



Available at

[www.ElsevierComputerScience.com](http://www.ElsevierComputerScience.com)

POWERED BY SCIENCE @ DIRECT®

Theoretical Computer Science 313 (2004) 427–446

Theoretical  
Computer Science

[www.elsevier.com/locate/tcs](http://www.elsevier.com/locate/tcs)

# On the lengths of symmetry breaking–preserving games on graphs

F. Harary<sup>a,\*</sup>, W. Slany<sup>b,1</sup>, O. Verbitsky<sup>c</sup>

<sup>a</sup>Computer Science Department, New Mexico State University, Las Cruces, NM 88003-8001, USA

<sup>b</sup>Institut für Informationssysteme, Technische Universität Wien, Favoritenstr. 9, A-1040 Wien, Austria

<sup>c</sup>Department of Mechanics & Mathematics, Lviv University, Universytetska 1, 79000 Lviv, Ukraine

Received 30 April 2002; received in revised form 21 September 2002; accepted 31 October 2002

## Abstract

Given a graph  $G$ , we consider a game where two players,  $\mathcal{A}$  and  $\mathcal{B}$ , alternately color edges of  $G$  in red and in blue, respectively. Let  $L_{\text{sym}}(G)$  be the maximum number of moves in which  $\mathcal{B}$  is able to keep the red and the blue subgraphs isomorphic, if  $\mathcal{A}$  plays optimally to destroy the isomorphism. This value is a lower bound for the duration of any *avoidance* game on  $G$  under the assumption that  $\mathcal{B}$  plays optimally. We prove that if  $G$  is a path or a cycle of odd length  $n$ , then  $\Omega(\log n) \leq L_{\text{sym}}(G) \leq O(\log^2 n)$ . The lower bound is based on relations with Ehrenfeucht–Fraïssé games from model theory. We also consider complete graphs and prove that  $L_{\text{sym}}(K_n) = O(1)$ .

© 2003 Elsevier B.V. All rights reserved.

*Keywords:* Symmetry breaking–preserving game; Ehrenfeucht–Fraïssé game

## 1. Introduction

The symmetry breaking–preserving game  $\text{SYM}(G)$  is played by two players on a graph  $G$ . The players,  $\mathcal{A}$  and  $\mathcal{B}$ , alternately color edges of  $G$  in red and in blue respectively, one edge per move. Player  $\mathcal{A}$  is first to move. A round of the game consists of a move of  $\mathcal{A}$  and the following move of  $\mathcal{B}$ . The objective of  $\mathcal{B}$  is to keep the red and the blue subgraphs of  $G$  isomorphic after every round. As soon as  $\mathcal{B}$  fails

\* Corresponding author.

*E-mail addresses:* [fnh@crl.nmsu.edu](mailto:fnh@crl.nmsu.edu) (F. Harary), [wsu@dbai.tuwien.ac.at](mailto:wsu@dbai.tuwien.ac.at) (W. Slany), [oleg@ov.litech.net](mailto:oleg@ov.litech.net) (O. Verbitsky).

<sup>1</sup> Research partly supported from Austrian Science Foundation Grant Z29-INF.

to do so, this is a win for  $\mathcal{A}$ . If  $\mathcal{B}$  succeeds until all the edges are colored, this is a win for him.

This game was introduced in [5] in the context of the graph avoidance games [4,2]. The game  $\text{AVOID}(G, F)$  is a two-person edge-coloring game on a graph  $G$  with the following ending condition: The player who first creates a monochromatic copy of a forbidden subgraph  $F$  loses. As easily seen, as long as  $\mathcal{B}$  does not lose in  $\text{SYM}(G)$ , he does not lose in  $\text{AVOID}(G, F)$  for any  $F$ .

In [5] we addressed the class  $\mathcal{C}_{\text{sym}}$  of those graphs  $G$  for which  $\mathcal{B}$  has a winning strategy in  $\text{SYM}(G)$ . We now consider a more general problem: Given  $G$ , how long is  $\mathcal{B}$  able to keep the red and the blue subgraphs isomorphic if both players play optimally? We define  $L_{\text{sym}}(G)$ , the *length* of the game  $\text{SYM}(G)$ , to be the maximum number of rounds in which  $\mathcal{B}$ , playing optimally, does not lose, independently of  $\mathcal{A}$ 's strategy (a precise definition is given in Section 2). This function of a graph  $G$  will be our main concern.

Note that  $G$  belongs to  $\mathcal{C}_{\text{sym}}$  iff  $L_{\text{sym}}(G)$  has the maximum possible value  $\lfloor m/2 \rfloor$ , where  $m$  is the size of  $G$ . In [5] we observe that  $\mathcal{C}_{\text{sym}}$  contains all graphs having an involutory automorphism without fixed edges. Though generally it is NP-hard to recognize if such an automorphism exists for a given graph, we nevertheless obtain many examples of graphs in  $\mathcal{C}_{\text{sym}}$ . The simplest examples are even cycles and paths. Let  $C_n$  (resp.  $P_n$ ) denote the cycle (resp. path) of size  $n$ . Thus, we have  $L_{\text{sym}}(C_n) = L_{\text{sym}}(P_n) = n/2$  for  $n$  even.

In the present paper we treat odd cycles and paths. For odd  $n$ , we prove that

$$\Omega(\log n) \leq L_{\text{sym}}(P_n) \leq O(\log^2 n), \quad \Omega(\log n) \leq L_{\text{sym}}(C_n) \leq O(\log^2 n).$$

Our proof of the lower bound is based on the connections with Ehrenfeucht–Fraïssé games known in model theory [1], what may be of independent interest. In particular, we use the well-known fact that the length of the Ehrenfeucht–Fraïssé game on the pair of paths  $P_n$  and  $P_{n+1}$  equals  $\log n$  up to an additive constant and the same is true for the pair of cycles  $C_n$  and  $C_{n+1}$ .

We also consider symmetry breaking–preserving games on complete graphs. As implicitly shown in [5],  $L_{\text{sym}}(K_n) \leq n - 2$ . We now improve this estimate showing that  $L_{\text{sym}}(K_n)$  is for all  $n$  bounded by an absolute constant.

Note that all the upper (resp. lower) bounds proven here are based on efficiently computable strategies for the player  $\mathcal{A}$  (resp.  $\mathcal{B}$ ).

In the next section we give the definitions and state some useful facts. We estimate the asymptotics of  $L_{\text{sym}}(G)$  for odd paths and cycles in Section 3 and for complete graphs in Section 4.

## 2. Preliminaries

Given a graph  $G$ , we denote its vertex set by  $\mathcal{V}(G)$  and its edge set by  $\mathcal{E}(G)$ .

The *symmetry breaking–preserving game* on a graph  $G$ , denoted by  $\text{SYM}(G)$ , is a two-person positional game of the following kind. Two players,  $\mathcal{A}$  and  $\mathcal{B}$ , alternately color edges of a graph  $G$  in red and in blue, respectively. Player  $\mathcal{A}$  starts the game. In

a *move*, a player colors an edge that was so far uncolored. The *i*th round consists of the *i*th move of  $\mathcal{A}$  and the *i*th move of  $\mathcal{B}$ . Let  $a_i$  (resp.  $b_i$ ) denote an edge colored by  $\mathcal{A}$  (resp.  $\mathcal{B}$ ) in the *i*th round. Let  $A_i = \{a_1, \dots, a_i\}$  (resp.  $B_i = \{b_1, \dots, b_i\}$ ) consist of the red (resp. blue) edges colored up to the *i*th round. Player  $\mathcal{B}$  wins in  $\text{SYM}(G)$  if the subgraphs  $A_i$  and  $B_i$  are isomorphic for every  $i \leq |\mathcal{E}(G)|/2$ . As soon as an isomorphism between  $A_i$  and  $B_i$  is violated, this is a win for  $\mathcal{A}$ .

A *strategy* for a player determines the edge to be colored by him at every round of the game. Formally, let  $\varepsilon$  denote the empty sequence. A strategy of  $\mathcal{A}$  is a function  $S_1$  that maps every, possibly empty, sequence of pairwise distinct edges  $e_1, \dots, e_i$  into an edge different from  $e_1, \dots, e_i$  and from  $S_1(\varepsilon), S_1(e_1), S_1(e_1, e_2), \dots, S_1(e_1, \dots, e_{i-1})$ . A strategy of  $\mathcal{B}$  is a function  $S_2$  that maps every nonempty sequence of pairwise distinct edges  $e_1, \dots, e_i$  into an edge different from  $e_1, \dots, e_i$  and from  $S_2(e_1), S_2(e_1, e_2), \dots, S_2(e_1, \dots, e_{i-1})$ . If  $\mathcal{A}$  follows a strategy  $S_1$  and  $\mathcal{B}$  follows a strategy  $S_2$ , then  $a_i = S_1(b_1, \dots, b_{i-1})$  and  $b_i = S_2(a_1, \dots, a_i)$ .

The *length* of the game is the total number of rounds under the condition that the players play optimally. To be more precise, assume that  $\mathcal{A}$  follows a strategy  $S_1$  and  $\mathcal{B}$  follows a strategy  $S_2$  and let  $l(S_1, S_2)$  denote the maximum  $l$  such that  $A_i$  and  $B_i$  are isomorphic for every  $i \leq l$ . We denote the length of  $\text{SYM}(G)$  by  $L_{\text{sym}}(G)$  and define it by

$$L_{\text{sym}}(G) = \max_{S_2} \min_{S_1} l(S_1, S_2).$$

An alternative definition could be

$$L'_{\text{sym}}(G) = \min_{S_1} \max_{S_2} l(S_1, S_2).$$

Observe that the definitions are equivalent.

**Proposition 1.**  $L_{\text{sym}}(G) = L'_{\text{sym}}(G)$ .

**Proof.** The inequality  $L_{\text{sym}}(G) \leq L'_{\text{sym}}(G)$  holds true by the universal min-max relation. To prove the reverse inequality, define a game  $\text{SYM}_r(G)$  to be a variant of  $\text{SYM}(G)$  in which  $\mathcal{B}$  wins if he does not lose the first  $r$  rounds. A strategy of a player is *winning* if it beats every strategy of the opponent. Since  $\text{SYM}_r(G)$  is a finite perfect information game with no draws, in this game one of the players has a winning strategy. Assume that  $L_{\text{sym}}(G) = l$  and  $l < \lfloor |\mathcal{E}(G)|/2 \rfloor$ . This means that  $\mathcal{B}$  has no winning strategy in  $\text{SYM}_{l+1}(G)$ . Hence the winning strategy in  $\text{SYM}_{l+1}(G)$  exists for  $\mathcal{A}$ , which implies that  $L'_{\text{sym}}(G) \leq l$ .  $\square$

We will refer to the following observation that follows from [5].

**Proposition 2.** *If  $G$  has an involutory automorphism without fixed edges, then*

$$L_{\text{sym}}(G) = \lfloor |\mathcal{E}(G)|/2 \rfloor.$$

**Proof.** If  $\phi : \mathcal{V}(G) \rightarrow \mathcal{V}(G)$  is an automorphism of  $G$ , it determines a permutation  $\phi' : \mathcal{E}(G) \rightarrow \mathcal{E}(G)$  by  $\phi'(\{u, v\}) = \{\phi(u), \phi(v)\}$ . We assume that  $\phi$  is involutory and

$\phi'$  has no fixed element. Then the edge set  $\mathcal{E}(G)$  is partitioned into 2-subsets of the form  $\{e, \phi'(e)\}$ . This gives  $\mathcal{B}$  the following winning strategy: Whenever  $\mathcal{A}$  chooses an edge  $e$ ,  $\mathcal{B}$  chooses the edge  $\phi'(e)$ . After every round of the game, an isomorphism between the red and the blue subgraphs is induced by  $\phi$ .  $\square$

The *Ehrenfeucht–Fraïssé game* can be played on an arbitrary structure. We give a definition conformably to graphs. Assume that graphs  $G_0$  and  $G_1$  have disjoint vertex sets. In the Ehrenfeucht–Fraïssé game on  $G_0$  and  $G_1$ , denoted further on by  $\text{EF}(G_0, G_1)$ , the players  $\mathcal{A}$  and  $\mathcal{B}$  alternately pick up vertices of either  $G_0$  or  $G_1$ , one vertex per move.  $\mathcal{A}$  starts the game. Let  $u_i$  (resp.  $v_i$ ) be the vertex picked up by  $\mathcal{A}$  (resp. by  $\mathcal{B}$ ) in his  $i$ th move. In each round the objective of  $\mathcal{B}$  is to obey the following conditions:

- If  $u_i \in \mathcal{V}(G_a)$ , then  $v_i \in \mathcal{V}(G_{1-a})$ .
- The correspondence “ $u_i$  to  $v_i$ ” is a partial isomorphism between  $G_0$  and  $G_1$ , i.e., an isomorphism between the subgraphs of  $G_0$  and  $G_1$  induced by the chosen vertices.

The maximum number of rounds in which  $\mathcal{B}$ , irrespective of  $\mathcal{A}$ 's strategy, is able to obey these two conditions is denoted by  $L_{\text{EF}}(G_0, G_1)$  and is formally defined similarly to  $L_{\text{sym}}(G)$ .

Let  $\log n$  denote logarithm base 2. We will use the following folklore result.

**Proposition 3.** *For every  $n$ ,*

- (1)  $\log n - 2 < L_{\text{EF}}(P_n, P_{n+1}) < \log n + 2$ .
- (2)  $\log n - 1 < L_{\text{EF}}(C_n, C_{n+1}) < \log n + 1$ .

The proof can be found in [6, Theorems 2.1.2 and 2.1.3] for the case of paths. The case of cycles can be treated similarly (cf. [1, Example 2.3.8]).

### 3. Games on paths and cycles

Given two functions  $f(n)$  and  $g(n)$ , we use notation  $f(n) = \Omega(g(n))$  whenever  $f(n) \geq c \cdot g(n)$  for some  $c > 0$  and all  $n$ .

The main result of this section estimates the asymptotics of  $L_{\text{sym}}(G)$  for odd paths and cycles. It should be contrasted with even paths and cycles, for which  $L_{\text{sym}}(P_n) = L_{\text{sym}}(C_n) = n/2$  by Proposition 2.

**Theorem 4.** *If  $n$  is odd, then*

- (1)  $L_{\text{sym}}(P_n) = \Omega(\log n)$  and  $L_{\text{sym}}(C_n) = \Omega(\log n)$ ,
- (2)  $L_{\text{sym}}(P_n) = O(\log^2 n)$  and  $L_{\text{sym}}(C_n) = O(\log^2 n)$ .

The proof of the theorem is given in the rest of this section.

#### 3.1. Lower bound

To prove the lower bounds for  $L_{\text{sym}}(P_n)$  and  $L_{\text{sym}}(C_n)$ , we relate the symmetry breaking–preserving game with the Ehrenfeucht–Fraïssé game. We are actually able to

prove Claim 1 of Theorem 4 in two different ways, both using Proposition 3. We will refer to one way as the logical approach and to the other way as the combinatorial approach.

We start with the brief overview of the logical approach for the case of paths; the case of cycles is virtually identical. Given an odd path  $P_n$ , we consider also the even path  $P_{n+1}$  for which we know that  $L_{\text{sym}}(P_{n+1}) = (n+1)/2$ . As known from model theory, the length of  $\text{EF}(P_n, P_{n+1})$  tells us to which extent the properties of  $P_{n+1}$  expressible in first-order logic hold true for  $P_n$  (see Lemma 5). On the other hand, Lemma 6 tells us to which extent the property of a graph  $G$  that  $L_{\text{sym}}(G) \geq k$  is first-order expressible. Putting it together, we see that, as the property  $L_{\text{sym}}(G) = \Omega(\log n)$  is true for  $P_{n+1}$ , it must be true for  $P_n$  too (cf. Proposition 7). Curiously, this method proves the existence of the desired strategy for player  $\mathcal{B}$  without yielding it explicitly.

The combinatorial approach does not exploit the logical aspects of the Ehrenfeucht–Fraïssé game. Instead, it directly exploits the partial isomorphism constructed during the course of  $\text{EF}(P_n, P_{n-1})$  in order to translate, as long as possible, the winning strategy of  $\mathcal{B}$  from  $\text{SYM}(P_{n+1})$  into  $\text{SYM}(P_n)$  (see Proposition 8).

The combinatorial approach gives us a bound twice as good as the logical approach. What is more important than the gain in a multiplicative constant, the former approach provides us with an efficiently computable strategy for  $\mathcal{B}$ . Nevertheless, though the combinatorial approach is more preferable to the logical one in the particular cases of odd paths and cycles, generally it has a more restrictive applicability range. We do not exclude that both techniques may be useful in the analysis of other games on graphs (cf. Remark 9).

We now present both the proof methods in detail, starting from the logical one.

From the logical point of view a graph  $G$  is a structure consisting of a single binary predicate  $E$  on  $\mathcal{V}(G)$  such that  $E(u, v)$  iff  $u$  and  $v$  are adjacent. Every closed first-order formula over vocabulary  $\{E, =\}$  is either true or false on  $G$ .

**Lemma 5** (Ehrenfeucht see [1, Theorem 2.2.8]).  *$G_0$  and  $G_1$  satisfy precisely the same first-order sentences with at most  $L_{\text{EF}}(G_0, G_1)$  quantifiers.*

Observe that the sentence “ $L_{\text{sym}}(G) \geq k$ ” is expressible with  $4k$  quantifiers.

**Lemma 6.** *There is a first-order formula  $\Phi_k$  with  $4k$  quantifiers that is true on  $G$  of size at least  $2k$  iff  $L_{\text{sym}}(G) \geq k$ .*

**Proof.** Let  $\text{DIST}(x_1, x_2, y_1, y_2)$  express the property that two pairs of vertices  $\{x_1, x_2\}$  and  $\{y_1, y_2\}$  are distinct. Formally,

$$\text{DIST}(x_1, x_2, y_1, y_2) \stackrel{\text{def}}{=} \neg((x_1 = y_1 \wedge x_2 = y_2) \vee (x_1 = y_2 \wedge x_2 = y_1)).$$

Let  $u_{1,1}, u_{1,2}, \dots, u_{k,1}, u_{k,2}$  and  $v_{1,1}, v_{1,2}, \dots, v_{k,1}, v_{k,2}$  be variables ranging over  $\mathcal{V}(G)$  with meaning that in the  $i$ th round  $\mathcal{A}$  chooses an edge  $\{u_{i,1}, u_{i,2}\}$  and  $\mathcal{B}$  chooses an edge  $\{v_{i,1}, v_{i,2}\}$ . We also need a formula  $\text{ISO}_j$  to express the fact that the subgraphs consisting

of the edges chosen by the players during the first  $j$  rounds are isomorphic:

$$\begin{aligned} & \text{ISO}_j(u_{1,1}, u_{1,2}, \dots, u_{j,1}, u_{j,2}, v_{1,1}, v_{1,2}, \dots, v_{j,1}, v_{j,2}) \\ & \stackrel{\text{def}}{=} \bigvee_f \bigwedge_{\substack{1 \leq i, i' \leq j \\ 1 \leq a, a' \leq 2}} (u_{i,a} = u_{i',a'} \leftrightarrow v_{f(i,a)} = v_{f(i',a')}), \end{aligned}$$

where the disjunction is over all permutations of the index set  $\{1, \dots, j\} \times \{1, 2\}$  with the property that if  $f(i, 1) = (m, a)$ , then  $f(i, 2) = (m, 3 - a)$  for all  $i \leq j$ . The permutation  $f$  should be thought of as a map from the multiset  $\{u_{i,a}\}_{i \leq j, a \leq 2}$  to the multiset  $\{v_{i,a}\}_{i \leq j, a \leq 2}$  taking every edge  $\{u_{i,1}, u_{i,2}\}$  to some edge  $\{v_{m,1}, v_{m,2}\}$ . Such a permutation is a subgraph isomorphism if it takes equal  $u$ 's to equal  $v$ 's and distinct  $u$ 's to distinct  $v$ 's.

Define formulas

$$A_j \stackrel{\text{def}}{=} E(u_{j,1}, u_{j,2}) \wedge \bigwedge_{i=1}^{j-1} \text{DIST}(u_{j,1}, u_{j,2}, u_{i,1}, u_{i,2}) \wedge \bigwedge_{i=1}^{j-1} \text{DIST}(u_{j,1}, u_{j,2}, v_{i,1}, v_{i,2})$$

and

$$B_j \stackrel{\text{def}}{=} E(v_{j,1}, v_{j,2}) \wedge \bigwedge_{i=1}^j \text{DIST}(v_{j,1}, v_{j,2}, u_{i,1}, u_{i,2}) \wedge \bigwedge_{i=1}^{j-1} \text{DIST}(v_{j,1}, v_{j,2}, v_{i,1}, v_{i,2})$$

saying that  $\{u_{j,1}, u_{j,2}\}$  and, respectively,  $\{v_{j,1}, v_{j,2}\}$  are edges different from the edges chosen by the players previously. The formula

$$\begin{aligned} \Phi_k & \stackrel{\text{def}}{=} \forall u_{1,1} \forall u_{1,2} \exists v_{1,1} \exists v_{1,2} \dots \forall u_{k,1} \forall u_{k,2} \exists v_{k,1} \exists v_{k,2} \left( \bigwedge_{j=1}^k A_j \right. \\ & \left. \rightarrow \bigwedge_{j=1}^k B_j \wedge \bigwedge_{j=1}^k \text{ISO}_j(u_{1,1}, u_{1,2}, \dots, u_{j,1}, u_{j,2}, v_{1,1}, v_{1,2}, \dots, v_{j,1}, v_{j,2}) \right) \end{aligned}$$

is as desired. Indeed, assume that  $\mathcal{B}$  has a strategy nonlosing  $k$  rounds. Then  $\Phi_k$  is true because, if all  $u_{j,1}, u_{j,2}$  are chosen so that the antecedent in  $\Phi_k$  is satisfied, then  $v_{j,1}, v_{j,2}$  satisfying the consequent are provided by  $\mathcal{B}$ 's strategy.

On the other hand, if  $\Phi_k$  is true, then the following strategy of  $\mathcal{B}$  does not lose  $k$  rounds to any strategy of  $\mathcal{A}$ . We describe the  $j$ th move of  $\mathcal{B}$ . Assume that  $\mathcal{A}$  and  $\mathcal{B}$  have previously chosen edges  $\{u_{1,1}, u_{1,2}\}, \dots, \{u_{j,1}, u_{j,2}\}$  and  $\{v_{1,1}, v_{1,2}\}, \dots, \{v_{j-1,1}, v_{j-1,2}\}$ . In particular,  $u_{1,1}, u_{1,2}, \dots, u_{j,1}, u_{j,2}$  satisfy the antecedent in  $\Phi_k$ . Then  $\mathcal{B}$  chooses an edge  $\{v_{j,1}, v_{j,2}\}$  with vertices  $v_{j,1}$  and  $v_{j,2}$  whose existence is claimed by  $\Phi_k$ . Such  $v_{j,1}$  and  $v_{j,2}$  satisfy the members  $B_j$  and  $\text{ISO}_j$  of the consequent in  $\Phi_k$  because otherwise one could choose the subsequent  $u_{j+1,1}, u_{j+1,2}, \dots, u_{k,1}, u_{k,2}$  satisfying the antecedent in  $\Phi_k$  and therewith falsify the implication. It follows that this move of  $\mathcal{B}$  is legitimate and successful.  $\square$

**Proposition 7** (logical approach).

$$L_{\text{sym}}(G_1) \geq \min \left\{ \frac{1}{4} L_{\text{EF}}(G_0, G_1), L_{\text{sym}}(G_0) \right\}.$$

**Proof.** Assume that  $L_{\text{sym}}(G_0) \geq k$  and  $L_{\text{EF}}(G_0, G_1) \geq 4k$ . The former inequality implies that  $G_0$  has size at least  $2k$ . By the latter inequality, the same must be also true for  $G_1$ . By Lemma 6,  $G_0$  satisfies  $\Phi_k$ . By Lemma 5,  $G_1$  also satisfies  $\Phi_k$  and therefore, again by Lemma 6,  $L_{\text{sym}}(G_1) \geq k$ .  $\square$

We now turn to the combinatorial approach to the proof of Theorem 4(1).

Given a graph  $H$ , let  $\mathcal{L}(H)$  denote its *line graph*. Recall that  $\mathcal{V}(\mathcal{L}(H)) = \mathcal{E}(H)$  and two vertices  $e_1$  and  $e_2$  of  $\mathcal{L}(H)$  are connected by an edge in this graph iff they have a common vertex in  $H$ . Two graphs  $H_1$  and  $H_2$  are *edge-isomorphic* if there is a one-to-one map from  $\mathcal{E}(H_1)$  onto  $\mathcal{E}(H_2)$  preserving the adjacency of edges. In other words,  $H_1$  and  $H_2$  are edge-isomorphic iff  $\mathcal{L}(H_1)$  and  $\mathcal{L}(H_2)$  are isomorphic. If two graphs are isomorphic, then they are obviously edge-isomorphic. The Whitney theorem [3, Theorem 8.3] says that the converse implication is also true for all connected  $H_1$  and  $H_2$  unless one of them is  $K_3$  and the other  $K_{1,3}$ .

To avoid ambiguity, in the next proposition we keep the names  $\mathcal{A}$  and  $\mathcal{B}$  for the players in the game  $\text{SYM}(G_1)$ , but rename them  $\mathcal{A}_0$  and  $\mathcal{B}_0$  in  $\text{SYM}(G_0)$ , and *spoiler* and *duplicator* in  $\text{EF}(G_0, G_1)$ .

**Proposition 8** (combinatorial approach). *If  $G_1$  does not contain a subgraph  $K_3$ , then  $L_{\text{sym}}(G_1) \geq \min \left\{ \frac{1}{2} L_{\text{EF}}(\mathcal{L}(G_0), \mathcal{L}(G_1)), L_{\text{sym}}(G_0) \right\}$ . Moreover, the player  $\mathcal{B}$  in  $\text{SYM}(G_1)$  has an efficiently computable strategy  $S$  with oracle access to a strategy  $D$  of the duplicator in  $\text{EF}(\mathcal{L}(G_0), \mathcal{L}(G_1))$  and to a strategy  $S_0$  of  $\mathcal{B}_0$  in  $\text{SYM}(G_0)$  such that, if  $D$  does not lose  $l$  rounds irrespective of the spoiler's strategy and  $S_0$  does not lose  $m$  rounds irrespective of  $\mathcal{A}_0$ 's strategy, then  $S(D, S_0)$  does not lose at least  $\min \left\{ \frac{1}{2} l, m \right\}$  rounds irrespective of  $\mathcal{A}$ 's strategy.*

**Proof.** To make a move according to  $S(D, S_0)$ , in each round of  $\text{SYM}(G_1)$  the player  $\mathcal{B}$  simulates one round of  $\text{SYM}(G_0)$  following  $S_0$  and two rounds of  $\text{EF}(\mathcal{L}(G_0), \mathcal{L}(G_1))$  following  $D$ . Before describing  $S(D, S_0)$ , we introduce some notation. Let  $A_i, B_i \subset \mathcal{E}(G_1)$  consist of the edges colored by  $\mathcal{A}$  and  $\mathcal{B}$ , respectively, up to the  $i$ th round of  $\text{SYM}(G_1)$  and  $A'_i, B'_i \subset \mathcal{E}(G_0)$  consist of the edges colored by  $\mathcal{A}_0$  and  $\mathcal{B}_0$ , respectively, up to the  $i$ th round of the simulated game  $\text{SYM}(G_0)$ . Initially  $A_0 = B_0 = A'_0 = B'_0 = \emptyset$ . It will be the case that, up to the  $(2i-1)$ th round of the simulated game  $\text{EF}(\mathcal{L}(G_0), \mathcal{L}(G_1))$ , the spoiler and the duplicator choose exactly the vertices in  $A_i \cup A'_i \cup B_{i-1} \cup B'_{i-1}$  and, up to the  $(2i)$ th round, they choose the vertices in  $A_i \cup A'_i \cup B_i \cup B'_i$ .

Assume that  $S_0$  succeeds in  $i$  rounds of  $\text{SYM}(G_0)$  and  $D$  succeeds in  $2i$  rounds of  $\text{EF}(\mathcal{L}(G_0), \mathcal{L}(G_1))$  irrespective of the other players's strategies. Under this assumption, we describe the move of  $\mathcal{B}$  in the  $i$ th round of  $\text{SYM}(G_1)$  and then show that this move is successful.

Assume that  $\mathcal{A}$  colors an edge  $a$  and hence  $A_i = A_{i-1} \cup \{a\}$ . Simulating the  $(2i-1)$ th round of  $\text{EF}(\mathcal{L}(G_0), \mathcal{L}(G_1))$ , the player  $\mathcal{B}$  makes the spoiler choose  $a$ , a vertex in  $\mathcal{L}(G_1)$ , and then makes the duplicator apply the strategy  $D$ . Let  $a'$  denote the vertex chosen by the duplicator in  $\mathcal{L}(G_0)$ . Simulating the  $i$ th round of  $\text{SYM}(G_0)$ , the player  $\mathcal{B}$  makes  $\mathcal{A}_0$  color the edge  $a'$  thereby setting  $A'_i = A'_{i-1} \cup \{a'\}$  and then makes  $\mathcal{B}_0$  apply the strategy  $S_0$ . Let  $b'$  denote the edge colored by  $\mathcal{B}_0$  and  $B'_i = B'_{i-1} \cup \{b'\}$ . Next

$\mathcal{B}$  simulates the  $(2i)$ th round of  $\text{EF}(\mathcal{L}(G_0), \mathcal{L}(G_1))$ . He makes the spoiler choose  $b'$ , a vertex in  $\mathcal{L}(G_0)$ , and then makes the duplicator apply  $D$ . Let  $b$  denote the vertex chosen by the duplicator in  $\mathcal{L}(G_1)$ . Finally,  $\mathcal{B}$  colors the edge  $b$  and hence  $B_i = B_{i-1} \cup \{b\}$ .

We now have to show that  $S(D, S_0)$  succeeds in the  $i$ th round irrespective of the player  $\mathcal{A}$ 's strategy. Since by our assumption  $S_0$  succeeds against any strategy of  $\mathcal{A}_0$ , the subgraphs  $A'_i$  and  $B'_i$  of  $G_0$  are isomorphic. By the definition of a line graph, the subgraphs of  $\mathcal{L}(G_0)$  induced by the vertex sets  $A'_i$  and  $B'_i$  are isomorphic too. As easily seen from the description of  $S(D, S_0)$ , the duplicator constructs  $A'_i$  from  $A_i$  and  $B_i$  from  $B'_i$ . Since by our assumption  $D$  succeeds against any strategy of the spoiler, the subgraphs induced by  $A_i$  in  $\mathcal{L}(G_1)$  and by  $A'_i$  in  $\mathcal{L}(G_0)$  are isomorphic, as well as the subgraphs induced by  $B_i$  and  $B'_i$  are isomorphic. It follows that the subgraphs of  $\mathcal{L}(G_1)$  induced by  $A_i$  and  $B_i$  are isomorphic. Since these are the line graphs of the subgraphs  $A_i$  and  $B_i$  of  $G_1$ , the latter two are edge-isomorphic. By the condition imposed on  $G_1$ , neither  $A_i$  nor  $B_i$  have a connected component  $K_3$ . By the Whitney theorem, we conclude that  $A_i$  and  $B_i$  are isomorphic and therefore the strategy  $S(D, S_0)$  of  $\mathcal{B}$  does succeed in  $\text{SYM}(G_1)$  independently of  $\mathcal{A}$ 's strategy.  $\square$

**Remark 9.** Propositions 7 and 8 actually hold true for any edge-coloring game in place of  $\text{SYM}(G)$  if this game has isomorphism-invariant winning conditions.

We are now prepared to prove Claim 1 of Theorem 4.

**Corollary 10.** For odd  $n$ ,  $L_{\text{sym}}(P_n) > \frac{1}{2} \log(n-1) - 1$  and  $L_{\text{sym}}(C_n) > \frac{1}{2} \log n - \frac{1}{2}$ .

**Proof.** By Proposition 2,  $L_{\text{sym}}(P_{n+1}) = (n+1)/2$ . A weaker bound  $L_{\text{sym}}(P_n) > \frac{1}{4} \log n - \frac{1}{2}$  follows from Proposition 7, with  $G_1 = P_n$  and  $G_0 = P_{n+1}$ , and from Proposition 3. To obtain the bound claimed, notice that  $\mathcal{L}(P_m) = P_{m-1}$  and apply Proposition 8 instead of Proposition 7. For cycles the proof is the same and uses the fact that  $\mathcal{L}(C_m) = C_m$ .  $\square$

### 3.2. Upper bound

We now prove Claim 2 of Theorem 4.

**Proposition 11.** If  $n$  is odd,

$$L_{\text{sym}}(P_n) \leq (3.5 + o(1)) \log^2 n \quad \text{and} \quad L_{\text{sym}}(C_n) \leq (3.5 + o(1)) \log^2 n.$$

**Proof.** We prove the proposition for paths in full detail and then briefly notice what should be changed for cycles.

Given a subgraph  $A$  of  $P_n$ , we denote its size by  $|A|$ . The distance between two subgraphs  $A$  and  $B$ , denoted by  $d(A, B)$ , is the minimum distance between vertices  $u \in \mathcal{V}(A)$  and  $v \in \mathcal{V}(B)$ .



We describe a strategy of  $\mathcal{A}$  that aims to destroy the isomorphism between the red and the blue subgraphs possibly sooner. All moves of  $\mathcal{A}$  are split into consecutive series. The first move of each series creates a new component of the red subgraph and every subsequent move of the series prolongs the component in one edge. The component created by  $\mathcal{A}$  during the  $j$ th series will be denoted by  $A_j$ . It will be always the case that

$$|A_{j+1}| < |A_j|. \tag{1}$$

*Convention:* Throughout our description of  $\mathcal{A}$ 's strategy, we assume that  $\mathcal{B}$  plays optimally against this strategy, that is, keeps the isomorphism between the red and the blue subgraphs as long as possible. This, together with condition (1), implies that in the first move of every series,  $\mathcal{B}$  also must start constructing a new component of the blue subgraph and in each subsequent move of the series he must extend this component (or otherwise  $\mathcal{B}$  violates the isomorphism and loses immediately). The component created by  $\mathcal{B}$  during the  $j$ th series will be denoted by  $A'_j$ .

**Definition 12.** In any position of the game such that it is  $\mathcal{A}$ 's turn, we call two red components  $A_i$  and  $A_j$  a *distinctive pair* if

- (1)  $d(A_i, A_j) \neq 2$ ,
- (2) no edge between  $A_i$  and  $A_j$  has been chosen by the players,
- (3)  $d(A'_i, A'_j) \neq d(A_i, A_j)$  or between  $A'_i$  and  $A'_j$  there is at least one edge chosen by the players.

*Notation:* We number all edges of  $P_n$  from one end edge to the other end edge. For notational convenience we identify the edges with their numbers  $1, 2, \dots, n$ . By  $s_j$  and  $f_j$  ( $s'_j$  and  $f'_j$  resp.) we denote the edges chosen by  $\mathcal{A}$  ( $\mathcal{B}$  resp.) in the first and the last moves of the  $j$ th series. Note that it is unnecessary that  $s_j$  and  $f_j$  are the end edges of  $A_j$  but most often this will be so.

Set

$$t = 4\lceil \log n \rceil + 22. \tag{2}$$

With the exception of a few last series, the number of moves in the  $j$ th series and hence the length of  $A_j$  will be  $t - j$ . The parameter  $t$  is chosen large enough to ensure that, until the end of the game,  $t - j$  is a positive number; the proof is given by Claim 1 below.

To avoid separately handling several exceptional cases of small  $n$ , we just assume  $n$  to be sufficiently large to satisfy the inequality

$$n > 14t. \tag{3}$$

The cases of smaller  $n$  are covered by the  $o(1)$  term in the statement of Proposition 11.

**The first series of moves by  $\mathcal{A}$ .** In the first move  $\mathcal{A}$  chooses the middle edge of  $P_n$ , that is,  $s_1 = (n + 1)/2$ . Without loss of generality assume that  $s'_1 < s_1$ . Then in the next moves  $\mathcal{A}$  chooses the edges  $s_1 + 1, s_1 + 2, \dots, s_1 + t - 2 = f_1$ .

In our further description of  $\mathcal{A}$ 's strategy, we distinguish two phases of the game.

**Phase 1:**  $\mathcal{A}$  enforces appearance of a distinctive pair  $\{C_0, D_0\}$ .

**The  $j$ th series of moves,  $j > 1$ .** We assume that after completion of the preceding series the following conditions are met for  $m = j - 1$ .

*Condition 0:*  $|A_1| > |A_2| > \dots > |A_{m-1}| > |A_m|$ .

*Condition 1:*  $n - f_m > 2$ .

*Condition 2:* No vertex on the right from  $f_m$  has been chosen by the players.

*Condition 3:* The edges of  $A_1, A_2, \dots, A_m$  have been chosen in the ascending order.

In particular,

$$s_1 < f_1 < s_2 < f_2 < \dots < s_m < f_m$$

and  $s_p$  and  $f_p$  are the end edges of  $A_p$ , for every  $p \leq m$ .

*Condition 4:* For notational convenience, define  $A_0$  to be the subgraph of  $P_n$  induced by the first vertex of  $P_n$  ( $A_0$  has a single vertex and no edge). Let  $q$  be such that  $A'_m$  is between  $A_{q-1}$  and  $A_q$ . Then  $A'_q, A'_{q+1}, \dots, A'_{m-1}$  all are also between  $A_{q-1}$  and  $A_q$ , exactly in this order (in the direction either from  $A_{q-1}$  to  $A_q$  or from  $A_q$  to  $A_{q-1}$ ). In addition,  $d(A'_{p-1}, A'_p) = d(A_{p-1}, A_p)$  for every  $q < p \leq m$ .

*Condition 5:*  $n - (s_p - 1) > d(A_{p-1}, A_p)$  for every  $1 < p \leq m$ .

Observe that Conditions 0–5 are obeyed after the first series of moves, that is, they are true for  $m = 1$ . Condition 1 follows from (3); Conditions 2 and 3 follow from the description of the first series; Conditions 0, 4, and 5 for the first series are trivial. The fulfillment of Conditions 0–5 for every series excepting the last series of Phase 1 will be proven in Claim 1 below.

Define a function  $\phi$  by  $\phi(x) = x + \lceil (n - x)/2 \rceil$ . In the first move of the  $j$ th series  $\mathcal{A}$  chooses

$$s_j = \phi(f_{j-1}). \quad (4)$$

*Comment:* This choice of  $s_j$  implies Condition 5 for  $m = j$ , unless the  $j$ th series concludes Phase 1. The function  $\phi$  provides the smallest value of  $s_j$  with this property. Note also that  $s_j$  really starts a new component due to Condition 1 for  $m = j - 1$ .

The further moves of  $\mathcal{A}$  depend much on the first move of  $\mathcal{B}$  in the series.

*Case 1:*  $s'_j < s_j$ .

$\mathcal{A}$  continues the series choosing  $s_j + 1, s_j + 2, \dots, f_j$ . The last edge in the series is determined from the following rules.

*Rule 1:* If  $n - (s_j - 1) \leq t - j$ , then  $f_j = n$ . Otherwise:

*Rule 2:* If  $n - (\phi(s_j + t - j - 1) - 1) \geq t - (j + 1)$ , then  $f_j = s_j + t - j - 1$ .

*Rule 3:* If  $n - (\phi(s_j + t - j - 1) - 1) < t - (j + 1)$ , then  $f_j$  is the smallest number such that  $n - \phi(f_j) < f_j - s_j$ .

*Comment:* These rules can be reformulated as follows.  $A_j$  is constructed edge by edge in the ascending order starting from  $s_j$  so that  $|A_j| = t - j$  with two exceptional cases:

- (i) Assignment  $|A_j| = t - j$  and starting the next component  $A_{j+1}$  from  $s_{j+1} = \phi(f_j)$  in the ascending order could not give  $|A_{j+1}| = t - (j + 1)$  because the final edge

$n$  would be reached earlier than in  $t - (j + 1)$  moves. In this case  $|A_j|$  is taken as smaller than  $t - j$  as possible to keep the relation  $|A_{j+1}| < |A_j|$ , where  $A_{j+1}$  starts at  $\phi(f_j)$  and finishes at  $n$ .

(ii)  $|A_j|$  is shorter because the last edge  $n$  is reached.

Case (i) corresponds to Rule 3, and Case (ii) corresponds to Rule 1.

It is also useful to make the following observation. Assume that Case 1 occurs in the  $(j - 1)$ th and in the  $j$ th series. Then, if Case (i) occurs in the  $(j - 1)$ th series, Case (ii) must occur in the  $j$ th series. Vice versa, if Case (ii) occurs in the  $j$ th series, then Case (i) must occur in the  $(j - 1)$ th series.

*Subcase 1a:*  $f_{j-1} < s'_j < s_j$ . If  $f_j < n$ , then Phase 1 of the game continues and  $\mathcal{A}$  starts the  $(j + 1)$ th series. If  $f_j = n$ , then  $\mathcal{B}$  by (4) has not enough room to make  $A'_j$  so long as  $A_j$ . Therefore the isomorphism is violated and  $\mathcal{B}$  loses.

*Subcase 1b:*  $s'_j < s_{j-1}$ .

Let  $q$  be as in Condition 4 for  $m = j - 1$ . If  $A'_j$  is not between  $A_{q-1}$  and  $A_q$ , then  $A_{j-1}$  and  $A_j$  are a distinctive pair. Items 2 and 3 of Definition 12 are clear, and Item 1 is proved in Claim 2 below.  $\mathcal{A}$  therefore terminates Phase 1 and takes  $C_0 = A_{j-1}$ ,  $D_0 = A_j$ . Suppose that  $A'_j$  is between  $A_{q-1}$  and  $A_q$ .

If  $d(A'_j, A'_{j-1}) \neq d(A_j, A_{j-1})$ , then  $A_{j-1}$  and  $A_j$  again are a distinctive pair and  $\mathcal{A}$  terminates Phase 1 with  $C_0 = A_{j-1}$ ,  $D_0 = A_j$ . Suppose that  $d(A'_j, A'_{j-1}) = d(A_j, A_{j-1})$ . If Condition 4 is violated for  $m = j$ , then  $A'_j$  must be between  $A'_{j-1}$  and  $A'_{j-2}$  and therefore  $A_{j-1}$  and  $A_{j-2}$  become a distinctive pair. In this case  $\mathcal{A}$  terminates Phase 1 and takes  $C_0 = A_{j-1}$ ,  $D_0 = A_{j-2}$ . A distinctive pair does not exist in the only case that Condition 4 holds true for  $m = j$ . In this case Phase 1 of the game continues and  $\mathcal{A}$  starts the  $(j + 1)$ th series of moves.

Let us pay special attention to the case when  $f_j = n$ . Condition 4 then cannot happen in view of Condition 5 for  $p = q$ . Therefore, a distinctive pair exists and the game goes to Phase 2.

*Case 2:*  $s'_j > s_j$ .  $\mathcal{A}$  continues the series choosing  $s_j - 1, s_j - 2, \dots, f_j = \max\{f_{j-1} + 2, s_j - t + j + 1\}$  unless this maximum equals  $f_{j-1} + 3$ . In the latter case the series is shorter in one move and  $f_j = f_{j-1} + 4$ . The series  $A_{j-1}$  and  $A_j$  are a distinctive pair and  $\mathcal{A}$  terminates Phase 1 with  $C_0 = A_{j-1}$ ,  $D_0 = A_j$ .

*Comment 3:* In other words,  $A_j$  is constructed starting from  $s_j$ , edge by edge in the descending order, until  $|A_j| = t - j$  or  $d(A_j, A_{j-1}) = 1$ . Special care is taken to ensure that  $d(A_j, A_{j-1}) \neq 2$ , one of the defining properties of a distinctive pair.

The above program for  $\mathcal{A}$  makes sense as long as  $t - j$ , the length assigned to  $A_j$ , is a positive number, which is the case for at most  $t - 1$  series of moves. In fact, we prove that  $\lceil \log n \rceil$  series suffice for  $\mathcal{A}$  to terminate Phase 1, that is, either to win the game or to find a pair  $\{C_0, D_0\}$  (Item 3 of Claim 1 below). We will prove that the pair  $\{C_0, D_0\}$  found by  $\mathcal{A}$  is indeed distinctive (Claim 2). We also should prove our assumption that Conditions 0–5 hold true at the start of every series of moves within Phase 1 (Claim 1, Item 6). To verify (1), in addition to Condition 0 we need to prove the inequality  $|A_j| < |A_{j-1}|$  for components created in the last two series of Phase 1 (Claim 1, Item 4).

**Claim 1.** Let  $\mathcal{A}$  play as described above and  $\mathcal{B}$  play optimally against this strategy of  $\mathcal{A}$ . Suppose that  $\mathcal{A}$  has made the  $j$ th series of moves in Phase 1. Then

- (1)  $d(A_j, A_{j-1}) < \frac{1}{2}d(A_{j-1}, A_{j-2})$ , if  $j \geq 3$ .
- (2)  $d(A_j, A_{j-1}) < \left(\frac{1}{2}\right)^{j-1} n$ , if  $j \geq 2$ .
- (3)  $j \leq \lceil \log n \rceil$ .
- (4)  $|A_j| < |A_{j-1}|$ , if  $j \geq 2$ .
- (5)  $|A_j| > \log n + 4$ .
- (6) If the  $j$ th series is not last in Phase 1, then Conditions 0–5 hold true for  $m = j$ .

**Proof.** We proceed by induction on  $j$ . Consider two base cases  $j = 1, 2$ . Item 1 is trivial. Item 2 reads  $d(A_2, A_1) < \frac{1}{2}n$  and is straightforward by the description of  $\mathcal{A}$ 's strategy. Item 3 is equivalent to  $n > 2$  and follows from (3) and (2). By (3) and by the description of  $\mathcal{A}$ 's strategy,  $|A_1| = t - 1$  and  $|A_2| = t - 2$  and hence Items 4 and 5 are true. Taking into account (3), it is also easy to check Item 6.

Assume that Items 1–6 are true for the  $(j - 1)$ th series and prove each of them for the  $j$ th series.

*Item 1:* By the induction assumption applied to Item 5 we have

$$|A_{j-1}| > 4. \quad (5)$$

Assume that  $A_j$  is created in Case 1. Then

$$\begin{aligned} s_{j-2} < f_{j-2} < s_{j-1} < f_{j-1} < s_j < f_j, \\ d(A_j, A_{j-1}) = s_j - f_{j-1} - 1, \quad d(A_{j-1}, A_{j-2}) = s_{j-1} - f_{j-2} - 1. \end{aligned}$$

By the choice of  $s_{j-1}$  (see (4)),

$$d(A_{j-2}, A_{j-1}) + 2 \geq |A_{j-1}| + d(A_j, A_{j-1}) + |A_j| + (n - f_j).$$

By the choice of  $s_j$ ,

$$|A_j| + (n - f_j) = n - s_j + 1 > d(A_j, A_{j-1}).$$

Taking into account (5), we infer that

$$d(A_{j-2}, A_{j-1}) > 2d(A_j, A_{j-1}). \quad (6)$$

Assume now that  $A_j$  is created in Case 2. Then

$$\begin{aligned} s_{j-2} < f_{j-2} < s_{j-1} < f_{j-1} < f_j < s_j, \\ d(A_j, A_{j-1}) = f_j - f_{j-1} - 1, \quad d(A_{j-1}, A_{j-2}) = s_{j-1} - f_{j-2} - 1. \end{aligned}$$

By the choice of  $s_{j-1}$ ,

$$d(A_{j-2}, A_{j-1}) + 2 \geq |A_{j-1}| + d(A_j, A_{j-1}) + |A_j| + (n - s_j).$$

By the choice of  $s_j$ ,

$$n - s_j + 1 > d(A_j, A_{j-1}) + (|A_j| - 1).$$

Taking into account (5), we again easily infer (6).

Item 2: By the induction assumption,

$$d(A_{j-1}, A_{j-2}) < \left(\frac{1}{2}\right)^{j-2} n.$$

Using Item 1, we derive

$$d(A_j, A_{j-1}) < \frac{1}{2}d(A_{j-1}, A_{j-2}) < \left(\frac{1}{2}\right)^{j-1} n$$

as required.

Item 3: As  $d(A_j, A_{j-1}) \geq 1$ , this is a consequence of Item 2.

Item 4: Note that the component  $A_{j-1}$  followed by  $A_j$  can be constructed according to one of six scenarios.

Scenario 1. Case 1, Rule 2 followed by Case 1, Rule 2.

Scenario 2. Case 1, Rule 2 followed by Case 1, Rule 1.

Scenario 3. Case 1, Rule 2 followed by Case 1, Rule 3.

Scenario 4. Case 1, Rule 2 followed by Case 2.

Scenario 5. Case 1, Rule 3 followed by Case 1, Rule 1.

Scenario 6. Case 1, Rule 3 followed by Case 2.

In Scenarios 1–4 we have  $|A_{j-1}| = t - (j - 1)$  and  $|A_j| \leq t - j$ . In Scenario 5,  $|A_j| = n - (s_j - 1)$  by Rule 2 and the inequality  $|A_j| < |A_{j-1}|$  is enforced by Rule 3. In Scenario 6,  $|A_j|$  is even shorter than in Scenario 5 because  $|A_j| \leq s_j - f_{j-1} - 1$  and  $s_j - f_{j-1} - 1 < n - (s_j - 1)$  by the choice of  $s_j$ .

Item 5: We distinguish the same six scenarios as above.

Scenarios 1 and 2. We have  $|A_j| = t - j$  and the claim follows from Item 3 and (2).

Scenario 3 will be considered a bit later.

Scenario 4. We have  $|A_{j-1}| = t - (j - 1)$ . Since  $A_{j-1}$  is constructed according to Rule 2, we have  $n - (s_j - 1) \geq t - j$ . Together with (4), this implies that  $s_j - f_{j-1} - 1 \geq t - j - 2$ . As in Case 2  $|A_j| \geq \min\{t - j - 1, s_j - f_{j-1} - 1\}$ , we have  $|A_j| \geq t - j - 2$ . The claim now follows from Item 3 and (2).

Scenario 5. According to Rule 3,

$$n - \phi(f_{j-1}) < f_{j-1} - s_{j-1} \tag{7}$$

and

$$n - \phi(f_{j-1} - 1) \geq (f_{j-1} - 1) - s_{j-1}. \tag{8}$$

From (7) we infer

$$n - s_{j-1} < 3(f_{j-1} - s_{j-1}) + 1 \tag{9}$$

and from (8) we infer

$$n - f_{j-1} + 1 \geq 2(f_{j-1} - s_{j-1} - 1). \tag{10}$$

As  $A_{j-1}$  is constructed according to Rule 3 and therefore the assumption of Rule 1 is false, we have

$$n - (s_{j-1} - 1) > t - (j - 1). \tag{11}$$

From (9) and (11) we conclude that

$$f_{j-1} - s_{j-1} > (t - j - 1)/3. \quad (12)$$

By Rule 1,  $|A_j|$  is equal to

$$n - (s_j - 1) = n - \phi(f_{j-1}) + 1 \geq (n - f_{j-1} + 1)/2.$$

Using (10) and (12), we obtain

$$n - (s_j - 1) \geq f_{j-1} - s_{j-1} - 1 > (t - j - 1)/3 - 1. \quad (13)$$

Hence  $|A_j| > (t - j - 4)/3$  and the claim follows from Item 3 and (2).

Scenario 6. Since  $A_{j-1}$  is constructed according to Rule 3, we have

$$(s_j - 1) - f_{j-1} \leq n - (s_j - 1) - 1 \leq (t - j) - 2.$$

In Case 2 we have  $|A_j| \geq \min\{t - j - 1, s_j - f_{j-1} - 1\}$  and hence  $|A_j| \geq s_j - f_{j-1} - 1$ . The latter value, by the choice of  $s_j$ , is no less than  $n - (s_j - 1) - 2$ . Similarly to Scenario 5, relation (13) is true and hence

$$|A_j| \geq n - (s_j - 1) - 2 > (t - j - 10)/3.$$

It remains to apply Item 3 and (2).

Scenario 3. By Rule 3,  $|A_j| > n - (s_{j+1} - 1)$ . Applying precisely the same argument as in Scenario 5, similarly to (13) we derive

$$n - (s_{j+1} - 1) > (t - (j + 1) - 4)/3.$$

It remains to apply Item 3 and (2).

*Item 6:* The assumption made in this item implies that for the  $j$ th series we have Case 1 and  $f_j < n$ , that is, either Rule 2 or Rule 3 was applied. Condition 0 follows from the induction assumption and Item 4. Condition 1 is ensured by Rules 2 and 3. Conditions 2 and 3 can be violated only in Case 2 which always terminates Phase 1. Condition 4 is obvious if  $q = m$ . If  $q \leq m - 1$ , the condition follows from the induction assumption (see explanations accompanying the description of Case 1). Condition 5 follows from the choice of  $s_p$  (see (4)) and Condition 3.  $\square$

**Claim 2.** *If  $\mathcal{A}$  finishes Phase 1 with some  $C_0$  and  $D_0$ , then these components are a distinctive pair.*

**Proof.** If  $l$  is the number of series in Phase 1, then either  $\{C_0, D_0\} = \{A_l, A_{l-1}\}$  or  $\{C_0, D_0\} = \{A_{l-1}, A_{l-2}\}$ . It is easy to check that this pair is always chosen so that items 2 and 3 of Definition 12 are true. Let us check item 1, i.e.,  $d(C_0, D_0) \neq 2$ .

If  $\{C_0, D_0\}$  is chosen in Case 1, then  $d(C_0, D_0)$  equals either  $s_l - f_{l-1} - 1$  or  $s_{l-1} - f_{l-2} - 1$ . By choice (4) of  $s_j$  these values are not less than  $|A_l| - 2$  and  $|A_{l-1}| - 2$ , respectively. By Item 5 of Claim 1,  $d(C_0, D_0) > 2$ .

If  $\{C_0, D_0\} = \{A_l, A_{l-1}\}$  is chosen in Case 2, then the inequality  $d(C_0, D_0) \neq 2$  is true by the choice of the length  $|A_l|$  in Case 2.  $\square$

*Notation:* In the sequel we denote the number of series in Phase 1 by  $l$ . Let  $t' = |A_l|$ , the length of the shortest component created in Phase 1.

**Claim 3.** (1)  $l \leq \lceil \log n \rceil$ .  
 (2)  $t' > \log n + 4$ .

**Proof.** The claim is a direct corollary of Items 3 and 5 of Claim 1.  $\square$

**Phase 2:**  $\mathcal{A}$  reduces the distance between components of a distinctive pair to 1.

**The  $(l + j)$ th series of moves (the  $j$ th series in Phase 2).** Let  $\{C_{j-1}, D_{j-1}\}$  be the distinctive pair created in the preceding series. In particular,  $\{C_0, D_0\}$  is the output of Phase 1. If  $d(C_{j-1}, D_{j-1}) = 1$ ,  $\mathcal{A}$  makes the last move described below. Suppose that  $s(C_{j-1}, D_{j-1}) \geq 3$ . Let  $a$  and  $b$  be two nearest edges in  $C_{j-1}$  and  $D_{j-1}$ , respectively. Without loss of generality, assume  $a < b$ . In the first move of the series  $\mathcal{A}$  chooses the medium edge  $s_{l+j} = \lceil (a + b - 1)/2 \rceil$ . The further moves of  $\mathcal{A}$  depend on the first move of B.

*Case 1:*  $a < s'_{l+j} < s_{l+j}$ .  $\mathcal{A}$  continues the series choosing  $s_{l+j} + 1, s_{l+j} + 2, \dots, f_{l+j} = \min\{b - 2, s_{l+j} + (t' - j - 1)\}$  unless  $s_{l+j} + (t' - j - 1) = b - 3$ . In the latter case  $\mathcal{A}$  stops one move earlier at  $f_{l+j} = b - 4$ . The new distinctive pair is  $C_j = A_{l+j}$  and  $D_j = D_{j-1}$ .

*Case 2:*  $s_{l+j} < s'_{l+j} < b$ .  $\mathcal{A}$  continues the series choosing  $s_{l+j} - 1, s_{l+j} - 2, \dots, f_{l+j} = \max\{a + 2, s_{l+j} - (t' - j - 1)\}$  unless  $s_{l+j} - (t' - j - 1) = a + 3$ . In the latter case  $\mathcal{A}$  stops one move earlier at  $f_{l+j} = a + 4$ . The new distinctive pair is  $C_j = C_{j-1}$  and  $D_j = A_{l+j}$ .

*Case 3:*  $s'_{l+j}$  is not between  $C_{j-1}$  and  $D_{j-1}$ .  $\mathcal{A}$  continues the series choosing  $s_{l+j} + 1, s_{l+j} - 1, s_{l+j} + 2, s_{l+j} - 2$  and so on until one of the following situations happens:

- (1)  $d(C_{j-1}, A_{l+j}) = d(A_{l+j}, D_{j-1}) = 1$ .
- (2)  $|A_{l+j}| = t' - j$  but  $d(C_{j-1}, A_{l+j}) \neq 2$  and  $d(A_{l+j}, D_{j-1}) \neq 2$ .
- (3)  $t' - j - 3 \leq |A_{l+j}| < t' - j$  and  $d(C_{j-1}, A_{l+j}) = d(A_{l+j}, D_{j-1}) = 3$ .

As shown in Claim 5 below, at least one of the pairs  $\{C_{j-1}, A_{l+j}\}$  or  $\{A_{l+j}, D_{j-1}\}$  is distinctive and  $\mathcal{A}$  takes it as  $\{C_j, D_j\}$ .

**The last move.** As soon as  $\mathcal{A}$  creates a distinctive pair  $\{C, D\}$  with  $d(C, D) = 1$ , he chooses the edge between  $C$  and  $D$  and wins. To keep isomorphism,  $\mathcal{B}$  should make, in place of two corresponding blue components  $C'$  and  $D'$ , a new component of length  $|C'| + |D'| + 1$ . This task is impossible to implement as  $d(C', D') > 1$ .

The description of  $\mathcal{A}$ 's strategy in Phase 2 makes sense as long as the value assigned to the length of a series is a positive number. The smallest value that can be assigned for the  $(l + j)$ th series, if it is not last in Phase 2, is  $t' - j - 3$ . Hence the condition  $j < t' - 4$  is required. The same condition is required also in order to pass by the forbidden distance  $d(C_j, D_j) = 2$  in cases  $d(C_{j-1}, D_{j-1}) = 4, 5, 6$ . We will show that  $t' - 4$  series are indeed enough for  $\mathcal{A}$  to finish Phase 2 (Claim 4, Item 3) and that  $\{C, D\}$ , the outcome of Phase 2, is indeed a distinctive pair with  $d(C, D) = 1$  (Claim 5). We also will verify (1) for the components created during Phase 2 (Claim 4, Item 4).

**Claim 4.** Let  $\mathcal{A}$  play as described above and let  $\mathcal{B}$  play optimally against this strategy of  $\mathcal{A}$ . Suppose that  $\mathcal{A}$  has made the  $(l+j)$ th series of moves in Phase 2. Then

- (1)  $d(C_j, D_j) \leq \frac{1}{2}d(C_{j-1}, D_{j-1})$ .
- (2)  $d(C_j, D_j) \leq \left(\frac{1}{2}\right)^j d(C_0, D_0)$ .
- (3)  $j < \log n - 1$ .
- (4)  $|A_{l+j}| < |A_{l+(j-1)}|$ .

**Proof.** Item 1 is clear from  $\mathcal{A}$ 's strategy. Item 2 follows from Item 1. Item 3 follows from Item 2, because  $d(C_j, D_j) \geq 1$  and  $d(C_0, D_0) < \frac{1}{2}n$ .

Item 4 is given by easy inspection of  $\mathcal{A}$ 's strategy. If the  $(l+j)$ th series is neither last nor last but one in Phase 2, then  $|A_{l+(j-1)}| = t' - (j-1)$  and  $|A_{l+j}| = t' - j$ . If the  $(l+j)$ th series is last but one, then  $|A_{l+(j-1)}| = t' - (j-1)$  and  $t' - j - 3 \leq |A_{l+j}| \leq t' - j$ . If the  $(l+j)$ th series is last, then either  $|A_{l+(j-1)}| = t' - (j-1)$  and  $|A_{l+j}| \leq t' - j$  or  $t' - (j-1) - 3 \leq |A_{l+(j-1)}| \leq t' - (j-1) - 1$  and  $|A_{l+j}| = 1$ . In the latter case Item 4 follows from Item 3 and Claim 3.  $\square$

**Claim 5.** Let  $\mathcal{A}$  play as described above and let  $\mathcal{B}$  play optimally against this strategy of  $\mathcal{A}$ . Suppose that  $\mathcal{A}$  has made the  $(l+j)$ th series of moves in Phase 2. Then  $\{C_j, D_j\}$  is a distinctive pair and eventually  $d(C_j, D_j) = 1$ .

**Proof.** Conditions 1 and 2 in Definition 12 are directly enforced by  $\mathcal{A}$ 's strategy. Condition 3 is easy to see if the  $(l+j)$ th series of moves was done in Case 1 or 2. Assume it was done in Case 3. We have to prove that at least one of the pairs  $\{C_{j-1}, A_{l+j}\}$  or  $\{A_{l+j}, D_{j-1}\}$  is distinctive. Suppose, to the contrary, that they both are not. This implies for the corresponding blue components  $C', D', A'_{l+j}$  that no edge is chosen between  $C', A'_{l+j}$  and between  $A'_{l+j}, D'$ , and that  $d(C', A'_{l+j}) = d(C_{j-1}, A_{l+j})$  and  $d(A'_{l+j}, D') = d(A_{l+j}, D_{j-1})$ . It follows that before the  $(l+j)$ th series no edge was chosen between  $C'$  and  $D'$  and  $d(C', D') = d(C_{j-1}, D_{j-1})$ . Thus,  $C_{j-1}$  and  $D_{j-1}$  could not be a distinctive pair, a contradiction.  $\square$

It remains to estimate the total number of moves in the game if  $\mathcal{A}$  follows the above strategy. Since  $\{C_0, D_0\}$  is always either  $\{A_l, A_{l-1}\}$  or  $\{A_{l-1}, A_{l-2}\}$ , we have  $d(C_0, D_0) \leq d(A_{l-1}, A_{l-2})$ . By Item 2 of Claim 4 and Item 2 of Claim 1, in the last  $(l+k)$ th series of moves of Phase 2 we have  $d(C_k, D_k) < \left(\frac{1}{2}\right)^{l+k-2} n$  and therefore  $l+k \leq \lceil \log n \rceil + 2$ . As the  $j$ th series has at most  $t-j$  moves, the total number of rounds in the game does not exceed  $\sum_{j=1}^{\lceil \log n \rceil + 2} (t-j) + 1 = 3.5 \log^2 n + O(\log n)$ .

The proof of Proposition 11 for paths is complete.  $\square$

**Proof of Proposition 11** (for cycles (Sketch)). We employ the same idea as for paths and refer to strategies in Phases 1 and 2 described above. The moves of  $\mathcal{A}$  are split into series, and in a series  $\mathcal{A}$  creates a component of the red subgraph. The  $j$ th series consists of  $t-j$  moves, with a few possible exceptions in the end of the game.



We adopt the notion of a distinctive pair of components with only one refinement: The distance  $d(A, B)$  between two components  $A$  and  $B$  is the minimum length of a path that joins a vertex in  $A$  and a vertex in  $B$  and that consists of edges unchosen so far. Thus,  $d(A, B)$  may differ from the standard distance in a graph. The goal of  $\mathcal{A}$  is to create a distinctive pair and then to apply the strategy of Phase 2 literally. However, creation of a distinctive pair in cycles is a bit more complicated task. Namely, before applying the strategy of Phase 1 some additional efforts are needed.

In the first series of moves  $\mathcal{A}$  creates the component  $A_1$  and  $\mathcal{B}$ , not to lose immediately, creates the component  $A'_1$  of the same length. Denote two paths connecting  $A_1$  and  $A'_1$  by  $I_1$  and  $I_2$ , and their lengths by  $l_1$  and  $l_2$ . Note that one number of  $l_1$  and  $l_2$  is odd and the other is even. Without loss of generality assume  $l_1 > l_2 \geq 0$ .

*Case 1:*  $l_1$  is odd.  $\mathcal{A}$  starts the second series choosing  $s_2$ , the middle edge of  $I_1$ . If  $\mathcal{B}$  chooses  $s'_2$  in  $I_1$  between  $s_2$  and  $A'_1$ , then  $\mathcal{A}$  completes  $A_2$  and  $\{A_1, A_2\}$  is a distinctive pair. If  $\mathcal{B}$  chooses  $s'_2$  in  $I_1$  between  $A_1$  and  $s_2$ , then  $\mathcal{A}$  continues to play on  $I_1$  in the direction towards  $A'_1$  applying the strategy of Phase 1.

$\mathcal{B}$  has another possibility to try to avoid creating a distinctive pair: he can choose  $s'_2$  in  $I_2$  at the same distance from  $A'_1$  as between  $A_1$  and  $s_2$ . In this case, if  $\mathcal{A}$  continues to play Phase 1 on  $I_1$  in the direction towards  $A'_1$ ,  $\mathcal{B}$  can copy moves of  $\mathcal{A}$  in  $I_2$ . Nevertheless, since  $l_2 < l_1$ , eventually either the isomorphism will be violated, or a distinctive pair appears, or  $\mathcal{B}$  will be forced to switch back to  $I_1$ .

*Case 2:*  $l_1$  is even. Playing on  $I_1$  gives no gain for  $\mathcal{A}$  because  $\mathcal{B}$  can keep isomorphism using the involutory fixed-edge-free automorphism of  $I_1$  (this is a difference between the cases of paths and cycles). Therefore,  $\mathcal{A}$  should play in  $I_2$ . However, it is impossible for  $\mathcal{A}$  to adapt the strategy of Phase 1 directly because  $\mathcal{B}$  can just copy moves of  $\mathcal{A}$  in  $I_1$ . To prevent this,  $\mathcal{A}$  chooses  $s_2$  in  $I_1$  at distance  $(l_2 - 1)/2$  from  $A_1$ .

If  $\mathcal{B}$  chooses  $s'_2$  in  $I_1$  at the same distance from  $A'_1$ , then  $\mathcal{A}$  completes  $A_2$  and starts the third series choosing  $s_3$  at the center of  $I_2$ . After this everything goes through as in Case 1 with roles of  $I_1$  and  $I_2$  interchanged. Note that  $\mathcal{B}$  is not able to choose  $s'_3$  in  $I_1$  at the same distance from  $A'_1$  as between  $s_3$  and  $A_1$  because the corresponding edge is already occupied in the preceding series.

If  $\mathcal{B}$  chooses  $s'_2$  in  $I_1$  between  $s_2$  and  $A'_1$  but not at distance  $(l_2 - 1)/2$  from  $A'_1$ , then  $\mathcal{A}$  completes  $A_2$  so that  $\{A_1, A_2\}$  is a distinctive pair.

If  $\mathcal{B}$  chooses  $s'_2$  in  $I_1$  between  $A_1$  and  $s_2$  or chooses  $s'_2$  to be the middle edge of  $I_2$ , then  $\mathcal{A}$  continues the game on  $I_1$  as in Case 1.

**Remark 13.** Notice an essential difference between the Ehrenfeucht–Fraïssé game and the symmetry breaking–preserving game. While in each round of the former game  $\mathcal{B}$  is obliged to extend the isomorphism established in the preceding round, in the latter game no dependence between isomorphisms in two successive rounds is required. Eliminating this difference, let  $\text{SYM}+(G)$  be a modification of the symmetry breaking–preserving game  $\text{SYM}(G)$  in which  $\mathcal{B}$  not merely keeps the red and the blue subgraphs isomorphic but, moreover, extends the isomorphism between them from round to round. Clearly,  $L_{\text{sym}+}(G) \leq L_{\text{sym}}(G)$ .

Propositions 7 and 8 hold true for  $\text{SYM}+(G)$  with minor changes in the proofs. In particular, in the proof of Proposition 8 we need to apply a stronger form of

the Whitney theorem asserting that, with a few exceptions excluded by prohibiting a subgraph  $K_3$ , an isomorphism between  $\mathcal{L}(H_1)$  and  $\mathcal{L}(H_2)$  is induced by an isomorphism between  $H_1$  and  $H_2$  and, moreover, the latter is unique for graphs of size more than 1. Since Proposition 2 holds true for  $\text{SYM}+(G)$  as well, we obtain the same lower bounds  $L_{\text{sym}+}(P_n) = \Omega(\log n)$  and  $L_{\text{sym}+}(C_n) = \Omega(\log n)$  for odd paths and cycles. The upper bound of Proposition 11 can be improved to  $L_{\text{sym}+}(P_n) = O(\log n)$  and  $L_{\text{sym}+}(C_n) = O(\log n)$  for  $n$  odd. The proof becomes much simpler because now the red and blue components  $A_j$  and  $A'_j$  correspond to one another by the rules of the game rather than by having the same distinctive length. In particular,  $\mathcal{A}$  can now make each series of moves being of constant length.

It would be interesting to know how much the values of  $L_{\text{sym}+}(G)$  and  $L_{\text{sym}}(G)$  can differ from each other.

#### 4. Games on complete graphs

In this section we analyze the symmetry breaking–preserving game on the complete graph of order  $n$ . Unlike the preceding section where we used the knowledge of  $L_{\text{sym}}(P_n)$  and  $L_{\text{sym}}(C_n)$  for even  $n$ , we now have to estimate  $L_{\text{sym}}(K_n)$  for all  $n$ . The relation with the Ehrenfeucht–Fraïssé game can still give us some information. Similarly to Proposition 7 one can prove that, if  $L_{\text{EF}}(G_0, G_1) \geq 4L_{\text{sym}}(G_0) + 4$ , then  $L_{\text{sym}}(G_1) \leq L_{\text{sym}}(G_0)$ . Since  $L_{\text{EF}}(K_n, K_{n+1}) = n$ , it follows that either  $L_{\text{sym}}(K_n) > n/4 - 1$  for all  $n$  or  $L_{\text{sym}}(K_n) = O(1)$ . We here prove the latter alternative.

**Theorem 14.**  $L_{\text{sym}}(K_n) \leq 6$  for all  $n$ .

**Proof.** If  $n \leq 5$ , the assertion is trivial. We assume that  $n \geq 6$  and describe a strategy of  $\mathcal{A}$  breaking the isomorphism in at most 7 moves. In the first three rounds  $\mathcal{A}$  creates a 3-star in such a way that  $\mathcal{B}$  is not able to choose any edge connecting leafs of this star without immediately losing. This can be done so that one of the five positions in Fig. 1 occurs.

The next move of  $\mathcal{A}$  from Position 1 creates a triangle and simultaneously blocks creating a triangle by  $\mathcal{B}$ . In Positions 2 and 3 the player  $\mathcal{A}$  is able in the next three moves to create a  $K_4$  in such a way that  $\mathcal{B}$  cannot do the same.

*Game from Position 4:* In the next two rounds  $\mathcal{A}$  chooses the edges  $\{v_1, v_2\}$  and  $\{v_2, v_3\}$ . If  $\mathcal{B}$  in these rounds chooses two edges of the triangle  $T = \{u_1, u_2, u_3\}$  with the common vertex  $u_i$ , then  $\mathcal{A}$  chooses  $\{u_i, v_2\}$  and wins. Otherwise in the 6th and 7th moves  $\mathcal{A}$  chooses two edges of  $T$  and wins.

*Game from Position 5:* In the 4th and 5th rounds  $\mathcal{A}$  chooses the edges  $\{u_3, v_2\}$  and  $\{v_2, v_1\}$ , respectively. If  $\mathcal{B}$  in these rounds chooses edges not both in  $T$ , in the next two moves  $\mathcal{A}$  chooses two edges of  $T$  and wins. Assume therefore that in the 4th and 5th rounds  $\mathcal{B}$  chooses  $\{u_3, u_2\}$  and  $\{u_2, u_1\}$  (the choice of  $\{u_3, u_1\}$  and  $\{u_1, u_2\}$  is symmetric). In the next round  $\mathcal{A}$  chooses  $\{u_1, v_1\}$  and  $\mathcal{B}$  is forced to choose  $\{u_1, v_2\}$ . Finally,  $\mathcal{A}$  chooses  $\{u_2, v_1\}$  and wins.  $\square$

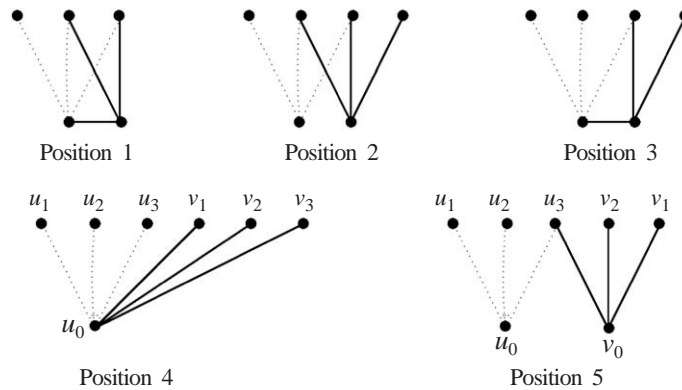


Fig. 1. The first three rounds of  $\text{SYM}(K_n)$ .  $\mathcal{A}$ 's edges are dotted and  $\mathcal{B}$ 's edges are continuous.

**Remark 15.** A more lengthy and complicated analysis allows us to lower the bound 6 of Theorem 14 to 5.

Finally we briefly discuss the case of complete bipartite graphs. If at least one of  $m$  and  $l$  is even, then  $K_{m,l}$  has an involutory automorphism without fixed edges and, by Proposition 2,  $L_{\text{sym}}(K_{m,l})$  is maximum possible for graphs of this size. If both  $m$  and  $l$  are odd,  $K_{m,l}$  has no involutory fixed-edge-free automorphism but removal of one edge from  $K_{m,l}$  leads to a graph  $K_{m,l} - e$  with such an automorphism. It is therefore interesting to estimate  $L_{\text{sym}}(K_{m,l})$  for  $ml$  odd.

An easy lower bound is

$$L_{\text{sym}}(K_{m,l}) \geq \max \left\{ \frac{m-1}{2}, \frac{l-1}{2} \right\}. \tag{14}$$

The appropriate strategy of  $\mathcal{B}$  is based on a partial involutory automorphism of  $K_{m,l}$  constructed during the course of the game. The automorphism leaves one vertex class fixed. Whenever during the course of the game in the other vertex class a new vertex of the red subgraph appears, the automorphism interchanges it with an arbitrary vertex in this class that is unchosen so far.

Note that, if  $ml$  is odd, then  $K_{m,l} - e$ ,  $K_{m-1,l}$ ,  $K_{m,l-1}$ , and  $K_{m-1,l-1}$  all have involutory automorphisms without fixed edges. One could therefore try to apply Propositions 7 and 8 with  $G_1 = K_{m,l}$  and  $G_0$  one of these graphs. However, in all the cases  $L_{\text{EF}}(G_0, G_1)$  and  $L_{\text{EF}}(\mathcal{L}(G_0), \mathcal{L}(G_1))$  are not large enough to give us anything better than (14).

**Problem 16.** What are the asymptotics of  $L_{\text{sym}}(K_{n,n})$  for odd  $n$ ?

*Note added in proof:* Problem 16 was recently solved by Oleg Pikhurko who proved that  $L_{\text{sym}}(K_{n,n}) \leq 2n + 38$  for odd  $n \geq 51$ . This matches up to a constant factor the lower bound (14) that reads  $L_{\text{sym}}(K_{n,n}) \geq (n-1)/2$ . Pikhurko's result is actually more general and implies that, if  $m \leq l \leq m^{O(1)}$ , then  $L_{\text{sym}}(K_{m,l}) = O(l)$ .

## Acknowledgements

We thank Helmut Veith for helpful discussions on the Ehrenfeucht–Fraïssé games, Oleg Pikhurko for useful comments on the paper, and an anonymous referee for a considerable simplification of our original proof of Theorem 14.

## References

- [1] H.-D. Ebbinghaus, J. Flum, *Finite Model Theory*, 2nd revised Edition, Springer, Berlin, 1999.
- [2] M. Erickson, F. Harary, Generalized Ramsey theory XV: achievement and avoidance games for bipartite graphs, in: *Graph theory, Proc. 1st Southeast Asian Coll. Singapore, 1983, Lecture Notes in Mathematics*, Vol. 1073, Springer, Berlin, 1984, pp. 212–216.
- [3] F. Harary, *Graph Theory*, Addison-Wesley, Reading, MA, 1969.
- [4] F. Harary, Achievement and avoidance games for graphs, *Ann. Discrete Math.* 13 (1982) 111–119.
- [5] F. Harary, W. Slany, O. Verbitsky, A symmetric strategy in graph avoidance games, in: R.J. Nowakowski (Ed.), *More Games of No Chance, Mathematical Sciences Research Institute Publications*, Vol. 42, Cambridge University Press, Cambridge, 2002, pp. 369–381.
- [6] J. Spencer, *The Strange Logic of Random Graphs*, Springer, Berlin, 2001.