \tilde{c}_i + \beta_i \quad \text{on } \overline{\Omega} \times [-\tau_i, 0], \quad i = 1, 2, 3,
\]

where the positive constants \(\alpha_i\) and \(\beta_i\) satisfy

\[
0 < \alpha_i < \tilde{c}_i \quad (i = 1, 2, 3), \quad b_{12}\alpha_2 < b_{11}\beta_1, \quad b_{12}\beta_2 < b_{11}\alpha_1, \quad b_{32}\beta_2 < b_{33}\beta_3, \quad b_{32}\alpha_2 < b_{33}\alpha_3, \quad b_{21}\beta_1 + b_{23}\alpha_3 < b_{22}\beta_2, \quad b_{21}\alpha_1 + b_{23}\beta_3 < b_{22}\alpha_2. \quad (1.11)
\]
The conclusions (a)–(c) and (e) in Theorem A are given in [10], while the conclusions (d) and (f) are presented in [2]. As pointed out in [2], the conclusions (d) and (f) improve the corresponding conclusions in [10] by enlarging the ranges of the initial functions \( \eta_i(x, t) \) \( (i = 1, 2, 3) \). However, we see that in the conclusions (d) and (f) of Theorem A there is still a limitation of the initial functions. In this paper, we give a further investigation to the asymptotic behavior of the nonnegative solution of (1.1) for the cases (d) and (f) in Theorem A, and obtain the better results. Our specific goal is to give some simple and easily verifiable conditions on the rate constants so that

(i) for any nonnegative initial function \((\eta_1(x, t), \eta_2(x, t), \eta_3(x, t)) \) with \( \eta_3(x, 0) \equiv 0 \) and \( \eta_i(x, 0) \neq 0 \) \( (i = 1, 2) \), the corresponding nonnegative solution \((u_1(x, t), u_2(x, t), u_3(x, t)) \) of (1.1) converges uniformly to \((\hat{c}_1, \hat{c}_2, 0) \) as \( t \to \infty \), and

(ii) for any nonnegative initial function \((\eta_1(x, t), \eta_2(x, t), \eta_3(x, t)) \) with \( \eta_i(x, 0) \neq 0 \) \( (i = 1, 2, 3) \), the corresponding nonnegative solution \((u_1(x, t), u_2(x, t), u_3(x, t)) \) of (1.1) converges uniformly to \((\hat{c}_1, \hat{c}_2, 0) \) or \((\hat{c}_1, \hat{c}_2, \hat{c}_3) \) as \( t \to \infty \).

Clearly, the above conclusions improve the conclusions (d) and (f) in Theorem A by removing the limitation of the initial functions. Our conditions for the above conclusions involve only the rate constants, and are independent of the diffusion coefficients and time delays. This property leads to the same conclusions for the corresponding parabolic-ordinary differential system \((d_i = 0 \) for some or all \( i) \) and the corresponding system without time delays. To achieve the above goal we use the method of upper and lower solutions and its associated monotone iterations, together with some comparisons of scalar reaction–diffusion equations.

The outline of the paper is as follows. In Section 2, we state our main results. Section 3 is devoted to some preliminary results for a general reaction–diffusion system. In the final section, we give the proofs of the main results.

2. The main results

The main results of the paper are given as follows.

Theorem 2.1. Assume that condition (1.6) is satisfied. Then for any nonnegative initial function \((\eta_1(x, t), \eta_2(x, t), \eta_3(x, t)) \) with \( \eta_3(x, 0) \equiv 0 \) and \( \eta_i(x, 0) \neq 0 \) \( (i = 1, 2) \), the corresponding nonnegative solution \((u_1(x, t), u_2(x, t), u_3(x, t)) \) of (1.1) converges uniformly to \((\hat{c}_1, \hat{c}_2, 0) \) as \( t \to \infty \).

Theorem 2.2. For any nonnegative initial function \((\eta_1(x, t), \eta_2(x, t), \eta_3(x, t)) \) with \( \eta_i(x, 0) \neq 0 \) \( (i = 1, 2, 3) \), we have that

(a) if (1.6) holds and

\[ a_1 b_{11} b_{32} < a_2 b_{11} b_{32} + a_3 b_{11} b_{22}, \]  

then the corresponding nonnegative solution \((u_1(x, t), u_2(x, t), u_3(x, t)) \) of (1.1) converges uniformly to \((\hat{c}_1, \hat{c}_2, 0) \) as \( t \to \infty \);

(b) if (1.9) holds and

\[ a_2 b_{11} b_{32} + a_3 b_{11} b_{22} \leq a_1 b_{21} b_{32} \leq a_2 b_{11} b_{32} + a_3 (b_{11} b_{22} + b_{12} b_{21}), \]  

(2.2)
then the corresponding nonnegative solution \((u_1(x,t), u_2(x,t), u_3(x,t))\) of (1.1) converges uniformly to \((\hat{c}_1, \hat{c}_2, 0)\) as \(t \to \infty\).

**Theorem 2.3.** Assume that condition (1.9) is satisfied, and
\[
\begin{align*}
a_1b_2b_{32} > a_2b_{11}b_{32} + a_3\left(b_{11}b_{22} + \frac{b_{12}b_{21}b_{11}b_{22}b_{33}}{b_{11}b_{22}b_{33} - b_{12}b_{21}b_{33} - b_{11}b_{23}b_{32}}\right). 
\end{align*}
\] (2.3)

Then for any nonnegative initial function \((\eta_1(x,t), \eta_2(x,t), \eta_3(x,t))\) with \(\eta_i(x,0) \neq 0\) \((i = 1, 2, 3)\), the corresponding nonnegative solution \((u_1(x,t), u_2(x,t), u_3(x,t))\) of (1.1) converges uniformly to \((\hat{c}_1, \hat{c}_2, \hat{c}_3)\) as \(t \to \infty\).

**Remark 2.1.** (a) Theorem 2.2 implies that under conditions (1.6) and (2.1) or conditions (1.9) and (2.2), the semitrivial constant steady-state solution \((\hat{c}_1, \hat{c}_2, 0)\) is globally asymptotically stable (with respect to nonnegative initial perturbations).

(b) Theorem 2.3 implies that under conditions (1.9) and (2.3), the constant steady-state solution \((\hat{c}_1, \hat{c}_2, \hat{c}_3)\) is globally asymptotically stable.

As two immediate consequences of Theorems 2.2 and 2.3 we have the following results.

**Corollary 2.1.** Under conditions (1.6) and (2.1) or conditions (1.9) and (2.2), problem (1.1) has no positive steady-state solution.

**Corollary 2.2.** Under conditions (1.9) and (2.3), problem (1.1) has a unique positive steady-state solution \((\hat{c}_1, \hat{c}_2, \hat{c}_3)\).

It is seen that the conditions for the above conclusions involve only the rate constants, and are independent of the diffusion coefficients and time delays. This means that all these conclusions are directly applicable to the corresponding parabolic-ordinary differential system \((d_i = 0\) for some or all \(i\)) and the corresponding system without time delays. In particular, for the corresponding ordinary differential system of (1.1) in the form
\[
\begin{align*}
du_1/dt &= u_1(a_1 - b_{11}u_1 - b_{12}u_2(t - \tau_2)), \\
du_2/dt &= u_2(-a_2 + b_{21}u_1(t - \tau_1) - b_{22}u_2 - b_{23}u_3(t - \tau_3)), \\
du_3/dt &= u_3(-a_3 + b_{32}u_2(t - \tau_2) - b_{33}u_3), \quad t \in (0, \infty), \\
u_i(t) &= \eta_i(t) \geq 0, \quad t \in [-\tau_i, 0] \quad (i = 1, 2, 3),
\end{align*}
\] (2.4)

we have the following results.

**Theorem 2.4.** Let \((u_1(t), u_2(t), u_3(t))\) be the unique nonnegative solution of (2.4). We have

(a) if \(\eta_1(0) = 0\), then \((u_1(t), u_2(t), u_3(t))\) converges to \((0, 0, 0)\) as \(t \to \infty\);

(b) if \(\eta_2(0) = 0\) and \(\eta_1(0) > 0\), then \((u_1(t), u_2(t), u_3(t))\) converges to \((a_1/b_{11}, 0, 0)\) as \(t \to \infty\);

(c) if \(\eta_3(0) = 0\), \(\eta_1(0) > 0\), \(\eta_2(0) > 0\) and \(a_1b_{21} \leq a_2b_{11}\), then \((u_1(t), u_2(t), u_3(t))\) converges to \((a_1/b_{11}, 0, 0)\) as \(t \to \infty\);

(d) if \(\eta_3(0) = 0\), \(\eta_1(0) > 0\), \(\eta_2(0) > 0\) and condition (1.6) holds, then \((u_1(t), u_2(t), u_3(t))\) converges to \((\hat{c}_1, \hat{c}_2, 0)\) as \(t \to \infty\);
(e) if \( \eta_i(0) > 0 \) \((i = 1, 2, 3) \) and \( a_1 b_{21} \leq a_2 b_{11} \), then \( (u_1(t), u_2(t), u_3(t)) \) converges to \((a_1/b_{11}, 0, 0)\) as \( t \to \infty \);

(f) if \( \eta_i(0) > 0 \) \((i = 1, 2, 3) \) and either conditions \((1.6)\) and \((2.1)\) or conditions \((1.9)\) and \((2.2)\) hold, then \( (u_1(t), u_2(t), u_3(t)) \) converges to \((\hat{c}_1, \hat{c}_2, 0)\) as \( t \to \infty \);

(g) if \( \eta_i(0) > 0 \) \((i = 1, 2, 3) \) and conditions \((1.9)\) and \((2.3)\) hold, then \( (u_1(t), u_2(t), u_3(t)) \) converges to \((\hat{c}_1, \hat{c}_2, \hat{c}_3)\) as \( t \to \infty \).

The above conclusions hold true for system \((2.4)\) without time delays.

3. Preliminary results for a general system

In this section, we present some preliminary results for the following more general system:

\[
\begin{aligned}
\dot{u}_i / \partial t &= d_i \Delta u_i + f_i(u, u_t), \\
\frac{\partial u_i}{\partial v} &= 0, \\
u_i(x, t) &= \eta_i(x, t) \geq 0,
\end{aligned}
\tag{3.1}
\]

where \( u = (u_1, u_2, \ldots, u_N) \) and \( u_t = ((u_1)_t, (u_2)_t, \ldots, (u_N)_t) \) with \( (u_i)_t = u_i(x, t - \tau_i) \) and \( \tau_i > 0 \). For each \( i = 1, 2, \ldots, N \), \( d_i \geq 0 \) and \( f_i(u, v) \) is a \( C^1 \)-function of \( u \) and \( v \) in a suitable subset \( \Lambda \) of \( \mathbb{R}^N \) (see \(3.5)\)). In the above system we allow \( d_i = 0 \) (and without the corresponding boundary condition) for some or all \( i \). This implies that problem \((3.1)\) includes the corresponding ordinary differential system which has been investigated by many investigators (cf. \[3,6,7,21\] and references therein).

By writing the vectors \( u, v \) in the split form

\[
u = (u_i, [u]_{\mu_i}, [u]_{\mu_i'}), \quad v = ([v_i]_{\nu_i}, [v]_{\nu_i'}),
\tag{3.2}
\]

where \( \mu_i, \mu_i', \nu_i \) and \( \nu_i' \) are nonnegative integers satisfying

\[
\mu_i + \mu_i' = N - 1, \quad \nu_i + \nu_i' = N, \quad i = 1, 2, \ldots, N,
\tag{3.3}
\]

and \([w]_\sigma\) denotes a vector with \( \sigma \) number of components of \( w \), we write

\[
f_i(u, v) = f_i(u_i, [u]_{\mu_i}, [u]_{\mu_i'}, [v]_{\nu_i}, [v]_{\nu_i'}), \quad i = 1, 2, \ldots, N.
\tag{3.4}
\]

Motivated by system \((1.1)\) we assume that for each \( i = 1, 2, \ldots, N \), there exist nonnegative integers \( \mu_i, \mu_i', \nu_i \) and \( \nu_i' \) satisfying \((3.3)\) such that the function \( f_i(u_i, [u]_{\mu_i}, [u]_{\mu_i'}, [v]_{\nu_i}, [v]_{\nu_i'}) \) is monotone nondecreasing in \([u]_{\mu_i}\) and \([v]_{\nu_i}\), and is monotone nonincreasing in \([u]_{\mu_i'}\) and \([v]_{\nu_i'}\) for all \( u, v \) in a suitable subset \( \Lambda \) of \( \mathbb{R}^N \). The subset \( \Lambda \) is taken as

\[
\Lambda = \{ u \in \mathbb{R}^N; \; \underline{c} \leq u \leq \overline{c} \},
\tag{3.5}
\]

where \( \overline{c} = (\overline{c}_1, \ldots, \overline{c}_N) \) and \( \underline{c} = (\underline{c}_1, \ldots, \underline{c}_N) \) are a pair of constant vectors satisfying \( \overline{c} \geq \underline{c} \geq 0 \) and

\[
f_i((\overline{c}_i - \underline{c}_i), [\underline{c}]_{\mu_i}, [\overline{c}]_{\mu_i'}, [\underline{c}]_{\nu_i}, [\overline{c}]_{\nu_i'}) \leq 0 \leq f_i((\overline{c}_i - \underline{c}_i), [\overline{c}]_{\mu_i}, [\overline{c}]_{\mu_i'}, [\overline{c}]_{\nu_i}, [\overline{c}]_{\nu_i'}), \quad i = 1, 2, \ldots, N.
\tag{3.6}
\]

Let \( M_i \) be any positive constant satisfying

\[
M_i \geq \max \left\{ -\frac{\partial f_i}{\partial u_i}(u, v); \; \underline{c} \leq u, v \leq \overline{c} \right\}, \quad i = 1, 2, \ldots, N.
\tag{3.7}
\]
Starting from \( \tilde{c}^{(0)} = \tilde{c} \) and \( \xi^{(0)} = \xi \) we construct two sequences of constant vectors \( \{\tilde{c}^{(m)}\} = \{(\tilde{c}_1^{(m)}, \tilde{c}_2^{(m)}, \ldots, \tilde{c}_N^{(m)})\} \) and \( \{\xi^{(m)}\} = \{(\xi_1^{(m)}, \xi_2^{(m)}, \ldots, \xi_N^{(m)})\} \) from the recursion relation
\[
\begin{align*}
\tilde{c}_i^{(m)} &= \tilde{c}_i^{(m-1)} + f_i(\tilde{c}_i^{(m-1)}, [\xi_i^{(m-1)}]_{\mu_i}, [\tilde{c}_i^{(m-1)}]_{\mu_i'}, [\xi_i^{(m-1)}]_{v_i}, [\tilde{c}_i^{(m-1)}]_{v_i'})/M_i, \\
\tilde{c}_i^{(m)} &= \tilde{c}_i^{(m-1)} + f_i(\xi_i^{(m-1)}, [\xi_i^{(m-1)}]_{\mu_i}, [\tilde{c}_i^{(m-1)}]_{\mu_i'}, [\xi_i^{(m-1)}]_{v_i}, [\tilde{c}_i^{(m-1)}]_{v_i'})/M_i, 
\end{align*}
\tag{3.8}
\]
where \( m = 1, 2, \ldots \).

The following lemma from [17] shows that these two sequences converge monotonically to the respective limits \( \tilde{c}^* = (\tilde{c}_1^*, \tilde{c}_2^*, \ldots, \tilde{c}_N^*) \) and \( \xi^* = (\xi_1^*, \xi_2^*, \ldots, \xi_N^*) \) that satisfy the equations
\[
\begin{align*}
f_i(\tilde{c}_i^*, [\xi_i^*]_{\mu_i}, [\xi_i^*]_{\mu_i'}, [\tilde{c}_i^*]_{v_i}, [\xi_i^*]_{v_i'}) &= 0, \\
f_i(\xi_i^*, [\xi_i^*]_{\mu_i}, [\xi_i^*]_{\mu_i'}, [\xi_i^*]_{v_i}, [\tilde{c}_i^*]_{v_i'}) &= 0, 
\end{align*}
\tag{3.9}
\]
\[ \text{Lemma 3.1.} \]
Let \( \tilde{c} = (\tilde{c}_1, \ldots, \tilde{c}_N), \ \xi = (\xi_1, \ldots, \xi_N) \) be a pair of constant vectors satisfying \( \tilde{c} \geq \xi \geq 0 \) and relation (3.6). Then the sequences \( \{\tilde{c}^{(m)}\} \) and \( \{\xi^{(m)}\} \) given by (3.8) with \( \tilde{c}^{(0)} = \tilde{c} \) and \( \xi^{(0)} = \xi \) converge monotonically to the respective limits \( \tilde{c}^* = (\tilde{c}_1^*, \tilde{c}_2^*, \ldots, \tilde{c}_N^*) \) and \( \xi^* = (\xi_1^*, \xi_2^*, \ldots, \xi_N^*) \) that satisfy the equation in (3.9). Moreover,
\[ \xi \leq c^{(m)} \leq c^{(m+1)} \leq c^* \leq \xi^{(m+1)} \leq c^{(m)} \leq \tilde{c}, \quad m = 0, 1, 2, \ldots \tag{3.10} \]

It is known that if \( \tilde{u} = (\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_N) \) and \( u = (u_1, u_2, \ldots, u_N) \) are a pair of coupled upper and lower solutions of (3.1) then system (3.1) has a unique solution \( u = u(x,t) \) such that \( u \leq \tilde{u} \leq u \) on \( \Omega \times [0, \infty) \) (cf. [16]). For the definition of coupled upper and lower solutions we refer to [14–17]. We observe that the constant pair \( \tilde{c}, \xi \) satisfying \( \tilde{c} \geq \xi \geq 0 \) and relation (3.6) are coupled upper and lower solutions of (3.1) whenever \( \xi \leq (\eta_1(x,t), \eta_2(x,t), \ldots, \eta_N(x,t)) \leq \tilde{c} \). By an application of Theorem 4.2 in [15] (also see [16,17]) we have the following result.

\[ \text{Theorem 3.1.} \]
Let the conditions in Lemma 3.1 hold, and let \( \tilde{c}^* \) and \( c^* \) be the limits in Lemma 3.1. Also let \( u(x,t) = (u_1(x,t), \ldots, \nu_N(x,t)) \) be the solution of (3.1) with an arbitrary nonnegative initial function \( (\eta_1(x,t), \ldots, \eta_N(x,t)) \). If \( \tilde{c}^* = c^* (= c^*) \) and there exists \( t^* \geq 0 \) such that
\[ c_i \leq u_i(x,t) \leq \tilde{c}_i, \quad (x,t) \in \Omega \times [t^* - \tau_i, t^*], \quad i = 1, 2, \ldots, N, \tag{3.11} \]
then \( u(x,t) \) converges uniformly to \( c^* \) as \( t \to \infty \). The above conclusions hold true when \( d_i = 0 \) for some or all \( i \).

4. Proofs of the theorems

We first introduce some lemmas. The proofs of these lemmas are straightforward by some comparisons of scalar reaction–diffusion equations, and we omitted them here.

\[ \text{Lemma 4.1.} \]
Let \( u(x,t) \) be a positive function on \( \Omega \times [t_0, \infty) \) such that
\[
\begin{align*}
\frac{\partial u}{\partial t} &\leq d \Delta u + u(a - bu), \quad (x,t) \in \Omega \times (t_0, \infty), \\
\partial u/\partial \nu &\leq 0, \quad (x,t) \in \partial \Omega \times (t_0, \infty),
\end{align*}
\tag{4.1}
\]
where \( a, b, d \) and \( t_0 \) are some constants with \( a > 0, b > 0, d \geq 0 \) and \( t_0 \geq 0 \). Then for arbitrary positive number \( \epsilon \), there exists a finite \( t^* > t_0 \) such that
Lemma 4.2. Let \( u(x,t) \) be a positive function on \( \overline{\Omega} \times [t_0, \infty) \) \((t_0 \geq 0)\) such that the reversed inequalities in (4.1) hold. Then for arbitrary positive number \( \epsilon \), there exists a finite \( t^* > t_0 \) such that
\[
u(x,t) \geq \frac{a}{b} - \epsilon, \quad (x,t) \in \overline{\Omega} \times [t^*, \infty).
\]

Lemma 4.3. Let \( u(x,t) \) be a positive function on \( \overline{\Omega} \times [t_0, \infty) \) such that
\[
\begin{align*}
\partial u / \partial t & \leq d \Delta u - au, \quad (x,t) \in \Omega \times (t_0, \infty), \\
\partial u / \partial v & \leq 0, \quad (x,t) \in \partial \Omega \times (t_0, \infty),
\end{align*}
\]
where the constants \( a, d \) and \( t_0 \) satisfy \( a > 0, d \geq 0 \) and \( t_0 \geq 0 \). Then for arbitrary positive number \( \epsilon \), there exists a finite \( t^* > t_0 \) such that
\[
u(x,t) \leq \epsilon, \quad (x,t) \in \overline{\Omega} \times [t^*, \infty).
\]

Proof of Theorem 2.1. Since \( \eta_3(x,0) \equiv 0 \) and \( \eta_i(x,0) \neq 0 \) for \( i = 1, 2 \), we have \( u_3(x,t) \equiv 0 \) and \( u_i(x,t) > 0 \) for \( i = 1, 2 \) on \( \overline{\Omega} \times (0, \infty) \) (cf. [14]). In this situation, system (1.1) is reduced to the following subsystem:
\[
\begin{align*}
\partial u_1 / \partial t &= d_1 \Delta u_1 + u_1(a_1 - b_{11}u_1 - b_{12}u_2(x,t-\tau_2)), \\
\partial u_2 / \partial t &= d_2 \Delta u_2 + u_2(-a_2 + b_{21}u_1(x,t-\tau_1) - b_{22}u_2), \\
\partial u_1 / \partial v &= \partial u_2 / \partial v = 0, \quad (x,t) \in \partial \Omega \times (0, \infty), \\
u_i(x,t) &= \eta_i(x,t) \geq 0, \quad (x,t) \in \Omega \times [-\tau_i, 0] \ (i = 1, 2).
\end{align*}
\]
Consider system (4.6) as a special case of (3.1) with \( N = 2 \) and \( f_i(u,v) \) given by
\[
f_1(u,v) = u_1(a_1 - b_{11}u_1 - b_{12}v_2), \quad f_2(u,v) = u_2(-a_2 + b_{21}v_1 - b_{22}u_2).
\]
In this situation, condition (3.6) for the constant pair \( \xi = (\xi_1, \xi_2) \) and \( \xi = (\xi_1, \xi_2) \) is reduced to
\[
\begin{align*}
\tilde{\xi}_1(a_1 - b_{11}\tilde{\xi}_1 - b_{12}\tilde{\xi}_2) & \leq 0 \leq \xi_1(a_1 - b_{11}c_1 - b_{12}c_2), \\
\tilde{\xi}_2(-a_2 + b_{21}\tilde{\xi}_1 - b_{22}\tilde{\xi}_2) & \leq 0 \leq \xi_2(-a_2 + b_{21}c_1 - b_{22}c_2).
\end{align*}
\]
We divide the proof into three parts.

PART I. We construct a pair constant vectors \( \xi = (\xi_1, \xi_2) \) and \( \xi = (\xi_1, \xi_2) \) such that \( \xi \geq \xi > 0 \) and condition (4.8) holds.

Choose a positive number \( \epsilon \) such that
\[
0 < \epsilon < \min \left\{ \frac{a_1(b_1 b_{22} - b_{12} b_{21}) + a_2 b_{11} b_{12}}{b_{11} (b_{12} b_{21} + b_{12} b_{22} + b_{11} b_{22})}, \frac{(a_1 b_{21} - a_2 b_{11}) (b_{11} b_{22} - b_{12} b_{21})}{b_{11} (b_{21} + b_{22}) (b_{11} b_{22} + b_{12} b_{21})} \right\},
\]
and define
\[
\begin{align*}
\tilde{\xi}_1 &= \frac{a_1}{b_{11}} + \epsilon, \\
\tilde{\xi}_2 &= \frac{a_1 b_{21} - a_2 b_{11}}{b_{11} b_{21}} + \frac{b_{21} + b_{22}}{b_{22}} \epsilon, \\
\xi_1 &= \frac{a_1 (b_{11} b_{22} - b_{12} b_{21}) + a_2 b_{11} b_{12}}{b_{11} (b_{12} b_{21} + b_{12} b_{22} + b_{11} b_{22})} - \frac{b_{12} b_{21} + b_{12} b_{22} + b_{11} b_{22}}{b_{11} b_{22}} \epsilon, \\
\xi_2 &= \frac{(a_1 b_{21} - a_2 b_{11}) (b_{11} b_{22} - b_{12} b_{21})}{b_{11} b_{22}^2} - \frac{b_{21} + b_{22}}{b_{11} b_{22}} \epsilon.
\end{align*}
\]
By condition (1.6), the above $\varepsilon$ is well defined. A simple calculation shows that the pair $\tilde{c} = (\tilde{c}_1, \tilde{c}_2)$, $\varrho = (\varrho_1, \varrho_2)$ and

$$
\begin{align*}
\begin{cases}
a_1 - b_{11} \tilde{c}_1 = -b_{11} \varepsilon < 0, \\
a_1 - b_{11} \varrho_1 - b_{12} \tilde{c}_2 = b_{11} \varepsilon > 0,
\end{cases}
\end{align*}
$$

This implies that the above pair satisfy condition (4.8). In addition, we have from (4.10) that

$$
\begin{align*}
-a_2 + b_{21} \tilde{c}_1 &\geq -a_2 + b_{21} \varrho_1 > 0, \\
a_1 - b_{12} \tilde{c}_2 &> 0.
\end{align*}
$$

\text{PART II.} We prove that there exists a finite $t^* > 0$ such that

$$
\varrho_i \leq u_i(x, t) \leq \tilde{c}_i, \quad (x, t) \in \overline{\Omega} \times [t^* - \tau_1, t^*], \quad i = 1, 2,
$$

where $\tilde{c} = (\tilde{c}_1, \tilde{c}_2)$ and $\varrho = (\varrho_1, \varrho_2)$ are defined by (4.9).

In view of the positive property of $u_1(x, t)$ and $u_2(x, t)$ on $\overline{\Omega} \times (0, \infty)$, we have from (4.6) that for some finite $t_0 > 0$,

$$
\begin{align*}
\begin{cases}
\partial u_1/\partial t \leq d_1 \Delta u_1 + u_1(a_1 - b_{11} u_1), \\
\partial u_1/\partial \nu = 0,
\end{cases} \quad (x, t) \in \Omega \times (t_0, \infty),
\end{align*}
$$

By Lemma 4.1, there exists a finite $t_1 > t_0$ such that

$$
u_1(x, t) \leq \frac{a_1}{b_{11}} + \varepsilon = \tilde{c}_1, \quad (x, t) \in \overline{\Omega} \times [t_1, \infty).$$

Using this estimate we have from (4.6) that

$$
\begin{align*}
\begin{cases}
\partial u_2/\partial t \leq d_2 \Delta u_2 + u_2(-a_2 + b_{21} \tilde{c}_1 - b_{22} u_2), \\
\partial u_2/\partial \nu = 0,
\end{cases} \quad (x, t) \in \Omega \times (t_1 + \tau_1, \infty),
\end{align*}
$$

Since $-a_2 + b_{21} \tilde{c}_1 > 0$ (see (4.11)), we have from Lemma 4.1 that there exists a finite $t_2 > t_1 + \tau_1 > 0$ such that

$$
u_2(x, t) \leq \frac{-a_2 + b_{21} \tilde{c}_1}{b_{22}} + \varepsilon = \tilde{c}_2, \quad (x, t) \in \overline{\Omega} \times [t_2, \infty).$$

By this bound and (4.6) we obtain

$$
\begin{align*}
\begin{cases}
\partial u_1/\partial t \geq d_1 \Delta u_1 + u_1(a_1 - b_{12} \tilde{c}_2 - b_{11} u_1), \\
\partial u_1/\partial \nu = 0,
\end{cases} \quad (x, t) \in \overline{\Omega} \times (t_2 + \tau_2, \infty),
\end{align*}
$$

Since $a_1 - b_{12} \tilde{c}_2 > 0$ (see (4.11)), an application of Lemma 4.2 gives that there exists a finite $t_3 > t_2 + \tau_2 > 0$ such that

$$
u_1(x, t) \geq \frac{a_1 - b_{12} \tilde{c}_2}{b_{11}} - \varepsilon = \varrho_1, \quad (x, t) \in \overline{\Omega} \times [t_3, \infty).$$

This estimate and (4.6) imply that

$$
\begin{align*}
\begin{cases}
\partial u_2/\partial t \geq d_2 \Delta u_2 + u_2(-a_2 + b_{21} \varrho_1 - b_{22} u_2), \\
\partial u_2/\partial \nu = 0,
\end{cases} \quad (x, t) \in \overline{\Omega} \times (t_3 + \tau_1, \infty),
\end{align*}
$$

By $-a_2 + b_{21} \varrho_1 > 0$ (see (4.11)) and Lemma 4.2, we obtain that there exists a finite $t_4 > t_3 + \tau_1 > 0$ such that

$$
u_2(x, t) \geq \frac{-a_2 + b_{21} \varrho_1}{b_{22}} - \varepsilon = \varrho_2, \quad (x, t) \in \overline{\Omega} \times [t_4, \infty).$$
Define $t^* = t_4 + \max_i \tau_i$. Then $t^* - \tau_i \geq t_4$ for all $i = 1, 2$. By (4.14), (4.16), (4.18) and (4.20), we conclude (4.12).

**PART III.** We prove that the nonnegative solution $(u_1(x, t), u_2(x, t))$ of (4.6) converges uniformly to $(\hat{c}_1, \hat{c}_2)$ as $t \to \infty$.

By Lemma 3.1, the sequences $\{\tilde{c}^{(m)}\} = \{(\tilde{c}_1^{(m)}, \tilde{c}_2^{(m)})\}$ and $\{\hat{c}^{(m)}\} = \{((\hat{c}_1^{(m)}, \hat{c}_2^{(m)})\}$ defined by (3.8) with $N = 2$, $(\tilde{e}^{(0)}, \tilde{e}^{(0)}) = (\tilde{e}, \tilde{e})$ defined by (4.9) and $(f_1, f_2)$ given by (4.7) converge monotonically to the limits $\tilde{c}^* = (\tilde{c}_1^*, \tilde{c}_2^*)$ and $\hat{c}^* = (\hat{c}_1^*, \hat{c}_2^*)$ that satisfy $\tilde{c}^* \geq \hat{c}^* \geq \hat{e} > 0$ and the equations

\[
\begin{cases}
  a_1 - b_{11} \tilde{c}_1^* - b_{12} \tilde{c}_2^* = 0, & a_1 - b_{11} \hat{c}_1^* - b_{12} \hat{c}_2^* = 0, \\
  -a_2 + b_{21} \tilde{c}_1^* - b_{22} \tilde{c}_2^* = 0, & -a_2 + b_{21} \hat{c}_1^* - b_{22} \hat{c}_2^* = 0
\end{cases}
\]  

(4.21)

(see (3.9)). Solving the above system leads to $(\tilde{c}_1^*, \tilde{c}_2^*) = (\hat{c}_1^*, \hat{c}_2^*)$. Finally, using the results in Parts I and II we have from Theorem 3.1 that the nonnegative solution $(u_1(x, t), u_2(x, t))$ of (4.6) converges uniformly to $(\hat{c}_1, \hat{c}_2)$ as $t \to \infty$. □

**Proof of Theorem 2.2.** The proof follows from the similar argument as that in the proof of Theorem 2.1 and we give a sketch. Since $\eta_i(x, 0) \neq 0$, we have $u_i(x, t) > 0$ on $\Omega \times (0, \infty)$ ($i = 1, 2, 3$) (cf. [14]). Consider system (1.1) as a special case of (3.1) with $N = 3$ and $f_i(u, v)$ given by

\[
\begin{align*}
  f_1(u, v) &= u_1(a_1 - b_{11} u_1 - b_{12} v_2), & f_2(u, v) &= u_2(-a_2 + b_{21} v_1 - b_{22} u_2 - b_{23} v_3), \\
  f_3(u, v) &= u_3(-a_3 + b_{32} v_2 - b_{33} u_3).
\end{align*}
\]  

(4.22)

In this case, condition (3.6) for the constant pair $\tilde{c} = (\tilde{c}_1, \tilde{c}_2, \tilde{c}_3)$, $\hat{c} = (\hat{c}_1, \hat{c}_2, \hat{c}_3)$ is reduced to

\[
\begin{align*}
  \tilde{c}_1(a_1 - b_{11} \tilde{c}_1 - b_{12} \tilde{c}_2) &\leq 0 \leq \tilde{c}_1(a_1 - b_{11} \tilde{c}_1 - b_{12} \hat{c}_2), \\
  \tilde{c}_2(-a_2 + b_{21} \tilde{c}_1 - b_{22} \tilde{c}_2 - b_{23} \tilde{c}_3) &\leq 0 \leq \tilde{c}_2(-a_2 + b_{21} \hat{c}_1 - b_{22} \hat{c}_2 - b_{23} \hat{c}_3), \\
  \tilde{c}_3(-a_3 + b_{32} \tilde{c}_2 - b_{33} \tilde{c}_3) &\leq 0 \leq \tilde{c}_3(-a_3 + b_{32} \hat{c}_2 - b_{33} \hat{c}_3).
\end{align*}
\]  

(4.23)

**Proof of (a).** Define

\[
\begin{align*}
  \Sigma_0 &= \frac{a_1(b_{11} b_{22} - b_{12} b_{21}) + a_2 b_{11} b_{12}}{b_{11}(b_{12} b_{21} + b_{12} b_{22} + b_{11} b_{22})}, \\
  \Sigma_1 &= \frac{(a_1 b_{21} - a_2 b_{11})(b_{11} b_{22} - b_{12} b_{21})}{b_{11}(b_{21} + b_{22})(b_{11} b_{22} + b_{11} b_{22} + b_{12} b_{23})}, \\
  \Sigma_2 &= \frac{a_2 b_{11} b_{32} + a_3 b_{11} b_{22} - a_1 b_{21} b_{32}}{b_{11} b_{32}(b_{21} + b_{22})},
\end{align*}
\]  

and choose a positive number $\varepsilon$ such that

\begin{align*}
  0 < \varepsilon < \min\{\Sigma_0, \Sigma_1, \Sigma_2\}.
\end{align*}

(4.25)

Conditions (1.6) and (2.1) ensure that the above $\varepsilon$ is well defined. Let
\[
\begin{aligned}
\tilde{c}_1 &= \frac{a_1}{b_{11}} + \varepsilon, \quad \tilde{c}_2 = \frac{a_1 b_{21} - a_2 b_{11}}{b_{11} b_{22}} + \frac{b_{21} + b_{22}}{b_{22}} \varepsilon, \quad \tilde{c}_3 = \varepsilon, \\
\xi_1 &= \frac{a_1 (b_{11} b_{22} - b_{12} b_{21}) + a_2 b_{11} b_{22}}{b_{11} b_{22}} - \frac{b_{12} b_{21} + b_{12} b_{22} + b_{11} b_{22}}{b_{12} b_{21}} \varepsilon, \\
\xi_2 &= \frac{(a_1 b_{21} - a_2 b_{11}) (b_{11} b_{22} - b_{12} b_{21})}{b_{11} b_{22}^2} - \frac{(b_{21} + b_{22}) (b_{11} b_{22} + b_{12} b_{21}) + b_{11} b_{22} b_{23}}{b_{12} b_{21}} \varepsilon, \\
\xi_3 &= 0.
\end{aligned}
\]

(4.26)

It follows from the choice of \( \varepsilon \) and conditions (1.6) and (2.1) that the pair \( \tilde{c} = (\tilde{c}_1, \tilde{c}_2, \tilde{c}_3) \) and \( \xi = (\xi_1, \xi_2, \xi_3) \) satisfy \( \tilde{c} \geq \xi \geq 0 \) and

\[
\begin{aligned}
a_1 - b_{11} \tilde{c}_1 &= -b_{11} \varepsilon < 0, \\
a_2 + b_{21} \tilde{c}_1 - b_{22} \tilde{c}_2 &= -b_{22} \varepsilon < 0, \\
a_3 + b_{32} \tilde{c}_2 < 0, \\
a_1 - b_{11} \xi_1 - b_{12} \xi_2 &= b_{11} \varepsilon > 0, \\
a_2 + b_{21} \xi_1 - b_{22} \xi_2 - b_{23} \xi_3 &= b_{22} \varepsilon > 0.
\end{aligned}
\]

(4.27)

Therefore, condition (4.23) is fulfilled by the above pair \( \tilde{c} \) and \( \xi \).

Using the similar argument as that in the proof of Theorem 2.1, we have that there exist a finite \( t_1 > 0 \) and a finite \( t_2 > t_1 + \tau_1 \) such that

\[
u_1(x,t) \leq \frac{a_1}{b_{11}} + \varepsilon = \tilde{c}_1, \quad (x,t) \in \bar{\Omega} \times [t_1, \infty),
\]

(4.28)

and

\[
u_2(x,t) \leq \frac{-a_2 + b_{21} \tilde{c}_1}{b_{22}} + \varepsilon = \tilde{c}_2, \quad (x,t) \in \bar{\Omega} \times [t_2, \infty).
\]

(4.29)

By estimate (4.29) and system (1.1), we obtain

\[
\begin{aligned}
\frac{\partial u_3}{\partial t} &\leq d_3 \Delta u_3 - (a_3 - b_{32} \tilde{c}_2) u_3, \quad (x,t) \in \Omega \times (t_2 + \tau_2, \infty), \\
\frac{\partial u_3}{\partial v} &= 0, \quad (x,t) \in \partial \Omega \times (t_2 + \tau_2, \infty).
\end{aligned}
\]

(4.30)

Since \( a_3 - b_{32} \tilde{c}_2 > 0 \) (see (4.27)), an application of Lemma 4.3 gives that there exists a finite \( t_3 > t_2 + \tau_2 > 0 \) such that

\[
u_3(x,t) \leq \varepsilon, \quad (x,t) \in \bar{\Omega} \times [t_3, \infty).
\]

(4.31)

Again by the argument used in the proof of Theorem 2.1, we obtain that there exist a finite \( t_4 > t_2 + \tau_2 \) and a finite \( t_5 > \max\{t_4 + \tau_1, t_3 + \tau_3\} \) such that

\[
u_1(x,t) \geq \frac{a_1 - b_{12} \tilde{c}_2}{b_{11}} - \varepsilon = \xi_1, \quad (x,t) \in \bar{\Omega} \times [t_4, \infty),
\]

(4.32)

and

\[
u_2(x,t) \geq \frac{-a_2 + b_{21} \xi_1 - b_{23} \tilde{c}_3}{b_{22}} - \varepsilon = \xi_2, \quad (x,t) \in \bar{\Omega} \times [t_5, \infty).
\]

(4.33)

Define \( t^* = t_5 + \max_i \tau_i \). Then we have

\[
\xi_i \leq \nu_i(x,t) \leq \tilde{c}_i, \quad (x,t) \in \bar{\Omega} \times [t^* - \tau_i, t^*], \quad i = 1, 2, 3.
\]

(4.34)

Using \( (\tilde{c}_1^{(0)}, \tilde{c}_2^{(0)}, \tilde{c}_3^{(0)}) = (\tilde{c}_1, \tilde{c}_2, \tilde{c}_3) \) and \( (\xi_1^{(0)}, \xi_2^{(0)}, \xi_3^{(0)}) = (\xi_1, \xi_2, \xi_3) \) in the iteration process (3.8), where \( N = 3 \) and \( (f_1, f_2, f_3) \) is given by (4.22), we obtain two sequences of constant vectors \( \{\tilde{c}^{(n)}\} = \{(\tilde{c}_1^{(m)}, \tilde{c}_2^{(m)}, \tilde{c}_3^{(m)})\} \) and \( \{\xi^{(n)}\} = \{(\xi_1^{(m)}, \xi_2^{(m)}, \xi_3^{(m)})\} \). By Lemma 3.1,
these two sequences converge monotonically to respective limits \((\bar{c}_1^*, \bar{c}_2^*, \bar{c}_3^*)\) and \((\bar{c}_1^*, \bar{c}_2^*, \bar{c}_3^*)\) that satisfy

\[
(\bar{c}_1, \bar{c}_2, \bar{c}_3) \geq (\bar{c}_1^*, \bar{c}_2^*, \bar{c}_3^*) \geq (\bar{c}_1, \bar{c}_2, \bar{c}_3)
\]

and the equations (see (3.9))

\[
\begin{align*}
\bar{c}_1^* (a_1 - b_{11} \bar{c}_1^* - b_{12} \bar{c}_2^*) &= 0, \\
\bar{c}_2^* (-a_2 + b_{21} \bar{c}_1^* - b_{22} \bar{c}_2^* - b_{23} \bar{c}_3^*) &= 0, \\
\bar{c}_3^* (-a_3 + b_{32} \bar{c}_2^* - b_{33} \bar{c}_3^*) &= 0,
\end{align*}
\]

(4.35)

Since \(c_1 > 0, c_2 > 0\) and \(c_3 = 0\), we have that \(\bar{c}_1^* \geq \bar{c}_1 > 0, \bar{c}_2^* \geq \bar{c}_2 > 0, \bar{c}_3^* = \bar{c}_3 = 0,\) and \(\bar{c}_3^{(m)} = 0\) for every \(m\) which implies \(\bar{c}_3^* = 0\). On the other hand, we have from (4.35) that \(\bar{c}_3^* = 0\)

because of \(-a_3 + b_{32} \bar{c}_2^* - b_{33} \bar{c}_3^* \leq -a_3 + b_{32} \bar{c}_2^* < 0\) (see (4.27)). Therefore, system (4.35) is reduced to (4.21). This implies \((\bar{c}_1^*, \bar{c}_2^*) = (\hat{c}_1, \hat{c}_2)\). Finally by an application of Theorem 3.1 the solution \((u_1(x, t), u_2(x, t), u_3(x, t))\) converges uniformly to \((\hat{c}_1, \hat{c}_2, 0)\) as \(t \to \infty\).

**Proof of (b).** Define

\[
\begin{align*}
\Sigma_3 &= b_{11} b_{22} b_{33} - b_{12} b_{21} b_{33} - b_{11} b_{23} b_{32}, \\
\Sigma_4 &= b_{11} b_{22} b_{33} + b_{12} b_{21} b_{33} + b_{11} b_{23} b_{32}, \\
\Sigma_5 &= a_1 (b_{11} b_{22} - b_{12} b_{21}) + a_2 b_{11} b_{12} b_{11} b_{22} + b_{11} b_{12} + b_{12} b_{11}, \\
\Sigma_6 &= (a_1 b_{21} - a_2 b_{11}) \Sigma_3 + a_3 b_{11}^2 b_{22} b_{23} b_{33}, \\
\end{align*}
\]

(4.36)

and choose a positive number \(\varepsilon\) such that

\[
0 < \varepsilon < \min\{\Sigma_5, \Sigma_6\}.
\]

(4.37)

Conditions (1.9) and (2.2) ensure that the above \(\varepsilon\) is well defined. Let

\[
\begin{align*}
\bar{c}_1 &= \frac{a_1}{b_{11}} + \varepsilon, \\
\bar{c}_2 &= \frac{a_1 b_{21} - a_2 b_{11}}{b_{11} b_{22}} + \frac{b_{21} + b_{22}}{b_{22}} \varepsilon, \\
\bar{c}_3 &= \frac{a_1 b_{21} b_{32} - a_2 b_{11} b_{32} - a_3 b_{11} b_{22}}{b_{11} b_{22} b_{33}} + \frac{b_{21} b_{32} + b_{22} b_{32} + b_{22} b_{33}}{b_{22} b_{33}} \varepsilon, \\
\bar{c}_1 &= \frac{a_1(b_{11} b_{22} - b_{12} b_{21}) + a_2 b_{11} b_{12}}{b_{11} b_{22}} + \frac{b_{12} b_{21} + b_{12} b_{22} + b_{11} b_{22}}{b_{11} b_{22}} \varepsilon, \\
\bar{c}_2 &= \frac{(a_1 b_{21} - a_2 b_{11}) \Sigma_3 + a_3 b_{11}^2 b_{22} b_{23} b_{33}}{b_{11} b_{22} b_{33}} - \frac{(b_{21} + b_{22}) \Sigma_4 + b_{11} b_{22} b_{33} b_{23}}{b_{11} b_{22} b_{33}} \varepsilon, \\
\bar{c}_3 &= 0.
\end{align*}
\]

(4.38)

It is easy to see from the choice of \(\varepsilon\) and conditions (1.9) and (2.2) that the pair \(\bar{c} = (\bar{c}_1, \bar{c}_2, \bar{c}_3), \bar{c} = (\bar{c}_1, \bar{c}_2, \bar{c}_3)\) satisfies \(\bar{c} \geq \bar{c} \geq 0\). By a similar argument as that in the proof of Theorem 2.1, we have that this pair satisfies condition (4.23), and there exists a finite \(t^* > 0\) such that

\[
\bar{c}_i \leq u_i(x, t) \leq \bar{c}_i, \quad (x, t) \in \Omega \times [t^* - \tau_i, t^*], \quad i = 1, 2, 3.
\]

(4.39)

Using \((\bar{c}_1, \bar{c}_2, \bar{c}_3) = (\bar{c}_1, \bar{c}_2, \bar{c}_3)\) and \((\bar{c}_1, \bar{c}_2, \bar{c}_3) = (\bar{c}_1, \bar{c}_2, \bar{c}_3)\) in the iteration process (3.8), where \(N = 3\) and \((f_1, f_2, f_3)\) is given by (4.22), we obtain two sequences of constant
vectors \( \{ \tilde{c}^{(m)} \} = \{ (c_1^{(m)}, c_2^{(m)}, c_3^{(m)}) \} \) and \( \{ c^{(m)} \} = \{ (c_1^{(m)}, c_2^{(m)}, c_3^{(m)}) \} \). Lemma 3.1 ensures that these two sequences converge monotonically to respective limits \((c_1^*, c_2^*, c_3^*)\) and \((c_1^*, c_2^*, c_3^*)\) that satisfy
\[
(c_1, c_2, c_3) \geq (c_1^*, c_2^*, c_3^*) \geq (c_1^*, c_2^*, c_3^*) > (c_1, c_2, c_3)
\]
and system (4.35). Since \(c_1 > 0, c_2 > 0\) and \(c_3 = 0\), we have that \(c_1^* \geq c_1 > 0, c_2^* \geq c_2 > 0, c_3^* \geq c_3^* \geq 0\), and \(c_3^{(m)} = 0\) for every \(m\) which gives \(c_3^* = 0\). If \(c_3^* > 0\) then system (4.35) is reduced to
\[
\begin{align*}
& a_1 - b_{11} c_1^* - b_{12} c_2^* = 0, \quad a_1 - b_{11} c_1^* - b_{12} c_2^* = 0, \\
& -a_2 + b_{21} c_1^* - b_{22} c_2^* = 0, \quad -a_2 + b_{21} c_1^* - b_{22} c_2^* - b_{23} c_3^* = 0, \\
& -a_3 + b_{32} c_2^* - b_{33} c_3^* = 0.
\end{align*}
\]
(4.40)
Solving the above system we have
\[
\tilde{c}_3^* = \frac{b_{32}(a_1 b_{21} - a_2 b_{11}) \Sigma_3 + a_3 b_{11}^2 b_{22}(b_{23} b_{32} - 2 b_{22} b_{33}) - a_1 b_{21} b_{32})}{b_{12} b_{21}(b_{12} b_{21} b_{33} + b_{11} b_{23} b_{32}) - b_{11}^2 b_{22}^2 b_{33}}.
\]
By conditions (1.9) and (2.2), \(c_3^* \leq 0\) which leads to a contradiction. This proves \(c_3^* = c_3^* = 0\), and so \((c_1^*, c_2^*, c_3^*)\) satisfies system (4.21), which implies \((c_1^*, c_2^*) = (c_1^*, c_2^*) = (\tilde{c}_1, \tilde{c}_2)\). Finally by an application of Theorem 3.1 the solution \((u_1(x, t), u_2(x, t), u_3(x, t))\) converges uniformly to \((\tilde{c}_1, \tilde{c}_2, 0)\) as \(t \to \infty\). \(\Box\)

**Proof of Theorem 2.3.** Define
\[
\Sigma_7 = \frac{b_{32}(a_1 b_{21} - a_2 b_{11}) \Sigma_3 + a_3 b_{11}^2 b_{22}(b_{23} b_{32} - 2 b_{22} b_{33})}{b_{11} b_{21}(b_{21} + b_{22}) \Sigma_4 + b_{11}^2 b_{22} b_{33}(b_{32} b_{32} + 2 b_{32} b_{33})},
\]
and choose a positive number \(\varepsilon\) such that
\[
0 < \varepsilon < \min\{ \Sigma_5, \Sigma_6, \Sigma_7 \},
\]
(4.42)
where \(\Sigma_3, \Sigma_4, \Sigma_5\) and \(\Sigma_6\) are defined by (4.36). Conditions (1.9) and (2.3) ensure that \(\varepsilon\) is well defined. Let \(\tilde{c} = (\tilde{c}_1, \tilde{c}_2, \tilde{c}_3)\) and \(\xi = (c_1, c_2, c_3)\) be given by (4.38) except with \(c_3\) being replaced by
\[
\xi_3 = \frac{b_{32}(a_1 b_{21} - a_2 b_{11}) \Sigma_3 + a_3 b_{11}^2 b_{22}(b_{23} b_{32} - 2 b_{22} b_{33})}{b_{11}^2 b_{22}^2 b_{33}} - \frac{b_{31}(b_{21} + b_{22}) \Sigma_4 + b_{11} b_{22} b_{33}(b_{32} b_{32} + 2 b_{32} b_{33})}{b_{11}^2 b_{22}^2 b_{33}} \varepsilon.
\]
(4.43)
Then the proof follows from the similar argument as that in the proof of Theorem 2.1 by using the above pair. Details are omitted. \(\Box\)

**Proof of Corollary 2.1.** Assume that \((v_1(x), v_2(x), v_3(x))\) is a positive steady-state solution of (1.1). Let \(u_i(x, t) = v_i(x)\) for \((x, t) \in \mathbb{R} \times [-\tau_i, \infty)\) \((i = 1, 2, 3)\). Then \((u_1(x, t), u_2(x, t), u_3(x, t))\) is the solution of (1.1) with the initial function \((\eta_1(x, t), \eta_2(x, t), \eta_3(x, t)) = (v_1(x), v_2(x), v_3(x))\). It follows from Theorem 2.2 that \((u_1(x, t), u_2(x, t), u_3(x, t))\) converges uniformly to \((\tilde{c}_1, \tilde{c}_2, 0)\) as \(t \to \infty\), and therefore \(v_3(x) \equiv 0\) which gives a contradiction. \(\Box\)
Proof of Corollary 2.2. Assume that \((v_1(x), v_2(x), v_3(x))\) is a positive steady-state solution of (1.1). Let \(u_i(x,t) = v_i(x)\) for \((x,t) \in \Omega \times [-\tau_i, \infty)\) \((i = 1, 2, 3)\). An application of Theorem 2.3 shows that \((u_1(x,t), u_2(x,t), u_3(x,t))\) converges uniformly to \((\tilde{c}_1, \tilde{c}_2, \tilde{c}_3)\) as \(t \to \infty\), and therefore \((v_1(x), v_2(x), v_3(x)) \equiv (\tilde{c}_1, \tilde{c}_2, \tilde{c}_3)\).

Proof of Theorem 2.4. The conclusions of the theorem follow from Theorem A and Theorems 2.1, 2.2 and 2.3 with \(d_i = 0\) \((i = 1, 2, 3)\) and without the boundary conditions.

References