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Stress-strength reliability for designs based on large historic values of stress

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Abstract

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Some new ideas and thoughts concerning the definition and calculation of reliability for stress-strength models for failure are presented. In particular, calculations are carried out and statistical inference is made for systems whose design is made on the basis of the past data with emphasis on extremes and excesses. This is done based on the observation that for such designs reliability estimation can be viewed as the statistical problem of comparing future values, with large values of the past for a single distribution. It is discussed that this approach could particularly prove useful when no basis exists for assuming any specific distributions for either stress or strength or both, but when design is made or experimentation has been performed yielding sufficient information to assume a certain functional relationship between distributions for stress and strength. Some ideas from information theory are also brought in to provide a guideline for defining reliability on the basis of an "equivalent" system. Finally, a simple demonstrating example is also included using a set of published data.

Keywords: Stress-strength, reliability, functional relationship, extreme, *J*-divergence, equivalent system.

1. Introduction

Suppose R is the strength of a system or a structural element subjected to a sequence of stresses $S_1, S_2, \dots$. In general, a system fails when the stress exceeds the strength, see [9] for details and properties of the stress-strength model. Let T denote the random lifetime of the system and suppose that the occurrence of stress is governed by a counting process $\{N(t),\ t \geqslant 0\}$. The the survival distribution of T is given by

$$P(T>t)=\sum_{h=0}^{\infty} P(N(t)=h)\bar{P}(h), \tag{1}$$

where $\bar{P}(0)=1$ and $\bar{P}(h)=P(\max(S_1,\ S_2,\dots,S_h)<R)$, $h=1, 2,\dots$ (see, e.g., [6]).

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Let $F_S(\cdot)$ and $F_R(\cdot)$ denote distribution functions for stress and strengths, respectively; then noting that the distribution function of $\max(S_1, S_2, \ldots, S_h)$ is given by $(F_S(x))^h$ we have

$$\bar{P}(h) = \int_0^\infty (F_S(x))^h \, dF_R(x). \tag{2}$$

Moreover, since $\{T > t\}$ if and only if $\{\max(S_1, S_2, \ldots, S_h) < R\}$, it follows that

$$P(T>t) = \sum_{h=0}^{\infty} P(N(t)=h) \int_0^\infty (F_S(x))^h \, dF_R(x)$$
$$= \int_0^\infty \sum_{h=0}^{\infty} P(N(t)=h)(F_S(x))^h \, dF_R(x). \tag{3}$$

If, for example, stresses occurring by time t form a nonhomogeneous Poisson process with time-dependent rate $\lambda(t) > 0$, then letting $\Lambda(t) = \int_0^t \lambda(u)\, du$, we have

$$P(T>t) = \sum_{h=0}^{\infty} e^{-\Lambda(t)} \frac{[\Lambda(t)]^h}{h!} \int_0^\infty F_S^h(x) \, dF_R(x) = \int_0^\infty \exp(-\Lambda(t)(1-F_S(x))) \, dF_R(x). \tag{4}$$

Now, it is clear that for a given $\{N(t)\}$ the application of (3) or (4) requires knowledge of both $F_R(\cdot)$ and $F_S(\cdot)$. In practice, however, there are many cases of prime importance where the nature of stress and strength random variables are not completely known and no basis exists for assuming any specific distribution for both variables. Thus, an alternative solution should be sought. Considering this, the object of this study is to develop an appropriate technique for reliability calculation based on the amount of information available. In fact, as will be demonstrated, using properties of some well-known designs, it is possible to assume a functional relationship between $F_R(\cdot)$ to $F_S(\cdot)$, and develop a methodology for reliability calculations. Taking the viewpoint that the strength of a component is measured by the stress required to cause failure, this is a reasonable approach. One advantage of this approach is that reliability calculation can be carried out in the absence of any distributional assumptions concerning S and R. In fact, unlike most classical methods which rely either on parametric or nonparametric approaches (see [9] for details), here a semiparametric solution will be presented.

2. Problem formulation and the proposed methodology

Let $y = F_S(x)$ and $G(\cdot)$ be a function relating $F_S(\cdot)$ to $F_R(\cdot)$, that is, assume that

$$F_R(x) = G(F_S(x)),$$

so that

$$\bar{P}(h) = \int_0^\infty F_S^h(x) \, dF_R(x) = \int_0^\infty F_S^h(x) \, dG(F_S(x)) = \int_0^1 y^h \, dG(y). \tag{5}$$

Note that (5) involves only a single function $G(\cdot)$, and if it is given, the reliability can easily be determined. Considering this, we propose to determine $G(\cdot)$ based on design consideration or in the absence of any information by fitting a regression function. To clarify this, consider a design on the basis of the largest stress of the past, e.g., consider a structure that is built to resist the largest of the k earthquakes in the history of the region. Since for this case $F_R(x) = (F_S(x))^k$, we have

$$G(y) = y^k,$$

and therefore

$$\bar{P}(h) = \int_0^1 y^h \, \mathrm{d} y^k = \frac{k}{k+h} = \frac{1}{1+h/k} = \frac{1}{1+\omega}, \tag{6}$$

where $\omega = h/k$. Note that for $h = k$, $\bar{P}(h) = \frac{1}{2}$ as expected.

The following theorem summarizes the characteristics of the function G.

Theorem 1. *Let R and S be continuous random variables on* $[0, \infty)$; *then $G(\cdot)$ exists, is unique and is a distribution function on* $[0, 1]$.

Proof. Define $G = F_R \circ F_S^{-1}$; then

$$G(F_S) = (F_R \circ F_S^{-1}) \circ F_S = F_R.$$

Also since F_S and F_R are nondecreasing, it follows that G is a nondecreasing function on $[0, 1]$. Moreover,

$$F_R(-\infty) = G(F_S(-\infty)) = G(0) = 0, \qquad F_R(\infty) = G(F_S(\infty)) = G(1) = 1,$$

and this complete the proof. □

Note that for random variables whose supports have a partial overlap, G represents only a part of a distribution function defined on the overlapping interval.

We now consider few examples to demonstrate the usefulness of the proposed methodology.

Example 2. Suppose that the process $\{N(t)\}$ has been observed throughout the time interval $(-\tau, 0)$ and further stress values are referred to the largest stress occurring therein. Then

$$F_R(x) = P(R < x) = \sum_{h=0}^{\infty} P(\max(S_1, S_2, \ldots, S_h) < x \mid N(\tau) = h) P(N(\tau) = h).$$

For nonhomogeneous Poisson $\{N(t)\}$, this equals to

$$\sum_{h=0}^{\infty} \exp(-\Lambda(\tau)) \frac{[\Lambda(\tau)]^h}{h!} (F_S(x))^h = \exp(-\Lambda(\tau)(1 - F_S(x))).$$

That is, for this case,

$$G(y) = \exp(-\Lambda(\tau)(1-y)),$$

and hence

$$P(T > t) = \Lambda(\tau) \frac{1 - \exp(-[\Lambda(\tau) + \Lambda(t)])}{\Lambda(\tau) + \Lambda(t)}. \tag{7}$$

Example 3. Suppose that the designer knows the critical values of the stress (e.g., sea wave height, wind speed or flood amount) that lead to damage and his interest is centered in the frequencies of excess of such values. Let ζ denote the value of the mth largest of the k past stresses and let x denote the (unknown) number of excesses of ζ in the next (so far unobserved) h stresses. Then, it is known that

$$P(x)=P(X=x)=\binom{h+k-m-x}{k-m}\binom{x+m-1}{m-1}\Big/\binom{h+k}{k}, \quad x=0,1,\dots,h, \tag{8}$$

[8, p.59]. Consider now a less conservative design based on ζ. For this case the probability of no excess of ζ over the next h stresses equals $P(x=0)$ and can be obtained from (8). As an example, if we have 100 years of data and set our design standard equal to the second largest, there is about a 70% chance that this will survive the next 20 years. Noting that the distribution function of the mth largest order statistics for a sample of size k from $F_S(x)$ is given by

$$\sum_{j=k-m+1}^{k}\binom{k}{j}(F_S(x))^j(1-F_S(x))^{k-j},$$

it follows that for the above design we have

$$G(y)=\sum_{j=k-m+1}^{k}\binom{k}{j}y^j(1-y)^{k-j}, \quad 0\leqslant y\leqslant 1,$$

and $P(h)$ is given by (8). Note that for $m=1$, this equals

$$\bar{P}(h)=\frac{k}{k+h}=\frac{1}{1+h/k}=\frac{1}{1+\omega}.$$

Example 4. Rather than considering the maxima for stress measurements one could accomplish a design based on the minima of the strength measurements. This is also intuitively appealing since failures are often due to the weak component of a system (weakest link principal, see [11, p.267]), e.g., a series system. Noting that the distribution of minima for a sample of size n from $F(x)$ is given by $1-(1-F(x))^n$, we have

$$F_S(x)=1-(1-F_R(x))^n,$$

and the required relation is given by

$$G(y)=1-(1-y)^{1/n}, \quad 0<y<1.$$

It is worth noting that this model represents a proportional failure rate (Lehman alternative), see, e.g., [2]. The exponential distribution and the Weibull distributions with common shape parameter satisfy this requirement. We end this section by noting that each of the examples introduced above preserves the distributional form of a certain class of the distributions.

3. Reliability inference when $G(y)=y^\alpha$

We now concentrate on the case $G(y)=y^\alpha$, $0<y<1$, which includes design for the largest stress of the past. As noted earlier since the strength is measured by the stress required to cause failure, and failure happens in the incidence of the largest stress so far observed, this is a reasonable assumption for most practical purposes. Also, as is demonstrated in Section 4, for any given system there is an equivalent system having the same reliability but being designed for the largest stress of the past. There are, also, other circumstances which lead to design based on the largest stress. As an example, consider a situation where it is required to maintain a reliability of at least $1-p$ for a certain period of time. What should the design policy be? Let $p = P(S > R)$ and N be the number of stresses that the system would experience during the period of interest. Then the probability of failure in the incidence of the nth stress equals to

$$P(N=n) = p(1-p)^{n-1}, \quad n = 1, 2, \ldots,$$

and that $E(N) = 1/p$. If we exclude the stress causing the first failure, then $(1/p) - 1$ presents the expected number of stresses up to the next failure. Now, to maintain an average reliability of at least $1-p$ we could adopt the policy of designing for the largest of $h(1/p-1) = k$ stresses. Since for this case $G(y) = y^k$, reliability for h applications of the load is given by

$$\bar{P}(h) = \int_0^\infty F_S^h(x)\,\mathrm{d}F_R(x) = \int_0^\infty F_S^h(x)\,\mathrm{d}F_S^k(x) = \frac{k}{h+k} = \frac{h(1/p-1)}{h(1/p)} = 1-p. \tag{9}$$

Thus, the required reliability will be maintained.

4. Reliability, *J*-divergence rate and equivalent systems

Consider once more the case $F_R(x) = F_S^k(x)$ for a fixed k. Then $k/(k+h) = \rho$ is the reliability of a system that will experience exactly h stresses (shocks) during its lifetime. Now it is evident that increasing k will result in larger ρ, whereas increasing h will lead to a smaller ρ. Now, to have a better insight we could use the ideas from information theory, and define a divergence measure between $F_S^h(x)$ and $F_S^k(x)$, which in turn lead to meaningful and useful measure for reliability. Let $f_R(\cdot)$ and $f_S(\cdot)$ denote the probability density function of R and S, respectively, and consider the expressions

$$I_1 = \int_{-\infty}^{\infty} f_S \log\frac{f_S}{f_R}\,\mathrm{d}x, \qquad I_2 = \int_{-\infty}^{\infty} f_R \log\frac{f_R}{f_S}\,\mathrm{d}x.$$

These are called the Kullback–Leibler discrimination information rate [10] for discriminating in favor of f_S over f_R and f_R over f_S, respectively. Also their sum $J = I_1 + I_2$ is called J-divergence rate, which measures the degree of separation between $F_S(\cdot)$ and $F_R(\cdot)$. For example, the divergence rate for two distributions $F_S^h(x)$ and $F_S^k(x)$ is given by

$$J = \frac{h}{k} + \frac{k}{h} - 2 = \omega + \frac{1}{\omega} - 2, \tag{10}$$

where $h/k = \omega$. This is a monotonic increasing function of ω with minimum occurring at $\omega = 1$. Note that (10) is also the divergence rate between $(l - F_S(x))^h$ and $(l - F_S(x))^k$. Also since $\rho = 1/(1+\omega)$, it follows that

$$J = \frac{(1-2\rho)^2}{\rho(1-\rho)}. \tag{11}$$

This function has its minimum for $\rho = \frac{1}{2}$, and its maximum (∞) for $\rho = 1$ and is a monotonic increasing function of ρ in $\frac{1}{2} \leqslant \rho \leqslant 1$. Thus we see that reliability of a system can also be defined on the basis of the divergence rate between distributions for strength and stress.

Now when $F_R(x) = F_S^k(x)$, the reliability of the system for single application of stress ($h = 1$) and the corresponding divergence rate are respectively $\rho = k/(1+k)$ and J given by (11). If for a given system, J_0 denotes the divergence rate between the distribution for strength and stress, then since $k + 1/k - 2$ is monotonic increasing in k, it follows that there exists a unique k, $1 \leqslant k \leqslant \infty$, such that $k + 1/k - 2 = J_0$. This means that corresponding to any given system, there is a unique equivalent system possessing the property $F_R(x) = F_S^k(x)$ and having the same reliability as the system under consideration. This is because J is also a monotonic increasing function of ρ. Using this point the reliability estimation for a given system can be carried out by estimating J or alternatively by estimating the parameter k in the equivalent system. This leads to define the reliability as the maximum stress that a component can stand. For noninteger k we can think of the equivalent system as a system that can stand the largest of $[k]$ stresses but not the largest of $[k]+1$ where $[k]$ denotes the integer part of k. Hence, an estimate of k for any given system could furnish an estimate for reliability of that system for a single application of the stress. Note that since for a system experiencing h stresses (shocks), reliability equals $1/(1+h/k)$, the equivalent system, being subject to a single stress, is the one with the property $F_R(x) = F_S^{k/h}(x) = F_S^{1/\omega}(x)$. The next section considers the problem of statistical inference for ω and for the reliability ρ.

5. Statistical inference

In this section the problem of making inference about reliability for a system described in previous sections will be considered. Since, as was discussed, the value of ω could be specified based on design considerations, the statistical test of a hypothesis for a specified value of ω (or ρ) versus an appropriate alternative will also be included in our discussion. When the hypothesis is not rejected, the value assigned to ω could be used as an estimate for the reliability.

We start by noting that in most of the studies concerning the stress-strength model for failure, it is assumed that the distribution of S (*or* both of S and R) are known except for a few unknown parameters. This problem is considered in [3,7,12,13] under the assumption that S and R follow a normal distribution. Basu [1] has used distributions such as exponential and gamma. Dargahi-Noubary [4] has considered a combined Power–Pareto for both stress and strength distributions. Here we estimate the unknown parameter ω and later use that to estimate the reliability. It is of interest to note that both $\hat{\omega}$ and $\hat{\rho}$ are distribution-free estimates.

Suppose that $x_1, x_2, \ldots, x_n$ are stress measurements and $y_1, y_2, \ldots, y_m$ are strength measurements taken at random over a certain period of time. Under the present set-up, $F_R(x) = [F_S(x)]^{1/\omega}$, $0 < \omega \leqslant 1$. First we estimate ω and then test the hypothesis that

$$H_0: F_R = (F_S)^{1/\omega_0} \quad \text{vs.} \quad H_a: F_R = (F_S)^{1/\omega},$$

where $0 < \omega$, $\omega_0 \leqslant 1$ with $\omega < \omega_0$. Note that reliability is given by $\rho = 1/(1+\omega)$.

Estimation

The likelihood function for ω under the alternative hypothesis is given by

$$L(\omega)=\prod_{i=1}^{n} f_S(x_i)\prod_{j=1}^{m}\left\{\frac{1}{\omega}\left[F_S(y_j)\right]^{1/\omega-1} f_S(y_j)\right\}. \tag{12}$$

By differentiating with respect to ω and then setting the derivative to zero, we obtain the maximum likelihood estimate $\hat{\omega}$ of ω as

$$\hat{\omega}=-\frac{1}{m}\sum_{j=1}^{m}\log F_S(y_j). \tag{13}$$

In reliability theory, the interest centers on testing the hypothesis that H_0: $\rho=\rho_0$ vs. H_a: $\rho>\rho_0$ and this is equivalent to testing the hypothesis H_0: $\omega=\omega_0$ vs. H_a: $\omega<\omega_0$. In fact, designers often claim and wish to demonstrate that the designed system has a reliability of at least ρ_0. In this regard, we need the distributions of $\hat{\omega}$ under both null and alternative hypothesis.

Lemma 5. *Under the null hypothesis, $m\hat{\omega}$ follows a gamma (m, ω_0) distribution.*

Proof. Under the null hypothesis, $[F_R(y_j)]^{\omega_0}=F_S(y_j)$, and hence

$$P\{-\log F_S(y_j)\leqslant x\}=P\{F_S(y_j)\geqslant e^{-x}\}=P\{F_R(y_j)\geqslant e^{-x/\omega_0}\}=1-e^{-x/\omega_0}.$$

This shows that $-\log F_S(y_j)$ follows an exponential distribution with parameter ω_0. Since y_i's are independent and identically distributed random variables, $m\hat{\omega}=-\sum_{j=1}^{m}\log F_S(y_j)$ follows a gamma distribution with parameters m and ω_0. This completes the proof. □

Corollary 6. *Under the null hypothesis, $m(1/\hat{\rho}-1)$ follows a gamma (m, ω_0) distribution.*

Lemma 7 gives the distribution $\hat{\omega}$ under the alternative hypothesis. From this, the distribution of $\hat{\rho}$ under the alternative hypothesis is also obtained.

Lemma 7. *Under the alternative hypothesis, $m\hat{\omega}$ follows a gamma (m, ω)distribution.*

Proof. Same as the proof of Lemma 5, except for changing ω_0 to ω. □

Corollary 8. *Under the alternative hypothesis, $m(1/\hat{\rho}-1)$ follows a gamma (m, ω) distribution.*

Test of hypothesis

Here our interest is to test the hypothesis that

$$H_0:\ \omega=\omega_0 \quad \text{vs.} \quad H_a:\ \omega<\omega.$$

From the Neyman–Pearson lemma H_0 is rejected if $\Lambda = \mathrm{Sup}_\omega L_{H_a}/L_{H_0} > C^*$, where C^* is the critical value. Here we have

$$\sup_\omega(\log L_{H_a}) = \sum_{i=1}^n \log f_S(x_i) + \sum_{j=1}^m \log f_S(y_j) - m\log\hat{\omega} + \left(\frac{1}{\omega} - 1\right)\sum_{j=1}^m \log F_S(y_j),$$

$$\log L_{H_0} = \sum_{i=1}^n \log f_S(x_i) + \sum_{j=1}^m \log f_S(y_j) - m\log\omega_0 + \left(\frac{1}{\omega_0} - 1\right)\sum_{j=1}^m \log F_S(y_j). \tag{14}$$

Now, combining (13) and (14), we get

$$\log\Lambda = m\left(\frac{\hat{\omega}}{\omega_0} - \log\frac{\hat{\omega}}{\omega_0}\right) - m,$$

and the decision rule is to reject H_0 if $\log\Lambda > \log C^*$. Now let $\lambda = \omega/\omega_0$ and $\hat{\lambda} = \hat{\omega}/\omega_0$ and consider the following function:

$$H(\lambda) = \lambda - \log\lambda.$$

Since $0 < \lambda \leqslant 1$, it follows that $H'(\lambda) = 1 - \lambda^{-1} \leqslant 0$ and therefore an equivalent decision rule is to reject H_0 if $m\hat{\omega} < c$, where c is a constant determined by the following equation:

$$\int_0^c \frac{x^{m-1}\,\mathrm{e}^{-x/\omega_0}}{\omega_0^m\Gamma(m)}\,\mathrm{d}x = \alpha.$$

Here α denotes the significant level of the test. The power of the test can be calculated as follows:

$$\text{Power} = P\{m\hat{\omega} < c\} = \int_0^c \frac{x^{m-1}\,\mathrm{e}^{-x/\omega}}{\omega^m\Gamma(m)}\mathrm{d}x = \int_0^{c/\omega} \frac{t^{m-1}\,\mathrm{e}^{-t}}{\Gamma(m)}\mathrm{d}t.$$

Since

$$\frac{\partial\,\text{Power}}{\partial\omega} = \frac{-(c/\omega)^{m-1}\,\mathrm{e}^{-c/\omega}}{\Gamma(m)}c\omega^{-2} \leqslant 0, \tag{15}$$

we see that the maximum power occurs when $\omega = 0$, and the power decreases as we increase the value of ω.

Shortest confidence intervals for reliability

We have already shown that under the alternative hypothesis, $m((1/\hat{\rho}) - 1)$ follows a gamma distribution having m and ω as parameters. We set up the $100(1-\alpha)\%$ confidence interval for ρ in such a way that

$$P\{c_1 \leqslant \hat{\rho} \leqslant c_2\} = 1 - \alpha \tag{16}$$

and

$$c_2 - c_1 = \text{shortest distance}. \tag{17}$$

We use the following lemma to find the limits of the shortest confidence interval.

Lemma 9. *The shortest interval* (c_1, c_2) *such that* $P\{c_1 \leqslant \hat{\rho} \leqslant c_2\} = 1-\alpha$ *satisfies the condition*

$$(c_1^{-1}-1)^{m-1}\,\mathrm{e}^{-(m/\omega)c_1^{-1}}c_1^{-2} = (c_2^{-1}-1)^{m-1}\,\mathrm{e}^{-(m/\omega)c_2^{-1}}c_2^{-2},$$

where $\hat{\rho}$ *is the estimate of the reliability.*

Proof. Note that

$$P\{c_1 \leqslant \hat{\rho} \leqslant c_2\} = P\{c_2^{-1} \leqslant 1/\hat{\rho} \leqslant c_1^{-1}\} = P\{m(c_2^{-1}-1) \leqslant m(1/\hat{\rho}-1) \leqslant m(c_1^{-1}-1)\}$$

$$= \int_{(m/\omega)(c_2^{-1}-1)}^{(m/\omega)(c_1^{-1}-1)} \frac{t^{m-1}\,\mathrm{e}^{-t}}{\Gamma(m)}\,\mathrm{d}t = 1-\alpha. \tag{18}$$

Differentiation with respect to c_1 and setting the derivative equal to zero yields

$$0 = \left[\frac{m}{\omega}(c_1^{-1}-1)\right]^{m-1} \frac{\mathrm{e}^{-[m/\omega(c_1^{-1}-1)]}}{\Gamma(m)}(-1)\frac{m}{\omega}c_1^{-2}$$

$$+\left[\frac{m}{\omega}(c_2^{-1}-1)\right]^{m-1} \frac{\mathrm{e}^{-[m/\omega(c_2^{-1}-1)]}}{\Gamma(m)}\frac{m}{\omega}c_2^{-2}\frac{\mathrm{d}c_2}{\mathrm{d}c_1}. \tag{19}$$

However, when c_1 and c_2 are placed at the shortest distance, $\mathrm{d}c_2/\mathrm{d}c_1$ equals 1. This immediately gives the condition. □

Also is it of interest to note that (18) can be rewritten as

$$\sum_{j=0}^{m-1}\left[\frac{m}{\omega}(c_2^{-1}-1)\right]^j \frac{\mathrm{e}^{-[m/\omega(c_2^{-1}-1)]}}{j!} - \sum_{j=0}^{m-1}\left[\frac{m}{\omega}(c_1^{-1}-1)\right]^j \frac{\mathrm{e}^{-[m/\omega(c_1^{-1}-1)]}}{j!} = 1-\alpha. \tag{20}$$

We use (20) and the result of the previous lemma to compute c_1 and c_2.

6. Example of application

Consider the reliability data presented in Table 1, for 15 pairs of stress and strength measurements. For this data, Basu [1] has found $\hat{\rho} = 0.9639$ assuming exponential distribution

Table 1
Stress S and strength R

S	0.0352	0.0397	0.0677	0.0233	0.087
R	1.77	0.9457	1.8985	2.6121	1.0929
S	0.1156	0.0286	0.0200	0.0793	0.0072
R	0.0362	1.0615	2.3895	0.0982	0.7971
S	0.0245	0.0251	0.0469	0.0838	0.0796
R	0.8316	3.2304	0.4373	2.5648	0.6377

Table 2
95% confidence interval

Distribution	$(\hat{\omega})$	$(\hat{\rho})$	Lower	Upper
Exponential	0.0555	0.948	0.922	0.972
Normal	0.0802	0.926	0.892	0.960

for the stress and strength measurements. Ebrahimi [5] has found the reliability to be 0.903 assuming normal distribution for the stress and IFRA for strength measurements. Using the method previously described, we computed $\hat{\omega}$, $\hat{\rho}$ and the p-value together with a 95% confidence limit for ρ assuming exponential and normal distributions for stress. These are summarized in Table 2. Note that for both distributions the p-value is equal to zero. Also, the value of ρ known from the past data is 0.95 [1].

We finish this section by noting that strength data are often rare and sometimes expensive to sample. However, since this is not usually the case for stress data, one could, as is demonstrated by the above example, consider a suitable parametric model for stress and apply the method described.

7. Conclusion

Designs are often made on the basis of the past data on stress and usually with particular emphasis on extremes and excesses. For such designs the methodology developed here can be used to calculate the reliability. The proposed method is particularly useful when the distributions of stress and strength random variables are not completely known. It proposes presenting the design information in the form of a functional relationship between distributions for stress and strength and it provides a simple method for reliability calculation. It also includes a technique for reliability estimation based on the idea of "equivalent" system and provides a guideline for defining reliability using J-divergence.

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