J. Math. Anal. Appl. 380 (2011) 814–830

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Long memory in a linear stochastic Volterra differential equation

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ARTICLE INFO

Article history:
Received 8 September 2010
Available online 9 March 2011
Submitted by Steven G. Krantz

Keywords:
Volterra integro-differential equations
Volterra difference equations
Itô–Volterra integro-differential equations
Differential resolvent
Asymptotic stability
Stationary solutions
Long memory
Long-range dependence
Regular variation
Subexponential

ABSTRACT

In this paper we consider a linear stochastic Volterra equation which has a stationary solution. We show that when the kernel of the fundamental solution is regularly varying at infinity with a log-convex tail integral, then the autocovariance function of the stationary solution is also regularly varying at infinity and its exact pointwise rate of decay can be determined. Moreover, it can be shown that this stationary process has either long memory in the sense that the autocovariance function is not integrable over the reals or is subexponential. Under certain conditions upon the kernel, even arbitrarily slow decay rates of the autocovariance function can be achieved. Analogous results are obtained for the corresponding discrete equation.

1. Introduction

In recent years, much attention in quantitative finance has centred on the question of whether financial markets are efficient, and whether there is a significant impact of past events on the current state of the system, see e.g. Cont [13]. A mathematical way in which this phenomenon can be captured is through the theory of *long range dependence*, or *long memory*. For continuous time processes, this is measured by the autocovariance function of a stationary process being non-integrable and polynomially decaying, so it must decay more slowly than exponentially. Processes with long memory also arise in other areas of science such as data network traffic or hydrology see e.g. Doukhan et al. [14].

In this paper, we describe a class of processes, both in discrete and continuous time which exhibit long range dependence through non-exponential convergence of their autocovariance functions. In the continuous case, these are solutions of scalar affine stochastic Volterra equations of the form

$$dX(t) = \left(aX(t) + \int_0^t k(t-s)X(s)\,ds \right) dt + \sigma\,dB(t) \quad \text{for } t \geqslant 0, \tag{1.1}$$

where B is standard Brownian motion and k is an integrable function. Applications of such equations stochastic Volterra equations arise in physics and mathematical finance. In physics, for example, the behaviour of viscoelastic materials under external stochastic loads has been analysed using Itô–Volterra equations (cf., e.g. Drozdov and Kolmanovskiĭ [15]). In finan-

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2011.03.010

cial mathematics, the presence of inefficiency in real markets can be modelled by using stochastic functional differential equations. Anh et al. [1,2] have posited models for the evolution of asset returns using stochastic Volterra equations with infinite memory.

For affine stochastic functional differential equations with bounded delay, it has been shown that stationary solutions always have exponentially fading autocovariance function, see e.g. Gushchin and Küchler [21], Riedle [32]. This is a consequence of the fact that, if an autonomous linear differential equation with finite delay is stable, then its resolvent converges to zero at an exponentially fast rate, see Hale and Lunel [22].

In order to obtain polynomial convergence results for linear autonomous Volterra equations, it is necessary to consider kernels k which decay non-exponentially, both for deterministic and stochastic equations. While a substantial literature exists in the deterministic case (see e.g. [34,19,25,7,5,6]) only a few results for non-exponential convergence phenomena of linear stochastic autonomous Volterra equations exist, and those that do concern the asymptotic stability of point equilibria. Examples of such papers include Appleby [3,4] for pointwise convergence rates, Appleby and Riedle [9] for convergence rates in weighted L^p-spaces, and Mao and Riedle for mean square convergence rates [28]. In particular, polynomial convergence rates of the autocovariance function of (1.1) have not been recorded.

In this paper, we examine the asymptotic behaviour of the autocovariance function of asymptotically stationary solutions of (1.1). To do this, our first class of results concerns the exact rate of convergence to zero of the solution of the differential resolvent associated with (1.1), namely

$$r'(t) = ar(t) + \int_0^t k(t-s)r(s)\,ds \quad \text{for } t \geqslant 0, \qquad r(0) = 1. \tag{1.2}$$

We consider first equations for which the kernel k is positive and integrable with infinite first moment. In this case it is only known to date that the resolvent r converges to zero and is not integrable.

In this paper we first show that if the kernel k additionally satisfies $a + \int_0^\infty k(s)\,ds = 0$ and the tail integral $\lambda(t) := \int_t^\infty k(s)\,ds$ is a log-convex regularly varying function with index α, then the solution r is decays at a hyperbolic rate, according to

$$\lim_{t\to\infty} r(t)t^{1-\alpha}L(t) = \frac{\sin\alpha\pi}{\pi}, \tag{1.3}$$

where L is a slowly varying function related to k. Corresponding asymptotic results are established in discrete time. The discrete analogue of Eq. (1.2) with positive summable kernel of infinite moment corresponds to the renewal sequence of a null-recurrent Markov chain [20], and under similar additional assumptions on the kernel, the hyperbolic decay of the sequence relies upon well-known results by Garsia and Lamperti [18] and Isaac [23].

Our second class of results in this paper employ the convergence rate of the resolvent r to investigate the long memory properties of the solution of the Itô–Volterra differential equation (1.1) and its discrete analogue. It turns out, that under the same conditions on the kernel k, Eq. (1.1) possesses an asymptotically stationary solution for $0 < \alpha < 1/2$. There also exists a limiting equation which is stationary and its autocovariance function c obeys

$$\lim_{t\to\infty} c(t)L^2(t)t^{1-2\alpha} = \sigma^2 \frac{\Gamma(1-2\alpha)\Gamma(\alpha)}{\Gamma(1-\alpha)} \cdot \frac{\sin^2(\pi\alpha)}{\pi^2}. \tag{1.4}$$

Moreover, because c is non-integrable, the process has long memory. Again, corresponding results hold in discrete time.

If $\alpha > 1/2$, no stationary solutions exist and the case $\alpha = 1/2$ turns out to be critical. In this situation, we give necessary and sufficient conditions for the existence of a stationary solution and show not only that its autocovariance function has long memory, but that it can also decay at an arbitrarily slow rate in the class of slowly varying functions.

In order to give a complete characterization the asymptotic behaviour of the autocovariance function of (1.1), we also treat the cases $a + \int_0^\infty k(s)\,ds < 0$ and $a + \int_0^\infty k(s)\,ds > 0$. While in the latter case no stationary solution exists, we show in the first case, that under weaker assumptions on the kernel k, the autocovariance function of the stationary solution is integrable. Nevertheless, its decay is very slow: the rate of convergence to zero is the same as the decay rate of k, that is hyperbolic.

Although we have mentioned discrete results only briefly in this introduction, there are many reasons to formulate the models (1.2) and (1.1) in discrete time. When modelling dynamic real-world phenomena, it is desirable that properties formulated in discrete or continuous time should be consistent. In this paper, our results demonstrate that the long or subexponential memory are general properties of the Volterra model and do not depend on the continuity assumption. Secondly, by applying for example a constant step size Euler–Maruyama scheme to the continuous equation (1.1), we obtain consistent estimates of the decay rate of the autocovariance function. These decay estimates stabilise appropriately to those obtained in the continuous case in the limit as the step size tends to zero.

2. Discrete and continuous stochastic Volterra equations

2.1. Mathematical preliminaries

We denote the spaces of real-valued continuous functions by $C([0,\infty);\mathbb{R})$. Let $L^p([0,\infty);\mathbb{R})$, (ℓ^p), $p \geqslant 1$, denote the space of real-valued measurable functions f (sequences $(f_n)_{n\in\mathbb{N}}$) satisfying

$$\int_0^\infty |f(t)|^p\,dt < \infty \quad \left(\sum_{n=0}^\infty |f_n|^p < \infty\right).$$

We write $f \sim g$ for $x \to x_0 \in \mathbb{R}\cup\{\pm\infty\}$ if $\lim_{x\to x_0} f(x)/g(x) = 1$.

A function $L:[0,\infty)\to(0,\infty)$ is *slowly varying at infinity* if for all $x>0$,

$$\lim_{t\to\infty}\frac{L(xt)}{L(t)} = 1. \tag{2.1}$$

A function f *varies regularly with index* $\alpha\in\mathbb{R}$, $f\in \mathrm{RV}_\infty(\alpha)$, if it is of the form

$$f(t) = t^\alpha L(t) \tag{2.2}$$

with L slowly varying, see e.g. Feller [17, Chapter VIII.8].

The definition of a regularly varying sequence is a counterpart of the continuous definition [12]: a sequence of positive numbers $(c_n)_{n\in\mathbb{N}}$ is said to be *regularly varying of index* $\rho\in\mathbb{R}$ (c is *slowly varying* if $\rho=0$), if

$$\lim_{n\to\infty}\frac{c_{[\lambda n]}}{c_n} = \lambda^\rho, \quad \text{for every } \lambda>0,$$

where $[x]$ denotes the integer part of $x\in\mathbb{R}_+$. A regularly varying sequence is embeddable as the integer values of a regularly varying function: the function $c(\cdot)$, defined on $[0,\infty)$ by $c(x) := c_{[x]}$ is regularly varying of index ρ.

2.2. Continuous-time Gaussian Volterra equations

We first turn our attention to the deterministic Volterra equation in $\mathbb{R}$:

$$x'(t) = ax(t) + \int_0^t k(t-s)x(s)\,ds \quad \text{for } t\geqslant 0, \qquad x(0)=x_0. \tag{2.3}$$

For any $x_0\in\mathbb{R}$ there is a unique $\mathbb{R}$-valued function x which satisfies (2.3) on $[0,\infty)$. The so-called *fundamental solution or resolvent of* (2.3) is the real-valued function $r:[0,\infty)\to\mathbb{R}$, which is the unique solution of Eq. (1.2).

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space equipped with a filtration $(\mathcal{F}_t)_{t\geqslant 0}$, and let $B=\{B(t):\ t\geqslant 0\}$ be a one-dimensional Brownian motion on this probability space. We will consider the stochastic integro-differential equation of the form

$$dX(t) = \left(aX(t) + \int_0^t k(t-s)X(s)\,ds\right)dt + \sigma\,dB(t) \quad \text{for } t\geqslant 0,$$
$$X(0) = X_0, \tag{2.4}$$

where k is a continuous, integrable real-valued function, and σ is a non-zero real constant. The initial condition X_0 is a real-valued, $\mathcal{F}_0$-measurable random variable with $\mathbb{E}|X_0|^2<\infty$ which is independent of B. The existence and uniqueness of a continuous solution X of (2.4) with $X(0)=X_0$ $\mathbb{P}$-a.s. is covered in Berger and Mizel [10], for instance. Independently, the existence and uniqueness of solutions of stochastic functional equations was established in Itô and Nisio [24] and Mohammed [30]. In fact, X has the variation of constants representation

$$X(t) = r(t)X_0 + \int_0^t r(t-s)\sigma\,dB(s), \quad t\geqslant 0. \tag{2.5}$$

We first discuss the existence of asymptotically stationary solutions of (2.4). It transpires that the critical condition to guarantee stationarity is that the fundamental solution r of (2.3) is in $L^2([0,\infty);\mathbb{R})$.

Theorem 2.1. *Let $k \in L^1([0,\infty);\mathbb{R}) \cap C([0,\infty);\mathbb{R})$. Suppose the fundamental solution r of* (2.3) *obeys $r \in L^2([0,\infty);\mathbb{R})$. Let $\sigma \in \mathbb{R}\setminus\{0\}$. Let X be the solution of* (2.4)*. Then there exists a real-valued function c such that*

$$c(t) := \lim_{s\to\infty} \mathrm{Cov}\big(X(s), X(s+t)\big) = \sigma^2 \int_0^\infty r(s)r(s+t)\,ds, \quad t \geqslant 0. \tag{2.6}$$

The result follows directly from (2.5), and the fact that X_0 is independent of B.

The following theorem shows that (2.4) has a limiting equation which possesses a stationary, rather than an asymptotically stationary solution. To this end, let $B_1 = \{B_1(t)\colon t \geqslant 0\}$ and $B_2 = \{B_2(t)\colon t \geqslant 0\}$ be independent standard Brownian motions, and consider the process $B = \{B(t)\colon t \in \mathbb{R}\}$ defined by

$$B(t) = \begin{cases} B_1(t), & t > 0, \\ B_2(-t), & t \leqslant 0. \end{cases} \tag{2.7}$$

Then B is a standard Brownian motion defined on the whole line.

Theorem 2.2. *Let $k \in L^1([0,\infty);\mathbb{R}) \cap C([0,\infty);\mathbb{R})$. Suppose the fundamental solution r of* (2.3) *obeys $r \in L^2([0,\infty);\mathbb{R})$. Let $\sigma \in \mathbb{R}\setminus\{0\}$. Let $B = \{B(t)\colon t \in \mathbb{R}\}$ be the standard one-dimensional Brownian motion defined by* (2.7)*. Then the unique continuous adapted process which obeys*

$$\begin{aligned} dX(t) &= \left(aX(t) + \int_0^\infty k(s)X(t-s)\,ds\right)dt + \sigma\,dB(t), \quad t > 0; \\ X(t) &= \int_{-\infty}^t r(t-s)\sigma\,dB(s), \quad t \leqslant 0, \end{aligned} \tag{2.8}$$

is given by

$$X(t) = \int_{-\infty}^t r(t-s)\sigma\,dB(s), \quad t \in \mathbb{R}. \tag{2.9}$$

Moreover, X is a stationary zero mean Gaussian process with autocovariance function given by

$$c(t) = \mathrm{Cov}\big(X(s), X(s+t)\big) = \sigma^2 \int_0^\infty r(s)r(s+t)\,ds. \tag{2.10}$$

It is clear that if r is in $L^2([0,\infty);\mathbb{R})$ that X defined by (2.9) is a stationary zero mean Gaussian process with autocovariance function given by (2.10). To show that X satisfies (2.8) requires more work, and a proof is given in Section 6.

Theorem 2.2 provides direction for the investigations in this paper. It is readily seen that $r \in L^1([0,\infty);\mathbb{R})$ implies $c \in L^1([0,\infty);\mathbb{R})$. Therefore in order to possess long memory but still to have stationary solutions, we need to consider conditions on the kernel k in (2.3) such that the fundamental solution r of (2.3) obeys $r \in L^2([0,\infty);\mathbb{R})$ but $r \notin L^1([0,\infty);\mathbb{R})$.

Section 3 gives an example of how this can be achieved. The crucial hypotheses on k is that it is regularly varying and its tail integral is log-convex: this enables us to prove that r is regularly varying and to determine the exact rate of decay of r. We then show how the asymptotic behaviour of c can be inferred from r when r is regularly varying in such a way that $r \in L^2([0,\infty);\mathbb{R})$ but $r \notin L^1([0,\infty);\mathbb{R})$. The results enable us to determine the exact rate of decay of the autocovariance function c in terms of the rate of decay of k.

2.3. Discrete-time Volterra equations

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$. We consider the discrete version of (2.4):

$$\begin{aligned} X_{n+1} - X_n &= aX_n + \sum_{j=1}^n k_j X_{n-j} + \xi_{n+1}, \quad n \geqslant 0, \\ X_0 &= x_0, \end{aligned} \tag{2.11}$$

where k is a positive summable kernel, $a := -\sum_{j=1}^{\infty} k_j$ and $\xi = \{\xi_n\colon n \in \mathbb{N}\}$ is a sequence of independent, identically distributed random variables with $\mathbb{E}(\xi_n) = 0$, $\mathbb{E}(\xi_n^2) = \sigma^2 > 0$ for all $n \in \mathbb{N}$. x_0 is an $\mathcal{F}_0$-measurable random variable with $\mathbb{E}(x_0^2) < \infty$ which is independent of ξ. Let $r = \{r_n\colon n \in \mathbb{N}\}$ denote the fundamental solution of (2.11), i.e., the unique solution of

$$r_{n+1} - r_n = ar_n + \sum_{j=1}^{n} k_j r_{n-j}, \quad n \geqslant 1, \qquad r_0 = 1. \tag{2.12}$$

For more information on Volterra difference equations, the reader is referred to the book of Elaydi [16]. An analogous result to Theorem 2.2 holds for (2.11):

Theorem 2.3. *Suppose that $k \in \ell^1$ and the fundamental solution* (2.12) *obeys $r \in \ell^2$. Then there is a unique adapted process X which obeys*

$$X_{n+1} - X_n = aX_n + \sum_{j=1}^{\infty} k_j X_{n-j} + \xi_{n+1}, \quad n \geqslant 0;$$
$$X_n = \sum_{j=-\infty}^{n} r_{n-j}\xi_j, \quad n < 0, \tag{2.13}$$

where ξ is extended to $n \in \mathbb{Z}$ by taking an independent copy ξ^1 of ξ (defined on the same probability space) and setting $\xi_{-n} = \xi_n^1$, $n \in \mathbb{N}$. X is a stationary zero mean process with autocovariance function given by

$$c(h) = \operatorname{Cov}(X_n, X_{n+h}) = \sigma^2 \sum_{n=0}^{\infty} r_n r_{n+h}, \quad h \in \mathbb{N}. \tag{2.14}$$

Again, we are able to show that if $(k_n)_{n\in\mathbb{N}}$ is a so-called Kaluza-sequence, then r satisfies $r \in \ell^2$ but $r \notin \ell^1$ with exact rate of decay specified. From (2.14) we can deduce the exact asymptotic behaviour of the autocovariance function of the stationary solution.

3. Long memory in the continuous equation

3.1. Asymptotic behaviour of the deterministic resolvent

This section gives the exact rate of decay of the solution of a scalar linear Volterra differential equation with a non-integrable solution r which nonetheless obeys $r(t) \to 0$ as $t \to \infty$. Suppose that $a + \int_0^{\infty} k(s)\,ds = 0$ and let k satisfy the following conditions

(C1) $k \in L^1([0,\infty); (0,\infty)) \cap C([0,\infty); (0,\infty))$,
(C2) $t \mapsto \log\lambda(t)$ is a convex function, where

$$\lambda(t) := \int_t^{\infty} k(s)\,ds, \tag{3.1}$$

(C3) $\lambda(t) = L(t)t^{-\alpha}$ with $\alpha \in (0,1)$ and a slowly varying at infinity function L.

Remark 3.1. The last two conditions are satisfied, if k is a completely monotone function such that $k \in \mathrm{RV}_\infty(-1-\alpha)$. Condition (C2) is equivalent to

(C2*) $\frac{\lambda(t)}{\lambda(t+T)}$ is non-increasing in t for all $T > 0$.

Proofs can be found in Miller [29].

Condition (C1) implies existence of a unique continuous function r which is a solution of the integro-differential equation (1.2). In particular, it follows from (C3) that k obeys

$$\int_0^{\infty} sk(s)\,ds = \infty. \tag{3.2}$$

In this case it is only known that the differential resolvent r satisfies

$$\lim_{t\to\infty} r(t) = 0, \quad r \notin L^1\big((0,\infty);(0,\infty)\big). \tag{3.3}$$

Theorem 3.2. *Suppose that k obeys* (C1)–(C3)*. If r is the unique continuous solution of* (1.2)*, then*

$$\lim_{t\to\infty} r(t)t^{1-\alpha}L(t) = \frac{\sin\alpha\pi}{\pi}. \tag{3.4}$$

Hence for $\alpha \in (0,1/2)$ we have $r \in L^2([0,\infty);(0,\infty))$ but $r \notin L^1([0,\infty);(0,\infty))$ due to $r \in \mathrm{RV}_\infty(\mu)$ for $\mu = \alpha - 1 \in (-1,-1/2)$.

Proof of Theorem 3.2. We note that $\lambda \in C^1((0,\infty);(0,\infty))$. Evidently λ is positive, non-increasing, satisfies $\lambda(t) \to 0$ as $t\to\infty$. Though, by virtue of (C3) this happens so slowly that $\lambda \notin L^1([0,\infty);\mathbb{R})$.

Since $r \in C^1((0,\infty);(0,\infty))$, we can also introduce the function $\rho = -r'$.

By differentiation of the function $f(t) = r(t) + \int_0^t \lambda(t-s)r(s)\,ds$, and using (1.2), we see that $f'(t) = 0$. Since $f(0) = r(0) = 1$, we have

$$r(t) + \int_0^t \lambda(t-s)r(s)\,ds = 1, \quad t \geqslant 0. \tag{3.5}$$

Therefore,

$$\rho(t) = -r'(t) = \frac{d}{dt}\left(-1 + \int_0^t \lambda(s)r(t-s)\,ds\right)$$
$$= \int_0^t \lambda(s)r'(t-s)\,ds + \lambda(t)r(0) = \lambda(t) - \int_0^t \lambda(t-s)\rho(s)\,ds.$$

Hence ρ is the integral resolvent of λ. Now by (C2) and Theorem 1.2 in [27], it follows that

$$0 \leqslant \rho(t) \leqslant \lambda(t) \quad \text{for all } t>0, \qquad \int_0^\infty \rho(t)\,dt = 1, \tag{3.6}$$

particularly implying $0 \leqslant r(t) \leqslant 1$ for all $t \geqslant 0$. Since $\lambda(t) \geqslant 0$, we may define a measure Λ by $\Lambda([0,t]) = \int_0^t \lambda(s)\,ds$. Then

$$\omega_\Lambda(z) := \int_0^\infty e^{-zt}\,\Lambda(dt) = \hat{\lambda}(z).$$

By (C3), it follows that $\Lambda \in \mathrm{RV}_\infty(1-\alpha)$, so as $1-\alpha > 0$, we can apply Theorem XIII.5.1 in [17] to get

$$\hat{\lambda}(\tau) = \omega_\Lambda(\tau) \sim \Gamma(-\alpha+2)\Lambda(1/\tau), \quad \text{as } \tau \to 0. \tag{3.7}$$

Next, as $r(t) > 0$ for all $t \geqslant 0$, we may define the measure U by $U([0,t]) = \int_0^t r(s)\,ds$. Then $u(t) := U'(t) = r(t)$ obeys $u'(t) = r'(t) = -\rho(t) \leqslant 0$ for all $t \geqslant 0$. Furthermore

$$\omega_U(z) := \int_0^\infty e^{-zt}\,U(dt) = \int_0^\infty e^{-zt} r(t)\,dt = \hat{r}(z).$$

Since $\lambda(t) \to 0$ and $r(t) \to 0$ as $t\to\infty$, $\hat{\lambda}(z)$ and $\hat{r}(z)$ exist for $\Re(z) > 0$. Therefore, by (3.5), we have

$$\hat{r}(z) + \hat{\lambda}(z)\hat{r}(z) = \frac{1}{z}, \quad \Re(z) > 0.$$

Therefore, for $\tau > 0$,

$$\omega_U(\tau) = \hat{r}(\tau) = \frac{1}{\tau + \tau\hat{\lambda}(\tau)}.$$

Now, by (3.7)

$$\tau\hat{\lambda}(\tau) \sim \Gamma(-\alpha+2)\tau\Lambda(1/\tau), \quad \text{as } \tau \to 0.$$

Because $\Lambda \in \mathrm{RV}_\infty(-\alpha+1)$, $\Lambda_1(\tau) := \tau\Lambda(1/\tau)$ obeys $\Lambda_1 \in \mathrm{RV}_0(\alpha)$. Since $\alpha \in (0,1)$, $\tau + \tau\hat{\lambda}(\tau) \sim \Gamma(2-\alpha)\Lambda_1(\tau) = \Gamma(2-\alpha)\tau\Lambda(1/\tau)$ as $\tau \to 0$. Thus

$$\omega_U(\tau) = \frac{1}{\tau + \tau\hat{\lambda}(\tau)} \sim \frac{1}{\Gamma(2-\alpha)\tau\Lambda(1/\tau)} = \frac{1}{\tau^\alpha}\tilde{L}(1/\tau), \quad \text{as } \tau \to 0, \tag{3.8}$$

where

$$\tilde{L}(1/\tau) := \frac{1}{\Gamma(2-\alpha)}\frac{\tau^{\alpha-1}}{\Lambda(1/\tau)}, \quad \tau > 0,$$

which is a slowly varying function by virtue of the fact that $\Lambda \in \mathrm{RV}_\infty(-\alpha+1)$. Then, as U has a monotone derivative u, and (3.8) holds, Theorem XIII.5.4 in [17] implies that

$$u(t) \sim \frac{1}{\Gamma(\alpha)}t^{\alpha-1}\tilde{L}(t), \quad \text{as } t \to \infty.$$

Since $u(t) = r(t)$, by the definition of $\tilde{L}$

$$r(t) \sim \frac{1}{\Gamma(\alpha)}t^{\alpha-1} \cdot \frac{1}{\Gamma(2-\alpha)}\frac{t^{-\alpha+1}}{\Lambda(t)} = \frac{1}{\Gamma(\alpha)\Gamma(2-\alpha)}\frac{1}{\Lambda(t)}, \quad \text{as } t \to \infty.$$

Moreover, we have from Proposition 1.5.8 in [11], that

$$\Lambda(t) = \int_0^t s^{-\alpha}L(s)\,ds \sim \frac{1}{1-\alpha}t^{1-\alpha}L(t), \quad \text{as } t \to \infty.$$

Hence,

$$\lim_{t\to\infty} r(t)t^{1-\alpha}L(t) = \frac{1-\alpha}{\Gamma(\alpha)\Gamma(2-\alpha)} = \frac{\sin\alpha\pi}{\pi},$$

as required. □

For the sake of completeness, we also study the case where λ, defined as in (3.1), satisfies $\lambda \in \mathrm{RV}_\infty(-\alpha)$ with $\alpha > 1$. It turns out that in this case r converges to a positive limit and hence cannot be asymptotically stable.

Corollary 3.3. *Suppose that k satisfies* (C1) *and* (C3) *with $\alpha > 1$ and that $a + \int_0^\infty k(s)\,ds = 0$ holds true. Then, $\int_0^\infty sk(s)\,ds < \infty$ and*

$$\lim_{t\to\infty} r(t) = \left(1 + \int_0^\infty sk(s)\,ds\right)^{-1}. \tag{3.9}$$

Proof. Since λ is continuous satisfying $\lambda(0) = \int_0^\infty k(s)\,ds < \infty$ and $\lambda \in \mathrm{RV}_\infty(-\alpha)$ with $\alpha > 1$, we also have $\lambda \in L^1([0,\infty);(0,\infty)) \cap C([0,\infty);(0,\infty))$. Moreover

$$\int_0^\infty \lambda(s)\,ds = \int_0^\infty sk(s)\,ds < \infty.$$

Then, Theorem 4.2 in [8] yields (3.9). □

3.2. Asymptotic behaviour of the autocovariance function

In this section we state our second main result, Theorem 3.4, which characterizes completely the asymptotic rate of convergence of the autocovariance function $c(t)$ of the solution of (2.8) for the case when $a = -\int_0^\infty k(s)\,ds$. In the case where $0 < \alpha < 1/2$, it turns out that for the kernels k satisfying (C1)–(C3), $c(t)$ resembles the power law function $t^{2\alpha-1}$ for large values of t and hence exhibits long memory. The case where $\alpha = 1/2$ is more subtle; indeed, for some such k we have $r \notin L^2([0,\infty);\mathbb{R})$. If $r \in L^2([0,\infty);\mathbb{R})$, it is still possible to determine the rate of decay of c, which continues to exhibit long memory. Perhaps the most interesting aspect of this result is that arbitrarily slow rates of decay of c in $\mathrm{RV}_\infty(0)$ can be obtained.

Theorem 3.4. *Suppose that k satisfies* (C1)–(C3) *with* $\alpha \in (0, 1/2)$*. Let r be the solution of* (1.2)*. Let* $\sigma \in \mathbb{R} \setminus \{0\}$ *and* $B = \{B(t)\colon t \in \mathbb{R}\}$ *be the standard one-dimensional Brownian motion defined by* (2.7)*. Then there is a unique stationary Gaussian process X which obeys* (2.8):

$$dX(t) = \left(aX(t) + \int_0^\infty k(s)X(t-s)\,ds\right)dt + \sigma\,dB(t), \quad t > 0;$$

$$X(t) = \int_{-\infty}^t r(t-s)\sigma\,dB(s), \quad t \leqslant 0.$$

The autocovariance function $c(\cdot) = \mathrm{Cov}(X(s), X(s+\cdot))$ *satisfies*

$$\lim_{t\to\infty} c(t)L^2(t)t^{1-2\alpha} = \sigma^2 \frac{\Gamma(1-2\alpha)\Gamma(\alpha)}{\Gamma(1-\alpha)} \cdot \frac{\sin^2(\pi\alpha)}{\pi^2}. \tag{3.10}$$

Proof. The proof of the theorem can be found in Section 7. □

Example 3.5. Let $\alpha \in (0, 1/2)$ and

$$k(t) = \frac{1}{(1+t)^{\alpha+1}}, \quad t \geqslant 0. \tag{3.11}$$

Then, $\lambda(t) = 1/(\alpha(1+t)^\alpha)$, $t \geqslant 0$, and since $L(t) \to 1/\alpha$ as $t \to \infty$, we obtain the following convergence rate of the autocovariance function:

$$\lim_{t\to\infty} \frac{c(t)}{t^{2\alpha-1}} = \sigma^2 \frac{\sin(\alpha\pi)\Gamma(1-2\alpha)}{\pi\,\Gamma(-\alpha)^2}.$$

We now consider the interesting and critical case where $\alpha = 1/2$. Depending on the properties of the slowly varying function L, both $r \notin L^2([0,\infty);\mathbb{R})$ as well as $r \in L^2([0,\infty);\mathbb{R})$ is possible. In the latter case, we can achieve arbitrary slow decay rates of the autocovariance function. We first determine the rate of convergence of the autocovariance function.

Theorem 3.6. *Suppose that k satisfies* (C1), (C2) *and* $\lambda(t) = L(t)t^{-1/2}$, $t \geqslant 0$*, with a slowly varying function L. Then,* $r \in L^2([0,\infty);\mathbb{R})$ *if and only if*

$$\int_1^\infty \frac{1}{tL(t)^2}\,dt < \infty. \tag{3.12}$$

Moreover, if (3.12) *holds true, then*

$$c(t) \sim \frac{\sigma^2}{\pi^2} \int_t^\infty \frac{1}{sL(s)^2}\,ds, \quad t \to \infty. \tag{3.13}$$

Proof. Theorem 3.2 yields that

$$\lim_{t\to\infty} r(t)t^{1/2}L(t) = \lim_{t\to\infty} r(t)\lambda(t)t = \frac{1}{\pi}. \tag{3.14}$$

Since r is continuous on $[0,\infty)$, $r \in L^2([0,\infty);\mathbb{R})$ if and only if

$$\int_1^\infty \frac{1}{t^2\lambda(s)^2}\,dt = \int_1^\infty \frac{1}{tL(t)^2}\,dt < \infty.$$

In this case we denote by

$$f(t) := \frac{\sigma^2}{\pi^2} \int_t^\infty \frac{1}{s^2\lambda(s)^2}\,ds, \quad t \geqslant 0.$$

The integrand of f is regularly varying with index -1. Then, by Karamata's theorem (see e.g. [11, Theorem 1.5.11]) we obtain

$$\frac{t}{t^2\lambda^2(t)f(t)} \to 0, \quad \text{for } t \to \infty. \tag{3.15}$$

Moreover, with (3.14) and (3.15) it holds that

$$\lim_{t\to\infty} \frac{tr(t)^2}{f(t)} = \lim_{t\to\infty} r(t)^2 t^2 \lambda(t)^2 \lim_{t\to\infty} \frac{t}{t^2\lambda(t)^2 f(t)} = 0. \tag{3.16}$$

We write

$$\frac{c(t)}{f(t)} = \frac{\sigma^2}{f(t)} \int_0^t r(s)r(t+s)\,ds + \frac{\sigma^2}{f(t)} \int_t^\infty r(s)r(t+s)\,ds =: I_1(t) + I_2(t), \quad t \geqslant 0.$$

By (3.6), r is positive and non-increasing, hence we obtain the following upper bound for $I_2(t)$:

$$I_2(t) \leqslant \frac{\sigma^2}{f(t)} \int_t^\infty r(s)^2\,ds, \quad t \geqslant 0. \tag{3.17}$$

The denominator and the numerator in (3.17) tend to zero as t tends to infinity, therefore, we may apply L'Hôspital's rule to obtain

$$\lim_{t\to\infty} \frac{\sigma^2}{f(t)} \int_t^\infty r(s)^2\,ds = \lim_{t\to\infty} \pi^2 r(t)^2 t^2 \lambda(t)^2 = 1. \tag{3.18}$$

On the other hand,

$$I_2(t) \geqslant \frac{\sigma^2}{f(t)} \int_t^\infty r(s+t)^2\,ds = \frac{\sigma^2}{f(t)} \int_{2t}^\infty r(s)^2\,ds, \quad t \geqslant 0. \tag{3.19}$$

By (3.16) we have

$$\lim_{t\to\infty} \frac{1}{f(t)} \int_t^{2t} r(s)^2\,ds \leqslant \lim_{t\to\infty} \frac{tr(t)^2}{f(t)} = 0. \tag{3.20}$$

Combining (3.17), (3.18), (3.19) and (3.20) we obtain $\lim_{t\to\infty} I_2(t) = 1$. The term $I_1(t)$ vanishes as t tends to infinity: applying Karamata's theorem to $r \in \mathrm{RV}_\infty(-1/2)$ and using (3.16), we obtain

$$\lim_{t\to\infty} \frac{I_1(t)}{\sigma^2} \leqslant \lim_{t\to\infty} \frac{r(t)}{f(t)} \int_0^t r(s)\,ds = \lim_{t\to\infty} \frac{\int_0^t r(s)\,ds}{tr(t)} \cdot \frac{tr(t)^2}{f(t)} = 2 \lim_{t\to\infty} \frac{tr(t)^2}{f(t)} = 0.$$

This completes the proof. □

To see that it is possible to obtain arbitrary rates of decay for c in the class of slowly varying functions which tend to zero, we consider such a function $\gamma \in \mathrm{RV}_\infty(0)$. We demonstrate this claim, under a mild technical assumption on γ.

Corollary 3.7. *Suppose that γ is in $C^1((0,\infty);(0,\infty))$, $\gamma(t) \to 0$ as $t \to \infty$ and that $-\gamma' \in \mathrm{RV}_\infty(-1)$. Then $\gamma \in \mathrm{RV}_\infty(0)$ and there exists $L \in \mathrm{RV}_\infty(0)$ which satisfies* (3.12) *and*

$$\int_t^\infty \frac{1}{sL^2(s)}\,ds \sim \gamma(t), \quad \text{as } t \to \infty. \tag{3.21}$$

Proof. For any $T > t \geqslant 0$, we have $\gamma(T) - \gamma(t) = \int_t^T \gamma'(s)\,ds$. Letting $T \to \infty$, we see that $\gamma(t) = \int_t^\infty -\gamma'(s)\,ds$. $-\gamma'$ is integrable because $\gamma(t) \to 0$ as $t \to \infty$. The fact that $-\gamma' \in \mathrm{RV}_\infty(-1)$ and is integrable forces γ to be in $\mathrm{RV}_\infty(0)$. Define the function $L:[1,\infty)\to(0,\infty)$ by

$$L^2(t)=\frac{-1}{t\gamma'(t)}. \tag{3.22}$$

Clearly $L\in \mathrm{RV}_\infty(0)$. Moreover for any $T\geqslant 1$

$$\int_1^T \frac{1}{sL^2(s)}\,ds=\int_1^T -\gamma'(s)\,ds=\gamma(1)-\gamma(T).$$

Since $\gamma(T)\to 0$ as $T\to\infty$, it follows that L obeys (3.12). The asymptotic relation (3.21) is an obvious consequence of the construction of L. □

Remark 3.8. By applying Theorem 3.6, it can be seen that if $k(t)\sim t^{-3/2}L(t)$ as $t\to\infty$, where L is given by (3.22), then $c(t)\sim \sigma^2/\pi^2\gamma(t)$ as $t\to\infty$. Therefore, functions k exist such that the rate of convergence of the autocovariance function is an (essentially) arbitrary function in $\mathrm{RV}_\infty(0)$. For example, $c(t)$ can decay to zero at a rate asymptotic to $(\log\log\log\cdots\log t)^{-1}$ as $t\to\infty$, where there are finitely but arbitrarily many compositions of logarithms.

4. Long memory in the discrete equation

In this section we study the discrete counterparts to Eqs. (1.2) and (2.4) for some summable kernels k with infinite mean.

4.1. Asymptotic behaviour of the deterministic resolvent

Let us first consider the deterministic equation (2.12) with $a+\sum_{j=1}^\infty k_j=0$. If $1+a>0$ and $(k_n)_{n\geqslant 1}$ has infinite mean, the classical renewal theorem yields that r_n converges to zero as n tends to infinity. If $(k_n)_{n\geqslant 1}$ has a regularly varying tail (Garsia and Lamperti [18, Theorem 1.1]) and $(r_n)_{n\in\mathbb{N}}$ is monotone non-increasing (Isaac [23, Theorem 3.1]), the exact convergence rates are also known.

In this section we prove that if the tail $(\sum_{j=n}^\infty k_j)_{n\geqslant 1}$ is a so-called Kaluza sequence, which is a discrete analogue of log-convexity, then the sequence $(r_n)_{n\in\mathbb{N}}$ is monotone non-increasing and we can apply the above mentioned theorems.

Theorem 4.1. *Let $(k_n)_{n\geqslant 1}$ be a positive sequence such that $\sum_{j=1}^\infty k_j\leqslant 1$. Moreover, let $\lambda_n:=\sum_{j=n}^\infty k_j$, $n\geqslant 1$, satisfy:*

(C2′) *$(\lambda_n)_{n\geqslant 1}$ is a Kaluza sequence, that is $\lambda_n^2\leqslant\lambda_{n-1}\lambda_{n+1}$ for all $n\geqslant 1$,*
(C3′) *$\lambda_n=L(n)n^{-\alpha}$, where $0<\alpha<1$ and $L(n)$ is a slowly varying sequence.*

Then

$$\lim_{n\to\infty} n^{1-\alpha}L(n)r_n=\frac{\sin\alpha\pi}{\pi}.$$

Proof. Since $(L(n))_{n\in\mathbb{N}}$ is slowly varying, so is the function $x\mapsto L([x])$. Since $1+a\geqslant 0$, we can apply Theorem 1.1 in [18] to obtain the result for $1/2<\alpha<1$. For $\alpha\leqslant 1/2$ the claim follows from [23, Theorem 3.1] if the sequence $(r_n)_{n\geqslant 0}$ is monotone non-increasing. To show this, we define

$$a_n:=r_n+\sum_{j=1}^{n-1} r_j\lambda_{n+1-j},\quad n\geqslant 0,$$

to obtain

$$\begin{aligned}a_{n+1}-a_n&=r_{n+1}-r_n+\sum_{j=0}^{n-1}(\lambda_{n+1-j}-\lambda_{n-j})r_j+r_n\lambda_1\\&=r_{n+1}-r_n-\sum_{j=0}^{n-1}k_{n-j}r_j+r_na\\&=0.\end{aligned}$$

Hence, $(a_n)_{n\geqslant 0}$ is a constant sequence and equals $a_0=r_0=1$. With $\Delta_n:=-(r_n-r_{n-1})$ we have

$$0 = a_n - a_{n-1} = -\Delta_n + \sum_{j=0}^{n-1} r_j \lambda_{n-j} - \sum_{j=0}^{n-2} r_j \lambda_{n-1-j}$$
$$= -\Delta_n + \sum_{j=1}^{n-1} \lambda_{n-j}(r_j - r_{j-1}) + \lambda_n$$
$$= -\Delta_n - \sum_{j=1}^{n-1} \lambda_{n-j}\Delta_j + \lambda_n.$$

Therefore, $(\Delta_n)_{n\geqslant 0}$ satisfies the recurrence relation

$$\Delta_n = \lambda_n - \sum_{j=1}^{n-1} \lambda_{n-j}\Delta_j. \tag{4.1}$$

Since $(\lambda_n)_{n\geqslant 0}$ is a Kaluza sequence, it follows from [33] that Δ_n is non-negative for all $n \geqslant 0$. Hence, the sequence $(r_n)_{n\geqslant 0}$ is non-increasing and the claim follows. □

4.2. Asymptotic behaviour of the autocovariance function

Now we are able to state the discrete analogue of Theorem 3.4:

Theorem 4.2. *Suppose that k satisfies the assumptions of Theorem 4.1 with $\alpha \in (0, 1/2)$. Let r be the solution of (2.12) and $\xi = \{\xi_n\colon n \in \mathbb{Z}\}$ be a sequence of random variables defined as in Theorem 2.3. Then there is a unique stationary process X which obeys (2.13):*

$$X_{n+1} - X_n = -aX_n + \sum_{j=1}^{\infty} k_j X_{n-j} + \xi_{n+1}, \quad n \geqslant 0;$$
$$X_n = \sum_{j=-\infty}^{n} r_{n-j}\xi_j, \quad n < 0.$$

The autocovariance function $c(\cdot) = \mathrm{Cov}(X_n, X_{n+\cdot})$ obeys

$$\lim_{h\to\infty} c(h)L^2(h)h^{1-2\alpha} = \sigma^2 \frac{\Gamma(1-2\alpha)\Gamma(\alpha)}{\Gamma(1-\alpha)} \cdot \frac{\sin^2(\pi\alpha)}{\pi^2}. \tag{4.2}$$

Proof. The stationary solution is given by $X(n) = \sum_{j=-\infty}^{n} r_{n-j}\xi_j$, $n \in \mathbb{Z}$, and its autocovariance function obviously satisfies (2.14). Since the sequence $(L(n))_{n\in\mathbb{N}}$ is slowly varying we obtain with Theorem 4.1 for all $\lambda > 0$

$$\lim_{n\to\infty} \frac{r_{[\lambda n]}}{r_n} = \lim_{n\to\infty} \frac{L(n)n^{1-\alpha}}{[\lambda n]^{1-\alpha} L([\lambda n])} = \lim_{n\to\infty} \frac{n^{1-\alpha}}{[\lambda n]^{1-\alpha}}$$
$$= \lim_{n\to\infty} \left(\lambda + \frac{[\lambda n] - \lambda n}{n}\right)^{\alpha-1} = \lambda^{\alpha-1}.$$

Hence the positive sequence $(r_n)_{n\in\mathbb{N}}$ is regularly varying with index $\alpha - 1$. Therefore, as mentioned in Section 2.1, the function $r(x) := r_{[x]}$, $x \geqslant 0$, is also regularly varying and we may write

$$c(h) = \sigma^2 \int_0^{\infty} r(x)r(x+h), \quad h \in \mathbb{N}.$$

With Theorem 7.1 we obtain

$$\lim_{h\to\infty} \frac{c(h)}{hr_h} = L.$$

Following the steps of the proof of Theorem 3.4 we obtain (4.2). □

5. Subexponential decay of the autocovariance function

In this section we study the properties of the autocovariance function of the stationary solution of the main continuous- and discrete-time equations (1.1) and (2.11) if the kernel k is again regularly varying with index $-1-\alpha$, $\alpha > 0$ but now $a + \int_0^\infty k(s)\,ds < 0$ or $a + \sum_{n=1}^\infty k_n < 0$ holds respectively.

Then, k is a subexponential function or sequence in the sense of Appleby et al. [5,6]. In this case, the fundamental solution in both discrete- (Theorem 3.2 in [6]) and continuous-time (Theorem 15 in [5]) decays at the same rate as the kernel k. Since k is regularly varying with parameter $-\alpha-1 < -1$, $r \in L^2([0,\infty);\mathbb{R}) \cap L^1([0,\infty);\mathbb{R})$. This implies that the autocovariance function of the stationary solution is integrable. The next results show that nevertheless the autocovariance function decays very slowly: it converges to zero at same rate as the kernel k, that is polynomially.

Remark 5.1. If $a + \int_0^\infty k(s)\,ds > 0$, then the fundamental solution grows exponentially: The characteristic function of r, a function h which satisfies $\hat{r}(z) = 1/h(z)$ is given by $h(z) = z - a - \hat{k}(z)$, $z \in \mathbb{C}$, and satisfies $h(0) = -a - \int_0^\infty k(s)\,ds < 0$. Since k is positive, we obtain for $x > 0$

$$h(x) = x - a - \int_0^\infty e^{-xs} k(s)\,ds \geqslant x - a - \int_0^\infty k(s)\,ds,$$

which is positive if $x > a + \int_0^\infty k(s)\,ds > 0$. Therefore, by the intermediate value theorem, there exists a positive root of the characteristic function. By the standard theory of Volterra equations this implies that the fundamental solution grows exponentially. Hence, the case $a + \int_0^\infty k(s)\,ds > 0$ is not interesting for our research.

5.1. Continuous-time stochastic equation with subexponentially decaying memory

Suppose $k \in C([0,\infty);(0,\infty))$ satisfies

(S1) $k \in \mathrm{RV}_\infty(-1-\alpha)$ for $\alpha > 0$,
(S2) $a + \int_0^\infty k(s)\,ds < 0$.

Theorem 15 in [5] yields, that the fundamental solution of (1.2) converges to zero at the same rate as k:

$$\lim_{t\to\infty} \frac{r(t)}{k(t)} = \frac{1}{(a + \int_0^\infty k(s)\,ds)^2} =: L_c. \tag{5.1}$$

Moreover, r is also subexponential. Since r is also square integrable, the stationary solution of (2.8) exists and the exact rate of decay of the autocovariance function can be determined.

Theorem 5.2. *Suppose k satisfies* (S1) *and* (S2). *Let r be solution of* (1.2). *Let $\sigma \in \mathbb{R}\setminus\{0\}$ and B be the Brownian motion defined by* (2.7). *Then, the autocovariance function $c(\cdot) = \mathrm{Cov}(X(s), X(s+\cdot))$ of the stationary solution of* (2.8) *satisfies*

$$\lim_{t\to\infty} \frac{c(t)}{k(t)} = \frac{\sigma^2}{(-a - \int_0^\infty k(s)\,ds)^3} > 0. \tag{5.2}$$

Proof. The autocovariance function of the stationary solution is again given by (2.10). Theorem 1.8.3 in [11] yields, that there exists a decaying function λ with $k(t) \sim \lambda(t)$ for $t \to \infty$. Since r is integrable, we choose for an arbitrary $\epsilon > 0$ a sufficiently large $T > 0$, so that $2L_c \int_T^\infty |r(s)|\,ds < \epsilon$. We now write

$$\int_0^\infty \frac{r(t+s)r(s)}{k(t)}\,ds = \int_0^T \frac{r(t+s)}{k(t+s)} \frac{k(t+s)}{k(t)} r(s)\,ds + \int_T^\infty \frac{\lambda(t)}{k(t)} \frac{r(t+s)}{\lambda(t+s)} \frac{\lambda(t+s)}{\lambda(t)} r(s)\,ds. \tag{5.3}$$

The second integral is negligible: since λ is decreasing and $r(t)/\lambda(t) \to L_c$ for $t \to \infty$, the integrand is bounded for sufficiently large t by $2L_c|r(s)|$. Hence

$$\limsup_{t\to\infty} \left| \int_T^\infty \frac{r(t+s)r(s)}{k(t)}\,ds \right| \leqslant 2L_c \int_T^\infty \left|r(s)\right|ds < \epsilon.$$

Let us now consider the first integral in (5.3). With Potter's bound (cf. [11, Theorem 1.5.6]) we obtain

$$\frac{k(t+s)}{k(t)} \to 1, \quad t \to \infty, \tag{5.4}$$

uniformly in s for all $s < T$. Therefore for all sufficiently large t

$$\sup_{s\leqslant T}\left|\frac{k(t+s)}{k(t)}\right| \leqslant 2 \quad \text{and} \quad \sup_{s>0}\left|\frac{r(t+s)}{k(t+s)}\right| \leqslant 2L_c. \tag{5.5}$$

Using dominated convergence theorem we obtain

$$\lim_{t\to\infty}\int_0^T \frac{r(t+s)}{k(t+s)}\frac{k(t+s)}{k(t)}r(s)\,ds = L_c\int_0^T r(s)\,ds.$$

Hence, the left-hand side of (5.2) converges to $L_c\int_0^\infty r(s)\,ds$ and the claim follows from the fact that $\int_0^\infty r(s)\,ds = -1/(a+\int_0^\infty k(s)\,ds)$. $\square$

Example 5.3. Let $\alpha > 0$ and

$$k(t) = \frac{1}{(1+t)^{\alpha+1}}, \quad t \geqslant 0. \tag{5.6}$$

We obtain the following convergence rate of the autocovariance function:

$$\lim_{t\to\infty} c(t)t^{1+\alpha} = \frac{\sigma^2}{(-a-1/\alpha)^3}.$$

Remark 5.4. Examples 3.5 and 5.3 make clear that there is a very different impact on the rate of convergence of the autocovariance function from the decay rate of the kernel k according as to whether we are in the long-memory or subexponential case. In the latter case, the rate of decay of the autocovariance function c is proportional to the rate of decay of the kernel k, so slow decay in the memory as measured by the rate of decay of k is reflected exactly in the statistical memory, as measured by c. On the contrary, in the long-memory case, a faster rate of decay of the kernel k results in a slower rate of decay of c.

5.2. Discrete-time stochastic equation with subexponentially decaying memory

Let us now consider Eq. (2.11) with a discrete kernel $k = \{k_n\colon n \geqslant 1\}$ satisfying

(S1′) k is a regularly varying sequence with index $-1-\alpha$ for $\alpha > 0$,
(S2′) $a + \sum_{j=1}^\infty k_j < 0$.

Then k satisfies the assumptions of Theorem 3.2 in [6] and the fundamental solution of (1.2) converges to zero at the same rate as k:

$$\lim_{n\to\infty}\frac{r_n}{k_n} = \frac{1}{(a+\sum_{j=1}^\infty k_j)^2} =: L_d. \tag{5.7}$$

Again, the stationary solution of (2.11) exists and the exact rate of decay of the autocovariance function can be determined.

Theorem 5.5. *Suppose k satisfy* (S1′) *and* (S2′). *Let r be solution of* (2.12). *Let $\xi = \{\xi_n\colon n \in \mathbb{Z}\}$ be a sequence of random variables defined as in Theorem* 2.3. *Then, the autocovariance function $c(\cdot) = \mathrm{Cov}(X_n, X_{n+\cdot})$ of the stationary process defined in* (2.13) *satisfies*

$$\lim_{h\to\infty}\frac{c(h)}{k_h} = \frac{\sigma^2}{(-a-\sum_{j=1}^\infty k_j)^3} > 0. \tag{5.8}$$

Proof. The autocovariance function of the stationary solution is again given by (2.14). Since $(k_n)_{n\in\mathbb{N}}$ is a regularly varying sequence, the function $x \mapsto k(x) := k_{[x]}$ is a regularly varying function with index $-1-\alpha$. Hence, we may choose the function λ as in the proof of Theorem 5.2. Since r is absolutely summable, we choose for an arbitrary $\epsilon > 0$ a sufficiently large N, so that $2L_d\sum_{n=N+1}^\infty |r_n| < \epsilon$. Similarly to the continuous case, we split the sum and study each term separately:

$$\sum_{n=0}^\infty \frac{r_{n+h}r_n}{k_h} = \sum_{n=0}^N \frac{r_{n+h}}{k_{n+h}}\frac{k(n+h)}{k(h)}r_n + \sum_{n=N+1}^\infty \frac{\lambda(h)}{k(h)}\frac{r_{n+h}}{\lambda(n+h)}\frac{\lambda(n+h)}{\lambda(h)}r_n. \tag{5.9}$$

The sequence $r_h/\lambda(h)$ converges to L_d as $h\to\infty$, so the terms of the second sum are bounded for sufficiently large h

by $2L_d|r_n|$. Therefore,

$$\limsup_{h\to\infty}\left|\sum_{n=N+1}^{\infty}\frac{r_{n+h}r_n}{k_n}\right| \leqslant 2L_d\sum_{n=N+1}^{\infty}|r_n| < \epsilon.$$

Let us now consider the first term in (5.9). Applying Potter's bound to the function $k(x)$ as in (5.4) we obtain the discrete version of (5.5). Thus,

$$\lim_{h\to\infty}\sum_{n=0}^{N}\frac{r_{n+h}r_n}{k_h} = L_d\sum_{n=0}^{N}r_n.$$

Similarly, $\sum_{n=0}^{\infty} r_n = -1/(a+\sum_{j=1}^{\infty}k_j)$ and the claim follows. □

6. Proof of Theorem 2.2

First we show that the process defined by (2.9) has a continuous modification. Applying Itô's lemma, Cauchy–Schwarz inequality and Fubini's theorem we obtain

$$\begin{aligned}
\mathbb{E}\big(\big(X(t)-X(u)\big)^2\big) &= \sigma^2\int_{\mathbb{R}}\big(r(t-s)\mathbf{1}_{\{s\leqslant t\}} - r(u-s)\mathbf{1}_{\{s\leqslant u\}}\big)^2\,ds \\
&= \sigma^2\int_{\mathbb{R}}\left(\int_u^t r'(v-s)\,dv\right)^2 ds \\
&\leqslant \sigma^2(t-u)\int_{\mathbb{R}}\int_u^t \big(r'(v-s)\big)^2\,dv\,ds \\
&= \sigma^2(t-u)\int_u^t\int_{\mathbb{R}}\big(r'(s)\big)^2\,ds\,dv\,\sigma^2(t-u)^2\int_{\mathbb{R}}\big(r'(s)\big)^2\,ds.
\end{aligned}$$

Now, r is square integrable and with $\|k*r\|_{L^2} \leqslant \|k\|_{L^1}\|r\|_{L^2}$, we have $r' \in L^2((0,\infty);\mathbb{R})$. Here, $(k*r)(\cdot)$ denotes the convolution of k and r, given by $\int_0^{\cdot} k(s)r(\cdot - s)\,ds$. The Kolmogorov–Chentsov theorem (see e.g. [26, Theorem 2.8]) yields that X has a continuous modification. It remains to show that the process defined by (2.9) solves (2.8). We write, using $r(t)=0$ for $t<0$,

$$\begin{aligned}
X(t)-X(0) &= \sigma\int_{-\infty}^{0}\big(r(t-s)-r(-s)\big)\,dB(s) + \sigma\int_0^t r(t-s)\,dB(s) \\
&= \sigma\int_{-\infty}^{t}\int_0^t r'(u-s)\,du\,dB(s) + \sigma B(t) \\
&= \sigma\int_{-\infty}^{t}\int_0^t\left(ar(u-s) + \int_0^{u-s} r(u-s-v)k(v)\,dv\right)du\,dB(s) + \sigma B(t) \\
&= \sigma\int_0^t\int_{-\infty}^{u} ar(u-s)\,dB(s)\,du + \sigma\int_0^t\int_0^{\infty}\int_{-\infty}^{u-v} r(u-s-v)\,dB(s)\,k(v)\,dv\,du + \sigma B(t) \\
&= \int_0^t aX(u)\,du + \int_0^t\int_0^{\infty} k(v)X(u-v)\,dv\,du + \sigma B(t).
\end{aligned}$$

Since r and B are continuous, we are able to apply stochastic Fubini's theorem (e.g. [31, Chapter IV.6, Theorem 65]), if

$$\int_{-\infty}^{t}\int_0^t r(u-s)^2\,du\,ds < \infty, \qquad \int_{-\infty}^{t}\int_0^t\left(\int_0^{u-s} r(u-s-v)k(v)\,dv\right)^2 du\,ds < \infty.$$

The statement follows from classical Fubini's theorem and the fact that $r \in L^2([0,\infty);\mathbb{R})$ and $\|k*r\|_{L^2} \leqslant \|k\|_{L^1}\|r\|_{L^2}$.

7. Proof of Theorem 3.4

Suppose that $r \in C([0,\infty);(0,\infty))$ obeys

$$r \in \mathrm{RV}_\infty(\mu) \quad \text{for some } \mu \in (-1,-1/2). \tag{7.1}$$

Since $r \in L^2([0,\infty);\mathbb{R})$, there exists $c:[0,\infty) \to (0,\infty)$ such that

$$c(t) = \int_0^\infty r(s)r(s+t)\,ds, \quad t \geqslant 0. \tag{7.2}$$

By assuming (7.1), we exclude the possibility that $r \in L^1([0,\infty);\mathbb{R})$. Our first result is the following rate of decay of c.

Theorem 7.1. *Suppose that r is a positive continuous function which obeys (7.1) for some $\mu \in (-1,-1/2)$. Then the function c in (7.2) is well defined and moreover obeys*

$$\lim_{t\to\infty} \frac{c(t)}{tr^2(t)} = \frac{\Gamma(-1-2\mu)\Gamma(1+\mu)}{\Gamma(-\mu)} =: L > 0. \tag{7.3}$$

Proof. For $\mu \in (-1,-1/2)$ we have $\int_0^\infty x^\mu(x+1)^\mu\,dx = L$. First we suppose that r is decreasing. In this case we choose for an arbitrary $0<\epsilon<1$ a $\delta=\delta(\epsilon)>0$ such that $\int_0^\delta x^\mu(x+1)^\mu\,dx<\epsilon$. The Uniform Convergence Theorem [11, Theorem 1.5.2] yields that

$$\frac{r(tx)}{r(t)} \to x^\mu, \quad \text{uniformly in } x, \text{ for all } x \geqslant \delta.$$

Hence, there exists a $t_0 = t_0(\delta)$ such that

$$\frac{r(tx)r(t(x+1))}{r(t)^2} \leqslant 2x^\mu(x+1)^\mu, \quad \text{for all } t \geqslant t_0,\ x>\delta.$$

The function on the right-hand side is integrable, hence, the dominated convergence theorem yields that

$$\lim_{t\to\infty}\int_\delta^\infty \frac{r(tx)r(t(x+1))}{r(t)^2}\,dx = \int_\delta^\infty \lim_{t\to\infty}\frac{r(tx)r(t(x+1))}{r(t)^2}\,dx = \int_\delta^\infty x^\mu(x+1)^\mu\,dx.$$

There exists a $t_1 = t_1(\delta) > t_0$ such that

$$\left| L - \int_{\delta t}^\infty \frac{r(s)r(t+s)}{tr(t)^2}\,ds \right| = \left| L - \int_\delta^\infty \frac{r(tx)r(t(x+1))}{r(t)^2}\,dx \right|$$
$$\leqslant \epsilon + \left| \int_\delta^\infty x^\mu(x+1)^\mu\,dx - \int_\delta^\infty \frac{r(tx)r(t(x+1))}{r(t)^2}\,dx \right| \leqslant 2\epsilon \tag{7.4}$$

for all $t \geqslant t_1$. On the other hand, using the monotonicity of r we obtain

$$\int_0^{\delta t} \frac{r(s)r(t+s)}{tr(t)^2}\,ds \leqslant \int_0^{\delta t}\frac{r(s)}{tr(t)}\,ds = \frac{R(t)}{r(t)t}\frac{R(\delta t)}{R(t)},$$

where $R(t) = \int_0^t r(s)\,ds \in \mathrm{RV}_\infty(\mu+1)$. It follows from Karamata's theorem [11, Theorem 1.5.11], that

$$\frac{R(t)}{tr(t)} \to \frac{1}{\mu+1}.$$

Choosing δ small enough and a $t_2(\delta) > t_1(\delta)$ large enough we obtain

$$\int_0^{\delta t}\frac{r(s)r(t+s)}{tr(t)^2}\,ds \leqslant 2\lim_{t\to\infty}\frac{R(t)}{r(t)t}\frac{R(\delta t)}{R(t)} = 2\frac{1}{\mu+1}\delta^{\mu+1} \leqslant \epsilon, \tag{7.5}$$

for all $t \geqslant t_2$. Hence, combining (7.5) and (7.4) we get for all $t \geqslant t_2$

$$\left| L - \int_0^\infty \frac{r(s)r(t+s)}{tr(t)^2}\,ds \right| \leqslant \left| L - \int_{\delta t}^\infty \frac{r(s)r(t+s)}{tr(t)^2}\,ds \right| + \int_0^{\delta t} \frac{r(s)r(t+s)}{tr(t)^2}\,ds \leqslant 3\epsilon.$$

Now, for arbitrary r obeying (7.1), let $\rho(t) := \sup\{r(t)\colon t \geqslant x\}$. Then ρ is a positive decreasing function, continuous on $[0,\infty)$ and satisfying $\rho(x) \sim r(x)$ for $x \to \infty$ [11, Theorem 1.5.3]. For an arbitrary $\epsilon > 0$ we choose $t_0 = t_0(\epsilon)$ such that for all $t > t_0$ we have

$$\left| \frac{1}{t\rho(t)^2} \int_0^\infty \rho(s)\rho(t+s)\,ds - L \right| \leqslant \epsilon.$$

Since $r(t)/\rho(t) \to 1$ as $t \to \infty$ for every $\varepsilon \in (0,1)$ there exists $t_1 = t_1(\varepsilon) \geqslant t_0$ such that $1-\varepsilon < r(t)/\rho(t) < 1+\varepsilon$ for all $t \geqslant t_1$. Therefore

$$(1-\varepsilon)^2 \leqslant \frac{\int_{t_1}^\infty r(s)r(s+t)\,ds}{\int_{t_1}^\infty \rho(s)\rho(s+t)\,ds} \leqslant (1+\varepsilon)^2. \tag{7.6}$$

For ϵ sufficiently small we obtain

$$\left| \frac{\int_{t_1}^\infty r(s)r(s+t)\,ds}{\int_{t_1}^\infty \rho(s)\rho(s+t)\,ds} - 1 \right| \leqslant 3\epsilon. \tag{7.7}$$

Now, since ρ is decreasing, we have for $t \geqslant t_1$

$$\frac{1}{\rho(t)} \int_0^{t_1} r(s)r(s+t)\,ds = \int_0^{t_1} r(s)\frac{r(s+t)}{\rho(s+t)}\frac{\rho(s+t)}{\rho(t)}\,ds \leqslant (1+\epsilon)\int_0^{t_1} r(s)\,ds. \tag{7.8}$$

Therefore as $t \mapsto t\rho(t)$ is in $\mathrm{RV}_\infty(\mu+1)$ and $\mu+1>0$, we have $t\rho(t) \to \infty$ as $t \to \infty$, and so there exists a $t_2 = t_2(\epsilon) \geqslant t_1$ such that

$$\left| \frac{1}{t\rho^2(t)} \int_0^{t_1} r(s)r(s+t)\,ds \right| \leqslant \epsilon \quad \text{and} \quad \left| \frac{1}{t\rho^2(t)} \int_0^{t_1} \rho(s)\rho(s+t)\,ds \right| \leqslant \epsilon$$

for all $t \geqslant t_2$. Therefore,

$$\begin{aligned}
&\left| \frac{1}{t\rho^2(t)} \int_0^\infty r(s)r(s+t)\,ds - L \right| \\
&\quad \leqslant \left| \frac{1}{t\rho^2(t)} \int_0^\infty \rho(s)\rho(s+t)\,ds - L \right| + \left| \frac{1}{t\rho^2(t)} \int_0^{t_1} r(s)r(s+t)\,ds \right| \\
&\qquad + \left| \frac{1}{t\rho^2(t)} \int_0^{t_1} \rho(s)\rho(s+t)\,ds \right| + \left| \frac{1}{t\rho^2(t)} \int_{t_1}^\infty r(s)r(s+t)\,ds - \frac{1}{t\rho^2(t)} \int_{t_1}^\infty \rho(s)\rho(s+t)\,ds \right| \\
&\quad \leqslant 3\epsilon + \frac{1}{t\rho^2(t)} \int_{t_1}^\infty \rho(s)\rho(s+t)\,ds \left| \frac{\int_{t_1}^\infty r(s)r(s+t)\,ds}{\int_{t_1}^\infty \rho(s)\rho(s+t)\,ds} - 1 \right| \\
&\quad \leqslant 3\epsilon + 3\epsilon L = (3+3L)\epsilon.
\end{aligned}$$

Finally we note that

$$\lim_{t\to\infty} \frac{c(t)}{r(t)^2} = \lim_{t\to\infty} \frac{c(t)}{\rho(t)^2}\frac{\rho(t)^2}{r(t)^2} = \lim_{t\to\infty} \frac{c(t)}{\rho(t)^2}. \qquad \square$$

We now explicitly connect the result of Theorem 7.1 to the autocovariance function of the stationary solution of (2.8) in the case when $a = -\int_0^\infty k(s)\,ds$ to prove our main result.

Proof of Theorem 3.4. It follows from Theorem 3.2 that

$$\lim_{t\to\infty} tr(t)\cdot L^2(t)t^{1-2\alpha} = \frac{\sin^2\alpha\pi}{\pi^2}. \tag{7.9}$$

Since $\alpha \in (0,1/2)$ we have that $r \in L^2([0,\infty);(0,\infty)) \cap C([0,\infty);(0,\infty))$ and $r \in \mathrm{RV}_\infty(\alpha-1)$ with $\mu := \alpha - 1 \in (-1,-1/2)$. Therefore by Theorem 7.1 and Theorem 2.2 obtain

$$\begin{aligned}\lim_{t\to\infty} c(t)L^2(t)t^{1-2\alpha} &= \lim_{t\to\infty} \frac{c(t)}{tr^2(t)}\cdot r^2(t)L^2(t)t^{2-2\alpha}\\ &= \sigma^2\frac{\Gamma(1-2\alpha)\Gamma(\alpha)}{\Gamma(1-\alpha)}\cdot\frac{\sin^2(\pi\alpha)}{\pi^2},\end{aligned}$$

as claimed. □

Acknowledgments

John A.D. Appleby is partly supported by Science Foundation Ireland under the Mathematics Initiative 2007 grant 07/MI/008 "Edgeworth Centre for Financial Mathematics". Katja Krol is supported by the Deutsche Telekom Stiftung.

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