Circular Chromatic Numbers and Fractional Chromatic Numbers of Distance Graphs

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This paper studies circular chromatic numbers and fractional chromatic numbers of distance graphs $G(S, D)$ for various distance sets $D$. In particular, we determine these numbers for those $D$ sets of size two, for some special $D$ sets of size three, for $D = \{1, 2, \ldots, m, n\}$ with $1 \leq m < n$, for $D = \{q, q+1, \ldots, p\}$ with $q \leq p$, and for $D = \{1, 2, \ldots, m\} - \{k\}$ with $1 \leq k \leq m$.

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1. INTRODUCTION

Suppose $S$ is a subset of a metric space $\mathcal{M}$ with a metric $\delta$, and $D$ a subset of positive real numbers. The distance graph $G(S, D)$, with a distance set $D$, is the graph with vertex set $S$ in which two vertices $x$ and $y$ are adjacent iff $\delta(x, y) \in D$. Distance graphs, first studied by Eggleton et al. [7], were motivated by the well-known plane-coloring problem: What is the minimum number of colors needed to color all points of a euclidean plane so that points at unit distances are colored with different colors. This problem is equivalent to determining the chromatic number of the distance graph $G(R^2, \{1\})$. It is well-known that the chromatic number of this distance graph is between 4 and 7 (see [12, 15]). However, the exact number of colors needed remains unknown.

For distance graphs on the real line $R$ or the integer set $Z$, the problem of finding the chromatic numbers of $G(R, D)$ or $G(Z, D)$ for different $D$ sets has been studied extensively (see [3, 10, 13, 14, 17, 18, 20, 22]). Two recent papers [3, 14] related distance graphs to the $T$-coloring problem. Chromatic numbers and fractional chromatic numbers of distance graphs were used to derive bounds for $T$-spans of the corresponding $T$-colorings, and vice versa. In this paper, we study circular chromatic numbers and fractional chromatic numbers of distance graphs $G(Z, D)$ for various $D$ sets.

The circular chromatic number of a graph is a natural generalization of the chromatic number of a graph, introduced by Vince [16] under the name the ‘star chromatic number’ of a graph. Suppose $p$ and $q$ are positive integers such that $p \geq 2q$. A $(p, q)$-coloring of a graph $G = (V, E)$ is a mapping $c$ from $V$ to $\{0, 1, \ldots, p-1\}$ such that $\|c(x) - c(y)\|_p \geq q$ for any edge $xy$ in $E$, where $\|a\|_p = \min\{a, p-a\}$. The circular chromatic number $\chi_c(G)$ of $G$ is the infimum of the ratios $p/q$ for which there exist $(p, q)$-colorings of $G$.

Note that a $(p, 1)$-coloring of a graph $G$ is simply an ordinary $p$-coloring of $G$. Therefore, $\chi_c(G) \leq \chi(G)$ for any graph $G$. On the other hand, it has been shown [16] that for all graphs $G$, we have $\chi(G) - 1 < \chi_c(G)$. Therefore, $\chi(G) = \lceil \chi_c(G) \rceil$. In particular, two graphs with the same circular chromatic number also have the same chromatic number. However, two graphs with the same chromatic number may have different circular chromatic numbers. Thus $\chi_c(G)$ is a refinement of $\chi(G)$, and it contains more information about the structure of the graph. It is usually much more difficult to determine the circular chromatic number of a graph than to determine its chromatic number. The main results of this article determine the circular chromatic numbers of various distance graphs. These results may be viewed as improvements.

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on previous results concerning the chromatic numbers of these distance graphs presented in [3, 4, 7, 13, 17, 21].

The fractional chromatic number of a graph is another well-known variation of the chromatic number. A fractional coloring of a graph $G$ is a mapping $c$ from $Z(G)$, the set of all independent sets of $G$, to the interval $[0, 1]$ such that $\sum_{I \in Z(G)} c(I) \geq 1$ for all vertices $x$ in $G$. The fractional chromatic number $\chi_f(G)$ of $G$ is the infimum of the value $\sum_{I \in Z(G)} c(I)$ of a fractional coloring $c$ of $G$.

For any graph $G$, it is well known that

$$\max\{\omega(G), |G|/\alpha(G)\} \leq \chi_f(G) \leq \chi_c(G) \leq \lceil \chi_e(G) \rceil = \chi(G).$$  

(\*)

For simplicity, let $\omega(S, D), \alpha(S, D), \chi_f(S, D), \chi_c(S, D)$ and $\chi(S, D)$ denote, respectively, the clique number, the independence number, the fractional chromatic number, the circular chromatic number, and the chromatic number of a distance graph $G(S, D)$.

Chromatic numbers of distance graphs with distance sets $|D| \leq 2$ were determined by Chen et al. [4] and Voigt [17]. Chromatic numbers of distance graphs with $|D| = 3$ were determined by Zhu [21]. In Section 2, we use a ‘multiplier method’ to establish an upper bound for the circular chromatic number of a distance graph $G(Z, D)$ with an arbitrary distance set $D$. This upper bound is then used to determine the circular chromatic numbers and the fractional chromatic numbers of those distance graphs with distance sets $D$ for $|D| = 2$, for some special $D$ with $|D| = 3$, for $D = \{1, 2, \ldots, m, n\}$ with $1 \leq m < n$, and for $D = \{q, q + 1, \ldots, p\}$ with $q \leq p$. The chromatic number for $G(Z, D)$ with $D = \{q, q + 1, \ldots, p\}$ was determined in [7, 13].

Chromatic numbers of distance graphs with distance sets of the form $D_{m,k} = \{1, 2, \ldots, m\} - \{k\}$, with $1 \leq k \leq m$, were studied in [3, 7, 13, 14]. Partial results concerning chromatic numbers of such distance graphs were obtained in [7, 13, 14], and a complete solution was recently obtained by Chang et al. [3]. The authors of [3] also obtained circular chromatic numbers of such distance graphs for some special values of $m$ and $k$. In Section 3, we determine the circular chromatic numbers $\chi_c(Z, D_{m,k})$ for all integer pairs $m, k$.

2. Multiplier Method for $\chi_f(Z, D)$ and $\chi_c(Z, D)$

In this section we use a ‘multiplier method’ to establish an upper bound on $\chi_c(Z, D)$ for an arbitrary $D$ set. We then use this upper bound to determine circular chromatic numbers for some $D$ sets.

The multiplier method was used in [2] to study the density of $D$-sets, and was also used in [11] to study fractional chromatic numbers and circular chromatic numbers of circulant graphs. In taking distance graphs to be ‘infinite’ circulant graphs, Theorem 2.2 is parallel to a result in [11]. Half of the proof of Theorem 2.3 is parallel to an argument in [2].

**Lemma 2.1.** Suppose $D$ is a set of positive integers, and that $p$ and $r$ are positive integers. Let

$$d_D(p, r) = \min\|ri \mod p\| : i \in D.$$

If $d_D(p, r) \geq 1$, then $\chi_c(Z, D) \leq p/d_D(p, r)$.

**Proof.** It is straightforward to verify that the coloring defined as $c(i) = (ri \mod p)$ for $i \in Z$ is a $(p, d_D(p, r))$-coloring of the distance graph $G(Z, D)$.

$\square$
Let $f_D = \inf \{p/d_D(p, r) : d_D(p, r) \geq 1\}$. The function is well defined since $d_D(p, r)$ is always an integer between 0 and $\lfloor p/2 \rfloor$. Theorem 2.2 follows from Lemma 2.1.

**THEOREM 2.2.** For any set $D$ of positive integers, $\chi_e(Z, D) \leq f_D$.

It is known [4, 17] that if $D$ contains exactly two relatively prime integers, then $\chi(Z, D) = 2$ when the two integers are odd and $\chi(Z, D) = 3$ when the two integers have different parities. We first use $f_D$ to determine $\chi_e(Z, D)$ and $\chi_f(Z, D)$ for $D$ with $|D| = 2$.

**THEOREM 2.3.** If $D = \{a, b\}$ and $\gcd(a, b) = 1$, then

$$\chi_f(Z, D) = \chi_e(Z, D) = f_D = (a + b)/\lfloor (a + b)/2 \rfloor.$$  

**PROOF.** Suppose both $a$ and $b$ are odd. Since $2 \leq \omega(Z, D)$ and $d_D(2, 1) = 1$, the theorem follows from (*) and Theorem 2.2.

Suppose that $a$ and $b$ have different parities, i.e., $a + b$ is odd. Assume that $a + b = p$. Since $\gcd(p, b - a) = 1$, there exists a positive integer $r$ such that $r(b - a) \equiv 1 \pmod{p}$. Since $r(b + a) \equiv 0 \pmod{p}$, it follows that $2rb \equiv -2ra \equiv 1 \pmod{p}$. Hence, $ra \equiv -rb \equiv (p - 1)/2 \pmod{p}$, which implies that $d_D(p, r) = (p - 1)/2$. Hence, according to Theorem 2.2, $\chi_e(Z, D) \leq f_D \leq 2p/(p - 1) = (a + b)/\lfloor (a + b)/2 \rfloor$. On the other hand, it is easy to see that $G(Z, D)$ contains the odd cycle $C_p$. Thus, $2p/(p - 1) \leq p/\omega(C_p) \leq \chi_f(C_p) \leq \chi_f(Z, D) \leq \chi_e(Z, D)$. This completes the proof of the theorem.

Note that precisely the same arguments in the first two lines of the proof above also give that $\chi_f(Z, D) = \chi_e(Z, D) = f_D = 2$ if $D$ contains only odd integers.

We now consider circular chromatic numbers and fractional chromatic numbers of distance graphs $G(Z, D)$ with $|D| = 3$. Zhu [21] proved the following result for chromatic numbers, which provides a range for circular chromatic numbers.

**THEOREM 2.4 ([21]).** If $D = \{a, b, c\}$, where $a < b < c$ are positive integers with $\gcd(a, b, c) = 1$, then

$$\chi(Z, D) = \begin{cases} 2, & \text{if } a, b, c \text{ are odd}, \\ 4, & \text{if } a = 1 \text{ and } b = 2 \text{ and } c \equiv 0 \pmod{3}, \\ 4, & \text{if } a + b = c \text{ and } a \not\equiv b \pmod{3}, \\ 3, & \text{otherwise}. \end{cases}$$

**THEOREM 2.5.** If $D = \{a, a + 1, c\}$, with $a + 1 < c$, where $c + a = (2a + 1)k + r$, with $k \geq 1$ and $0 \leq r \leq 2a$, then

$$\chi_f(Z, D) \leq \chi_e(Z, D) \leq f_D \leq \begin{cases} (c + a)/(ak), & \text{if } 0 \leq r \leq a, \\ (c + a + 1)/(ak + r - a), & \text{if } a + 1 \leq r \leq 2a. \end{cases}$$

**PROOF.** Note that $ck \equiv -ak \pmod{c + a}$ and $c(k + 1) \equiv -(a + 1)(k + 1) \pmod{c + a + 1}$. Therefore, $d_D(c + a, k) = ak$ for all $r$, and $d_D(c + a + 1, k + 1) = ak + r - a$ when $a + 1 \leq r \leq 2a$. The theorem then follows.

**THEOREM 2.6.** If $D = \{a, a + 1, c\}$ with $a + 1 < c$ and $c + a \equiv 2a \pmod{2a + 1}$, then $\chi_f(Z, D) = \chi_e(Z, D) = f_D = 2 + 1/a$. 

**Proof.** Since $G(Z, D)$ contains the odd cycle $C_{2a+1}$, according to (\(*\)), $2 + 1/a = (2a + 1)/\alpha(C_{2a+1}) \leq \chi_f(C_{2a+1}) \leq \chi_f(Z, D)$. On the other hand, since $c + a \equiv 2a$ or $0 \pmod{2a + 1}$, it follows from Theorem 2.5 that $f_D \leq 2 + 1/a$. □

Denote the subgraph of $G(Z, D)$ induced by $V_i = \{0, 1, \ldots, i\}$ as $G_i$.

**Theorem 2.7.** If $D = \{2, 3, c\}$, with $3 < c$, where $c + 2 = 5k + r$, with $k \geq 1$ and $0 \leq r \leq 4$, then

$$
\chi_f(Z, D) = \chi_c(Z, D) = f_D = \begin{cases} 
(c + 2)/2k, & \text{if } r = 1, 2, \\
(c + 3)/(2k + 1), & \text{if } r = 3, \\
5/2, & \text{if } r = 4, 0.
\end{cases}
$$

**Proof.** The case in which $r = 4$ or $0$ follows from Theorem 2.6. For the other cases, Theorem 2.5 implies that $f_D \leq (c + 2)/2k$ when $r = 1, 2$, and $f_D \leq (c + 3)/(2k + 1)$ when $r = 3$. Therefore it suffices to show that $\alpha(G_{c+2}) \leq 2k$ when $r = 1, 2$ and $\alpha(G_{c+2}) \leq 2k + 1$ when $r = 3$.

Consider the graph $G_{c+2}$ for $r = 1, 2, 3$. Decompose the vertex set $\{0, 1, \ldots, c+2\}$ into $k+1$ subsets $I_i = \{5i, 5i+1, \ldots, 5i+4\}$ for $0 \leq i \leq k-1$, and $J = \{5k, 5k+1, \ldots, 5k+r = c+2\}$. Then, $J = \{c+1, c+2\}$ when $r = 1, J = \{c, c+1, c+2\}$ when $r = 2$, and $J = \{c-1, c, c+1, c+2\}$ when $r = 3$. Suppose that $G_{c+2}$ has an independent set $S$ of size $2k + 2$. We may assume that $0 \in S$ and then $c \notin S$. Since every five consecutive vertices in $G_{c+2}$ form a 5-cycle, we conclude that $|I_i \cap S| = |J \cap S| = 2$ for $0 \leq i \leq k - 1$. Then $c+1 \in S$, and hence, $1 \notin S$.

Since $|I_0 \cap S| = 2$, $2$ and $3$ are not in $S$. We therefore conclude that $4 \in S$. In a general step, using the fact that $|I_i \cap S| = 2$ and $5(i - 1) - 1 \in S$, it is straightforward to derive that $5k - 1 \in S$. Therefore, $5k - 1 \notin S$. Since $5k - 1 = c$ when $r = 1$, and $5k - 1$ is adjacent to $c + 1$ when $r = 2$ or $3$, we have contradictions. Hence, $\alpha(G_{c+2}) \leq 2k + 1$ for $r = 1, 2, 3$. Moreover, for the case in which $r = 1$ or $2$, any independent set $S'$ of $G_{c+2}$ of size $2k + 1$ that contains the vertex $0$ does not contain the vertex $c + 1$. Hence, $c + 2 \in S'$ and $\alpha(G_{c+1}) \leq 2k$. This completes the proof of the theorem. □

**Theorem 2.8.** Suppose $D = \{a, b, a + b\}$, with $0 < a < b$ and $\gcd(a, b) = 1$. If $a \equiv b \pmod{3}$, then $\chi_f(Z, D) = \chi_c(Z, D) = f_D = 3$.

**Proof.** Since $\gcd(a, b) = 1$ and $a \equiv b \pmod{3}$, we have $a, b, c \equiv 0 \pmod{3}$ and so $d_D(3, 1) = 1$. The theorem then follows from (\(*\)) and the fact that $\{0, a, a + b\}$ is a clique. □

**Theorem 2.9.** If $D = \{1, 2, \cdots, m, n\}$, with $1 \leq m < n$, then

$$
\chi_f(Z, D) = \chi_c(Z, D) = f_D = \begin{cases} 
m + 1, & \text{if } n \not\equiv 0 \pmod{m + 1}, \\
m + 1 + k/n, & \text{if } n \equiv k(m + 1).
\end{cases}
$$

**Proof.** Suppose $n \not\equiv 0 \pmod{m + 1}$. Since $m + 1 \leq \omega(G)$ and $d_D(m + 1, 1) = 1$, the theorem follows from (\(*\)) and Theorem 2.2.

Suppose $n = k(m + 1)$. Since every independent set of $G_n$ contains at most one vertex from any $m + 1$ consecutive vertices, and at most one vertex from $[0, n]$, $\alpha(G_n) = k$. Consequently, $m + 1 + 1/k = (n+1)/\alpha(G_n) \leq \chi_f(G_n) \leq \chi_f(Z, D)$. Also, $f_D \leq (n+1)/d_D(n+1, k) = m + 1 + 1/k$. The theorem then follows. □

**Corollary 2.10.** If $D = \{1, 2, 3k\}$, where $k \geq 1$, then $\chi_f(Z, D) = \chi_c(Z, D) = 3 + 1/k$. 
This is one of the two cases covered by Theorem 2.4 in which we have $\chi(Z, D) = 4$. The other is that in which $D = \{a, b, c\}$, $a + b = c$ and $a \not\equiv b \pmod 3$. We note that, in this case, the chromatic number of $G(Z, D)$ is easily determined. However, the circular chromatic numbers of $G(Z, D)$ are still unknown, except for some special values of $a$ and $b$.

We summarize the results for $D = \{a, b, c\}$ with $a < b < c$ and $\gcd(a, b, c) = 1$ in Table 1.

**Theorem 2.11.** If $D = \{q, q + 1, \ldots, p\}$, with $q \leq p$, then $\chi_f(Z, D) = \chi_c(Z, D) = f_D = 1 + p/q$.

**Proof.** Since $d_D(p + q, 1) = q$, we conclude that $f_D \leq (p + q)/q$. On the other hand, it is quite obvious that $\alpha(G_{p+q-1}) = q$. Hence, $\chi_f(Z, D) = \chi_c(Z, D) = f_D = 1 + p/q$. \qed

**Theorem 2.12.** If $D = \{1, r\}$, where $r$ is any real number greater than or equal to 1, then $\chi_f(R, D) = \chi_c(R, D) = 1 + r$.

**Proof.** We first consider the case in which $r = p/q$ is rational. Let $D' = \{q, q + 1, \ldots, p\}$. It is then straightforward to verify that each connected component of $G(R, D)$ is isomorphic to $G(Z, D')$. According to Theorem 2.11, $\chi_f(R, D) = \chi_c(R, D) = 1 + r$.

When $r$ is irrational, then let $(r_i : i = 1, 2, \ldots)$ and $(r'_i : i = 1, 2, \ldots)$ be sequences of rational numbers such that $r'_i \leq r \leq r_i$ for each $i$ and $\lim_{i \to \infty} r'_i = \lim_{i \to \infty} r_i = r$. The above argument then shows that $1 + r'_i \leq \chi_f(R, D) \leq \chi_c(R, D) \leq 1 + r_i$ for each $i$. Therefore, $\chi_f(R, D) = \chi_c(R, D) = 1 + r$. \qed

It was shown by Eggleton et al. [10] (Theorem 2) that if a prime distance graph has a proper $k$-coloring, then it has a periodic $k$-coloring. The proof in fact shows that any $k$-colorable distance graph has a periodic $k$-coloring. We remark that an argument parallel to the proof of Theorem 2 of [10] shows that if a distance graph $G(Z, D)$ has a $(p, q)$-coloring, then it has a periodic $(p, q)$-coloring. Also we note that a $(p, q)$-coloring derived by the multiplier method is always a periodic $(p, q)$-coloring.

**3. Circular Chromatic Number $\chi_c(Z, D_{m,k})$**

As mentioned in the introduction, Chang et al. [3] determined the chromatic number and the fractional chromatic number of the distance graph $G(Z, D_{m,k})$, where $D_{m,k} = \{1, 2, \ldots, m\}$.
They also determined the circular chromatic number of $G(Z, D_{m,k})$ for some pairs of integers $m$ and $k$.

Let $m + k + 1 = 2' m'$ and $k = 2' k'$, where $m'$ and $k'$ are both odd. Table 2 shows their results.

The circular chromatic numbers $\chi_c(Z, D_{m,k})$ remain unknown for those pairs of integers $m, k$ corresponding to the question mark in Table 2. In this section, we shall fill in the unknown part of Table 2 by showing that $\chi_c(Z, D_{m,k}) = \frac{m+k+2}{2}$ when $2k \leq m$ and $r \leq s$ and $\gcd(m + k + 1, k) \neq 1$.

The following lemma was proven in [16] and is used frequently in our proofs.

**Lemma 3.1 ([16]).** If $G$ has a circular chromatic number $\frac{p}{q}$ (where $p$ and $q$ are relatively prime), then $p \leq |V(G)|$, and any $(p, q)$-coloring $c$ of $G$ is an onto mapping from $V(G)$ to $\{0, 1, \ldots, p-1\}$.

As in the preceding section, we denote the subgraph of $G(Z, D_{m,k})$ induced by $V_i = \{0, 1, \ldots, i\}$ as $G_i$. We shall first derive a lower bound for $\chi_c(Z, D_{m,k})$.

**Lemma 3.2.** Suppose $2k \leq m$. Let $m + k + 1 = 2' m'$ and $k = 2' k'$, where $r$ and $s$ are non-negative integers and $m'$ and $k'$ are odd integers. If $1 \leq r \leq s$, then $\chi_c(G_{m+2k-1}) > \frac{m+k+1}{2}$.

**Proof.** Since $m + k + 1$ is even and $\chi_c(G_{m+2k-1}) > \chi(G_{m+2k-1}) - 1$, it suffices to show that $\chi(G_{m+2k-1}) < \frac{m+k+1}{2}$, assume to the contrary that $\chi(G_{m+2k-1}) \leq \frac{m+k+1}{2}$, and that $c$ is a $\frac{m+k+1}{2}$-coloring of $G_{m+2k-1}$.

For each integer $i$ with $0 \leq i \leq k - 2$, consider the subgraph of $G_{m+2k-1}$ induced by the $m + k + 1$ vertices $\{i, i + 1, \ldots, i + m + k\}$. This graph has an independence number $2$. Therefore, each of the $\frac{m+k+1}{2}$ colors is used at most, and thus exactly, twice in this subgraph. Consequently, vertices $i$ and $i + m + k + 1$ have the same colors for all $0 \leq i \leq k - 2$. Therefore, for each $j \in S := \{0, 1, \ldots, m + k\}$, the only possible vertices in $S$ having the same color as $j$ are $j + k$ and $j - k$.

Consider the circulant graph $C(m + k + 1, k)$, with vertex set $S$ and in which vertex $i$ is adjacent to vertex $j$ iff $j \equiv i + k$ or $i - k$ (mod $m + k + 1$). It follows from the discussion in the preceding paragraph that two vertices $x$ and $y$ in $S$ have the same color only if $xy$ is an edge of the circulant graph $C(m + k + 1, k)$. Since the intersection of each color class with $S$ contains exactly two vertices, the coloring induces a perfect matching of $C(m + k + 1, k)$. However, $C(m + k + 1, k)$ is the disjoint union of $d$ cycles of length $\frac{m+k+1}{d}$, where $d = \gcd(m+k+1, k)$.
We first give an \( \frac{m+k+1}{2} \) for \( 0 \leq r \leq s \), contrary to the assumption \( r \leq s \). Hence, \( \chi(G_{m+2k-1}) > \frac{m+k+1}{2} \).

**Lemma 3.3.** Suppose \( 2k \leq m \). If \( m + k + 1 \) is odd and \( \gcd(m + k + 1, k) \neq 1 \), then \( \chi_c(G_{m+k}) > \frac{m+k+1}{2} \), and hence, \( \chi_c(G_{m+2k-1}) > \frac{m+k+1}{2} \).

**Proof.** First, it is clear that \( \chi_c(G_{m+k}) \geq \frac{m+k+1}{2} \). Suppose \( \chi_c(G_{m+k}) = \frac{m+k+1}{2} \). Since \( m + k + 1 \) and 2 are relatively prime, every \((m + k + 1, 2)\)-coloring of \( G_{m+k} \) is onto and hence is one-to-one; i.e., there exists an ordering \( u, v, w, \ldots, x_{m+k} \) of \( V_{m+k} \) such that \( c(x_i) = i \) for \( 0 \leq i \leq m + k \). Therefore, \( X = (x_0, x_1, \ldots, x_{m+k}, x_0) \) is a cycle in the complement \( G' \) of \( G_{m+k} \).

Let \( m = ak + b \), where \( 0 \leq b < k \). Since all vertices of \( \{k-1, k, \ldots, m+1\} \) are of degree two in \( G' \), the following paths must be on the cycle \( X \):

\[
P_i : i, k + i, 2k + i, \ldots, ak + i, (a + 1)k + i \quad \text{for} \quad 0 \leq i \leq b;
\]
\[
P_j : j, k + j, 2k + j, \ldots, ak + j \quad \text{for} \quad b + 1 \leq j \leq k - 1.
\]

For each vertex \( x \), let \( N(x) = \{v \in V_{m+k} : uv \in E(G')\} \). Since \( N(k-1) = \{2k-1, m+k\} \) and \( m + k = \{a + 1\} + b \), we have that \( P_b P_{k-1} \) is a path of the cycle \( X \). Since \( N(k-2) = \{2k-2, m-k-1, m+k\} \) and vertex \( m + k \) is on the path \( P_b P_{k-2} \), we have that \( P_{b-1} P_{k-2} \) is a path of the cycle \( X \). Continuing this process, we have that \( P_i^r = P_{b+1+r} \), where the index \( b+1+r \) is taken modulo \( k \), is a path of the cycle \( X \) for \( 0 \leq t \leq k-1 \). Since \( \gcd(m + k + 1, k) \neq 1 \), we have \( \gcd(b+1, k) \neq 1 \). Therefore, these paths \( P_i^r \) form at least \( 2 \) disjoint cycles, contrary to our assumption that \( X \) is a cycle. Thus, the coloring \( c \) does not exist and \( \chi_c(G_{m+k}) > \frac{m+k+1}{2} \).

Since \( G_{m+k} \) is a subgraph of \( G_{m+2k-1} \), we conclude that \( \chi_c(G_{m+2k-1}) > \frac{m+k+1}{2} \).

**Theorem 3.4.** Suppose \( 2k \leq m \). Let \( m + k + 1 = 2m' \) and \( k = 2k' \), where \( r \) and \( s \) are non-negative integers and \( m' \) and \( k' \) are odd integers. If \( r \leq s \) and \( \gcd(m + k + 1, k) \neq 1 \), then \( \chi_c(Z, D_{m,k}) \geq \frac{m+k+2}{2} \).

**Proof.** Suppose \( \chi_c(G_{m+2k-1}) = \frac{p}{q} \), where \( p \) and \( q \) are relatively prime. Then, \( p \leq |V_{m+2k-1}| = m + 2k \) and \( \frac{p}{q} > \frac{m+k+1}{2} \) according to Lemmas 3.2 and 3.3. If \( q \geq 3 \), then \( p > \frac{q}{2}(m + k + 1) \geq \frac{q}{2}(m + k + 1) > m + 2k \), a contradiction. Hence, \( q \leq 2 \) and so \( \chi_c(Z, D_{m,k}) \geq \frac{p}{q} \geq \frac{m+k+2}{2} \).

Now we give an \((m + k + 2, 2)\)-coloring of \( G(Z, D_{m,k}) \) to show that \( \chi_c(Z, D_{m,k}) \leq \frac{m+k+2}{2} \). We first give an \((m + k + 2, 2)\)-coloring of \( G_{m+k} \) that is a variation of the coloring given in Theorem 2.1 after a shift operation. It is then extended to an \((m + k + 2, 2)\)-coloring of \( G(Z, D_{m,k}) \).

**Lemma 3.5.** If \( 2k \leq m \), then \( G_{m+k} \) has an \((m + k + 2, 2)\)-coloring \( c \) such that \( c(x) = c(x-k) + 1 \) for \( k \leq x \leq m + k \).

**Proof.** Suppose \( m + k + 1 = dm' \) and \( k = dk' \), where \( \gcd(m + k + 1, k) = d \). Since \( \gcd(m', k') = 1 \), there exists an integer \( n \) such that \( nk' = 1 \mod m' \). Let \( a_i = in \mod m' \) for \( 0 \leq i \leq m' - 1 \). Consider the mapping \( c \) from \( V_{m+k} \) to \( \{0, 1, \ldots, dm' - 1 = m+k\} \) defined by \( c(x) = a_i + jm' \), where \( x = id + (d - 1 - j) \), with \( 0 \leq i \leq m' - 1 \) and \( 0 \leq j \leq d - 1 \).

For any edge \( xy \) in \( G_{m+k} \), we shall prove that \( ||c(x) - c(y)||_{m+k+2} \geq 2 \). Suppose to the contrary that \( c(x) = c(y) \), or \( c(x) = c(y) + 1 \), or \( c(x) + 1 = c(y) \). Let \( x = i_1d + (d - 1 - j) \)
and \(y = i_2d + (d - 1 - j_2)\). For the case in which \(c(x) = c(y)\), we have \(a_{i_1} = a_{i_2}\) and \(j_1 = j_2\), which imply \(i_1 = i_2\) and \(x = y\), a contradiction to \(xy\) being an edge. For the case in which \(c(x) = c(y) + 1\), either (1) \(a_{i_1} = a_{i_2} + 1\) and \(j_1 = j_2\), or (2) \(a_{i_2} = 0\) and \(a_{i_2} = m' - 1\) and \(j_1 + j_2 + 1\). In subcase (1), we have \(i_1 = i_2 + k' \mod m'\). Thus, \(x - y = k\) or \(y - x = m + 1\), a contradiction. In subcase (2), we have \(i_1 = 0\) and \(i_2 = m' - k'\). Thus, \(y - x = m + 2\), a contradiction. Similarly, it is impossible that \(c(x) + 1 = c(y)\). This completes the proof of the lemma.

\[\text{THEOREM 3.6. If } 2k \leq m, \text{ then } \chi_c(Z, D_{m,k}) \leq \frac{m+k+2}{2}.\]

\[\text{PROOF. Let } c \text{ be the coloring of } G_{m+k} \text{ given in Lemma 3.5. Consider the mapping } c' \text{ of } G(Z, D_{m,k}) \text{ defined by}\]

\[
c'(x) = \begin{cases} c(x), & \text{for } 0 \leq x \leq m + k, \\ (c'(x-k) + 1) \mod (m + k + 2), & \text{for } m + k + 1 \leq x, \\ (c'(x+k) - 1) \mod (m + k + 2), & \text{for } 0 > x. \end{cases}
\]

We now show that \(c'\) is a proper \((m + k + 2, 2)\)-coloring of \(G(Z, D_{m,k})\) by induction. According to Lemma 3.5, \(c'\) is proper in \(G_{m+k}\). Suppose \(c'\) is proper in \(G_{m+k+1}\). Let \(xy\) be an edge in \(G_x\); i.e., \(y = x - i\) for some \(i \in D_{m,k}\). First, \(c'(y)\) is not equal to \(c'(x) + 1 \mod (m + k + 2)\) when \(i = 2k\), and \(y = x - i\) is adjacent to \(x - k\) in \(G_{m+k+1}\). Also, \(c'(y) - k\) is not equal to \(c'(x) + 1 \mod (m + k + 2)\). By induction, \(c'\) is proper for non-negative vertices in \(G(Z^+, D_{m,k})\). Similar arguments work for negative vertices. This completes the proof of the theorem.

Combining Theorems 3.4 and 3.6 and results in [3], we have

\[\text{THEOREM 3.7. Suppose } 2k \leq m. \text{ Let } m + k + 1 = 2'm' \text{ and } k = 2'k', \text{ where } r \text{ and } s \text{ are non-negative integers and } m' \text{ and } k' \text{ are odd integers. If } r \leq s \text{ and } \gcd(m + k + 1, k) \neq 1, \text{ then } \chi_c(Z, D_{m,k}) = \frac{m+k+2}{2}; \text{ otherwise, } \chi_c(Z, D_{m,k}) = \frac{m+k+1}{2}.\]

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