Homeomorphic measures on stationary Bratteli diagrams

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Abstract

We study the set $\mathcal{S}$ of ergodic probability Borel measures on stationary non-simple Bratteli diagrams which are invariant with respect to the tail equivalence relation $\mathcal{R}$. Equivalently, the set $\mathcal{S}$ is formed by ergodic probability measures invariant with respect to aperiodic substitution dynamical systems. The paper is devoted to the classification of measures μ from $\mathcal{S}$ with respect to a homeomorphism. The properties of the clopen values set $S(\mu)$ are studied. It is shown that for every measure $\mu \in \mathcal{S}$ there exists a subgroup $G \subset \mathbb{R}$ such that $S(\mu) = G \cap [0, 1]$. A criterion of goodness is proved for such measures. Based on this result, the measures from $\mathcal{S}$ are classified up to a homeomorphism. We prove that for every good measure $\mu \in \mathcal{S}$ there exist countably many measures $\{\mu_i\}_{i \in \mathbb{N}} \subset \mathcal{S}$ such that the measures μ and μ_i are homeomorphic but the tail equivalence relations on the corresponding Bratteli diagrams are not orbit equivalent.
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1. Introduction

In this paper, we are interested in the problem of classification of Borel probability measures on a Cantor set with respect to a homeomorphism. Two probability measures μ and ν defined on Borel subsets of a topological space X are called *homeomorphic* or *topologically equivalent* if there exists a self-homeomorphism h of X such that $\mu = \nu \circ h$, i.e. $\mu(E) = \nu(h(E))$ for

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every Borel subset E of X. In such a way, the set of all Borel probability measures on X is partitioned into equivalence classes. One may be interested in the structure of the equivalence relation defined by the classes of homeomorphic measures or in the study of a certain equivalence class.

The topological properties of the space X are important for the classification of measures up to a homeomorphism. For instance, the following theorem proved by Oxtoby and Ulam [16] holds: a non-atomic Borel probability measure μ on the finite-dimensional cube $[0, 1]^n$ is homeomorphic to the Lebesgue measure if and only if every non-empty open set has a positive measure (in other words, μ is full) and the boundary of the cube has measure 0. Later, Oxtoby and Prasad extended this result to the Hilbert cube $[0, 1]^{\mathbb{N}}$ (see [15]). Similar results were also obtained for various manifolds (see the book by Alpern and Prasad [6] for the details).

The current work has two sources. The first one is the article [9] where an explicit description of all ergodic (finite and infinite) measures on stationary Bratteli diagrams was found. The second one is a series of papers by Akin, Austin, Dougherty, Mauldin, Yingst [3,7,10,17] where Borel probability measures on zero-dimensional compact perfect metric spaces (Cantor sets) were extensively studied. In those papers, the major results were focused on the classification of Bernoulli measures up to a homeomorphism continuing the preceding investigations (see [13,14]). We should mention that it was Akin who initiated a systematic study of homeomorphic measures on a Cantor space [1,2]. It turns out that the situation in this case is much more difficult than for connected spaces. Though there is, up to a homeomorphism, only one Cantor set it is not hard to construct full non-atomic measures on the Cantor set which are not topologically equivalent (see [1] where more impressive results were proved). The following question naturally arises: find necessary and sufficient conditions under which measures on a Cantor space X are homeomorphic. It was noted in [1] that there exist continuum classes of equivalent full non-atomic probability measures on a Cantor set. This fact is based on the existence of a countable base of clopen subsets of a Cantor set. Akin [1] defined the *clopen values set* $S(\mu)$ as the set of values of measure μ on all clopen subsets of X. The set $S(\mu)$ is a countable dense subset of the unit interval, and this set provides an invariant for topologically equivalent measures, although it is not a complete invariant, in general. But for the class of the so-called *good* measures, $S(\mu)$ *is* a complete invariant. By definition, a full non-atomic probability measure μ is good if whenever U, V are clopen sets with $\mu(U) < \mu(V)$, there exists a clopen subset W of V such that $\mu(W) = \mu(U)$. It turns out that such measures are exactly invariant measures of uniquely ergodic minimal homeomorphisms of Cantor sets (see [2,11]).

It this paper, we consider stationary (non-simple) Bratteli diagrams and ergodic probability measures on their path spaces invariant with respect to the cofinal (tail) equivalence relation. It follows from [9] that, for every such measure μ, the key invariant $S(\mu)$ can be easily computed in terms of eigenvector entries and eigenvalues of the corresponding incidence matrices. This allows us to answer the questions about properties of μ and $S(\mu)$ and construct homeomorphic measures.

The proved results and organization of the paper are the following. In Section 2 we collect the definitions and statements about measures on a Cantor set that are used in the paper. Since we would like to make the paper self-contained, we include the main results from [2,3,9] in this section. We also discuss the main notions related to Bratteli diagrams. In contrast to the case of full measures on a Cantor set, we have to deal with several singular ergodic invariant measures on a path space of a stationary Bratteli diagram. As shown in [9], the support of every such measure is a closed (Cantor) subset and any open subset of the support has positive measure. This allows us to use the machinery developed for full measures. In Section 3, we first study the structure

of the clopen values set $S(\mu)$ for any ergodic probability $\mathcal{R}$-invariant measure μ on a stationary Bratteli diagram B and prove that this set is group-like, i.e., $S(\mu) = G \cap [0, 1]$ for some additive subgroup G of $\mathbb{R}$. It is worth to mention that in the main result of Section 3 we consider two cases: (i) $S(\mu)$ is a subset of $\mathbb{Q}$, and (ii) $S(\mu) \cap (\mathbb{R} \setminus \mathbb{Q}) \neq \emptyset$. The first case is relatively simple. But in the second case we have to use some methods of linear algebra and matrix theory. As proved in [9], every ergodic finite invariant measure is completely determined in terms of eigenvalues and eigenvectors of the matrix that defines the Bratteli diagram B. In this case, the eigenvector entries and eigenvalues admit their representations as vectors with rational entries. In the same section, we prove an easy checkable criterion for a measure from $\mathcal{S}$ to be good. In Section 4, we apply the found criterion answering the following question. Given a good measure μ on a stationary Bratteli diagram, how many measures from $\mathcal{S}$ are homeomorphic to μ? Is this class infinite? It is proved that there exist stationary Bratteli diagrams $\{B_i\}_{i=0}^{\infty}$ and good ergodic $\mathcal{R}_i$-invariant probability measures μ_i on B_i such that each measure μ_i is homeomorphic to the given measure μ, but the dynamical systems $(B_i, \mathcal{R}_i)$, $(B_j, \mathcal{R}_j)$ are topologically orbit equivalent if and only if $i = j$. The last section contains several examples that illustrate the results proved in the preceding sections. Namely, we give a class of stationary non-simple Bratteli diagrams such that in the simplex of probability $\mathcal{R}$-invariant measures only ergodic measures are good. Another example contains an explicit description of all good measures in a particular class of stationary Bratteli diagrams. It is also shown that given a measure ν on a stationary Bratteli diagram there exists a good measure μ such that $S(\mu) = S(\nu)$.

2. Preliminaries: Good measures and stationary Bratteli diagrams

In this section, we collect some necessary definitions and results that are used throughout the paper. We do this for the reader's convenience. Unless stated otherwise, all measures considered in the paper are *Borel probability non-atomic* measures and all Bratteli diagrams are assumed to be *stationary*.

2.1. Good measures

For a measure μ on a Cantor space X, define the *clopen values set*:

$$S(\mu) = \{\mu(U): U \text{ is clopen in } X\}.$$

For each probability measure μ on X, the set $S(\mu)$ is a dense subset of the unit interval containing 0 and 1 [1].

Let X_1, X_2 be two Cantor sets, $h : X_1 \to X_2$ a continuous map, and μ_1 a measure on X_1. Then the *image measure* $h_*\mu_1$ on X_2 is defined by

$$h_*\mu_1(B) = \mu_1(h^{-1}(B))$$

for all Borel subsets B of X_2. It is said that the measures μ_1 on X_1 and μ_2 on X_2 are *homeomorphic* if there exists a homeomorphism $h : X_1 \to X_2$ such that $h_*\mu_1 = \mu_2$. Clearly, $S(\mu_1) = S(\mu_2)$ for any homeomorphic measures μ_1 and μ_2.

A measure μ on a Cantor set X is called *full* if $\mu(V) > 0$ for any non-empty clopen subset V of X. If $\mu(V) > 0$, then one can define the relative measure μ_V on V setting

$$\mu_V(A) = \frac{\mu(A \cap V)}{\mu(V)}$$

where A is a Borel subset of X.

We recall below the definitions of *good, refinable, and weakly refinable measures* which are based on some natural properties of measures on a Cantor set. We follow here the papers [2,3,10].

A *partition basis* $\mathcal{B}$ for a Cantor set X is a collection of clopen subsets of X such that every non-empty clopen subset of X can be partitioned by elements of $\mathcal{B}$. A partition basis is a basis for the topology but not every basis is a partition basis.

Definition 2.1. Let μ be a full measure on a Cantor set X.

(1) A clopen subset V of X is called *good* for μ (or just good when the measure is understood) if for every clopen subset U of X with $\mu(U) < \mu(V)$, there exists a clopen set W such that $W \subset V$ and $\mu(W) = \mu(U)$. A measure μ is called *good* if every clopen subset of X is good for μ.

(2) A clopen subset U of X is called *refinable* for μ if $\alpha_1, \ldots, \alpha_k \in S(\mu)$ with $\alpha_1 + \cdots + \alpha_k = \mu(U)$ implies that there exists a clopen partition $\{U_1, \ldots, U_k\}$ of U with $\mu(U_i) = \alpha_i$ for $i = 1, \ldots, k$. A measure μ is called *refinable* if every clopen subset is refinable.

(3) A measure μ is called *weakly refinable* if there exists a partition basis $\mathcal{B}$ for X with $X \in \mathcal{B}$ consisting of refinable clopen subsets.

(4) A non-empty clopen subset U of X is called a *clopen set of μ type* when $S(\mu_U) = S(\mu)$.

(5) A measure μ is called a measure of *Bernoulli type* when there is a partition basis $\mathcal{B}$ for X consisting of clopen sets of μ type.

(6) It is said that a measure μ on a Cantor set X satisfies the *Quotient Condition* when every non-empty clopen subset U of X is of μ type.

(7) A subset S of the unit interval $I = [0, 1]$ is called *group-like (ring-like, field-like)* if $S = G \cap I$ where G is an additive subgroup (subring, subfield) of $\mathbb{R}$.

We note that "goodness" $\Longrightarrow$ "refinability" $\Longrightarrow$ "weak refinability". One can find refinable but not good measures. To the best of our knowledge, it is an open question whether the notions of refinability and weak refinability are equivalent.

It can be easily verified that for a countable subset S of the unit interval with $0, 1 \in S$ the set S is group-like (ring-like, field-like) if and only if $S + \mathbb{Z}$ is a subgroup (subring, subfield) of $\mathbb{R}$. Clearly, S is ring-like if and only if S is group-like and multiplicative. In fact, the following lemma proved by Akin [2] holds.

Lemma 2.2. *Let S be a subset of $[0, 1]$ with $0, 1 \in S$. Let $G(S)$ be the additive group of $\mathbb{R}$ generated by S. The following conditions on S are equivalent:*

(1) *S is group-like;*
(2) *$S + \mathbb{Z} = G(S)$;*
(3) *$S + \mathbb{Z}$ is an additive subgroup of $\mathbb{R}$;*
(4) *$\alpha, \beta \in S$ and $\alpha \leqslant \beta$ imply that $\beta - \alpha \in S$.*

If S is group-like and G is an additive subgroup of $\mathbb{R}$, then

$$S = G \cap [0, 1] \quad \Longleftrightarrow \quad S + \mathbb{Z} = G.$$

In the next theorem we collect the following results on good measures.

Theorem 2.3. *Let μ be a measure on a Cantor set X.*

(1) *If μ is good and a measure ν is homeomorphic to μ, then ν is good.*
(2) *In order that the measure μ on X be good, it suffices that there exists a partition basis $\mathcal{B}$ consisting of clopen sets which are good for μ. In particular, if a clopen set can be partitioned by good clopen sets, then it is itself good* [3].
(3) *The measure μ is good if and only if there is a uniquely ergodic, minimal homeomorphism of the Cantor set for which μ is the unique invariant measure* [2,11].
(4) *The direct product of a finite or infinite sequence of good measures is a good measure* [3].
(5) *If μ is good, then $S(\mu)$ is group-like. Conversely, if S is a group-like countable dense subset of $[0, 1]$, then there is a good measure μ on X such that $S = S(\mu)$ (μ is unique up to a homeomorphism).*
(6) *If μ is good and V is a non-empty clopen subset of X, then μ_V is a good measure on the Cantor set V and therefore $S(\mu_V)$ is group-like* [2].
(7) *The following statements are equivalent* [3]:
   (i) *μ is a good measure;*
   (ii) *μ is refinable and $S(\mu)$ is group-like;*
   (iii) *μ is weakly refinable and $S(\mu)$ is group-like.*

Corollary 2.4. *If μ and ν are good measures on Cantor sets X and Y and if $S(\mu) = S(\nu)$, then μ and ν are homeomorphic.*

Let D be a countable subset of the unit interval which contains 1. A number $\delta \in [0, 1]$ is called a *divisor* of D if for all $\alpha \in [0, 1]$

$$\alpha \in D \quad \Longleftrightarrow \quad \alpha \cdot \delta \in D.$$

The set of all divisors of D is denoted by $Div(D)$. The set $Div(D)$ is multiplicative and $1 \in Div(D)$.

This following theorem focuses on the properties of measures of Bernoulli type:

Theorem 2.5. *Let μ be a full measure on a Cantor set X and let G be the group generated by the clopen values set $S(\mu)$.*

(1) *If μ is good then μ is of Bernoulli type if and only if G is a subring of $\mathbb{R}$ such that every positive element of G is a sum of positive units of G.*
(2) *If μ is good then every non-empty clopen subset of X is a set of μ type if and only if G is a subfield of $\mathbb{R}$.*
(3) *If μ is of Bernoulli type then the clopen values set $S(\mu)$ is multiplicative and for every non-empty clopen subset U of X we have $S(\mu) \subset S(\mu_U)$.*
(4) *If μ satisfies the Quotient Condition then μ is a refinable measure of Bernoulli type, and, for every non-empty clopen $U \subset X$, the relative measure μ_U is homeomorphic to μ. The rationals $\mathbb{Q} \cap [0, 1]$ are contained in $S(\mu)$.*

(5) *Any two of the following conditions imply the third*:
  (i) *U is a clopen subset good for μ*;
  (ii) *U is a clopen subset of μ type*;
  (iii) $\mu(U) \in Div(S(\mu))$.

2.2. Bratteli diagrams

We recall here some basic definitions and facts about Bratteli diagrams. We mainly use the notation and results from [9].

Definition 2.6. A *Bratteli diagram* is an infinite graph $B = (V, E)$ such that the vertex set $V = \bigcup_{i \geq 0} V_i$ and the edge set $E = \bigcup_{i \geq 1} E_i$ are partitioned into disjoint subsets V_i and E_i such that

(i) $V_0 = \{v_0\}$ is a single point;
(ii) V_i and E_i are finite sets;
(iii) there exist a range map r and a source map s from E to V such that $r(E_i) = V_i$, $s(E_i) = V_{i-1}$, and $s^{-1}(v) \neq 0$, $r^{-1}(v') \neq 0$ for all $v \in V$ and $v' \in V \setminus V_0$.

The pair (V_i, E_i) or just V_i is called the i-th level of the diagram B. A finite or infinite sequence of edges $(e_i: e_i \in E_i)$ such that $r(e_i) = s(e_{i+1})$ is called a *finite* or *infinite path*, respectively. For a Bratteli diagram B, we denote by X_B the set of all infinite paths starting at the vertex v_0. We endow X_B with the topology generated by cylinder sets $U(e_1, \ldots, e_n) = \{x \in X_B: x_i = e_i, \ i = 1, \ldots, n\}$, where $(e_1, \ldots, e_n)$ is a finite path from B. We consider here such Bratteli diagrams B for which the path space X_B is a Cantor set.

Given a Bratteli diagram $B = (V, E)$, define a sequence of incidence matrices $F_n = (f_{vw}^{(n)})$ of B:

$$f_{vw}^{(n)} = \left| \left\{ e \in E_{n+1}: r(e) = v, \ s(e) = w \right\} \right|$$

where $v \in V_{n+1}$ and $w \in V_n$ and the size of F_n is $|V_{n+1}| \times |V_n|$. Here and thereafter $|\Lambda|$ denotes the cardinality of the set Λ.

A Bratteli diagram is called *stationary* if $F_n = F_1$ for every $n \geq 2$.

Observe that every vertex $v \in V$ is connected to v_0 by a finite path, and the set $E(v_0, v)$ of all such paths is finite. Set $h_v^{(n)} = |E(v_0, v)|$, $v \in V_n$. Then

$$h^{(n+1)} = F_n h^{(n)},$$

where $h^{(n)} = (h_w^{(n)})_{w \in V_n}$.

For $w \in V_n$, the set $E(v_0, w)$ defines the clopen subset $X_w^{(n)} := \{x = (x_i) \in X_B: r(x_n) = w\}$ of X_B. Then $\{X_w^{(n)}: w \in V_n\}$ is a clopen partition of X_B. Analogously, the sets $X_w^{(n)}(\bar{e}) := \{x = (x_i) \in X_B: x_i = e_i, \ i = 1, \ldots, n\}$ determine a clopen partition of $X_w^{(n)}$ where $\bar{e} = (e_1, \ldots, e_n) \in E(v_0, w)$, $n \geq 1$.

Definition 2.7. Let $B = (V, E)$ be a Bratteli diagram. Two infinite paths $x = (x_i)$ and $y = (y_i)$ from X_B are called *tail equivalent* if there exists i_0 such that $x_i = y_i$ for all $i \geq i_0$. Denote by $\mathcal{R}$ the tail equivalence relation on X_B.

Recall that a Bratteli diagram is called *simple* if the tail equivalence relation $\mathcal{R}$ is minimal.

We will consider Bratteli diagrams B for which $\mathcal{R}$ is a countable Borel equivalence relation on X_B. Any two paths x, y from X_B are $\mathcal{R}$-equivalent if and only if there exists $w \in V$ such that $x \in X_w^{(n)}(\bar{e})$ and $y \in X_w^{(n)}(\bar{e}')$ for some $\bar{e}, \bar{e}' \in E(v_0, w)$.

Recall that a measure μ on X_B is called $\mathcal{R}$-*invariant* if for any two paths $\bar{e}$ and $\bar{e}'$ from $E(v_0, w)$ and any vertex w, one has $\mu(X_w^{(n)}(\bar{e})) = \mu(X_w^{(n)}(\bar{e}'))$. Then

$$\mu\big(X_w^{(n)}(\bar{e})\big) = \frac{1}{h_w^{(n)}} \mu\big(X_w^{(n)}\big), \quad \bar{e} \in E(v_0, w).$$

In [9], all invariant ergodic measures on a stationary Bratteli diagram were described as follows. It was first shown that the study of any stationary Bratteli diagram $B = (V, E)$ (with $|V| = K$) can be reduced to the case when the incidence matrix F of size $K \times K$ has the form:

$$F = \begin{pmatrix} F_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & F_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & F_s & 0 & \cdots & 0 \\ X_{s+1,1} & X_{s+1,2} & \cdots & X_{s+1,s} & F_{s+1} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ X_{m,1} & X_{m,2} & \cdots & X_{m,s} & X_{m,s+1} & \cdots & F_m \end{pmatrix}. \tag{2.1}$$

The square non-zero matrices F_i on the main diagonal are irreducible (without loss of generality, one can assume that these matrices are strictly positive). For any $j = s + 1, \ldots, m$, at least one of the matrices $X_{j,k}$ is non-zero. The matrices F_i determine the partition of the vertex set V into subsets (classes) V_i of vertices. In their turn, these subsets generate subdiagrams B_i. The non-zero matrices $X_{j,k}$ indicate which subdiagrams are linked by some edges (or finite paths). Notice that each subdiagram B_i, $i = 1, \ldots, s$, corresponds to a minimal component of the cofinal equivalence relation $\mathcal{R}$.

We denote by F_α, $\alpha \in \Lambda$, the *non-zero* matrices on the main diagonal in (2.1). Let $\alpha \geqslant \beta$. It is said that the class of vertices α *has access* to a class β, in symbols $\alpha \succcurlyeq \beta$, if and only if either $\alpha = \beta$ or there is a finite path in the diagram from a vertex which belongs to β to a vertex from α. In other words, the matrix $X_{\alpha,\beta}$ is non-zero. A class α is called *final* (*initial*) if there is no class β such that $\alpha \succ \beta$ ($\beta \succ \alpha$).

Let ρ_α be the spectral radius of F_α. A class $\alpha \in \{1, \ldots, m\}$ is called *distinguished* if $\rho_\alpha > \rho_\beta$ whenever $\alpha \succ \beta$. Notice that all classes $\alpha = 1, \ldots, s$ are necessarily distinguished. A real number λ is called a *distinguished eigenvalue* if there exists a non-negative eigenvector x with $Fx = \lambda x$. A real number λ is a distinguished eigenvalue if and only if there exists a distinguished class α such that $\rho_\alpha = \lambda$. In this case we denote $\lambda_\alpha = \lambda$. If $x = (x_1, \ldots, x_K)^T$ is an eigenvector corresponding to a distinguished eigenvalue λ_α, then $x_i > 0$ if and only if $i \in \beta$ and $\alpha \succcurlyeq \beta$.

Let $\lambda_1, \ldots, \lambda_k$ be the distinguished eigenvalues of the matrix $A = F^T$ (we will keep this notation below). The main result of [9] asserts that there exist exactly k ergodic probability invariant measures defined by $\lambda_1, \ldots, \lambda_k$. More precisely, fix a distinguished eigenvalue λ and let $x = (x_1, \ldots, x_K)^T$ be the probability non-negative eigenvector corresponding to λ. Then the ergodic probability measure μ defined by λ and x satisfies the relation:

$$\mu\big(X_i^{(n)}(\bar{e})\big) = \frac{x_i}{\lambda^{n-1}} \tag{2.2}$$

where $i \in V_n$ and $\bar{e}$ is a finite path with $s(\bar{e}) = i$. Therefore, the clopen values set for μ has the form:

$$S(\mu) = \left\{ \sum_{i=1}^{K} k_i^{(n)} \frac{x_i}{\lambda^{n-1}} : 0 \leqslant k_i^{(n)} \leqslant h_i^{(n)}; \ n = 1, 2, \dots \right\}. \tag{2.3}$$

This relation is of extreme importance for us and will be used throughout the paper.

Let λ_α be a distinguished eigenvalue corresponding to the distinguished class α. In the next section, we will use the following asymptotics mentioned in [9]

$$\big(A^n\big)_{ij} \sim \lambda_\alpha^n, \quad n \to \infty, \ \text{for} \ i \in \beta, \ j \in \alpha, \ \text{with} \ \alpha \succcurlyeq \beta. \tag{2.4}$$

Here $\sim$ means that the ratio tends to a positive constant. On the other hand,

$$\big(A^n\big)_{ij} = o\big(\lambda_\alpha^n\big), \quad n \to \infty, \ \text{for} \ j \in \beta \prec \alpha. \tag{2.5}$$

If λ is a non-distinguished Perron–Frobenius eigenvalue for A, then the corresponding $\mathcal{R}$-invariant measure on X_B is infinite [9]. We do not study infinite measures in this paper.

2.3. Measure supports

Given the diagram B as above, let Y_α be the path space of the Bratteli subdiagram B_α, $\alpha \in \Lambda$. Define $X_\alpha = \mathcal{R}(Y_\alpha)$, that is, a path $x \in X_B$ belongs to X_α if it is $\mathcal{R}$-equivalent to a path $y \in Y_\alpha$. We see that $X_\alpha = Y_\alpha$ if and only if α is a distinguished class corresponding to a minimal component of $\mathcal{R}$. It follows from the structure of the diagram B, see (2.1), that $X_\alpha \cap X_\beta = \emptyset$ for $\alpha \neq \beta$, and $\{X_\alpha : \alpha \in \Lambda\}$ is a partition of X_B. It is also easy to see that for any $x \in X_\alpha$ the orbit $\mathcal{R}(x)$ is dense in $\bigcup_{\beta \prec \alpha} X_\beta$.

We describe here the support of measure μ_α defined by a distinguished eigenvalue λ_α and the corresponding eigenvector x_α of A. Given such a measure μ_α, we call the measure support, $supp(\mu_\alpha)$, the largest closed (Cantor) subset of X_B such that every open subset of $supp(\mu_\alpha)$ has positive measure. In other words, μ_α is full on $supp(\mu_\alpha)$.

It is clear that when α is a final class, then the support of μ_α is a Cantor set Y_α.

If α is not a final class, then the measure μ_α is supported on $X_\alpha = \mathcal{R}(Y_\alpha)$. Then $supp(\mu_\alpha)$ is the closure $\overline{X_\alpha}$ of X_α. That is, to obtain $\overline{X_\alpha}$, we need to add to X_α those minimal components of the diagram which are accessible from α.

It is obvious that all definitions (clopen values set, good measures, measures of Bernoulli type, etc.) given in Section 2.1 are applicable to the measures μ_α because μ_α is full on the corresponding Cantor sets $\overline{X_\alpha}$. In particular, we note that the ergodic measures corresponding to minimal components are automatically good: on a simple stationary Bratteli diagram any Vershik map is minimal and uniquely ergodic.

Finally, we remark that one can extend the mentioned above definitions to non-ergodic $\mathcal{R}$-invariant measures. Such measures form a simplex whose extreme points are ergodic $\mathcal{R}$-invariant measures. Therefore, any non-ergodic finite $\mathcal{R}$-invariant measure is supported on the closure of a finite disjoint union of some sets X_α.

3. Good measures on stationary Bratteli diagrams

In this section, we study finite ergodic $\mathcal{R}$-invariant measures on stationary Bratteli diagrams. We show that, for any such a measure, the clopen values set $S(\mu)$ is group-like. We also give the necessary and sufficient conditions under which a measure on a stationary Bratteli diagram is good.

3.1. Group-like clopen values set

Consider a stationary non-simple Bratteli diagram $B = (V, E)$. Let F be its incidence $K \times K$ matrix and $A = F^T$. Let μ be the measure defined by a distinguished class of vertices α and λ the corresponding distinguished eigenvalue of A. Denote by $(y_1, \ldots, y_K)^T$ the probability eigenvector of the matrix A corresponding to λ. Notice that the vector $(y_1, \ldots, y_K)^T$ may have zero entries. These zero entries are assigned to the vertices from B that are not accessible from the class α. Denote by $(x_1, \ldots, x_n)^T$ the positive vector obtained from $(y_1, \ldots, y_K)^T$ by crossing out zero entries. We call $(x_1, \ldots, x_n)^T$ the *reduced vector* corresponding to the measure μ. Recall that we consider the measure μ only on its support. This means that we can ignore the part of B formed by subdiagrams which are not accessible from the class α. Without loss of generality, we can think that the matrix $A = F^T$ satisfies the condition $Ax = \lambda x$.

Let H be the additive subgroup of $\mathbb{R}$ generated by $\{x_1, \ldots, x_n\}$.

Lemma 3.1. *Let B, μ, A, λ, H, and $(x_1, \ldots, x_n)^T$ be as above. Let G be the additive subgroup of $\mathbb{R}$ generated by the clopen values set $S(\mu)$. Then:*

1) $\lambda H \subset H$;
2) $G = \bigcup_{N \in \mathbb{N}} \frac{1}{\lambda^N} H$;
3) $S(\mu)$ *is group-like if and only if* $S(\mu) + \mathbb{Z} = \bigcup_{N \in \mathbb{N}} \frac{1}{\lambda^N} H$;
4) $\lambda^{-1} \in Div(G)$.

Proof. 1) Let $A = (a_{ij})_{i,j=1}^n$. Then $\sum_{j=1}^n a_{ij} x_j = \lambda x_i$, hence $\lambda x_i \in H$ for $i = 1, \ldots, n$. Since λH is generated by $\lambda x_1, \ldots, \lambda x_n$, we see that $\lambda H \subset H$.

2) It follows from (2.3) that $\frac{1}{\lambda^N} H \subset G$ because $\frac{x_1}{\lambda^N}, \ldots, \frac{x_n}{\lambda^N} \in G$ for any $N \in \mathbb{N}$. On the other hand, we see that $\bigcup_{N \in \mathbb{N}} \frac{1}{\lambda^N} H$ is a group and $S(\mu) \subset \bigcup_{N \in \mathbb{N}} \frac{1}{\lambda^N} H$. Hence, $G \subset \bigcup_{N \in \mathbb{N}} \frac{1}{\lambda^N} H$.

3) By Lemma 2.2, $S(\mu)$ is group-like if and only if $G = S(\mu) + \mathbb{Z}$.

4) It follows from the above results that $\lambda G = G$. □

Remark. Since $\sum_{k=1}^n x_k = 1$, we have $1 \in H$ and $\lambda^N \in H$ for $N \in \mathbb{N}$. It is clear that $\frac{1}{\lambda^M} H \subset \frac{1}{\lambda^{M+1}} H$, $M \in \mathbb{N}$.

One of the main results of this section is the following:

Theorem 3.2. *Let μ be an ergodic invariant measure on a stationary diagram B defined by a distinguished eigenvalue λ of the matrix $A = F^T$. Let $(x_1, \ldots, x_n)^T$ be the corresponding reduced vector and H the additive subgroup of $\mathbb{R}$ generated by $\{x_1, \ldots, x_n\}$. Then the clopen*

values set $S(\mu)$ is group-like and

$$S(\mu) = \left(\bigcup_{N=0}^{\infty} \frac{1}{\lambda^N} H \right) \cap [0, 1].$$

Proof. The proof is divided into two parts depending on the properties of λ. The first part deals with rational (hence integer) λ, and the second one contains the proof of the case of irrational (hence algebraic integer) λ.

1. Let $\lambda \in \mathbb{Q}$ and $x = (\frac{p_1}{q}, \ldots, \frac{p_n}{q})^T$ be the corresponding reduced probability vector. It follows from (2.3) that

$$S(\mu) = \left\{ \sum_{i=1}^{n} l_i^{(N)} \frac{p_i}{q\lambda^{N-1}} \;\middle|\; 0 \leqslant l_i^{(N)} \leqslant h_i^{(N)}, \; N = 1, 2, \ldots \right\}.$$

Hence, $S(\mu) \subset \{\frac{m}{q\lambda^N} : N \in \mathbb{N}, \; m = 0, 1, \ldots, q\lambda^N\}$. We need to prove the converse, that is, for every natural number N and every integer $0 \leqslant m \leqslant q\lambda^N$, there exist $M \in \mathbb{N}$ and integers $l_i^{(M)} \in [0, h_i^{(M)}]$, $i = 1, \ldots, n$, such that

$$\frac{m}{q\lambda^N} = \sum_{i=1}^{n} l_i^{(M)} \frac{p_i}{q\lambda^{M-1}} \tag{3.1}$$

or, equivalently,

$$m\lambda^R = \sum_{i=1}^{n} l_i^{(M)} p_i,$$

where $R = M - N - 1$. We may assume that $0 < m < q\lambda^N$ because the cases $m = 0$ and $m = q\lambda^N$ are trivial. We note also that if an integer M satisfying (3.1) exists then M can be chosen arbitrary large, in particular, $M > N$.

Let α be the class of vertices corresponding to λ and defining μ. If the measure μ is supported on a simple subdiagram of B corresponding to a minimal component, then there is nothing to prove since this measure is invariant for a uniquely ergodic homeomorphism of a Cantor set.

Without loss of generality, we may assume that the non-zero value $\frac{p_1}{q}$ is assigned to a vertex from a class β such that $\alpha \succ \beta$ (see (2.1)).

Since $\gcd(p_1, \ldots, p_n) = 1$, there exist integers $d_1, \ldots, d_n$ such that $d_1 p_1 + \cdots + d_n p_n = 1$. For the homogeneous equation

$$\sum_{i=1}^{n} z_i p_i = 0,$$

there are $n - 1$ independent parameters, say $(z_2, \ldots, z_n)$, amongst the solution of this equation. Then $z_1 = -\frac{1}{p_1} \sum_{i=2}^{n} z_i p_i$. It is obvious that we can choose parameters $\{z_i\}_{i=2}^{n}$ such that all the numbers $\{z_i\}_{i=1}^{n}$ are integers. From the above relations we obtain that

$$m\lambda^R = \sum_{i=1}^{n} y_i\, p_i,$$

where $y_i = m\lambda^R d_i + z_i$. We need to show that there exist $z_2, \ldots, z_n$ such that $y_i \in \mathbb{N}$ and $0 \leqslant y_i \leqslant h_i^{(M)}$ for $i = 1, \ldots, n$. We first note that

$$-m\lambda^R d_j \leqslant z_j \leqslant h_j^{(N+R+1)} - m\lambda^R d_j, \quad j = 2, \ldots, n. \tag{3.2}$$

Since y_1 must be in the interval $[0, h_1^{(M)}]$, the value z_1 must satisfy the inequalities

$$-m\lambda^R d_1 \leqslant z_1 \leqslant h_1^{(N+R+1)} - m\lambda^R d_1. \tag{3.3}$$

On the other hand, it follows from (3.2) that

$$\frac{m\lambda^R}{p_1} \sum_{j=2}^{n} d_j p_j - \sum_{j=2}^{n} \frac{h_j^{(N+R+1)} p_j}{p_1} \leqslant z_1 \leqslant \frac{m\lambda^R}{p_1} \sum_{j=2}^{n} d_j p_j. \tag{3.4}$$

Since $\sum_{j=2}^{n} d_j p_j = 1 - d_1 p_1$, we deduce from (3.4) that

$$\frac{m\lambda^R}{p_1} - m\lambda^R d_1 - \frac{1}{p_1} \sum_{j=2}^{n} h_j^{(N+R+1)} p_j \leqslant z_1 \leqslant \frac{m\lambda^R}{p_1} - m\lambda^R d_1. \tag{3.5}$$

Thus, z_1 must satisfy both inequalities (3.3) and (3.5). We show that if z_1 satisfies (3.3) then z_1 also satisfies (3.5) when R is sufficiently large.

To do this, we compare the left bounds of (3.5) and (3.3) and show that for sufficiently large R

$$\frac{m\lambda^R}{p_1} - \frac{1}{p_1} \sum_{j=2}^{n} h_j^{(N+R+1)} p_j = \frac{\lambda^{N+R} q}{p_1} \left(\frac{m}{q\lambda^N} - \sum_{j=2}^{n} h_j^{(N+R+1)} \frac{p_j}{\lambda^{N+R} q} \right) < 0. \tag{3.6}$$

Indeed, the vertex of the diagram corresponding to $\frac{p_1}{q}$ belongs to a final class of the vertices. Since

$$\mu(X_B) = \sum_{j=1}^{n} h_j^{(M)} \frac{p_j}{q\lambda^{M-1}} = 1, \quad M \in \mathbb{N},$$

we have from asymptotics (2.5) that $h_1^{(M)} \cdot \frac{p_1}{q\lambda^{M-1}} \to 0$ as $M \to \infty$. Thus,

$$\sum_{j=2}^{n} h_j^{(N+R+1)} \frac{p_j}{\lambda^{N+R} q} \to 1$$

as $R \to +\infty$. Since $m < \lambda^N q$, the expression in parentheses in (3.6) is negative for sufficiently large R as desired. Moreover, the absolute value of expression (3.6) tends to $+\infty$ as $R \to +\infty$.

Similarly, comparing the right bounds of (3.5) and (3.3), and using asymptotics $h_1^{(N+R+1)} = o(\lambda^R)$, we obtain that

$$\frac{m\lambda^R}{p_1} - h_1^{(N+R+1)} > 0 \tag{3.7}$$

when R is sufficiently large. Therefore, the interval defined by (3.3) lies in that defined by (3.5) and its length tends to infinity as $R \to +\infty$.

To finish the proof, we need to show that $z_2, \ldots, z_n$ can be chosen so that they satisfy simultaneously (3.2) and (3.3). We consider z_1 as a linear function of the parameters $z_2, \ldots, z_n$. The integer parameters such that z_1 is integer and Eqs. (3.2) hold form the domain of z_1. Relation (3.5) contains the range of z_1. The range of z_1 is a finite number of points. The largest distance between two neighboring points is bounded and does not depend on R. The length of subinterval (3.3) tends to infinity as R becomes infinitely large. Hence, we can find allowable parameters $\{z_j\}_{j=2}^n$ such that z_1 lies in the interval (3.3). Thus, we set

$$l_j^{(M)} = m\lambda^R d_j + z_j, \quad j = 1, \ldots, n$$

where $M = R + N + 1$. This proves the theorem in the rational case.

2. Let $\lambda \in \mathbb{R} \setminus \mathbb{Q}$ and $x = (x_1, \ldots, x_n)^T$ be the corresponding reduced vector.

To clarify the main idea of the proof, we first consider an example. Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

We have the eigenvalue $\lambda = \frac{3+\sqrt{5}}{2}$ and eigenvector $x = (\frac{3-\sqrt{5}}{2}, \frac{\sqrt{5}-1}{2})^T$ for A. Hence, $x_1 = 3 - \lambda$, $x_2 = \lambda - 2$ and $\frac{1}{\lambda} = 3 - \lambda$. Then, by (2.3),

$$S(\mu) = \left\{ l_1^{(N)}(3 - \lambda)^N + l_2^{(N)}(\lambda - 2)(3 - \lambda)^{N-1} \,\middle|\, 0 \leqslant l_i^{(N)} \leqslant h_i^{(N)}, \ i = 1, 2; \ N = 1, 2, \ldots \right\}.$$

The minimal polynomial for λ is $f(t) = t^2 - 3t + 1$. It can be proved that $(3 - \lambda)^N = -f_{2N-1}\lambda + f_{2N+1}$, where f_i is the i-th Fibonacci number. Hence $S(\mu)$ can be written in terms of polynomials of λ of first degree:

$$S(\mu) = \left\{ l_1^{(N)}(-f_{2N-1}\lambda + f_{2N+1}) + l_2^{(N)}(f_{2N-2}\lambda - f_{2N}), \ N \geqslant 1 \right\},$$

where $0 \leqslant l_i^{(N)} \leqslant h_i^{(N)}$, $i = 1, 2$. Instead of polynomials, we can work with vectors formed by their coefficients. Thus, we obtain a vector representation of any element from $S(\mu)$. Let $P_N = \{l_1^{(N)}(-f_{2N-1}\lambda + f_{2N+1}) + l_2^{(N)}(f_{2N-2}\lambda - f_{2N}) \,|\, 0 \leqslant l_i^{(N)} \leqslant h_i^{(N)}, \ i = 1, 2\}$. Then P_N is a part of the lattice in $\mathbb{R}^2$ generated by vectors $u_N = (f_{2N+1}, -f_{2N-1})^T$ and $v_N = (-f_{2N}, f_{2N-2})^T$ which includes all points with coordinates $\{(i, j) \mid 0 \leqslant i \leqslant h_1^{(N)}, \ 0 \leqslant j \leqslant h_2^{(N)}\}$ in the basis $\{u_N, v_N\}$. We see that $P_{N+1} \supset P_N$ and $S(\mu) = \bigcup_{i=1}^{\infty} P_N$. It can be proved that u_N, v_N both tend to the same line a in $\mathbb{R}^2$ generated by the vector $(-\lambda, 1)^T$ as $N \to \infty$. We also show that the norms of these vectors tend to infinity. This suffices to conclude that the points in $\mathbb{R}^2$ that represent $S(\mu)$ "uniformly" fill the gap between lines a and $a + (1, 0)^T$. This means that $S(\mu)$ is group-like.

Now we consider the general case. It suffices to prove that for any $u, v \in S(\mu)$ with $u + v \leqslant 1$ we have $u + v \in S(\mu)$. Indeed, let $u, v \in S(\mu)$ with $u < v$. Then $v - u = 1 - ((1 - v) + u) \in S(\mu)$ (Akin used similar arguments in [2]). Then $S(\mu)$ is group-like by Lemma 2.2. It follows from (2.3) that it suffices to prove that any number $s = \sum_{i=1}^{n} l_i \frac{x_i}{\lambda^{N-1}}$ from $[0, 1]$ such that $l_i \geqslant 0$ belongs to $S(\mu)$.

We will use a vector representation of algebraic numbers as we did in the above example. Since λ is a root of the characteristic polynomial $\Delta(x)$ of A, λ is an algebraic integer number. Suppose the degree of λ is k. Denote by

$$f(t) = t^k + m_{k-1} t^{k-1} + \cdots + m_1 t + m_0$$

the minimal polynomial of λ over $\mathbb{Q}$. Let $\mathbb{Q}(\lambda)$ denote the least field that contains both $\mathbb{Q}$ and λ. If $\mathbb{Q}[\lambda]$ stands for the least ring that contains both $\mathbb{Q}$ and λ, then $\mathbb{Q}[\lambda] = \mathbb{Q}(\lambda)$. Any element $y \in \mathbb{Q}(\lambda)$ can be uniquely represented as $a_0 + a_1 \lambda + \cdots + a_{k-1} \lambda^{k-1} = y(\lambda)$ where $a_i \in \mathbb{Q}$. Hence, there is a one-to-one correspondence between polynomials in λ and vectors formed by their coefficients:

$$a_0 + a_1 \lambda + \cdots + a_{k-1} \lambda^{k-1} \leftrightarrow (a_0, a_1, \ldots, a_{k-1})^T. \tag{3.8}$$

Since $S(\mu) \subset \mathbb{Q}(\lambda)$, every element of $S(\mu)$ can be also considered as a vector in the space $\mathbb{Q}^k$. Thus, if we need to emphasize that a number $y \in S(\mu)$ is considered as a vector from $\mathbb{Q}^k$ (or a polynomial from $\mathbb{Q}(\lambda)$), we will use the notation $\mathbf{y}$. This convention will be used throughout the paper.

It follows from (3.8) that the polynomials $\{1, \lambda, \ldots, \lambda^{k-1}\}$ correspond to the vectors $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_k\}$, the standard basis in $\mathbb{R}^k$ (or $\mathbb{Q}^k$). Denote by $\langle \mathbf{u}, \mathbf{v} \rangle$ the scalar product of vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^k$. Let $\mathbf{n}$ denote the vector $(1, \lambda, \ldots, \lambda^{k-1})^T \in \mathbb{R}^k$. Then, for any $y \in \mathbb{Q}^k$, the corresponding polynomial $y(\lambda)$ can be written as $y(\lambda) = \langle \mathbf{y}, \mathbf{n} \rangle \in \mathbb{R}$.

Take the entries $x_1, \ldots, x_n$ of the reduced vector, and for each x_i find its representation as a vector from $\mathbb{Q}^k \subset \mathbb{R}^k$:

$$\mathbf{x}_i = \left(a_0^{(i)}, a_1^{(i)}, \ldots, a_{k-1}^{(i)}\right)^T.$$

Denote by

$$A_0 = \begin{pmatrix} a_0^{(1)} & \cdots & a_0^{(n)} \\ a_1^{(1)} & \cdots & a_1^{(n)} \\ \vdots & \cdots & \vdots \\ a_{k-1}^{(1)} & \cdots & a_{k-1}^{(n)} \end{pmatrix}$$

the matrix formed by the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$. In other words, A_0 represents the transposed eigenvector $(x_1, \ldots, x_n)$.

Let $p(\lambda)$ be the polynomial in $\mathbb{Q}(\lambda)$ such that $\lambda^{-1} = p(\lambda)$. The map $y \mapsto p(\lambda)y$ in $\mathbb{Q}(\lambda)$ determines a linear transformation in the vector space $\mathbb{Q}^k$. Find the $k \times k$ matrix D which corresponds to this transformation. It is obvious that $D(\mathbf{e}_i) = \mathbf{e}_{i-1}$, $i = 2, \ldots, k$, and $D(\mathbf{e}_1)$ is the vector corresponding to $p(\lambda)$. Thus, the matrix D and the inverse matrix $C = D^{-1}$ have the form

$$
D = \begin{pmatrix} -\frac{m_1}{m_0} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{m_{k-1}}{m_0} & 0 & \cdots & 1 \\ -\frac{1}{m_0} & 0 & \cdots & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & \cdots & 0 & -m_0 \\ 1 & \cdots & 0 & -m_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & -m_{k-1} \end{pmatrix},
$$

where $m_0, m_1, \ldots, m_{k-1}$ are the coefficients of the minimal polynomial $f(t)$. Then the equation $Ax = \lambda x$ can be written in new terms as $CA_0 = A_0 A^T$ or $A_0 = DA_0 A^T$.

Now we use the found matrix D to represent the clopen values set in more convenient form:

$$
S(\mu) = \left\{ D^{N-1} \left(\sum_{i=1}^{n} k_i^{(N)} \mathbf{x}_i \right) \,\Big|\, 0 \leqslant k_i^{(N)} \leqslant h_i^{(N)}; \ N = 1, 2, \ldots \right\} \subset \mathbb{Q}^k.
$$

Denote by π the hyperplane in $\mathbb{R}^k$ which is specified by the relation $\pi = \{ \mathbf{y} : \langle \mathbf{y}, \mathbf{n} \rangle = 0 \}$. Let π_1 be the hyperplane which is obtained by the relation $\pi_1 = \{ \mathbf{y} : \langle \mathbf{y}, \mathbf{n} \rangle = 1 \}$ (hence $\pi_1 = \pi + \mathbf{e}_1$). Then all points of $S(\mu)$ lie in the stripe between π and π_1.

Consider the family of sets

$$
P_N = \left\{ \sum_{i=1}^{n} l_i^{(N)} D^{N-1}(\mathbf{x}_i) \,\Big|\, 0 \leqslant l_i^{(N)} \leqslant h_i^{(N)} \right\}, \quad N \in \mathbb{N}.
$$

Since

$$
\frac{x_i}{\lambda^N} = \sum_{i=1}^{n} a_{ij} \frac{x_j}{\lambda^{N+1}},
$$

we see that $P_{N+1} \supset P_N$ for $N \in \mathbb{N}$. Let $P = \bigcup_{N \in \mathbb{N}} P_N$. Clearly, $S(\mu) = P$.

We show that the vectors from P can be chosen arbitrary close to the hyperplane π and have arbitrary big length. Hence they "fill" the gap between hyperplanes π and π_1. Thus we show that for any two vectors $\mathbf{u}, \mathbf{v} \in P$ if $\mathbf{u} + \mathbf{v}$ lies in the stripe between hyperplanes π and π_1 then there exists $N \in \mathbb{N}$ such that $\mathbf{u} + \mathbf{v} \in P_N \subset P$.

Recall that the non-zero entries $x_1, \ldots, x_n$ are related to the vertices that are accessible from the distinguished class α defining the measure μ. Suppose that the vertices $m+1, \ldots, n$ belong to α. We will need the following lemma.

Lemma 3.3.

(1) $\det(C - \lambda I) = 0$ if and only if $f(\lambda) = 0$. All eigenvalues of C (and hence D) are distinct.
(2) *The rank of the matrix A_0 is k. Moreover, a set of k linearly independent columns of A_0 can be chosen amongst the vectors $\{\mathbf{x}_{m+1}, \ldots, \mathbf{x}_n\}$.*

Proof. In order to prove (1), it suffices to notice that $\det(C - tI) = (-1)^k f(t)$.

Because the eigenvector $(x_1, \ldots, x_n)^T$ is probability, we obtain that for any $M \in \mathbb{N}$

$$
\lambda^M = \sum_{i=1}^{n} \sum_{j=1}^{n} (A^M)_{ij} x_j = \sum_{j=1}^{n} x_j \sum_{i=1}^{n} (A^M)_{ij}, \tag{3.9}
$$

and therefore every basis vector $\mathbf{e}_i$ can be represented as a linear combination of vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ with non-negative coefficients. Let S be the subspace of $\mathbb{Q}^k$ generated by vectors $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$. First we show that $\dim(S) \geqslant k$. Indeed, relation (3.8) shows that the numbers $1, \lambda, \ldots, \lambda^{k-1}$ correspond to the standard basis in $\mathbb{Q}^k$. Then we deduce from (3.9) that this basis belongs to S and, therefore, $\mathrm{rank}(A_0) = k$.

Next, we use the fact that λ is also the Perron–Frobenius eigenvalue of the submatrix $A_\alpha = F_\alpha^T$. Applying (3.9) to A_α, we obtain that

$$\sum_{i=m+1}^{n} \sum_{j=m+1}^{n} \left(A^r\right)_{ij} x_j = \lambda^r \sum_{i=m+1}^{n} x_i \qquad (3.10)$$

for $r \in \mathbb{N}$. We denote $q(\lambda) = \sum_{i=m+1}^{n} x_i$. Then $q(\lambda) \in \mathbb{Q}[\lambda]$. It follows from (3.10) that, for any polynomial $h(\lambda) \in \mathbb{Q}[\lambda]$, the product $h(\lambda)q(\lambda)$ is a linear combination of vectors $\{\mathbf{x}_{m+1}, \ldots, \mathbf{x}_n\}$ with rational coefficients. Then, choose $h_r \in \mathbb{Q}[\lambda]$ such that $\lambda^r = h_r(\lambda)q(\lambda)$. Therefore, each λ^r, $r = 0, \ldots, k-1$, can be represented as a linear combination of $\{\mathbf{x}_{m+1}, \ldots, \mathbf{x}_n\}$ with rational coefficients. It follows that the set $\{\mathbf{x}_{m+1}, \ldots, \mathbf{x}_n\}$ contains k linearly independent vectors. $\square$

We continue the proof of the theorem. We note that $D^T \mathbf{n} = \frac{1}{\lambda} \mathbf{n}$. It is clear that $D(\pi) = C(\pi) = \pi$.

Since $(-1)^k f(x)$ is the characteristic polynomial for C, the matrix C has k distinct eigenvalues in $\mathbb{C}$. In particular, λ is an eigenvalue for C, and for every other eigenvalue ν we have $|\nu| < \lambda$. It is not hard to find the eigenvector $\mathbf{y}_1$ of C corresponding to λ:

$$\mathbf{y}_1 = \begin{pmatrix} 1 \\ \frac{1}{\lambda} + \frac{m_1}{m_0} \\ \vdots \\ (\frac{1}{\lambda})^{k-1} + \frac{m_1}{m_0}(\frac{1}{\lambda})^{k-2} + \cdots + \frac{m_{k-1}}{m_0} \end{pmatrix}.$$

This is the only eigenvector of C (and hence of D) which is not orthogonal to $\mathbf{n}$. Indeed, suppose $C\mathbf{z} = \nu\mathbf{z}$. Then

$$\nu \langle \mathbf{z}, \mathbf{n} \rangle = \langle C\mathbf{z}, \mathbf{n} \rangle = \langle \mathbf{z}, C^T \mathbf{n} \rangle = \lambda \langle \mathbf{z}, \mathbf{n} \rangle.$$

We see that either $\nu = \lambda$ or $\langle \mathbf{z}, \mathbf{n} \rangle = 0$. All eigenvalues of C are different, so we obtain that $\langle \mathbf{z}, \mathbf{n} \rangle = 0$. This means that all eigenvectors of C (or D) different from $\mathbf{y}_1$ belong to π.

Now we are ready to prove that the vectors of P can be chosen arbitrary close to the hyperplane π. Consider two cases when the eigenvalues of C are real or complex.

(I) Suppose all eigenvalues of C are real. Then C is diagonalizable in $\mathbb{R}^k$. Let $\mathbf{y}_1, \ldots, \mathbf{y}_k$ be the eigenvectors of C which correspond to the eigenvalues $\lambda_1 = \lambda, \ldots, \lambda_k$. For $\mathbf{y} \in \mathbb{R}^k$, there exist unique real numbers α_i such that $\mathbf{y} = \sum_{i=1}^{k} \alpha_i \mathbf{y}_i$. Then $C^N \mathbf{y} = \alpha_1 \lambda^N \mathbf{y}_1 + \mathbf{z}_N$ where $\mathbf{z}_N = \sum_{i=2}^{k} \alpha_i \lambda_i^N \mathbf{y}_i$ belongs to π, $N \in \mathbb{N}$. We prove that the angle between the lines generated by $C^N \mathbf{y}$ and $\mathbf{y}_1$ can be made arbitrary small when N tends to infinity. Indeed, it is obvious that

$$\frac{\|\mathbf{z}_N\|}{\|\alpha_1 \lambda^N \mathbf{y}_1\|} = \frac{\sqrt{\sum_{2 \leqslant i,j \leqslant k} \alpha_i \alpha_j \langle \mathbf{y}_i, \mathbf{y}_j \rangle \lambda_i^N \lambda_j^N}}{\|\alpha_1 \lambda^N \mathbf{y}_1\|} \to 0$$

as $N \to \infty$ because $|\lambda_i| < \lambda$ for $i = 2, \dots, n$.

(II) Suppose some of the eigenvalues of C are complex. Then they form the pairs of complex conjugate numbers; say, $\lambda_2 = \alpha + i\beta$, and $\lambda_3 = \alpha - i\beta$ where $\alpha, \beta \in \mathbb{R}$. Let $\mathbf{u} + i\mathbf{v}$ be the eigenvector of C corresponding to λ_2 where $\mathbf{u}, \mathbf{v} \in \mathbb{R}^k$. Then the subspace of $\mathbb{R}^k$ generated by $\mathbf{u}, \mathbf{v}$ is an invariant space for C and $C\mathbf{u} = \alpha\mathbf{u} - \beta\mathbf{v}$, $C\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{u}$. Since $|\lambda_2| < \lambda$, we have $|\alpha| < \lambda$ and $|\beta| < \lambda$.

We represent $\mathbf{y}$ as a linear combination of real eigenvectors and real components of complex eigenvectors of C. The proof now is analogous to that in case (I).

Thus, while the iterations of C drive any ray which is not in π to the limit ray generated by $\mathbf{y}_1$, the iterations of $D = C^{-1}$ do the opposite thing. Arguing as above, we can prove that the angle between the line generated by $D^N \mathbf{y}$ and π can be made arbitrary small when N tends to infinity.

Applying the iterations of D, we prove that the length of vectors from P can be made arbitrary long. Recall that the vertices $m + 1, \dots, n$ belong to the distinguished class α corresponding to μ. By Lemma 3.3, the vectors $\{x_i\}_{i=m+1}^n$ contain a basis of $\mathbb{R}^k$. We have $\langle D^N \mathbf{x}_i, \mathbf{n} \rangle \to 0$ as $N \to \infty$. Moreover, $\cos \angle (D^N \mathbf{x}_i, \mathbf{n}) \to 0$ as $N \to \infty$. Consider $\langle h_i^{(N+1)} D^N \mathbf{x}_i, \mathbf{n} \rangle$. For $i = m+1, \dots, n$, it follows from the asymptotics of $(A^N)_{ij}$ that $h_i^{(N+1)} \frac{x_i}{\lambda^N} \sim c_i x_i$ as $N \to \infty$ where c_i is a constant. Hence $\langle h_i^{(N+1)} D^N \mathbf{x}_i, \mathbf{n} \rangle \sim c_i x_i$ and finally $\|h_i^{(N+1)} D^N \mathbf{x}_i\| \to \infty$ as $N \to \infty$ because $\cos \angle (D^N \mathbf{x}_i, \mathbf{n}) \to 0$.

Recall that we need to prove that any number $s = \sum_{i=1}^n l_i \frac{x_i}{\lambda^{N-1}}$ from $[0, 1]$ such that $l_i \geqslant 0$ lies in $S(\mu)$. In the vector interpretation, this means that $\mathbf{s} = \sum_{i=1}^n l_i D^{N-1} \mathbf{x}_i$. We will use in our proof the fact that the measure μ is supported on the set X_α of all infinite paths that eventually go through vertices of the class α. We consider two cases.

(i) Let $0 \leqslant l_i \leqslant h_i^{(N)}$ for $i \notin \alpha$ and $0 < s = \langle \mathbf{s}, \mathbf{n} \rangle < 1$. We show that $s \in S(\mu)$. Indeed, the part $\mathbf{s}' = \sum_{i \notin \alpha} l_i D^{N-1} \mathbf{x}_i$ belongs to P_N because its coefficients l_i lie in the needed range. Clearly, $\mathbf{s}' \in P_M$ for $M > N$. The vector $\sum_{i \in \alpha} l_i D^{N-1} \mathbf{x}_i$ lies in the integer lattice generated by $D^M \mathbf{x}_{m+1}, \dots, D^M \mathbf{x}_m$ for $M \geqslant N$. Since $\|h_i^{(M)} D^{M-1} \mathbf{x}_i\| \to \infty$ as $M \to \infty$, $i = m+1, \dots, n$, the allowable linear combinations of $D^M \mathbf{x}_{m+1}, \dots, D^M \mathbf{x}_m$ eventually fill the stripe between π and π_1. Thus, the point $\mathbf{s}$ will be "covered" by an allowable combination.

(ii) Let $\mathbf{s} = \sum_{i=1}^n l_i^{(N)} D^{N-1} \mathbf{x}_i$, where $l_i^{(N)} \in \mathbb{N}$ and $0 < \langle \mathbf{s}, \mathbf{n} \rangle < 1$. We show that this case can be reduced to the previous one. Suppose there exist $l_i > h_i^{(N)}$ for some $i \in \{1, \dots, m\}$. By Lemma 3.3, there exist $\{p_{ij}\}_{j=m+1}^n \in \mathbb{Z}$ and $q_i \in \mathbb{N}$ such that

$$q_i \mathbf{x}_i = \sum_{j=m+1}^n p_{ij} \mathbf{x}_j \tag{3.11}$$

for $i = 1, \dots, m$. We can find $M > N$ such that $h_i^{(M)} > q_i$ for $i = 1, \dots, m$. We have $\mathbf{s} = \sum_{i=1}^n l_i^{(M)} D^{M-1} \mathbf{x}_i$ where $l_i^{(M)} = \sum_{j=1}^n l_j^{(N)} (A^{(M-N)})_{ji}$. Then if $l_i^{(M)} > h_i^{(M)}$ for some $i \in \{1, \dots, m\}$ we write down $l_i^{(M)} = t_i^{(M)} q_i + r_i^{(M)}$ where $t_i^{(M)} \in \mathbb{N}$ and $0 \leqslant r_i^{(M)} < q_i$. By (3.11), the vector $t_i^{(M)} q_i \mathbf{x}_i$ can be expressed as the integer combination of $\{\mathbf{x}_j\}_{j=m+1}^n$. Hence

$$\mathbf{s} = \sum_{i=1}^{m} r_i^{(M)} D^{M-1} \mathbf{x}_i + \sum_{i=m+1}^{n} \left(l_i^{(M)} + \sum_{j=1}^{m} p_{ji} t_j^{(M)} \right) D^{M-1} \mathbf{x}_i.$$

Since $r_j^{(M)} < q_j \leqslant h_j^{(M)}$ for $j = 1, \dots, m$, it suffices to show that the coefficients $(l_i^{(M)} + \sum_{j=1}^{m} p_{ji} t_j^{(M)})$, $i = m+1, \dots, n$, can be made positive for M large enough. It follows from the above relations that $l_i^{(M)} \sim \lambda^M$ as $M \to \infty$ for $i \in \alpha$. On the other hand, $t_j^{(M)} < l_j^{(M)}$ and $l_j^{(M)} \sim \bar{o}(\lambda^M)$ as $M \to \infty$ for $j = 1, \dots, m$. Hence the needed coefficients can be made positive. $\square$

3.2. Good measures

Now we consider the conditions under which an ergodic invariant measure on a stationary Bratteli diagram is good.

Lemma 3.4. *Let μ be an ergodic $\mathcal{R}$-invariant measure on a stationary Bratteli diagram B and let A be the matrix transposed to the incidence matrix of B. Denote by α the distinguished class of vertices that defines μ. Then μ is good if and only if all the clopen cylinder sets that end in the vertices of the class α are good.*

Proof. The "only if" part of this result is obvious.

To prove the "if" part, we consider any clopen sets $U, V \subset X_\alpha$ with $\mu(U) < \mu(V)$. Since cylinder sets form a partition basis for X_B, we may assume that V is a cylinder set (see Theorem 2.3). We must find a clopen subset $W \subset V$ with $\mu(U) = \mu(W)$. By definition of X_α, the finite path corresponding to V ends in a vertex of a class β that is accessible from α, i.e. $\alpha \succcurlyeq \beta$ (otherwise V would have zero measure). If $\beta = \alpha$, then there is nothing to prove. Suppose now that $\alpha \succ \beta$. Denote by N the length of the cylinder set V. Then V is a disjoint union of cylinder subsets of length $N + 1$. Their end vertices are either in the class α or in the classes that are accessible from α. Take the latter cylinder sets and represent each of them as a disjoint union of cylinder sets of length $N + 2$. We continue these partitions infinitely many times. Enumerate the cylinder subsets of V that end in vertices of α by V_k, $k \in \mathbb{N}$. Then

$$V \supset \bigsqcup_{k=1}^{\infty} V_k \quad \text{and} \quad \mu(V) = \sum_{k=1}^{\infty} \mu(V_k)$$

by asymptotics (2.4), (2.5). Since $\mu(U) < \mu(V)$, we can find m such that

$$\sum_{k=1}^{m} \mu(V_k) \leqslant \mu(U) < \sum_{k=1}^{m+1} \mu(V_k).$$

If the equality $\sum_{k=1}^{m} \mu(V_k) = \mu(U)$ holds, then we define $W = \bigsqcup_{k=1}^{m} V_k$. Otherwise consider the number $\mu(U) - \sum_{k=1}^{m} \mu(V_k)$. This number is contained in $S(\mu)$, since $S(\mu)$ is group-like by Theorem 3.2. The set V_{m+1} is good because it ends in a vertex of α. Hence, it contains a clopen subset U_1 such that $\mu(U_1) = \mu(U) - \sum_{k=1}^{m} \mu(V_k)$. Then we set $W = \bigsqcup_{k=1}^{m} V_k \sqcup U_1$. $\square$

Theorem 3.5. *Let μ be an ergodic $\mathcal{R}$-invariant measure on a stationary diagram B defined by a distinguished eigenvalue λ of the matrix $A = F^T$. Denote by $x = (x_1, \ldots, x_n)^T$ the corresponding reduced vector. Let the vertices $m + 1, \ldots, n$ belong to the distinguished class α corresponding to μ. Then μ is good if and only if there exists $R \in \mathbb{N}$ such that $\lambda^R x_1, \ldots, \lambda^R x_m$ belong to the additive group generated by $\{x_j\}_{j=m+1}^n$.*

If the clopen values set of μ is rational and $(\frac{p_1}{q}, \ldots, \frac{p_n}{q})^T$ is the corresponding reduced vector, then μ is good if and only if $\gcd(p_{m+1}, \ldots, p_n) \mid \lambda^R$ for some $R \in \mathbb{N}$.

Proof. If $m = 0$, then the Bratteli diagram B is simple and the measure μ is good by Theorem 2.3. Suppose $m > 0$. Consider the $(n - m) \times (n - m)$ block A_α of the matrix A whose entries count the edges between vertices of the class α. Set $\tilde{x} = \sum_{k=m+1}^n x_k$. Then $(\frac{x_{m+1}}{\tilde{x}}, \ldots, \frac{x_n}{\tilde{x}})^T$ is the probability eigenvector for A_α corresponding to the eigenvalue λ. Let B_α be the stationary subdiagram of B consisting of vertices from the class α and edges connecting them. Then A_α is the matrix transpose to the incidence matrix of the subdiagram B_α. Moreover, we can assume, without loss of generality, that B_α is a simple subdiagram (see [9]). Let $\tilde{\mu}$ be the (unique) ergodic $\mathcal{R}$-invariant measure on this diagram. If Y_α denotes the path space of the Bratteli diagram B_α, then $\mu(Y_\alpha) = \tilde{x}$ because Y_α is a complete section in X_α for the tail equivalence relation. Then $\tilde{\mu}$ can be regarded as a relative measure on the clopen subset Y_α of X_α. By Theorem 2.3, the measure $\tilde{\mu}$ is good.

Denote by $H(\frac{x_{m+1}}{\tilde{x}}, \ldots, \frac{x_n}{\tilde{x}})$ the group generated by $\frac{x_{m+1}}{\tilde{x}}, \ldots, \frac{x_n}{\tilde{x}}$. By Theorem 3.2,

$$S(\tilde{\mu}) = \left\{ \frac{r}{\lambda^N} \colon N \in \mathbb{N}, \ 0 \leqslant r \leqslant \lambda^N, \ r \in H\left(\frac{x_{m+1}}{\tilde{x}}, \ldots, \frac{x_n}{\tilde{x}}\right) \right\}. \tag{3.12}$$

Suppose that for any x_i, $i = 1, \ldots, m$ there exists $R \in \mathbb{N}$ such that x_i belongs to the additive group generated by $\{\frac{x_j}{\lambda^R}\}_{j=m+1}^n$. Then we can find one common R for them. By Lemma 3.4, it suffices to prove that any cylinder set that ends in the class α is good. Let V be a cylinder set whose terminal vertex v is in the class α. For any clopen set U with $\mu(U) < \mu(V)$ we must find a clopen subset $W \subset V$ such that $\mu(W) = \mu(U)$. Consider a cylinder set $\tilde{V}$ that ends at the same vertex v, but passes only through the vertices of the class α (i.e. $\tilde{V}$ is tail equivalent to V). Then $\tilde{V} \subset Y_\alpha$ and $\mu(\tilde{V}) = \mu(V)$. To prove the claim, it suffices to find a clopen set $\tilde{W} \subset \tilde{V}$ such that $\mu(\tilde{W}) = \mu(U)$. Indeed, if such a set $\tilde{W}$ exists then we can find a tail equivalent clopen subset $W \subset V$ such that $\mu(\tilde{W}) = \mu(W) = \mu(U)$. By Theorem 3.2, $\mu(V) = \frac{x_j}{\lambda^N}$, for some $N \in \mathbb{N}$ and $j \in \{m + 1, \ldots, n\}$. Since U is a clopen set, we have $\mu(U) = \frac{k}{\lambda^M}$ where $k \in H$. We can take $M \geqslant N$. For any subset $\tilde{W} \subset Y_\alpha$, we see that $\mu(\tilde{W}) = \tilde{x} \cdot \tilde{\mu}(\tilde{W})$. By (3.12), $\tilde{\mu}(\tilde{W}) = \frac{k_1}{\lambda^S}$ for some $S \in \mathbb{N}$, and $0 \leqslant k_1 \leqslant \lambda^S$, $k_1 \in H(\frac{x_{m+1}}{\tilde{x}}, \ldots, \frac{x_n}{\tilde{x}})$. Thus, we need to find S, k_1 such that

$$\mu(U) = \frac{k}{\lambda^M} = \frac{k_1}{\lambda^S} \cdot \tilde{x} = \mu(\tilde{W}). \tag{3.13}$$

Let $k = \sum_{i=1}^n d_i x_i$ and $k_1 = \sum_{i=m+1}^n c_i \frac{x_i}{\tilde{x}}$. Then it follows from (3.13) that

$$\sum_{i=1}^n d_i x_i = \frac{1}{\lambda^{S-M}} \sum_{j=m+1}^n c_j x_j.$$

Since $x_i \in H(\frac{x_{m+1}}{\lambda^{S-M}}, \ldots, \frac{x_n}{\lambda^{S-M}})$ for $S - M \geqslant R$, there exist integers $c_{m+1}, \ldots, c_n$ satisfying the above equation. Because $\tilde{\mu}$ is good, we can find a clopen subset $\tilde{W} \subset \tilde{V}$ with $\tilde{\mu}(\tilde{W}) = \frac{k_1}{\lambda^S}$, i.e. $\mu(\tilde{W}) = \mu(U)$.

Conversely, suppose that μ is a good measure. We can repeat the proof of the "if" part backwards to obtain the needed result.

In the rational case, $x_i = \frac{p_i}{q}$, $H = \frac{1}{q}\mathbb{Z}$. Then $H(x_{m+1}, \ldots, x_n) = \frac{a}{q}\mathbb{Z}$, where $a = \gcd(p_{m+1}, \ldots, p_n)$. Hence, μ is good if and only if $\gcd(p_{m+1}, \ldots, p_n) \mid \lambda^R$ for some $R \in \mathbb{N}$. $\square$

Corollary 3.6. *Let μ be the rational measure (i.e. $S(\mu) \subset \mathbb{Q}$) on a stationary diagram B defined by a distinguished eigenvalue λ of the matrix $A = F^T$. Denote by $(\frac{p_1}{q}, \ldots, \frac{p_n}{q})^T$ the corresponding reduced vector. Let the vertices $m + 1, \ldots, n$ belong to the distinguished class α corresponding to μ. If $\gcd(p_{m+1}, \ldots, p_n) = 1$, then μ is good.*

From Theorems 2.3 and 3.2 we obtain the following

Corollary 3.7. *For an ergodic invariant measure μ on a stationary Bratteli diagram the following are equivalent*:

(i) *μ is a good measure.*
(ii) *μ is refinable.*
(iii) *μ is weakly refinable.*

4. Homeomorphic measures on stationary diagrams

Let $\mathcal{D}$ be the set of all (non-simple) stationary Bratteli diagrams. Denote by $\mathcal{S}$ the set of all Borel probability measures on diagrams from $\mathcal{D}$ which are ergodic and invariant with respect to the tail equivalence relation. Recall that, in other words, a measure $\mu \in \mathcal{S}$ if and only if there exist an aperiodic substitution dynamical system (Y, φ) and an ergodic φ-invariant measure ν such that μ is homeomorphic to ν [8]. Clearly, $\mathcal{S}$ is a countable set.

Our goal is to show that for every good measure μ from $\mathcal{S}$ there are countably many measures $\mu_i \in \mathcal{S}$ on stationary Bratteli diagrams B_i such that μ_i is homeomorphic to μ, $i \in \mathbb{N}$. Moreover, the stationary Bratteli diagrams B_i can be chosen essentially different: the corresponding tail equivalence relations $\mathcal{R}_i$ are pairwise non-orbit equivalent.

Theorem 4.1. *Let μ be a good ergodic $\mathcal{R}$-invariant probability measure on a stationary (non-simple) Bratteli diagram B. Then there exist stationary Bratteli diagrams $\{B_i\}_{i=0}^\infty$ and good ergodic $\mathcal{R}_i$-invariant probability measures μ_i on B_i such that each measure μ_i is homeomorphic to μ and the dynamical systems $(B_i, \mathcal{R}_i)$, $(B_j, \mathcal{R}_j)$ are topologically orbit equivalent if and only if $i = j$. Moreover, the diagram B_i has exactly i minimal components for the tail equivalence relation $\mathcal{R}_i$, $i \in \mathbb{N}$.*

Proof. We divide the proof into two cases: (1) the set $S(\mu)$ has only rational values; (2) there are irrational values in $S(\mu)$.

1. Let $S = S(\mu) \subset \mathbb{Q}$. Then, as proved in Theorem 3.2, there exist natural numbers λ and q, greater than one, such that $S = \{\frac{m}{q\lambda^N} \mid m, N \in \mathbb{N}, \ 0 \leqslant m \leqslant q\lambda^N\}$.

We first construct a simple Bratteli diagram B_0 and an ergodic probability invariant measure μ_0 such that $S(\mu_0) = S$. H. Yuasa [18] used similar arguments in the study of orbit equivalence of substitution systems arising from primitive substitutions whose composition matrices have rational Perron–Frobenius eigenvalues. We take the probability vector $x = (\frac{1}{q}, \ldots, \frac{1}{q})^T$ and the $q \times q$ matrix

$$A_0 = \begin{pmatrix} \lambda - 1 & 1 & \ldots & 0 & 0 \\ 0 & \lambda - 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \lambda - 1 & 1 \\ 1 & 0 & \ldots & 0 & \lambda - 1 \end{pmatrix}.$$

Then $A_0 x = \lambda x$. Clearly, the stationary Bratteli diagram B_0 defined by the transpose to A_0 is simple. The unique ergodic probability measure μ_0 is good and $S(\mu_0) = S$. Hence, μ and μ_0 are homeomorphic by Corollary 2.4.

Next, fix $i \geqslant 1$ and construct Bratteli diagrams B_i and measures μ_i as follows. Set $\lambda_i = \lambda^{i+1}$ and define the probability vector $x_i = (\frac{1}{q\lambda_i}, \ldots, \frac{1}{q\lambda_i})^T$. Take a $q\lambda_i \times q\lambda_i$ non-negative matrix $A_i = (a_{lj}^{(i)})$ such that $A_i x_i = \lambda_i x_i$. Then for every $l = 1, \ldots, q\lambda_i$

$$\sum_{j=1}^{q\lambda_i} a_{lj}^{(i)} = \lambda_i. \tag{4.1}$$

By (4.1), exactly λ_i edges start from each vertex in the diagram B_i. Clearly, there are several matrices A_i that can satisfy the above conditions. For instance, we can choose

$$A_i = \begin{pmatrix} 2 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & \lambda_i - 2 \\ 0 & 2 & 0 & \ldots & 0 & 0 & 0 & \ldots & \lambda_i - 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 2 & 0 & 0 & \ldots & \lambda_i - 2 \\ 0 & 0 & 0 & \ldots & 0 & \lambda_i - 1 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 & 0 & \ldots & \lambda_i - 1 \end{pmatrix}$$

where the submatrix in the left upper corner with the number two on the main diagonal has the size $i \times i$. Let now B_i be the stationary Bratteli diagram defined by the incidence matrix transpose to A_i. The form of A_i means that every minimal component is a 2-odometer and there are exactly i such components. Therefore, the tail equivalence relations $\mathcal{R}_i$ and $\mathcal{R}_j$ are not orbit equivalent for diagrams B_i and B_j if $i \neq j$. Note also that the non-minimal component of B_i has the same form as the diagram B_0.

To finish the proof of first part, we conclude that if μ_i is the measure on B_i defined by the eigenvector x_i and the eigenvalue λ_i, then $S(\mu_i) = S$ and μ_i is good by Corollary 3.6. Therefore, μ_i and μ are homeomorphic for any i.

2. Let B and μ be as in the theorem and suppose μ is defined by the eigenvalue $\lambda \in \mathbb{R} \setminus \mathbb{Q}$ of $A = (a_{ij})_{i,j=1}^n$, the matrix transposed to the incidence matrix of B. To prove the theorem, it suffices to construct a stationary Bratteli diagram B' such that: (i) there is an ergodic invariant

probability good measure ν on B' such that $S(\nu) = S(\mu)$; (ii) B' has one more minimal component in comparison with B (in fact, we add another vertex to each level of the initial diagram B and this vertex will determine a minimal component for the tail equivalence relation $\mathcal{R}'$).

Denote by $(x_1, \ldots, x_n)^T$ the eigenvector corresponding to λ. Recall that we consider measures on their supports, hence we assume that all x_i are positive. Let $H = H(x_1, \ldots, x_n)$ denote the additive group generated by $x_1, \ldots, x_n$. Suppose that the vertices $m+1, \ldots, n$ belong to the distinguished class α of vertices that determine the measure μ. Since μ is good, there exists $R_0 \in \mathbb{N}$ such that for any integer $R \geqslant R_0$ the values $\lambda^R x_i$, $i = 1, \ldots, n$, belong to the additive group $H(x_{m+1}, \ldots, x_n)$ generated by $\{x_j\}_{j=m+1}^n$ (see Theorem 3.5).

Fix R such that $R \geqslant R_0$. We will construct a new diagram B' such that the matrix $Q = (q_{ij})_{i,j=1}^{n+1}$ transposed to the incidence matrix of B' has the eigenvector

$$z = \left(\frac{x_1}{\lambda^R}, \ldots, \frac{x_m}{\lambda^R}, \frac{\lambda^R - 1}{\lambda^R}, \frac{x_{m+1}}{\lambda^R}, \ldots, \frac{x_n}{\lambda^R} \right)^T$$

corresponding to the eigenvalue $\psi = \lambda^M$ where $M = R + N$, $N \in \mathbb{N}$ (M will be chosen below).

To define Q, take $A^M = (a_{ij}^{(M)})_{i,j=1}^n$ and insert in A^M the additional $(m+1)$-st row $(0, \ldots, 0, q_{m+1,m+1}, \ldots, q_{m+1,n+1})$ and $(m+1)$-st column $(0, \ldots, 0, q_{m+1,m+1}, 0, \ldots 0)^T$:

$$Q = \begin{pmatrix}
a_{11}^{(M)} & \cdots & a_{1,m}^{(M)} & 0 & a_{1,m+1}^{(M)} & \cdots & a_{1,n}^{(M)} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
a_{m1}^{(M)} & \cdots & a_{m,m}^{(M)} & 0 & a_{m,m+1}^{(M)} & \cdots & a_{m,n}^{(M)} \\
0 & \cdots & 0 & q_{m+1,m+1} & q_{m+1,m+2} & \cdots & q_{m+1,n+1} \\
0 & \cdots & 0 & 0 & a_{m+1,m+1}^{(M)} & \cdots & a_{m+1,n}^{(M)} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & a_{n,m+1}^{(M)} & \cdots & a_{n,n}^{(M)}
\end{pmatrix}$$

where $\{q_{m+1,j}\}_{j=m+1}^{n+1}$ are undefined non-negative integers yet. It is worth to mention that some of the entries $a_{ij}^{(M)}$ of A^M may be zero. But the submatrix of A^M formed by the rows enumerated from $m+1$ to n and the columns enumerated from 1 to m is zero matrix since we assumed that the measure μ is determined by the vertices of the class α.

It is obvious that if $q_{m+1,m+1} \geqslant 2$ and at least one of the numbers $\{q_{m+1,j}\}_{j=m+2}^{n+1}$ is non-zero, then the Bratteli diagram corresponding to Q has one more minimal component than the diagram B corresponding to A. Hence, the dynamical systems $(X_B, \mathcal{R})$ cannot be orbit equivalent to the system $(X_{B'}, \mathcal{R}')$.

Our goal now is to find non-negative integers $\{q_{m+1,j}\}_{j=m+1}^{n+1}$ such that $Qz = \psi z$. It is clear that this equality holds for the rows $1, \ldots, m, m+2, \ldots, n$ of the matrix Q since $A^M(\frac{x}{\lambda^R}) = \lambda^M \frac{x}{\lambda^R}$. Therefore, we need only to verify that the equation

$$q_{m+1,m+1} \frac{\lambda^R - 1}{\lambda^R} + \sum_{j=m+2}^{n+1} q_{m+1,j} \frac{x_{j-1}}{\lambda^R} = \lambda^N (\lambda^R - 1) \tag{4.2}$$

can be solved for non-negative integers $\{q_{m+1,j}\}_{j=m+1}^{n+1}$.

We will use the geometric representation of algebraic numbers as vectors over $\mathbb{Q}$ as we did in the proof of Theorem 3.2. Let λ be the algebraic integer of degree k. Suppose $\{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$ denote the standard basis in $\mathbb{R}^k$ corresponding to the numbers $1, \lambda, \ldots, \lambda^{k-1}$. Then (4.2) can be written as follows (we use here and below the notation from the proof of Theorem 3.2 where, in particular, matrices C and D were defined):

$$\sum_{j=m+2}^{n+1} q_{m+1,j} D^R \mathbf{x}_{j-1} = C^N \big(C^R \mathbf{e}_1 - \mathbf{e}_1 \big) - q_{m+1,m+1} \big(\mathbf{e}_1 - D^R \mathbf{e}_1 \big). \tag{4.3}$$

Then relation (4.3) can be considered as a $k \times (n-m)$ system of linear equations with respect to the unknowns $\{q_{m+1,j}\}_{j=m+2}^{n+1}$ and the parameter $q_{m+1,m+1}$. The matrix P of the system is formed by the columns $\mathbf{p}_1, \ldots, \mathbf{p}_{n-m}$ where $\mathbf{p}_1 = D^R \mathbf{x}_{m+1}, \ldots, \mathbf{p}_{n-m} = D^R \mathbf{x}_n$. Since μ is good, we see that $\lambda H \subset H \subset H(\frac{x_{m+1}}{\lambda^R}, \ldots, \frac{x_n}{\lambda^R})$ for $R \geqslant R_0$. It follows that $\lambda^N(\lambda^R - 1) \in H(\frac{x_{m+1}}{\lambda^R}, \ldots, \frac{x_n}{\lambda^R})$ for $R \geqslant R_0$ because $\lambda^R - 1 \in H$. Using the correspondence between the elements of $S(\mu)$ and $\mathbb{Q}^k$, we obtain that $C^N(C^R \mathbf{e}_1 - \mathbf{e}_1) \in H(D^R \mathbf{x}_{m+1}, \ldots, D^R \mathbf{x}_n)$. We show that there exists a natural number $q_{m+1,m+1}$ such that $q_{m+1,m+1}(\mathbf{e}_1 - D^R \mathbf{e}_1)$ belongs to $H(D^R \mathbf{x}_{m+1}, \ldots, D^R \mathbf{x}_n)$. Indeed, by Lemma 3.3, any vector in $\mathbb{Q}^k$ can be represented as a rational linear combination of vectors $D^R \mathbf{x}_{m+1}, \ldots, D^R \mathbf{x}_n$. Hence, there exist $t \in \mathbb{N}$ and $\{t_i\}_{i=1}^{n-m} \subset \mathbb{Z}$ such that

$$\mathbf{e}_1 - D^R \mathbf{e}_1 = \sum_{i=1}^{n-m} \frac{t_i}{t} D^R \mathbf{x}_{m+i}.$$

Therefore, $t(\mathbf{e}_1 - D^R \mathbf{e}_1) \in H(D^R \mathbf{x}_{m+1}, \ldots, D^R \mathbf{x}_n)$. It follows that there exist integers $\{q_{m+1,j}\}_{j=m+2}^{n+1}$ and a non-negative number $q_{m+1,m+1}$ such that relation (4.3) holds.

Let $K = \{\beta_1 D^R \mathbf{x}_{m+1} + \cdots + \beta_{n-m} D^R \mathbf{x}_n \colon \beta_1, \ldots, \beta_{n-m} \geqslant 0\}$. We consider two cases.

First, let $k = n - m$. Then the columns of P, the vectors $\{D^R \mathbf{x}_{m+1}, \ldots, D^R \mathbf{x}_n\}$, form a basis in $\mathbb{Q}^k$. Choose $q_{m+1,m+1}$ such that $q_{m+1,m+1}(\mathbf{e}_1 - D^R \mathbf{e}_1) \in H(D^R \mathbf{x}_{m+1}, \ldots, D^R \mathbf{x}_n)$. Then the vector in the right part of relation (4.3) has integer coordinates $\{q_{m+1,j}\}_{j=m+2}^{n+1}$ in the basis $\{D^R \mathbf{x}_{m+1}, \ldots, D^R \mathbf{x}_n\}$. We refer now to the proof of Theorem 3.2 where the behavior of the matrices C and D has been studied. Denote by $l(\mathbf{y})$ the line in $\mathbb{R}^k$ generated by a vector $\mathbf{y} \in \mathbb{R}^k$. Let $\mathbf{y}_1$ be the eigenvector of the matrix C (see the proof of Theorem 3.2). We can choose R sufficiently large such that the line $l(\mathbf{y}_1)$ lies in $K \cup (-K)$. Now we fix R and show that for sufficiently large N the numbers $\{q_{m+1,j}\}_{j=m+2}^{n+1}$ are non-negative. As N tends to infinity, the vector $C^N(C^R \mathbf{e}_1 - \mathbf{e}_1)$ approaches to the line $l(\mathbf{y}_1)$ in K. The norm of this vector tends to infinity. Hence, for sufficiently large N, the right part of (4.3) lies in K and is an integer combination of linearly independent vectors $\{D^R \mathbf{x}_{m+1}, \ldots, D^R \mathbf{x}_n\}$. Thus, the coefficients $\{q_{m+1,j}\}_{j=m+2}^{n+1}$ of this combination are non-negative integers.

Let $k < n - m$. To find non-negative integer solutions $\{q_{m+1,j}\}_{j=m+2}^{n+1}$ of (4.3), we use a vector generalization of the Frobenius Problem. The following lemma follows from [4,5]:

Lemma 4.2. *Let $A \in \mathbb{Z}^{m \times n}$, $1 \leqslant m < n$, be an integral $m \times n$ matrix satisfying*

(i) $\gcd(\det(\Omega_i) \colon \Omega_i$ *is an $m \times m$ minor of A) $= 1$,*
(ii) $\{\mathbf{y} \in \mathbb{R}_{\geqslant 0}^n \colon A\mathbf{y} = 0\} = \{0\}$.

Denote by $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{Z}^m$ *the columns of the matrix* A, *and let*

$$K = \{\beta_1 \mathbf{v}_1 + \cdots + \beta_n \mathbf{v}_n \colon \beta_1, \ldots, \beta_n \geqslant 0\}$$

be the cone generated by $\mathbf{v}_1, \ldots, \mathbf{v}_n$. *Set* $\mathbf{v} = \mathbf{v}_1 + \cdots + \mathbf{v}_n$. *Then there exists* $0 \leqslant t_0 < \infty$ *such that for* $t \geqslant t_0$ *and any vector* $\mathbf{b} \in \{t\mathbf{v} + K\} \cap \mathbb{Z}^m$ *there exists a vector* $\mathbf{y}$ *with non-negative integer entries such that* $A\mathbf{y} = \mathbf{b}$. (*The least possible number* t_0 *is called the diagonal Frobenius number* $g(A)$.)

Since the coordinates of vectors $D^R \mathbf{x}_{m+1}, \ldots, D^R \mathbf{x}_n$ in the standard basis $\mathbf{e}_1, \ldots, \mathbf{e}_k$ are rationals, it is clear that we can multiply the vectors $\mathbf{e}_1, \ldots, \mathbf{e}_k$ by some rational numbers to obtain a new basis $\{\mathbf{e}'_i\}_{i=1}^k$ in which the vectors $\mathbf{p}_1, \ldots, \mathbf{p}_{n-m}$ have integer coordinates. We will find a basis for which the matrix P written in this basis satisfies conditions (i), (ii) of Lemma 4.2.

Take the basis $\{\mathbf{e}'_i\}$ and denote by $\Lambda = \Lambda(D^R \mathbf{x}_{m+1}, \ldots, D^R \mathbf{x}_n)$ the sublattice of $\mathbb{Z}^k$ generated by the vectors $\{D^R \mathbf{x}_{m+1}, \ldots, D^R \mathbf{x}_n\}$. In order to have integer solutions for (4.3), the vector $C^N(C^R \mathbf{e}_1 - \mathbf{e}_1) - q_{m+1,m+1}(\mathbf{e}_1 - D^R \mathbf{e}_1)$ must belong to Λ. The lattice Λ has a basis of k elements which belong to Λ (see [12]). Then we can choose a new basis $\{\mathbf{f}_i\}_{i=1}^k$ of Λ such that $\Lambda(\mathbf{f}_1, \ldots, \mathbf{f}_k)$ is isomorphic to $\mathbb{Z}^k$. Denote by P_f the matrix P written in the basis $\{\mathbf{f}_i\}_{i=1}^k$. Since $D^R \mathbf{x}_{m+1}, \ldots, D^R \mathbf{x}_n \in \Lambda(\mathbf{f}_1, \ldots, \mathbf{f}_k)$ the matrix P_f has integer entries. We prove that P_f satisfies (i). Consider the identity matrix I (in the basis $\{\mathbf{f}_i\}_{i=1}^k$). Since $\Lambda(D^R \mathbf{x}_{m+1}, \ldots, D^R \mathbf{x}_n) = \Lambda(\mathbf{f}_1, \ldots, \mathbf{f}_k)$, we can express the vectors $\{\mathbf{f}_i\}_{i=1}^k$ as integer linear combinations of the vectors $\{D^R \mathbf{x}_i\}_{i=m+1}^n$. Since the determinant is a linear function of its columns, we obtain that $\det I = 1$ is an integer linear combination of the determinants $\det(\Omega_i)$ where Ω_i is a $k \times k$ minor of P_f. Hence, condition (*i*) holds for P_f.

It is not hard to prove that condition (ii) holds for P_f. The vectors $\mathbf{f}_1, \ldots, \mathbf{f}_k$ form the basis of $\mathbb{Q}^k$. Let J be the transition matrix from the basis $\{\mathbf{e}_i\}_{i=1}^k$ to the basis $\{\mathbf{f}_i\}_{i=1}^k$. Denote by P_e the matrix P in the basis $\{\mathbf{e}_i\}_{i=1}^k$. For a vector $\mathbf{y} \in \mathbb{Q}^k$ denote by $\mathbf{y}_f$ the vector $\mathbf{y}$ written in the basis $\{\mathbf{f}_i\}_{i=1}^k$. We keep the notation $\mathbf{y}$ for the vector $\mathbf{y}$ written in standard basis $\{\mathbf{e}_i\}_{i=1}^k$. The columns of P_e are vectors $D^R \mathbf{x}_{m+1}, \ldots, D^R \mathbf{x}_n$. Since $\mathbf{y}_f = J^{-1} \mathbf{y}$, we have $P_f = J^{-1} P_e$. Since $\det J \neq 0$, it suffices to prove (ii) for P_e. Using the fact that the vectors $D^R \mathbf{x}_{m+1}, \ldots, D^R \mathbf{x}_n$ belong to the same half-space (as shown in the proof of Theorem 3.2), we see that any non-trivial non-negative linear combination of vectors $D^R \mathbf{x}_{m+1}, \ldots, D^R \mathbf{x}_n$ is non-zero. Indeed, suppose there exist non-negative real numbers $\gamma_1, \ldots, \gamma_{n-m}$ such that $\sum_{i=1}^{n-m} \gamma_i D^R \mathbf{x}_{m+i} = 0$. We note that $\langle \sum_{i=1}^{n-m} \gamma_i D^R \mathbf{x}_{m+i}, \mathbf{n} \rangle = 0$ where $\mathbf{n}$ denotes the vector $(1, \lambda, \ldots, \lambda^{k-1})^T$. On the other hand, $\langle \sum_{i=1}^{n-m} \gamma_i D^R \mathbf{x}_{m+i}, \mathbf{n} \rangle = \sum_{i=1}^{n-m} \gamma_i \frac{x_{m+i}}{\lambda^R}$. Since all the values $\frac{x_{m+i}}{\lambda^R}$ are positive, the linear combination is equal to zero if and only if $\gamma_i = 0$ for all i.

By Lemma 4.2, for any right part of (4.3) which belongs to $\{t\mathbf{v} + K\} \cap \mathbb{Z}^k$ in the basis $\{\mathbf{f}_i\}_{i=1}^k$, there exists a non-negative solution $\{q_{m+1,j}\}_{j=m+2}^{n+1}$. Arguing as in the case $k = n - m$, we show that the vector $C^N(C^R \mathbf{e}_1 - \mathbf{e}_1) - q_{m+1,m+1}(\mathbf{e}_1 - D^R \mathbf{e}_1)$ belongs to $\{g(P)\mathbf{v} + K\} \cap \mathbb{Z}^k$ for sufficiently large N. The transformation to the basis $\{\mathbf{f}_i\}_{i=1}^k$ alters the coordinates of vectors and the entries of matrices but it doesn't change their properties used in the proof of the case $k = n - m$.

It is left to prove that the measure ν is good and $S(\nu) = S(\mu)$. Since μ is good, we have that $\lambda^R x_1, \ldots, \lambda^R x_n \in H(x_{m+1}, \ldots, x_n)$ for $R \geqslant R_0$. It follows from the relation $1, \lambda^R \in H(x_1, \ldots, x_n)$ that $\lambda^R(\lambda^R - 1) \in H(x_{m+1}, \ldots, x_n)$. Therefore,

$$\lambda^R \frac{x_1}{\lambda^R}, \ldots, \lambda^R \frac{x_n}{\lambda^R}, \lambda^R \frac{\lambda^R - 1}{\lambda^R} \in H\left(\frac{x_{m+1}}{\lambda^R}, \ldots, \frac{x_n}{\lambda^R}\right)$$

for $R \geqslant R_0$. Hence, we obtain that $\lambda^R z_1, \ldots, \lambda^R z_{n+1} \in H(z_{m+1}, \ldots, z_{n+1})$. Since $M > R$ we have $\psi z_1, \ldots, \psi z_{n+1} \in H(z_{m+1}, \ldots, z_{n+1})$ and ν is good by Theorem 3.5.

Finally, we conclude that

$$H(z_1, \ldots, z_{n+1}) = \frac{1}{\lambda^R} H(x_1, \ldots, x_n), \qquad \frac{1}{\psi} H(z_1, \ldots, z_{n+1}) = \frac{1}{\lambda^{R+M}} H(x_1, \ldots, x_n).$$

Since $S(\mu) = \bigcup_{N \in \mathbb{N}} \frac{1}{\lambda^N} H(x_1, \ldots, x_n)$ and $\lambda H \subset H$, we have $S(\nu) = S(\mu)$. $\square$

Remark. For $\lambda \notin \mathbb{Q}$, we can proceed as in the rational case by finding a measure ν on a simple stationary Bratteli diagram such that $S(\nu) = S(\mu)$. To do this, we construct a diagram with an $n \times n$ matrix $\tilde{A} = (\tilde{a}_{ij})$ transpose to the incidence matrix such that $\tilde{A}x = \lambda^M x$ for some $M \in \mathbb{N}$. We obtain the matrix $\tilde{A}$ by taking A^M for sufficiently large M and changing the zero block $\{a_{ij}^{(M)}\}_{i=m+1, j=1}^{n,m}$ to a non-zero one. We have $q_i \mathbf{x}_i = \sum_{j=m+1}^{n} p_{ij} \mathbf{x}_j$ for $i = 1, \ldots, m$ and some $q_i \in \mathbb{N}$, $p_{ij} \in \mathbb{Z}$. Since $A^M x = \lambda^M x$, we have $\sum_{i=m+1}^{n} a_{ij}^{(M)} x_j = \lambda^M x_i$ for $i = m+1, \ldots, n$. The block $\{a_{ij}\}_{i,j=m+1}^{n}$ of the matrix A is positive, hence we can make the numbers $\{a_{ij}^{(M)}\}_{i,j=m+1}^{n}$ arbitrary large. In particular, we take M such that $a_{m+1,j}^{(M)} \geqslant p_{1,j}$ for all $j = m+1, \ldots, n$. Then $q_1 x_1 + \sum_{j=m+1}^{n} (a_{m+1,j}^{(M)} - p_{1,j}) x_j = \sum_{j=m+1}^{n} a_{m+1,j}^{(M)} x_j = \lambda x_{m+1}$ and $\tilde{a}_{m+1,1} = q_1 > 0$. We proceed with other elements of the zero block in the similar way.

Thus, we obtain that for any finite ergodic invariant measure μ on a stationary Bratteli diagram there exists a good measure $\nu \in \mathcal{S}$ such that $S(\mu) = S(\nu)$. This result together with an example is discussed in the last section.

We will also construct two measures μ_1 and μ_2 such that $S(\mu_1) = S(\mu_2)$, the measure μ_1 is good and the measure μ_2 is not good. Hence these measures are not homeomorphic. The example can be found in the next section.

Now we consider two classes of measures on stationary Bratteli diagrams: (i) measures of Bernoulli type, (ii) measures satisfying the Quotient Condition.

Proposition 4.3. *Let μ be an ergodic $\mathcal{R}$-invariant measure on a stationary Bratteli diagram B. Then μ does not satisfy the Quotient Condition.*

Proof. Let $A = F^T$ be the $n \times n$ matrix transposed to the incidence matrix of the diagram B. Let λ and $x = (x_1, \ldots, x_n)^T$ be the eigenvalue and eigenvector of A which generate μ. Denote by H the additive subgroup of $\mathbb{R}$ generated by $\{x_1, \ldots, x_n\}$. Let eigenvalue λ be the algebraic number of degree k. If $k = 1$, the clopen values set $S(\mu)$ consists of rational numbers. To get a contradiction, we assume that μ satisfies the Quotient Condition. Then, by Theorems 2.5 and 3.2, we have

$$\mathbb{Q} \cap [0, 1] \subset S(\mu) = \left(\bigcup_{N=0}^{\infty} \frac{1}{\lambda^N} H\right) \cap [0, 1].$$

Then for any $l \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that $\frac{1}{l} \in \frac{1}{\lambda^N} H$. In the vector form, the latter is written as $\frac{1}{l}\mathbf{e}_1 \in H(D^N\mathbf{x}_1, \ldots, D^N\mathbf{x}_n)$. The matrix D (used in the proof of Theorem 3.2) has rational entries (d_{ij}). There exists non-negative integer t such that $d_{ij} = \frac{t_{ij}}{t}$ where $t_{ij} \in \mathbb{Z}$ for $i = 1, \ldots, k$, $j = 1, \ldots, n$. Denote by T the $k \times k$ matrix with entries (t_{ij}). Then $\frac{1}{l}\mathbf{e}_1 \in \frac{1}{t^N} H(T^N\mathbf{x}_1, \ldots, T^N\mathbf{x}_n)$. The set L of prime divisors of t and the denominators of the coordinates of $\mathbf{x}_1, \ldots, \mathbf{x}_n$ is finite. We can choose l as a prime number which does not belong to L. Then $\frac{1}{l}\mathbf{e}_1$ cannot belong to $H(D^N\mathbf{x}_1, \ldots, D^N\mathbf{x}_n)$ for any $N \in \mathbb{N}$. This is a contradiction. $\quad\square$

Now we focus on rational measures, i.e. on the case when the clopen values sets belong to $\mathbb{Q}$.

Proposition 4.4. *Let μ be a rational measure on a stationary diagram B defined by a distinguished eigenvalue λ of the matrix $A = F^T$. Denote by $(\frac{p_1}{q}, \ldots, \frac{p_n}{q})$ the corresponding reduced vector.*

(1) *A number $y \in S(\mu) + \mathbb{Z}$ if and only if there exists $N \in \mathbb{N}$ such that $y\lambda^N q \in \mathbb{Z}$.*
(2) *The set $S(\mu)$ is multiplicative if and only if there exists $K \in \mathbb{N}$ such that $q \mid \lambda^K$.*
(3) *If μ is good, then μ is of Bernoulli type if and only if there exists $K \in \mathbb{N}$ such that $q \mid \lambda^K$.*

Proof. 1. Suppose there exists $N \in \mathbb{N}$ such that $y\lambda^N q \in \mathbb{Z}$. Since $\gcd(p_1, \ldots, p_n) = 1$, there exist $u_1, \ldots, u_n \in \mathbb{Z}$ such that $u_1 p_1 + \cdots + u_n p_n = 1$. Then one can take $k_1, \ldots, k_n \in \mathbb{Z}$ such that $y\lambda^N q = k_1 p_1 + \cdots + k_n p_n$. Hence $y = \frac{1}{\lambda^N}(k_1 \frac{p_1}{q} + \cdots + k_n \frac{p_n}{q})$. By Theorem 3.2, $S(\mu)$ is group-like. Therefore $y \in S(\mu) + \mathbb{Z}$.

Conversely, let $y \in S(\mu) + \mathbb{Z}$. Then there exist $k_1, \ldots, k_n \in \mathbb{Z}$ and $N \in \mathbb{N}$ such that $y = \frac{1}{\lambda^N}(k_1 \frac{p_1}{q} + \cdots + k_n \frac{p_n}{q})$. Hence $y\lambda^N q = k_1 p_1 + \cdots + k_n p_n \in \mathbb{Z}$.

2. Suppose $S(\mu)$ is multiplicative. Take $u_1, \ldots, u_n \in \mathbb{Z}$ such that $u_1 p_1 + \cdots + u_n p_n = 1$. Since $S(\mu)$ is group-like, the fraction $\frac{1}{q} = u_1 \frac{p_1}{q} + \cdots + u_n \frac{p_n}{q}$ lies in $S(\mu)$. Then the fraction $\frac{1}{q^2}$ belongs to $S(\mu)$. Hence, $\frac{1}{q^2}\lambda^K q = \frac{1}{q}\lambda^K \in \mathbb{N}$ for some $K \in \mathbb{N}$.

Suppose $q \mid \lambda^K$. Let $y_1, y_2 \in S(\mu)$. Then $y_1\lambda^M q$ and $y_2\lambda^N q$ are integers for some $M, N \in \mathbb{N}$. Hence, $y_1 y_2 \lambda^{M+N} q^2 \in \mathbb{Z}$. Then $y_1 y_2 \lambda^{M+N+K} q$ is an integer. Therefore, $y_1 y_2 \in S(\mu)$.

3. Let $q \mid \lambda^K$ for some $K \in \mathbb{N}$. Then the set $S(\mu)$ is multiplicative, and $S(\mu)$ is ring-like. By Theorem 2.5, it suffices to show that every positive element y from $S(\mu) + \mathbb{Z}$ is a sum of positive units of $S(\mu) + \mathbb{Z}$. There exist $N \in \mathbb{N}$ and $k_1, \ldots, k_n$ such that $y = \frac{1}{\lambda^N q}(k_1 p_1 + \cdots + k_n p_n)$. The fractions $\frac{1}{\lambda^N}$ and $\frac{1}{q}$ are the positive units of the ring $S(\mu) + \mathbb{Z}$. Therefore, their product is a unit of the ring. Since $y > 0$, the integer $k_1 p_1 + \cdots + k_n p_n$ is non-negative. Thus, y is the sum of the positive unit $\frac{1}{q\lambda^N}$ taken $k_1 p_1 + \cdots + k_n p_n$ times.

Conversely, let μ be a good measure of Bernoulli type. Then $S(\mu)$ is ring-like. Hence, there exists $K \in \mathbb{N}$ such that $q \mid \lambda^K$. $\quad\square$
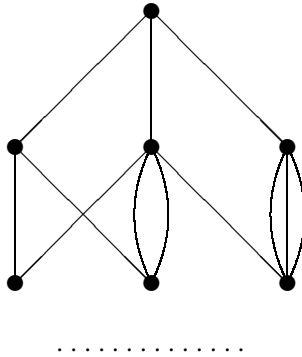
5. Examples

In this section, we consider several examples of ergodic invariant measures on some stationary Bratteli diagrams illustrating the results proved in Section 3.

Example 1. Let B be the stationary Bratteli diagram with incidence matrix

$$F = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix}.$$

This diagram B looks as follows:

This diagram has two simple stationary subdiagrams sitting on the first two and the third vertices, respectively. The left subdiagram is a minimal component for $\mathcal{R}$.

We consider the $\mathcal{R}$-invariant ergodic measures for the diagram B and compute their clopen values sets and prove some properties of these measures.

Denote $A = F^T$. The eigenvectors $x = (\frac{3-\sqrt{5}}{2}, \frac{\sqrt{5}-1}{2}, 0)^T$ and $y = (\frac{1}{4}, \frac{1}{2}, \frac{1}{4})^T$ of the matrix A correspond to the eigenvalues $\lambda_1 = \frac{3+\sqrt{5}}{2}$ and $\lambda_2 = 3$, respectively. Thus, there are two probability ergodic $\mathcal{R}$-invariant measures on the path space of the diagram B. Denote by μ_1 the measure generated by the vector x and the eigenvalue λ_1, and by μ_2 the measure generated by the vector y and the eigenvalue λ_2. We note that $S(\mu_1) \cap (\mathbb{R} \setminus \mathbb{Q}) \neq \emptyset$ and $S(\mu_2) \subset \mathbb{Q}$.

Let $h^{(n)} = (h_i^{(n)})$ denote the heights of towers corresponding to the vertices $i = 1, 2, 3$ enumerated from left to right. We have $h^{(1)} = (1, 1, 1)$ and

$$h^{(n+1)} = Fh^{(n)}. \tag{5.1}$$

In order to find the solutions $h^{(n)}$ of (5.1), we use generating functions. Set $f^{(i)}(s) = \sum_{n=0}^{\infty} h_i^{(n+1)} s^n$. It can be shown that

$$f^{(1)}(s) = \frac{1-s}{1-3s+s^2},$$

$$f^{(2)}(s) = \frac{1}{1-3s+s^2},$$

$$f^{(3)}(s) = \frac{(1-s)^2}{(1-3s+s^2)(1-3s)}.$$

Decomposing the generating functions into the series, we obtain

$$h_1^{(n)} = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\left(\frac{3+\sqrt{5}}{2}\right)^{n-1} - \frac{1-\sqrt{5}}{2}\left(\frac{3-\sqrt{5}}{2}\right)^{n-1}\right),$$

$$h_2^{(n)} = \frac{1}{\sqrt{5}}\left(\left(\frac{3+\sqrt{5}}{2}\right)^{n} - \left(\frac{3-\sqrt{5}}{2}\right)^{n}\right),$$

$$h_3^{(n)} = 4\cdot 3^n + \frac{7\sqrt{5}-15}{10}\left(\frac{3-\sqrt{5}}{2}\right)^{n-1} - \frac{7\sqrt{5}+15}{10}\left(\frac{3+\sqrt{5}}{2}\right)^{n-1}.$$

We see that $h_1^{(n)} = f_{2n-1}$ and $h_2^{(n)} = f_{2n}$ where f_i is the i-th Fibonacci number.

The above computation allows one to determine explicitly all elements of the sets $S(\mu_1)$ and $S(\mu_2)$. The ergodic measures μ_1 and μ_2 are good. Indeed, the measure μ_1 is supported on a simple subdiagram, hence μ_1 is good. It follows from Theorem 3.5 that μ_2 is also good. One can show (see Example 2) that in the simplex of all $\mathcal{R}$-invariant probability measures only these ergodic measures are good. In other words, the measure $\nu_\alpha = \alpha\mu_1 + (1-\alpha)\mu_2$ is not good for any $\alpha \in (0, 1)$. We remark also that, by Proposition 4.3, neither μ_1 nor μ_2 satisfy the Quotient Condition.

Proposition 5.1. *Let B, μ_1, μ_2 be as above. Then μ_1 is of Bernoulli type and μ_2 is not.*

Proof. We use Theorem 2.5 to prove that μ_1 is of Bernoulli type. Since $\frac{2}{3+\sqrt{5}} = \frac{3-\sqrt{5}}{2}$, we see that

$$S(\mu_1) = \left\{k_1^{(n)}\left(\frac{3-\sqrt{5}}{2}\right)^{n} + k_2^{(n)}\frac{\sqrt{5}-1}{2}\left(\frac{3-\sqrt{5}}{2}\right)^{n-1} : n \in \mathbb{N}\right\} \tag{5.2}$$

where $0 \leqslant k_i^{(n)} \leqslant h_i^{(n)}$.

Denote $G = S(\mu) + \mathbb{Z}$. By Theorem 2.3, G is an additive subgroup of $\mathbb{R}$. Since $(\frac{\sqrt{5}-1}{2})^2 = \frac{3-\sqrt{5}}{2}$, the group G is multiplicative. Hence G is a ring. We note that the number 1 is a positive unit of the ring G, and the fractions $\frac{\sqrt{5}-1}{2}$ and $\frac{3-\sqrt{5}}{2}$ are also positive units of G because $\frac{2}{\sqrt{5}-1} = \frac{\sqrt{5}-1}{2} + 1 \in G$ and $\frac{2}{3-\sqrt{5}} = 3 - \frac{3-\sqrt{5}}{2} \in G$. Therefore, every positive element of G is a sum of positive units of G. By Theorem 2.5, the measure μ_1 is of Bernoulli type.

We have already observed that the measure μ_2 is good. Then μ_2 is not of Bernoulli type by Proposition 4.4. $\quad\square$

Example 2. The following example is a generalization of Example 1. We consider a class of non-simple stationary Bratteli diagrams that have one minimal and one non-minimal component and have exactly two ergodic probability $\mathcal{R}$-invariant measures defined by these components. Let B be such a diagram. Suppose that $S(\mu_1)$ contains irrational numbers for the measure μ_1 supported on the minimal component, and $S(\mu_2) \subset \mathbb{Q}$ for the other ergodic measure μ_2. The Bratteli diagram in Example 1 belongs to this class. Since μ_1 and μ_2 are the only ergodic measures, any $\mathcal{R}$-invariant measure ν is of the form $\nu_\alpha = \alpha\mu_1 + (1-\alpha)\mu_2$ where $\alpha \in [0, 1]$. These measures

form the convex simplex of all $\mathcal{R}$-invariant probability measures on X_B. Our goal is to show that the measure ν_α is good only for $\alpha = 0, 1$.

Proposition 5.2. *Let μ_1, μ_2 and ν_α be as above. Then the measure ν_α on the Cantor space X_B is not good for any $\alpha \in (0, 1)$.*

Proof. Let $A = F^T$ be the $n \times n$ matrix transposed to the incidence matrix of the diagram B. Assume the vertices $1, \ldots, m$ belong to the minimal component of the diagram and the vertices $m + 1, \ldots, n$ are in the non-minimal component. Let μ_1 be generated by the eigenvalue $\lambda_1 \in \mathbb{R} \setminus \mathbb{Q}$ and the eigenvector $x = (x_1, \ldots, x_m, 0, \ldots, 0)^T$, and let μ_2 be generated by the eigenvalue $\lambda_2 \in \mathbb{N}$ and the eigenvector $y = (\frac{p_1}{q}, \ldots, \frac{p_n}{q})^T$ where $p_i, q \in \mathbb{N}$.

We first chose two particular clopen sets U and V such that $\nu_\alpha(U) < \nu_\alpha(V)$ and then we show that there is no clopen subset of V with measure $\nu_\alpha(U)$. Denote by V the cylinder set of length 1 that ends in a vertex of the non-minimal component. Without loss of generality, we assume that V ends in the vertex number n. Denote by U_N the cylinder set of length N which ends in the vertex 1 of the minimal component of B. Take N sufficiently large so that $\nu_\alpha(U_N) < \nu_\alpha(V)$. Let $U = U_N$.

Suppose ν_α is good for some $\alpha \in (0, 1)$. We have

$$\nu_\alpha(U) = \alpha \frac{x_1}{\lambda_1^{N-1}} + (1 - \alpha) \frac{p_1}{q \lambda_2^{N-1}} < \nu_\alpha(V) = \frac{p_n}{q}.$$

By assumption, there exists a subset $W \subset V$ such that $\nu_\alpha(U) = \nu_\alpha(W)$. Since $W \subset V$, there exist integers k, M (we can always choose $M > N$) such that

$$\nu_\alpha(W) = (1 - \alpha) \frac{k}{q \lambda_2^M}.$$

It follows from the equality $\nu_\alpha(W) = \nu_\alpha(U)$ that

$$k = \frac{\alpha}{1 - \alpha} \cdot \frac{x_1 q \lambda_2^M}{\lambda_1^{N-1}} + p_1 \lambda_2^{M-N+1}. \tag{5.3}$$

Since the numbers k, $p_1 \lambda_2^{M-N+1}$, $q \lambda_2^M$ are integers, we have

$$\frac{\alpha}{1 - \alpha} \cdot \frac{x_1}{\lambda_1^{N-1}} \in \mathbb{Q}.$$

We can repeat the same arguments for $N + 1$ instead of N and obtain that

$$\frac{\alpha}{1 - \alpha} \cdot \frac{x_1}{\lambda_1^N} \in \mathbb{Q}.$$

Hence, the ratio of the two above mentioned values should be rational. But this ratio equals $\lambda_1 \in \mathbb{R} \setminus \mathbb{Q}$. This is a contradiction. $\quad\square$

Example 3. We consider now a class of stationary Bratteli diagrams and determine which measures on them are good. Fix an integer $N \geqslant 3$ and let

$$F_N = \begin{pmatrix} 2 & 0 & 0 \\ 1 & N & 1 \\ 1 & 1 & N \end{pmatrix}$$

be the incidence matrix of the Bratteli diagram B_N. For $A_N = F_N^T$ we easily find the Perron–Frobenius eigenvalue $\lambda = N + 1$ and the corresponding probability eigenvector

$$x = \left(\frac{1}{N}, \frac{N-1}{2N}, \frac{N-1}{2N} \right)^T.$$

Let μ_N be the measure on B_N determined by λ and the eigenvector x. It follows from Theorem 3.5 that μ_N is a good measure if and only if for all sufficiently large R we have

$$\frac{(N+1)^R}{N} \in \frac{N-1}{2N} \mathbb{Z}$$

or, equivalently, $\frac{2(N+1)^R}{N-1}$ is an integer, $k \in \mathbb{N}$. This is possible if and only if $N = 2^k + 1$. For instance, the measure μ_N is good for $N = 3, 5$ but is not good for $N = 4$.

B_3: μ_3 is good B_4: μ_4 is not good B_5: μ_5 is good
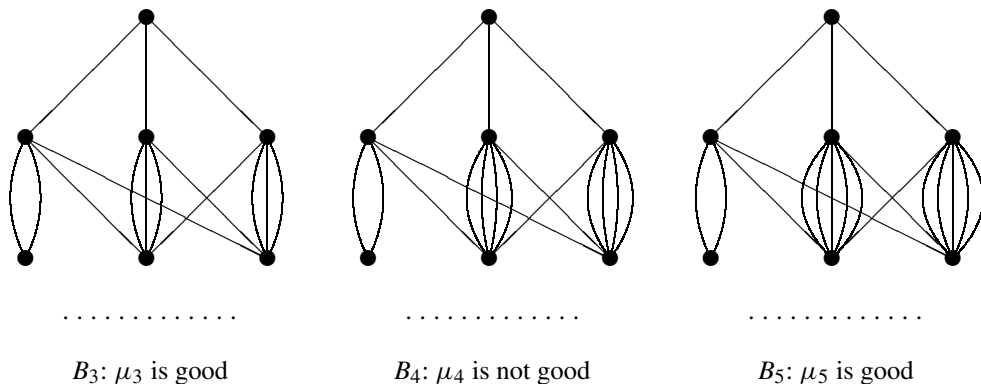
In the case when $N = 4$, we see that the eigenvector corresponding to $\lambda = 5$ is $(\frac{2}{8}, \frac{3}{8}, \frac{3}{8})$. The fact that μ_4 is not good can be also proved straightforwardly. Indeed, the measure of any cylinder set that ends in one of the last two vertices of the diagram B at the level n is $\frac{3}{8 \cdot 5^{n-1}}$. Hence, the measure $\mu_4(V)$ of any clopen set V combined from such cylinder sets is a rational number with factor 3 in the numerator. Let, for instance, U be the cylinder set of length 1 that ends at the first vertex of the diagram and V be the cylinder set of length 1 that ends at the second vertex. Then $\frac{2}{8} = \mu_4(U) < \mu_4(V) = \frac{3}{8}$ and the measure of any clopen subset of V is a rational irreducible fraction with the number 3 as a numerator factor. Thus, there is no clopen subset of V with measure $\mu_4(U) = \frac{2}{8}$ and μ_4 is not good.

On the other hand, it is easy to find a measure ν on a simple stationary Bratteli diagram such that $S(\mu_4) = S(\nu)$. In fact, the following general statement holds.

Proposition 5.3. *For any probability ergodic $\mathcal{R}$-invariant measure μ on a stationary Bratteli diagram there exists a good measure $\nu \in \mathcal{S}$ such that $S(\mu) = S(\nu)$.*

Proof. The proof immediately follows from Theorem 4.1 and the remark after the theorem: there exists a measure ν on a simple Bratteli diagram (hence ν is good) such that $S(\mu) = S(\nu)$. □

Another way to find such a measure ν is to find a non-negative integer solution $(p_{ij}) = P$ of the system $Py = \psi y$ where the eigenvector y and eigenvalue ψ generate the measure ν and are chosen such that $S(\nu) = S(\mu)$. For instance, if $\mu = \mu_4$, then we can take

$$P = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 1 \\ 9 & 3 & 2 \end{pmatrix}$$

where $y = (\frac{1}{8}, \frac{2}{8}, \frac{5}{8})$ and $\psi = 5$. It is not hard to see that the clopen values sets for $S(\nu)$ and $S(\mu_4)$ coincide, and the measure ν is good but μ_4 is not. Hence, by Theorem 2.3, the measures μ_4 and ν are not topologically equivalent.

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