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Existence and uniqueness of solutions for singular fourth-order boundary value problems ☆

Xiaoning Lin[a, b, *], Daqing Jiang[a], Xiaoyue Li[a]

[a] *School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, PR China*
[b] *School of Business, Northeast Normal University, Changchun 130024, PR China*

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Abstract

By mixed monotone method, the existence and uniqueness are established for singular fourth-order boundary value problems. The theorems obtained are very general and complement previous known results.

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Keywords: Mixed monotone operator; Fourth-order boundary value problem; Singular; Existence; Uniqueness

1. Introduction

In recent years, the study of fourth-order boundary value problems have been studied extensively in the literature (see for instance [1–3,7–12] and their references). In paper [9], the authors obtained some newest results for the singular fourth-order boundary value problems. But there is no result on the uniqueness of solution for singular fourth-order boundary value problems.

In this paper, first we get a unique fixed point theorem for a class of mixed monotone operators. Our idea comes from the fixed point theorems for mixed monotone operators (see [4–6]). In virtue of the theorem, we consider the following singular fourth-order boundary value problem:

$$\begin{cases} x^{(4)}(t) + \beta x''(t) = \lambda f(t, x), \quad 0 < t < 1, \quad \lambda > 0, \\ x(0) = x(1) = x''(0) = x''(1) = 0, \end{cases} \tag{1.1}$$

where $f(t, x) \in C((0, 1) \times (0, +\infty), (0, +\infty))$ and $\beta < \pi^2$.

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* Corresponding author. School of Business, Northeast Normal University, Changchun 130024, PR China. Tel.: +86 431 5515009.
E-mail address: linxn989@nenu.edu.cn (X. Lin).

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If $\beta = 0$, the existence of positive solutions of (1.1) has been studied in [10]. They show the existence of one positive solution when $f(t, x)$ is nonsingular and either superlinear or sublinear in x by employing a cone extension or compression theorem.

2. Preliminaries

Suppose that x is a positive solution of (1.1). Then

$$x(t) = \lambda \int_0^1 \int_0^1 G_1(t, \tau) G_2(\tau, s) f(s, x(s))\, \mathrm{d}s\, \mathrm{d}\tau, \quad 0 \leqslant t \leqslant 1, \tag{2.1}$$

where $G_1(t, s)$ is Green's function to $-x'' = 0, x(0) = x(1) = 0$, and $G_2(t, s)$ is Green's function to $-x'' - \beta x = 0, x(0) = x(1) = 0$. In particular,

$$G_1(t, s) = \begin{cases} t(1-s), & 0 \leqslant t \leqslant s \leqslant 1, \\ s(1-t), & 0 \leqslant s \leqslant t \leqslant 1, \end{cases}$$

and one can show that

$$\begin{aligned} &t(1-t)G_1(s, s) \leqslant G_1(t, s) \leqslant G_1(s, s) = s(1-s), \quad G_1(t, s) \leqslant t(1-t), \\ &\quad (t, s) \in [0, 1] \times [0, 1]. \end{aligned} \tag{2.2}$$

Set $\omega = \sqrt{|\beta|}$. If $\beta < 0$, then $G_2(t, s)$ is explicitly given by

$$G_2(t, s) = \begin{cases} \dfrac{\sinh \omega t \sinh \omega(1-s)}{\omega \sinh \omega}, & 0 \leqslant t \leqslant s \leqslant 1, \\ \dfrac{\sinh \omega s \sinh \omega(1-t)}{\omega \sinh \omega}, & 0 \leqslant s \leqslant t \leqslant 1. \end{cases}$$

If $\beta = 0$, then $G_2(t, s) = G_1(t, s)$. If $0 < \beta < \pi^2$, then $G_2(t, s)$ is explicitly given by

$$G_2(t, s) = \begin{cases} \dfrac{\sin \omega t \sin \omega(1-s)}{\omega \sin \omega}, & 0 \leqslant t \leqslant s \leqslant 1, \\ \dfrac{\sin \omega s \sin \omega(1-t)}{\omega \sin \omega}, & 0 \leqslant s \leqslant t \leqslant 1. \end{cases}$$

Clearly $G_2(t, s) > 0$ for $(t, s) \in (0, 1) \times (0, 1)$.

By using (2.1) and (2.2), we see that for every positive solution x of (1.1), one has

$$\begin{aligned} &\|x\| \leqslant \lambda \int_0^1 \int_0^1 G_1(\tau, \tau) G_2(\tau, s) f(s, x(s))\, \mathrm{d}s\, \mathrm{d}\tau, \\ &x(t) \geqslant t(1-t)\lambda \int_0^1 \int_0^1 G_1(\tau, \tau) G_2(\tau, s) f(s, x(s))\, \mathrm{d}s\, \mathrm{d}\tau \\ &\qquad \geqslant t(1-t)\|x\|, \end{aligned} \tag{2.3}$$

where $\|x\| = \sup\{|x(t)|;\ 0 \leqslant t \leqslant 1\}$.

Let

$$G(t, s) = \int_0^1 G_1(t, \tau) G_2(\tau, s)\, \mathrm{d}\tau, \tag{2.4}$$

thus by (2.1), one has

$$x(t) = \lambda \int_0^1 G(t, s) f(s, x(s))\, \mathrm{d}s, \quad 0 \leqslant t \leqslant 1, \tag{2.5}$$

and by (2.2) one has

$$t(1-t)\int_0^1 G_1(\tau,\tau)G_2(\tau,s)\,\mathrm{d}\tau \leqslant G(t,s) \leqslant t(1-t)\int_0^1 G_2(\tau,s)\,\mathrm{d}\tau. \tag{2.6}$$

Let P be a normal cone of a Banach space E, and $e \in P$ with $\|e\| \leqslant 1$, $e \neq \theta$. Define

$$Q_e = \{x \in P \mid \text{there exist constants } m, M > 0 \text{ such that } me \leqslant x \leqslant Me\}.$$

Now we give a definition (see [5]).

Definition 2.1. Assume $A : Q_e \times Q_e \to Q_e$. A is said to be mixed monotone if $A(x, y)$ is nondecreasing in x and nonincreasing in y, i.e., if $x_1 \leqslant x_2 (x_1, x_2 \in Q_e)$ implies $A(x_1, y) \leqslant A(x_2, y)$ for any $y \in Q_e$, and $y_1 \leqslant y_2 (y_1, y_2 \in Q_e)$ implies $A(x, y_1) \geqslant A(x, y_2)$ for any $x \in Q_e$. $x^* \in Q_e$ is said to be a fixed point of A if $A(x^*, x^*) = x^*$.

Theorem 2.1. *Suppose that* A: $Q_e \times Q_e \to Q_e$ *is a mixed monotone operator and* $\exists$ *a constant* α, $0 \leqslant \alpha < 1$, *such that*

$$A\left(tx, \frac{1}{t}y\right) \geqslant t^{\alpha} A(x, y), \quad \forall x, y \in Q_e, \quad 0 < t < 1. \tag{2.7}$$

Then A has a unique fixed point $x^* \in Q_e$. *Moreover, for any* $(x_0, y_0) \in Q_e \times Q_e$,

$$x_n = A(x_{n-1}, y_{n-1}), \quad y_n = A(y_{n-1}, x_{n-1}), \quad n = 1, 2, \ldots$$

satisfy

$$x_n \to x^*, \quad y_n \to x^*,$$

where

$$\|x_n - x^*\| = \mathrm{o}(1 - r^{\alpha^n}), \quad \|y_n - x^*\| = \mathrm{o}(1 - r^{\alpha^n}),$$

$0 < r < 1$, *r is a constant from* (x_0, y_0).

Proof. From (2.7),

$$A(x, y) = A(tt^{-1}x, t^{-1}ty) \geqslant t^{\alpha} A\left(\frac{x}{t}, ty\right), \quad x, y \in Q_e.$$

Then

$$A\left(\frac{x}{t}, ty\right) \leqslant t^{-\alpha} A(x, y), \quad \forall x, y \in Q_e, \quad 0 < t < 1. \tag{2.8}$$

For any $z_0 \in Q_e$, by virtue of $A(z_0, z_0) \in Q_e$, there exist constants $m, M > 0$, such that

$$me \leqslant A(z_0, z_0) \leqslant Me,$$

and there exists a small $0 < t_0 < 1$ such that

$$t_0^{(1-\alpha)/2} Me \leqslant z_0 \leqslant t_0^{-(1-\alpha)/2} me,$$

so we can obtain

$$z_0 t_0^{(1-\alpha)/2} \leqslant A(z_0, z_0) \leqslant z_0 t_0^{-(1-\alpha)/2}. \qquad \square \tag{2.9}$$

The following proof is the same as that in [5], we omit the proof.

Theorem 2.2 (*Guo [5]*). *Suppose that* A: $Q_e \times Q_e \to Q_e$ *is a mixed monotone operator and* $\exists$ *a constant* $\alpha \in (0, 1)$ *such that* (2.7) *holds. If* x_{λ}^* *is a unique solution of equation*

$$A(x, x) = \lambda x \quad (\lambda > 0)$$

in Q_e, *then* $\|x_\lambda^* - x_{\lambda_0}^*\| \to 0$, $\lambda \to \lambda_0$. *If* $0 < \alpha < \frac{1}{2}$, *then* $0 < \lambda_1 < \lambda_2$ *implies* $x_{\lambda_1}^* \geqslant x_{\lambda_2}^*$, $x_{\lambda_1}^* \neq x_{\lambda_2}^*$ *and*

$$\lim_{\lambda \to +\infty} \|x_\lambda^*\| = 0, \qquad \lim_{\lambda \to 0^+} \|x_\lambda^*\| = +\infty.$$

3. Singular fourth-order boundary value problem

This section discusses singular fourth-order boundary value problem

$$\begin{cases} x^{(4)}(t) + \beta x''(t) = \lambda f(t, x(t)), & 0 < t < 1, \quad \lambda > 0, \\ x(0) = x(1) = x''(0) = x''(1) = 0, \end{cases} \tag{3.1}$$

where $\beta < \pi^2$.

Throughout this section we assume that

$$f(t, x) = q(t)[g(x) + h(x)], \quad t \in (0, 1), \tag{3.2}$$

where

$$g : [0, +\infty) \to [0, +\infty) \text{ is continuous and nondecreasing;} \tag{3.3}$$

$$h : (0, +\infty) \to (0, +\infty) \text{ is continuous and nonincreasing.} \tag{3.4}$$

Let $P = \{x \in C[0, 1] | x(t) \geqslant 0, \forall t \in [0, 1]\}$. Obviously, P is a normal cone of Banach space $C[0, 1]$.

Theorem 3.1. *Suppose that there exists* $\alpha \in (0, 1)$ *such that*

$$g(tx) \geqslant t^\alpha g(x), \tag{3.5}$$

and

$$h(t^{-1}x) \geqslant t^\alpha h(x), \tag{3.6}$$

for any $t \in (0, 1)$ *and* $x > 0$, *and* $q \in C((0, 1), (0, \infty))$ *satisfies*

$$\int_0^1 s^{-\alpha}(1-s)^{-\alpha} q(s)\, \mathrm{d}s < +\infty. \tag{3.7}$$

Then (3.1) *has a unique positive solution* $x_\lambda^*(t)$. *And moreover*, $0 < \lambda_1 < \lambda_2$ *implies* $x_{\lambda_1}^* \leqslant x_{\lambda_2}^*$, $x_{\lambda_1}^* \neq x_{\lambda_2}^*$. *If* $\alpha \in (0, \frac{1}{2})$, *then*

$$\lim_{\lambda \to 0^+} \|x_\lambda^*\| = 0, \qquad \lim_{\lambda \to +\infty} \|x_\lambda^*\| = +\infty.$$

Proof. Since (3.6) holds, let $t^{-1}x = y$, one has

$$h(y) \geqslant t^\alpha h(ty).$$

Then

$$h(ty) \leqslant \frac{1}{t^\alpha} h(y), \quad \forall t \in (0, 1), \quad y > 0. \tag{3.8}$$

Let $y = 1$. The above inequality is

$$h(t) \leqslant \frac{1}{t^\alpha} h(1), \quad \forall t \in (0, 1). \tag{3.9}$$

From (3.6), (3.8) and (3.9), one has

$$h(t^{-1}x) \geqslant t^{\alpha} h(x), \quad h\left(\frac{1}{t}\right) \geqslant t^{\alpha} h(1),$$
$$h(tx) \leqslant \frac{1}{t^{\alpha}} h(x), \quad h(t) \leqslant \frac{1}{t^{\alpha}} h(1), \quad t \in (0,1), \quad x > 0. \tag{3.10}$$

Similarly, from (3.5), one has

$$g(tx) \geqslant t^{\alpha} g(x), \quad g(t) \geqslant t^{\alpha} g(1), \quad t \in (0,1), \quad x > 0. \tag{3.11}$$

Let $t = 1/x$, $x > 1$, one has

$$g(x) \leqslant x^{\alpha} g(1), \quad x \geqslant 1. \tag{3.12}$$

Let $e(t) = t(1-t)$, and we define

$$Q_e = \left\{ x \in C[0,1] \,\middle|\, \frac{1}{M}\, t(1-t) \leqslant x(t) \leqslant M t(1-t),\ t \in [0,1] \right\}, \tag{3.13}$$

where $M > 1$ is chosen such that

$$M > \max \left\{ \left\{ \int_0^1 \int_0^1 G_2(\tau, s)\, \mathrm{d}\tau q(s) \lambda [g(1) + s^{-\alpha}(1-s)^{-\alpha} h(1)]\, \mathrm{d}s \right\}^{1/(1-\alpha)}, \right.$$
$$\left. \left\{ \int_0^1 \int_0^1 G_1(\tau,\tau) G_2(\tau, s)\, \mathrm{d}\tau q(s) \lambda [h(1) + s^{\alpha}(1-s)^{\alpha} g(1)]\, \mathrm{d}s \right\}^{-1/(1-\alpha)} \right\}. \tag{3.14}$$

For any $x, y \in Q_e$, we define

$$A_\lambda(x, y)(t) = \lambda \int_0^1 G(t,s) q(s) [g(x(s)) + h(y(s))]\, \mathrm{d}s, \quad \forall t \in [0,1]. \tag{3.15}$$

First we show that $A_\lambda : Q_e \times Q_e \to Q_e$.
Let $x, y \in Q_e$, from (3.11) and (3.12), we have

$$g(x(t)) \leqslant g(Mt(1-t)) \leqslant g(M) \leqslant M^{\alpha} g(1), \quad t \in (0,1),$$

and from (3.10) we have

$$h(y(t)) \leqslant h\left(\frac{1}{M} t(1-t)\right) \leqslant t^{-\alpha}(1-t)^{-\alpha} h\left(\frac{1}{M}\right)$$
$$\leqslant M^{\alpha} t^{-\alpha}(1-t)^{-\alpha} h(1), \quad t \in (0,1).$$

Then, from (2.6) and (3.15), we have

$$A_\lambda(x,y)(t) \leqslant t(1-t) \int_0^1 \int_0^1 G_2(\tau, s)\, \mathrm{d}\tau q(s) M^{\alpha} \lambda [g(1) + s^{-\alpha}(1-s)^{-\alpha} h(1)]\, \mathrm{d}s$$
$$\leqslant M t(1-t), \quad t \in [0,1].$$

On the other hand, for any $x, y \in Q_e$, from (3.10) and (3.11), we have

$$g(x(t)) \geqslant g\left(\frac{1}{M} t(1-t)\right) \geqslant t^{\alpha}(1-t)^{\alpha} g\left(\frac{1}{M}\right) \geqslant t^{\alpha}(1-t)^{\alpha} \frac{1}{M^{\alpha}} g(1),$$

and

$$h(y(t)) \geqslant h(Mt(1-t)) \geqslant h(M) = h\left(\frac{1}{1/M}\right) \geqslant \frac{1}{M^{\alpha}} h(1), \quad t \in (0,1).$$

Thus, from (2.6) and (3.15), we have

$$A_\lambda(x,y)(t) \geqslant t(1-t)\int_0^1\int_0^1 G_1(\tau,\tau)G_2(\tau,s)\,\mathrm{d}\tau q(s)M^{-\alpha}\lambda[h(1)+s^\alpha(1-s)^\alpha g(1)]\,\mathrm{d}s$$
$$\geqslant \frac{1}{M}t(1-t), \quad t\in[0,1].$$

So, A_λ is well defined and $A_\lambda(Q_e\times Q_e)\subset Q_e$.

Next, for any $l\in(0,1)$, one has

$$A_\lambda(lx,l^{-1}y)(t) = \lambda\int_0^1 G(t,s)q(s)[g(lx(s))+h(l^{-1}y(s))]\,\mathrm{d}s$$
$$\geqslant \lambda\int_0^1 G(t,s)q(s)[l^\alpha g(x(s))+l^\alpha h(y(s))]\,\mathrm{d}s$$
$$= l^\alpha A_\lambda(x,y)(t), \quad t\in[0,1].$$

So the conditions of Theorems 2.1 and 2.2 hold. Therefore there exists a unique $x_\lambda^*\in Q_e$ such that $A_\lambda(x^*,x^*)=x_\lambda^*$. It is easy to check that x_λ^* is a unique positive solution of (3.1) for given $\lambda>0$. Moreover, Theorem 2.2 means that if $0<\lambda_1<\lambda_2$, then $x_{\lambda_1}^*(t)\leqslant x_{\lambda_2}^*(t)$, $x_{\lambda_1}^*(t)\neq x_{\lambda_2}^*(t)$ and if $\alpha\in(0,\frac{1}{2})$, then

$$\lim_{\lambda\to 0^+}\|x_\lambda^*\|=0, \qquad \lim_{\lambda\to+\infty}\|x_\lambda^*\|=+\infty.$$

This completes the proof. □

Example. Consider the following singular fourth-order boundary value problem:

$$\begin{cases} x^{(4)}(t)+\beta x''(t)=\lambda(\mu x^a+x^{-b}), & 0<t<1,\\ x(0)=x(1)=x''(0)=x''(1)=0, \end{cases} \tag{3.16}$$

where $\beta<\pi^2$, $\lambda, a, b>0$, $\mu\geqslant 0$.

Applying Theorem 3.1, we can find (3.16) has a unique positive solution $x_\lambda^*(t)$ provided

$$\max\{a,b\}<1. \tag{3.17}$$

In addition, $0<\lambda_1<\lambda_2$ implies $x_{\lambda_1}^*\leqslant x_{\lambda_2}^*$, $x_{\lambda_1}^*\neq x_{\lambda_2}^*$. If $\max\{a,b\}\in(0,\frac{1}{2})$, then

$$\lim_{\lambda\to 0^+}\|x_\lambda^*\|=0, \qquad \lim_{\lambda\to+\infty}\|x_\lambda^*\|=+\infty.$$

To see that, we put

$$\alpha=\max\{a,b\}, \quad q(t)=1, \quad g(x)=\mu x^a, \quad h(x)=x^{-b}.$$

Thus $0<\alpha<1$ and

$$g(tx)=t^a g(x)\geqslant t^\alpha g(x), \quad h(t^{-1}x)=t^b h(x)\geqslant t^\alpha h(x),$$

for any $t\in(0,1)$ and $x>0$, and

$$\int_0^1 s^{-\alpha}(1-s)^{-\alpha}\,\mathrm{d}s<+\infty,$$

thus all conditions in Theorem 3.1 are satisfied.

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