Smooth quiver representation spaces

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Abstract

In this paper we classify all quivers and corresponding dimension vectors having a smooth space of semisimple representation classes. The result is that these quiver settings can be reduced via some specific reduction steps to 3 simple types. © 2002 Elsevier Science (USA). All rights reserved.

1. Introduction and motivation

Many problems in representation theory can be reduced to representations of quivers. Suppose A is a finitely generated algebra and $\mathsf{Rep}_n A$ is the space of n-dimensional complex representations of A. On this space is an action of GL_α and one can divide out this action by taking the affine quotient to obtain a new space $\mathsf{iss}_n A := \mathsf{Rep}_n A / \mathsf{GL}_\alpha$ classifying the equivalence classes of n-dimensional semisimple representations of A (see [3]).

If $W \in \mathsf{Rep}_n A$ is a semisimple representation, one can wonder what the structure of $\mathsf{iss}_n A$ around the point p corresponding to the equivalence class of W looks like. If W is a smooth point in $\mathsf{Rep}_n A$ there is a neighborhood of p that is étale (or analytically) isomorphic to a neighborhood of the zero representation in the quotient space, $\mathsf{iss}_{\alpha_p}, Q_p$ of a quiver setting (Q_p, α_p) which is called the local quiver setting of p. This local quiver setting depends on the structure of W

as a direct sum of simple representations

$$W := S_1^{\oplus a_1} \oplus \cdots \oplus S_k^{\oplus a_k}.$$

(For the exact construction see [4].)

So if one for example asks whether $\mathrm{iss}_n A$ is smooth in the point p one can as well ask whether its local quiver setting has a quotient space that is smooth in zero. As we will see below this is the same as asking whether this quotient space is an affine space or whether the corresponding ring of invariant functions is a polynomial ring. Such quiver settings will be called coregular.

In this paper we present a method to determine if a random given quiver setting (Q, α) is indeed coregular. Because the quotient space $\mathrm{iss}_\alpha Q$ can be seen as the product of the quotient spaces of the strongly connected components of (Q, α) (see Lemma 2.4), we can restrict to strongly connected quiver settings.

The method will consist of a number of allowed reduction steps. Using these steps one attempts to simplify the quiver setting as much as possible. When this is done one has to check whether the reduced quiver setting is equal to one of 3 basic quiver settings that have a smooth quotient space. The main theorem we will prove can be formulated as follows.

Theorem 1.1. *Let (Q, α) be a genuine strongly connected quiver setting and (Q', α') is the quiver setting obtained after all possible reductions of the form*:
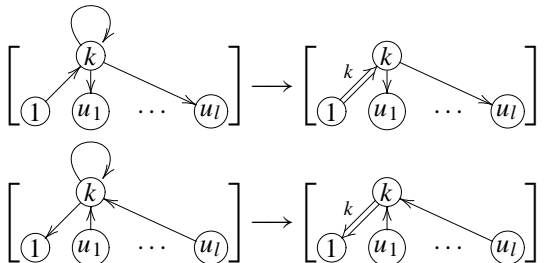
$\mathcal{R}_I$: *If $\sum_{j=1}^{k} i_j \leqslant \alpha_v$ or $\sum_{j=1}^{l} u_j \leqslant \alpha_v$ we delete the vertex v*:
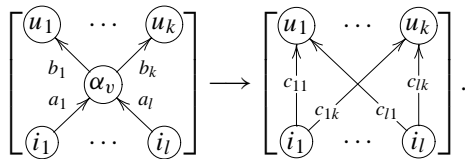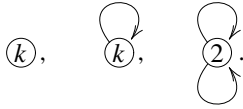

$\mathcal{R}_{II}$: *Remove the loops on a vertex with dimension 1*:
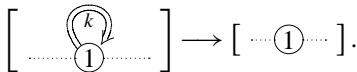
$\mathcal{R}_{III}$: *Remove the only loop on a vertex with dimension $k > 1$ which has a neighborhood like in one of the pictures below*:
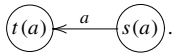
(Q, α) *is coregular if and only if* (Q', α') *is one of the three settings below*:

2. Quiver representations

In this section we recall some generalities about representations of quivers. A *quiver* $Q = (V, A, s, t)$ is a quadruple consisting of a set of vertices V, a set of arrows A and 2 maps $s, t : A \to V$ which assign to each arrow its starting and terminating vertex. We also denote this as

The *Euler form* of Q is the bilinear form $\chi_Q : \mathbb{Z}^{\#V} \times \mathbb{Z}^{\#V} \to \mathbb{Z}$ defined by the matrix

$$m_{ij} = \delta_{ij} - \#\big\{a \,\big|\, \text{(i)} \xleftarrow{\ a\ } \text{(j)}\big\},$$

where δ is the Kronecker delta. It is easy to see that a quiver is uniquely defined by its Euler form.

A *dimension vector* of a quiver is a map $\alpha : V \to \mathbb{N}$, the size of a dimension vector is defined as $|\alpha| := \sum_{v \in V} \alpha_v$. A couple (Q, α) consisting of a quiver and a dimension vector is called a *quiver setting* and for every vertex $v \in V$, α_v is referred to as the dimension of v. If no vertex has dimension zero the setting is called *genuine*. If we draw pictures of quiver settings we will put the dimension of a vertex inside that vertex.

An α-*dimensional complex representation* W of Q assigns to each vertex v a linear space $\mathbb{C}^{\alpha_v}$ and to each arrow a a matrix

$$W_a \in \mathsf{Mat}_{\alpha_{t(a)} \times \alpha_{s(a)}}(\mathbb{C}).$$

The space of all α-dimensional representations is denoted by $\mathsf{Rep}_\alpha Q$:

$$\mathsf{Rep}_\alpha Q := \bigoplus_{a \in A} \mathsf{Mat}_{\alpha_{t(a)} \times \alpha_{s(a)}}(\mathbb{C}).$$

To the dimension vector α we can also assign a reductive group

$$\mathsf{GL}_\alpha := \bigoplus_{v \in V} \mathsf{GL}_{\alpha_v}(\mathbb{C}).$$

This group can be considered as the group of base changes in the vector spaces associated to the vertices. Therefore every element of this group, g, has a natural action on $\mathsf{Rep}_\alpha Q$:

$$W := (W_a)_{a \in A}, \ W^g := \big(g_{t(a)} W_a g_{s(a)}^{-1}\big)_{a \in A}.$$

Two representations in $\mathsf{Rep}_\alpha Q$ are called equivalent, if they belong to the same orbit under the action of GL_α.

For every vertex we also define a special dimension vector

$$\epsilon_v : V \to \mathbb{N}, \quad w \mapsto \delta_{vw},$$

and an ϵ_v-dimensional representation S_v assigning to every arrow the zero matrix.

A representation W is called *simple* if the only collections of subspaces $(\mathsf{V}_v)_{v \in V}$, $\mathsf{V}_v \subseteq \mathbb{C}^{\alpha_v}$ having the property

$$\forall a \in A: \quad W_a \mathsf{V}_{s(a)} \subset \mathsf{V}_{t(a)}$$

are the trivial ones (i.e., the collection of zero-dimensional subspaces and $(\mathbb{C}^{\alpha_v})_{v \in V}$).

The direct sum $W \oplus W'$ of two representations W, W' has as dimension vector the sum of the two dimension vectors and as matrices $(W \oplus W')_a := W_a \oplus W'_a$. A representation equivalent to a direct sum of simple representations is called *semisimple*.

From the algebraic point of view one can look at the ring of polynomial functions over $\mathsf{Rep}_\alpha Q$ which is a polynomial ring denoted by $\mathbb{C}[\mathsf{Rep}_\alpha Q]$. On this ring there is a corresponding action of GL_α and one can look at the corresponding subring of functions that are invariant under this action:

$$\mathbb{C}[\mathsf{Rep}_\alpha Q]^{\mathsf{GL}_\alpha} := \big\{ f \in \mathbb{C}[\mathsf{Rep}_\alpha Q] \,\big|\, f^g = f \big\}.$$

The variety corresponding to this subring is denoted by $\mathsf{iss}_\alpha Q$ and by [1,3] this space classifies the equivalence classes semisimple α-dimensional representations of Q which are in fact the closed GL_α-orbits in $\mathsf{Rep}_\alpha Q$. The ring of invariants will also be denoted by $\mathbb{C}[\mathsf{iss}_\alpha Q]$.

If $\mathsf{iss}_\alpha Q$ is a smooth variety then it is an affine space, this follows immediately from [3, 4.3B, Lemma 1, p. 139] and [8, Proposition 2.4, p. 136].

Theorem 2.1. *Suppose V is a complex vector space with a linear action of a reductive group G. If the affine quotient V/G is smooth in the point corresponding to 0 then $V/\mathsf{G} = \mathbb{C}^t$ for a $t \in \mathbb{N}$. The corresponding ring of invariants $\mathbb{C}[V]^{\mathsf{G}}$ is then a polynomial ring.*

If we want to study the ring of invariants it is important to know by what functions it is generated. The solution to this problem is given in the article by Le Bruyn and Procesi about semisimple quiver representations [4].

A sequence of arrows $a_1, \ldots, a_p$ in a quiver Q is called a *path of length p* if $s(a_i) = t(a_{i+1})$, this path is called a cycle if $s(a_p) = t(a_1)$.

To a cycle we can associate a polynomial function

$$f_c : \mathsf{Rep}_\alpha Q \to \mathbb{C} : W \mapsto \mathsf{Tr}(W_{a_1} \cdots W_{a_p})$$

which is definitely GL_α-invariant. Two cycles that are a cyclic permutation of each other give the same polynomial invariant, because of the basic properties of the trace map. Two such cycles are called equivalent.

A cycle $a_1 \ldots a_p$ is called *primitive* if every arrow has a different starting vertex. This means that the cycle runs through each vertex at most 1 time. It is easy to see that every cycle has a decomposition in primitive cycles. It is however not true that the corresponding polynomial invariant decomposes to a product of the polynomial functions of the primitive cycles.

We will call a cycle *quasi-primitive* for a dimension vector α if the vertices that are ran through more than once, have dimension bigger than 1. By cyclicly permuting a cycle and splitting the trace of a product of two 1×1 matrices into a product of traces, we can always decompose an f_c into a product of traces of quasi-primitive cycles. We now have the following result

Theorem 2.2 (Le Bruyn–Procesi). $\mathbb{C}[\mathsf{iss}_\alpha Q]$ *is generated by all f_c where c is a quasi-primitive cycle with length smaller than $|\alpha|^2 + 1$. We can turn $\mathbb{C}[\mathsf{iss}_\alpha Q]$ into a graded ring by giving f_c the length of its cycle as degree.*

This result can be used to prove and interesting lemma about the coregularity of subquivers.

Definition 2.1. Define a partial ordering on the set of quivers in the following way. A quiver $Q' = (V', A', s', t')$ is smaller than $Q = (V, A, s, t)$ if (up to isomorphism)

$$V' \subseteq V, \qquad A' \subseteq A, \qquad s' = s|_{A'}, \quad \text{and} \quad t' = t|_{A'},$$

Q' is called a *subquiver* of Q.

Lemma 2.3. *If $\mathsf{iss}_\alpha Q$ is smooth and $Q' \leqslant Q$ then $\mathsf{iss}_{\alpha'} Q'$ is also smooth, where $\alpha' := \alpha|_{V'}$.*

Proof. We have an embedding

$$\mathsf{Rep}_{\alpha'} Q' \lhook\joinrel\longrightarrow \mathsf{Rep}_\alpha Q$$

by assigning to the additional arrows in Q zero matrices. So

$$\mathbb{C}[\mathsf{Rep}_\alpha Q] \longrightarrow\!\!\!\!\!\rightarrow \mathbb{C}[\mathsf{Rep}_{\alpha'} Q'] \quad \Rightarrow \quad \mathbb{C}[\mathsf{Rep}_\alpha Q]^{\mathsf{GL}_\alpha} \longrightarrow\!\!\!\!\!\rightarrow \mathbb{C}[\mathsf{Rep}_{\alpha'} Q']^{\mathsf{GL}_\alpha}.$$

Because the action of GL_α on $\mathsf{Rep}_{\alpha'} Q'$ reduces to that of $\mathsf{GL}_{\alpha'}$, $\mathbb{C}[\mathsf{iss}_{\alpha'} Q']$ is a quotient ring of $\mathbb{C}[\mathsf{iss}_\alpha Q] = \mathbb{C}[X_1, \ldots, X_n]$. The only relations that we have to divide out are the X_i that correspond to a cycle containing one of the additional arrows we put zero, so $\mathbb{C}[\mathsf{iss}_{\alpha'} Q']$ is just a polynomial ring with fewer variables. $\quad\square$

Two vertices v and w are said to be *strongly connected* if there is a path from v to w and vice versa. It is easy to check that this relation is an equivalence so we can divide the set of vertices into equivalence classes V_i. The subquiver Q_i having V_i as set of vertices, and as arrows all arrows between vertices of V_i is called a *strongly connected component* of Q.

Lemma 2.4. (1) *If (Q, α) is a quiver setting then*

$$\mathbb{C}[\mathsf{iss}_\alpha \, Q] := \bigotimes_i \mathbb{C}[\mathsf{iss}_{\alpha_i} \, Q_i]$$

where $Q_i = (V_i, A_i, s_i, t_i)$ are the strongly connected components of Q and $\alpha_i := \alpha|_{V_i}$.

(2) $\mathsf{iss}_\alpha \, Q$ *is smooth if and only if the $\mathsf{iss}_\alpha \, Q_i$ of all its strongly connected components are smooth.*

Proof. By Theorem 2.2 $\mathbb{C}[\mathsf{iss}_\alpha \, Q]$ is generated by the traces of cycles. Every cycle belongs to a certain connected component of Q. Between f_c's coming from cycles of different components there cannot be any relations, so we can consider the ring of invariants as a tensor-products of the rings of invariants different strongly connected components.

(2) If all the strongly connected components are coregular the ring of invariants of the total quiver setting will be the tensor product of polynomial rings and hence a polynomial ring. The inverse implication follows directly from Lemma 2.3. $\quad\square$

3. Reduction steps

As we stated in the introduction we want to apply some kind of reduction on quivers. By this we mean that if we start from a general quiver setting (Q, α), we want to construct a new quiver setting with fewer vertices or arrows but with the same or a closely related ring of invariants. In this section we will consider three different types of reductions.

First we have to recall a result from [3].

Theorem 3.1. *Consider the vector space $\mathsf{Mat}_{k \times l}(\mathbb{C}) \oplus \mathsf{Mat}_{l \times m}(\mathbb{C})$ together with an action of $\mathsf{GL}_l(\mathbb{C})$:*

$$(M_1, M_2)^g := \left(M_1 g, g^{-1} M_2 \right).$$

The quotient space $\mathsf{Mat}_{k \times l}(\mathbb{C}) \oplus \mathsf{Mat}_{l \times m}(\mathbb{C}) / \mathsf{GL}_l(\mathbb{C})$ is isomorphic to the space of all $k \times m$ matrices of rank smaller then l (so if $l \geqslant k$ or $l \geqslant m$ there is no restriction on the matrices and the quotient space is $\mathsf{Mat}_{k \times m}(\mathbb{C})$). Identification happens via the $\mathsf{GL}_l(\mathbb{C})$-invariant map
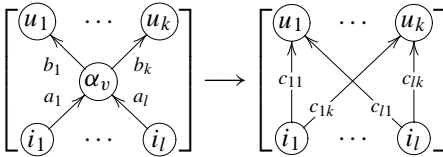
$$\pi : (M_1, M_2) \mapsto M_1 M_2.$$

This lemma can now be applied to quiver settings.

Lemma 3.2 (Reduction $\mathcal{R}_{\mathrm{I}}$: removing vertices). *Suppose (Q, α) is a quiver setting and v is a vertex without loops such that*

$$\chi_Q(\alpha, \epsilon_v) \geqslant 0 \quad or \quad \chi_Q(\epsilon_v, \alpha) \geqslant 0.$$

Construct a new quiver setting (Q', α') by changing Q:

(*some of the top and bottom vertices in the picture may be the same*). *These two quiver settings now have isomorphic rings of invariants.*

Proof. We can split up the representation space into the following direct sum:

$$\mathsf{Rep}_\alpha Q = \underbrace{\bigoplus_{a,\, s(a)=v} \mathsf{Mat}_{\alpha_{t(a)} \times \alpha_{s(a)}}(\mathbb{C})}_{\text{arrows starting in } v} \oplus \underbrace{\bigoplus_{a,\, t(a)=v} \mathsf{Mat}_{\alpha_{t(a)} \times \alpha_{s(a)}}(\mathbb{C})}_{\text{arrows terminating in } v} \oplus \text{Rest}$$

$$= \mathsf{Mat}_{\sum_{s(a)=v} \alpha_{t(a)} \times \alpha_v}(\mathbb{C}) \oplus \mathsf{Mat}_{\alpha_v \times \sum_{t(a)=v} \alpha_{s(a)}}(\mathbb{C}) \oplus \text{Rest}$$

$$= \mathsf{Mat}_{\alpha_v - \chi(\alpha, \epsilon_v) \times \alpha_v}(\mathbb{C}) \oplus \mathsf{Mat}_{\alpha_v \times \alpha_v - \chi(\epsilon_v, \alpha)}(\mathbb{C}) \oplus \text{Rest}.$$

The $\mathsf{GL}_{\alpha_v}(\mathbb{C})$-part only acts on the first two terms and not on the rest term. So if we take the quotient corresponding to $\mathsf{GL}_{\alpha_v}(\mathbb{C})$ we only have to consider the first two terms.

By the previous lemma and keeping in mind that either $\chi_Q(\alpha, \epsilon_v) \geqslant 0$ or $\chi_Q(\epsilon_v, \alpha) \geqslant 0$, the quotient space is equal to

$$\mathsf{Mat}_{\alpha_v - \chi(\alpha, \epsilon_v) \times \alpha_v - \chi(\epsilon_v, \alpha)}(\mathbb{C}) \oplus \text{Rest}.$$

This space can be decomposed in the following way:

$$\bigoplus_{a,\, t(a)=vb,\, s(b)=v} \mathsf{Mat}_{\alpha_{t(b)} \times \alpha_{s(a)}}(\mathbb{C}) \oplus \text{Rest}.$$

This direct sum is the same as the representation space of the new quiver setting (Q', α'). $\square$

Lemma 3.3 (Reduction $\mathcal{R}_{\mathrm{II}}$: removing loops of dimension 1). *Suppose that (Q, α) is a quiver setting and v a vertex with k loops and $\alpha_v = 1$. Take Q' the corresponding quiver without loops, then the following identity hold*
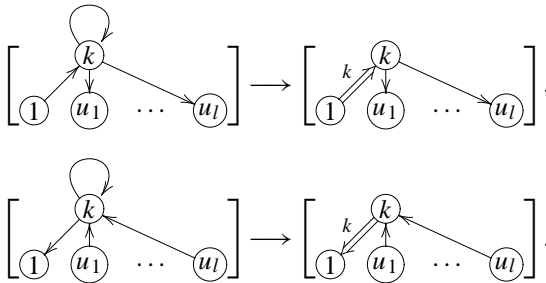
$$\mathbb{C}[\mathsf{iss}_\alpha Q] \cong \mathbb{C}[\mathsf{iss}_\alpha Q'] \otimes \mathbb{C}[X_1, \ldots, X_k].$$

Proof. This follows easily from Theorem 2.2 and the fact a cycle containing such a loop can never be quasi-primitive unless it is the loop itself. □

Lemma 3.4 (Reduction $\mathcal{R}_{\text{III}}$: removing a loop of higher dimension). *Suppose (Q, α) is a quiver setting and v is a vertex of dimension $k \geqslant 2$ with one loop such that*

$$\chi_Q(\alpha, \epsilon_v) = -1 \quad or \quad \chi_Q(\epsilon_v, \alpha) = -1.$$

Construct a new quiver setting (Q', α') by changing (Q, α):

We have the following identity:

$$\mathbb{C}[\text{iss}_\alpha Q] \cong \mathbb{C}[\text{iss}_{\alpha'} Q'] \otimes \mathbb{C}[X_1, \ldots, X_k].$$

Proof. We only prove this for the first case. Call the loop in the first quiver ℓ and the incoming arrow a. Call the incoming arrows in the second quiver c_i, $i = 0, \ldots, k-1$.

There is a map

$$\pi : \text{Rep}_\alpha Q \to \text{Rep}_{\alpha'} Q' \times \mathbb{C}^k, \quad V \mapsto \left(V', \text{Tr } V_\ell, \ldots, \text{Tr } V_\ell^k\right)$$

$$\text{with} \quad V'_{c_i} := V_\ell^i V_a.$$

Suppose $(V', x_1, \ldots, x_k) \in \text{Rep}_{\alpha'} Q' \times \mathbb{C}^k \in$ such that $(x_1, \ldots, x_k)$ corresponds to the traces of powers of an invertible diagonal matrix D with k different eigenvalues (λ_i: $i = 1, \ldots, k$) and the matrix A made of the columns (V_{c_i}: $i = 0, \ldots, k-1$) is invertible. The image of representation

$$V \in \text{Rep}_\alpha Q: \quad V_a = V'_{c_0},$$

$$V_\ell = A \begin{pmatrix} \lambda_1^0 & \cdots & \lambda_1^{k-1} \\ \vdots & & \vdots \\ \lambda_k^0 & \cdots & \lambda_k^{k-1} \end{pmatrix}^{-1} D \begin{pmatrix} \lambda_1^0 & \cdots & \lambda_1^{k-1} \\ \vdots & & \vdots \\ \lambda_k^0 & \cdots & \lambda_k^{k-1} \end{pmatrix} A^{-1}$$

under π is $(V', x_1, \ldots, x_k)$ because

$$V_\ell^i V_a = A \begin{pmatrix} \lambda_1^0 & \cdots & \lambda_1^{k-1} \\ \vdots & & \vdots \\ \lambda_k^0 & \cdots & \lambda_k^{k-1} \end{pmatrix}^{-1} D^i \begin{pmatrix} \lambda_1^0 & \cdots & \lambda_1^{k-1} \\ \vdots & & \vdots \\ \lambda_k^0 & \cdots & \lambda_k^{k-1} \end{pmatrix} A^{-1} V'_{c_0}$$

$$= A \begin{pmatrix} \lambda_1^0 & \cdots & \lambda_1^{k-1} \\ \vdots & & \vdots \\ \lambda_k^0 & \cdots & \lambda_k^{k-1} \end{pmatrix}^{-1} \begin{pmatrix} \lambda_1^i \\ \vdots \\ \lambda_k^i \end{pmatrix} = V_{c_i}$$

and the traces of V_ℓ are the same as the ones of D. The conditions we imposed on $(V', x_1, \ldots, x_k)$, imply that the image of π, U, is dense, and hence π is a dominant map.

We have a bijection between the generators of $\mathbb{C}[\text{iss}_\alpha Q]$ and $\mathbb{C}[\text{iss}_{\alpha'} Q'] \otimes \mathbb{C}[X_1, \ldots, X_k]$ by identifying

$$f_{\ell^i} \mapsto X_i, \quad i = 1, \ldots, k, \qquad f_{\ldots a \ell^i \ldots} \mapsto f_{\ldots c_i \ldots}, \quad i = 0, \ldots, k-1.$$

Notice that higher orders of ℓ do not occur because of the Caley Hamilton identity on V_ℓ. So if n is the number of generators of $\mathbb{C}[\text{iss}_\alpha Q]$, we have two maps

$$\phi : \mathbb{C}[Y_1, \ldots, Y_n] \to \mathbb{C}[\text{iss}_\alpha Q] \subset \mathbb{C}[\text{Rep}_\alpha Q],$$
$$\phi' : \mathbb{C}[Y_1, \ldots, Y_n] \to \mathbb{C}[\text{iss}_{\alpha'} Q'] \otimes \mathbb{C}[X_1, \ldots, X_k] \subset \mathbb{C}[\text{Rep}_{\alpha'} Q' \times \mathbb{C}^k].$$

Notice that we have that $\phi'(f) \circ \pi \equiv \phi(f)$ and $\phi(f) \circ \pi^{-1}|_U \equiv \phi'(f)|_U$. So if $\phi(f) = 0$ then also $\phi'(f)|_U = 0$. Because U is Zariski-open and dense in $\text{Rep}_{\alpha'} Q' \times \mathbb{C}^2$, $\phi'(f) \equiv 0$. A similar argument holds for the inverse implication so $\text{Ker}\, \phi = \text{Ker}\, \phi'$. $\square$

We have seen three possible reductions of a quiver setting which keep the ring of invariants intact or split of a tensor product with a polynomial ring. We can also apply the inverse steps of the reduction to add new vertices or loop while keeping the ring of invariants the same or tensoring it up with a polynomial ring. These inverse steps will be denoted as $\mathcal{R}^{-1}$.

The previous three lemma's can now be summarized as a theorem.

Theorem 3.5. *Suppose that (Q, α) and (Q', α') are two quiver settings that can be transformed into each other using consecutive steps of the form $\mathcal{R}_\text{I}$, $\mathcal{R}_\text{I}^{-1}$, $\mathcal{R}_\text{II}$, $\mathcal{R}_\text{II}^{-1}$, $\mathcal{R}_\text{III}$ or $\mathcal{R}_\text{III}^{-1}$. Then (Q, α) is coregular if and only if (Q', α') is coregular.*

Definition 3.1. A quiver setting (Q, α) such that there cannot be applied any reduction steps $\mathcal{R}_\text{I}$, $\mathcal{R}_\text{II}$ or $\mathcal{R}_\text{III}$ will be called *reduced*.

It remains now to search for the reduced coregular quiver settings. As we will see there are only a very limited number of them. But before we do that we must introduce some techniques that allow us to rule out non-coregular quiver settings.

4. Local quiver settings

The technique of local quiver settings is very useful to rule out quiver settings that are not coregular. If we want to prove that a certain (Q, α) is coregular, we have to check that $\text{iss}_\alpha Q$ is smooth in every point. Take a point $p \in \text{iss}_\alpha Q$, this point will correspond to the isomorphism class of a semisimple representation $V \in \text{Rep}_\alpha Q$ which can be decomposed as a direct sum of simple representations:

$$V = S_1^{\oplus a_1} \oplus \cdots \oplus S_k^{\oplus a_k}.$$

A theorem by Le Bruyn and Procesi [4, Theorem 4], based on the Luna-slice Theorem [5,7], states that we can build a new quiver setting with a similar quotient space, but having a simpler structure.

Theorem 4.1 (Le Bruyn–Procesi). *For a point $p \in \text{iss}_\alpha Q$ corresponding to a semisimple representation $V = S_1^{\oplus a_1} \oplus \cdots \oplus S_k^{\oplus a_k}$, there is a quiver setting (Q_p, α_p) called the* local quiver setting *such that we have an étale isomorphism between an open neighborhood of the zero representation in $\text{iss}_{\alpha_p} Q_p$ and an open neighborhood of p.*

Q_p has k vertices corresponding to the set $\{S_i\}$ of simple factors of V and between S_i and S_j the number of arrows equals

$$\delta_{ij} - \chi_Q(\alpha_i, \alpha_j)$$

where α_i is the dimension vector of the simple component S_i and χ_Q is the Euler form of the quiver Q. The dimension vector α_p is defined to be $(a_1, \ldots, a_k)$, where the a_i are the multiplicities of the simple components in V.

Suppose now that we want to find out whether a certain space $\text{iss}_\alpha Q$ is smooth. If this were the case we can choose a certain point p and look at it locally. Because of the étale isomorphism, the corresponding local quiver Q_p must have a quotient space $\text{iss}_{\alpha_p} Q_p$ that is smooth in the zero representation. Therefore by Theorem 2.1, $\mathbb{C}[\text{iss}_{\alpha_p} Q_p]$ must be a polynomial ring and hence (Q_p, α_p) is coregular. This must hold for every point so we have to check all possible points p.

Theorem 4.2. *(Q, α) is coregular if and only if for every possible semisimple α-dimensional representation V, the corresponding local quiver setting is coregular.*

One of the local quivers is equal to the original quiver, namely the one corresponding to the α-dimensional zero-representation

$$\bigoplus_{v \in V} S_v^{\oplus \alpha_v}.$$

This implies that we can only use this result to rule out quiver settings that are not coregular.

The structure of the local quiver setting only depends on the dimension vectors of the simple components. Therefore one can restrict to looking at decompositions of α into dimension vectors β_i if

$$\alpha = a_1\beta_1 + \cdots + a_k\beta_k \quad \text{(the } \beta_i \text{ need not to be different)}.$$

One can now ask whether there is a semisimple representation corresponding to such a decomposition. The answer to this question will be positive whenever for all the β_i there exist simple representations of that dimension vector and if there are two or more β_i equal, there are at least as many different simple representation classes with dimension vector β_i (otherwise you cannot make a direct sum with different simple representations having the same dimension vector).

To check the above conditions we must also have a characterization of the dimension vectors for which a quiver has simple representations. We recall a result from Le Bruyn and Procesi [4, Theorem 4].

Theorem 4.3. *Let (Q, α) be a genuine quiver setting. There exist simple representations of dimension vector α if and only if*

- *Q is of the form*
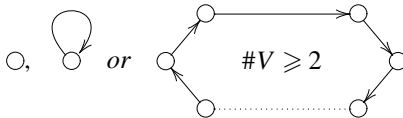
  and $\alpha = 1$ (this is the constant map from the vertices to 1).
- *Q is not of the form above, but strongly connected and*

$$\forall v \in V: \quad \chi_Q(\alpha, \epsilon_v) \leqslant 0 \quad and \quad \chi_Q(\epsilon_v, \alpha) \leqslant 0$$

  (we recall that a quiver is strongly connected if and only if between every two vertices there are paths connection them in both directions).

In both cases the dimension of $\mathrm{iss}_\alpha Q$ is given by $1 - \chi_Q(\alpha, \alpha)$. In all cases except for the one vertex without loops this dimension is bigger then 0, so then there are infinite classes of simples with that dimension vector. In the case of the one vertex v without loops, there is one unique simple representation S_v.

If (Q, α) is not genuine, the simple representations classes are in bijective correspondence to the simple representations classes of the genuine quiver setting obtained by deleting all vertices with dimension zero.

To rule out quiver settings that are not coregular we must find a local quiver setting that is not coregular or contains a non-coregular subquiver setting by Lemma 2.3.

For symmetric quiver settings, these are quiver settings with a symmetric Euler form, [2] gives us a complete classification of all possible quiver settings that are coregular.

Definition 4.1. A quiver $Q = (V, A, s, t)$ is said to be the *connected sum* of 2 subquivers $Q_1 = (V_1, A_1, s_1, t_1)$ and $Q_1 = (V_2, A_2, s_2, t_2)$ at the vertex v, if the two subquivers make up the whole quiver and only intersect in the vertex v. So in symbols $V = V_1 \cup V_2$, $A = A_1 \cup A_2$, $V_1 \cap V_2 = \{v\}$, and $A_1 \cap A_2 = \emptyset$.

$$Q_1 \overset{\#}{\underset{v}{}} Q_2 := Q_1 \quad \text{\textcircled{v}} \quad Q_2.$$
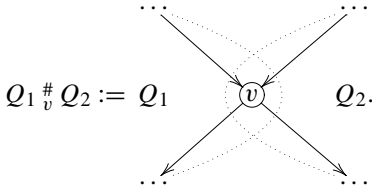
If we connect three or more components we write $Q_1 \overset{\#}{\underset{v}{}} Q_2 \overset{\#}{\underset{w}{}} Q_3$ instead of $(Q_1 \overset{\#}{\underset{v}{}} Q_2) \overset{\#}{\underset{w}{}} Q_3$ for sake of simplicity.

Theorem 4.4. *Let (Q, α) be a symmetric strongly connected quiver setting without. Then (Q, α) is coregular if and only if Q is a connected sum*

$$Q := Q_1 \overset{\#}{\underset{v_1}{}} Q_2 \overset{\#}{\underset{v_2}{}} \cdots \overset{\#}{\underset{v_{l-1}}{}} Q_l,$$

where the (Q_i, α_i) are of the form:

(I) $\text{\textcircled{n}} \quad \text{\textcircled{m}}$;

(II) $\text{\textcircled{1}} \overset{k}{\underset{k}{\rightleftarrows}} \text{\textcircled{n}}$, $k \leqslant n$;

(III) $\text{\textcircled{1}} \quad \text{\textcircled{n}} \quad \text{\textcircled{m}}$;

(IV) $\text{\textcircled{n}} \quad \text{\textcircled{2}} \quad \text{\textcircled{m}}$,

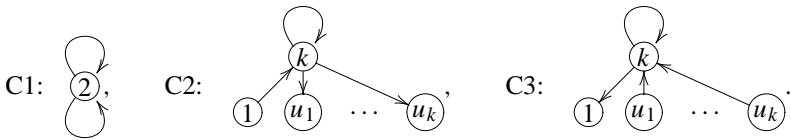*and $\alpha_{v_j} = 1$, $j = 1, \ldots, l - 1$.*

5. Reduced coregular quiver settings

First we look at the case of loops.

Lemma 5.1. *Suppose (Q, α) is a coregular strongly connected quiver setting such that*

$$\forall w \in V: \quad \chi_Q(\alpha, \epsilon_w) < 0 \quad \text{and} \quad \chi_Q(\epsilon_w, \alpha) < 0.$$

If v is a vertex with loops then $\alpha_v = 1$ or the neighborhood of v has the following form:

C1: ,   C2: ,   C3: .

**Proof.** (1) If $\alpha_v \geqslant 3$ there is only one loop in v.

Suppose that $\alpha_v \geqslant 3$ there are at least two loops in v. In this case we have a subquiver as shown below. This subquiver can be transformed into a symmetric quiver without loops using Lemma 3.2 (in both ways). By Theorem 4.4 this symmetric setting is not coregular, if $\alpha_v > 2$:
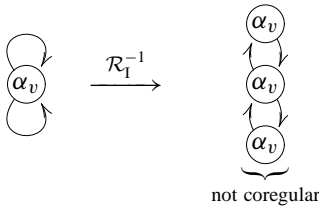
(2) If $\alpha_v = 2$ we are in C1 or there is only 1 loop in v.

If $\alpha_v = 2$ and we or not in C1, C2 or C3, Q has either at least 3 loops or either two loops and a cyclic path through v (this cyclic path can be constructed because Q is strongly connected and contains at least 2 vertices, otherwise $(Q, \alpha) = $ C1).

In both cases we can take again the corresponding subquivers and change them to a symmetric quiver without loops which is not coregular according to Theorem 4.4:
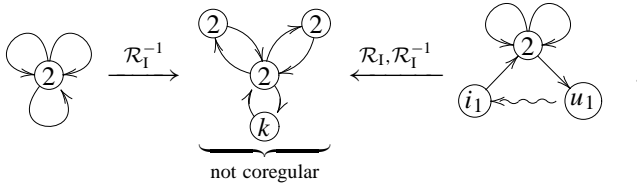
So the only possibility with more than one loop is C1.

(3) If $\alpha_v \geqslant 2$ and there is only 1 loop in v then we are in C2 or C3.

Suppose that the dimension in v is bigger than 1 and that there is only 1 loop. Consider the representation

$$W \oplus L \oplus \left(\bigoplus_{w \in V} S_w^{\oplus \alpha_w - 1 - \delta_{vw}} \right)$$

where W is a simple representation with dimension vector 1 which is the constant map assigning 1 to every vertex. Such a representation exists by Theorem 4.3 because Q is strongly connected and $\chi_Q(1, \epsilon_w) \leqslant 0$. S_w is the representation

with dimension vector ϵ_w which assigns to every arrow a zero matrix, while L is a representation with dimension vector ϵ_v which assigns to the loop in v a non-zero matrix.

For every vertex $w \neq v$ with dimension bigger than 1 the local quiver contains exactly one vertex corresponding to the simple representation S_w. For v there is at least one vertex in the local quiver coming from L, which has dimension 1. If $\alpha_v > 2$ there is an extra vertex from the S_v but we will not consider it because it does not change the proof.

The subquiver containing the vertices from L in S_w, $w \neq v$, is the same as in the original quiver because

$$\chi_Q(\epsilon_u, \epsilon_w) = \delta_{uw} - \#\{a \mid \textcircled{u} \xleftarrow{\ a\ } \textcircled{w}\}.$$

In the local quiver we will draw the additional vertex coming from W as a square. The number of arrows from another vertex coming from S_w to the vertex coming from W is equal to $-\chi_Q(1, \epsilon_w)$ and hence one less than the number of arrows leaving w in the original quiver. The same holds for the number of arrows in the opposite direction and for the arrows between L and W.

We will now look closely at the neighborhood of v.

$\chi_Q(\epsilon_v, 1) \leqslant -2$ and $\chi_Q(1, \epsilon_v) \leqslant -2$ is impossible.

The local quiver has a subquiver containing $\textcircled{1} \rightrightarrows \boxed{1}$, and (Q, α) is not coregular. For (Q, α) to be a coregular quiver setting, one can suppose that either $\chi_Q(\epsilon_v, 1) = -1$ or $\chi_Q(1, \epsilon_v) = -1$.

$\chi_Q(\epsilon_v, 1) = -1$ and $\chi_Q(1, \epsilon_v) \leqslant -2$ implies C2.

We claim that if w_1 is the unique vertex in Q such that $\chi_Q(\epsilon_v, \epsilon_{w_1}) = -1$ then $\alpha_{w_1} = 1$.

If this was not the case there is a vertex corresponding to S_{w_1} in the local quiver. If $\chi_Q(1, \epsilon_{w_1}) = 0$ then the dimension of the unique vertex w_2 with an arrow to w_1 has strictly bigger dimension than w_1, otherwise $\chi_Q(\alpha, \epsilon_{w_1}) \geqslant 0$. The vertex w_2 corresponds again to a vertex in the local quiver. If $\chi_Q(1, \epsilon_{w_2}) = 0$, the unique vertex w_3 with an arrow to w_2 has strictly bigger dimension than w_2. Proceeding this way one can find a sequence of vertices with increasing dimension, which attains a maximum in vertex w_k. Therefore $\chi_Q(1, \epsilon_{w_k}) \leqslant -1$. This last vertex is in the local quiver connected with W, so one has a path from 1 to ϵ_v:
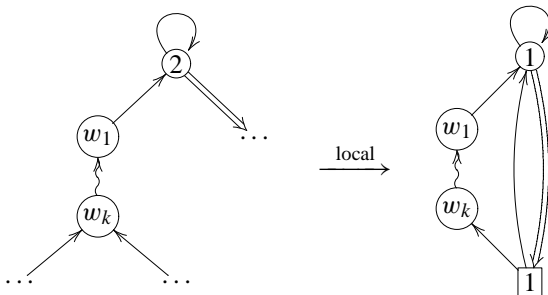
The local subquiver consisting of the vertices corresponding to W, S_v and the S_{w_i} is reducible via $\mathcal{R}_I$ to ①⊏⊐1. So if $\alpha_{w_1} > 1$, (Q, α) is not coregular.

$\chi_Q(\epsilon_v, 1) \leqslant -2$ and $\chi_Q(1, \epsilon_v) = -1$ implies C3.
This follows by symmetry.

$\chi_Q(\epsilon_v, 1) = -1$ and $\chi_Q(\epsilon_v, 1) = -1$ implies C2 or C3.
Suppose w_1 is the unique vertex in Q such that $\chi_Q(\epsilon_v, \epsilon_{w_1}) = -1$ and w_k is the unique vertex in Q such that $\chi_Q(\epsilon_{w_k}, \epsilon_v) = -1$, then either $\alpha_{w_1} = 1$ or $\alpha_{w_k} = 1$.

If this was not the case, consider the path connecting w_k and w_1 and call the intermediate vertices w_i, $1 < i < k$. Starting from w_1 we go back along the path until α_{w_i} reaches a maximum. At that point we know that $\chi_Q(1, \epsilon_{w_k}) \leqslant -1$, otherwise $\chi_Q(\alpha, \epsilon_{w_k}) \geqslant 0$. In the local quiver there is a path from the vertex corresponding to W over the ones from S_{w_i} to S_v. Doing the same thing starting from w_k we also have a path from the vertex from S_v over the ones of S_{w_j} to W.
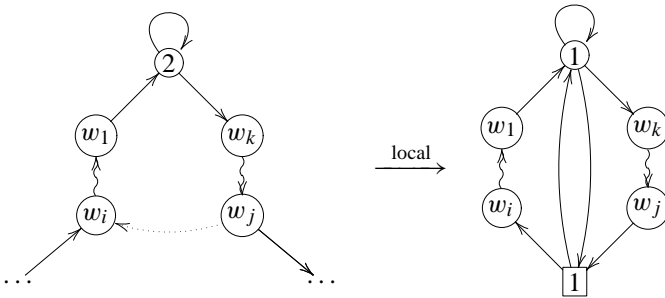
The subquiver consisting of 1, ϵ_v and the two paths through the ϵ_{w_i} is reducible to ①⊏⊐1. So if both $\alpha_{w_1} > 1$ and $\alpha_{w_k} > 1$, (Q, α) is not coregular. □

We will now look at the reduced quiver settings without loops.

Lemma 5.2. *A quiver setting with dimension vector* 1 *is coregular if and only if the number of primitive cycles equals the dimension of* $\mathbb{C}[\text{iss}_1 Q]$.

Proof. The condition is obviously sufficient. It is also necessary because if the number of cycles is bigger than the dimension then there will be a relation between the cycles. If $\mathbb{C}[\text{iss}_1 Q]$ is a polynomial ring, these relations must be of the form $Y = X_1 \ldots X_k$ but this is impossible because Y is a primitive cycle. □

Lemma 5.3. *A strongly connected reduced quiver setting without loops is never coregular.*

Proof. If $\alpha \neq 1$, consider the vertex v with the highest dimension. Then there exists indeed simple representations with dimension vector $\alpha - \epsilon_v$ because a reduced setting is never of the form:
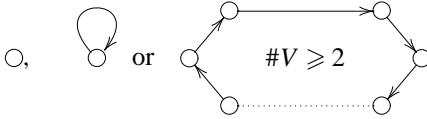
and $\alpha - \epsilon_v$ satisfies the second condition of Theorem 4.3:

- If there is no arrow from w to v, $\chi_Q(\alpha - \epsilon_v, \epsilon_w) = \chi_Q(\alpha, \epsilon_w) \leqslant -1$.
- If there are k arrows from w to v then $\chi_Q(\alpha, \epsilon_w) \leqslant \alpha_w - k\alpha_v \leqslant (1-k)\alpha_v$; so

$$\chi_Q(\alpha - \epsilon_v, \epsilon_w) \leqslant (1-k)\alpha_v + \chi_Q(\epsilon_v, \epsilon_w) = (1-k)\alpha_v - k \leqslant -1.$$

- Finally for $v = w$

$$\chi_Q(\alpha - \epsilon_v, \epsilon_v) = \chi(\alpha, \epsilon_v) - 1 < -1 \quad \text{and} \quad \chi_Q(\epsilon_v, \alpha - \epsilon_v) \leqslant -1.$$

For reasons of symmetry $\chi_Q(\epsilon, \alpha - \epsilon_v)$ will also be smaller than 0 for every $w \in V$.

Due to the inequality $\chi_Q(\epsilon_v, \alpha - \epsilon_v) \leqslant -1$, the local quiver of a decomposition of the form

$$(Q, \alpha - \epsilon_v) \oplus (Q, \epsilon_v)$$

will not be coregular.

Suppose thus $\alpha = 1$. Because (Q, α) is reduced, there are at least 2 arrows arriving and leaving every vertex. For a connected quiver without loops $\mathsf{Dim}\,\mathbb{C}[\mathrm{iss}_1 \, Q] = \#A - \#V + 1$ so we have to prove that for such quivers the number of primitive cycles is bigger than $\#A - \#V + 1$ or that Q contains a subquiver that is not coregular. We will do this by induction on the vertices.

- For $\#V = 2$ the statement is true because

- Suppose $\#V > 2$ and that we have a subquiver of the form:

If $k, l > 1$ we know that this subquiver is not coregular and hence neither is Q.

If both k and l are 1 then replace this subquiver by 1 vertex:

$$\left[\; \vdots \; \raisebox{-0.5em}{\text{①}}\;\raisebox{-0.5em}{\text{①}}\; \vdots \;\right] \longrightarrow \left[\; \vdots \; \raisebox{-0.5em}{\text{①}}\; \vdots \;\right].$$
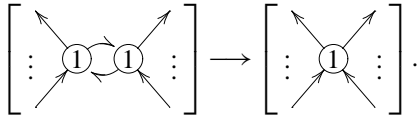
The new quiver Q' is again reduced without loops because there are at least 4 arrows arriving in one of the vertices of the subquiver and we only deleted 2, the same holds for the arrows leaving the subquiver. Q' has one primitive cycle less than the original. By induction we have that
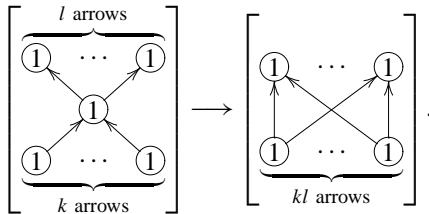
$$\text{Dim}\,\mathbb{C}[\text{iss}_1\, Q] = \text{Dim}\,\mathbb{C}[\text{iss}_1\, Q'] + 1 > (\#A' - \#V' + 1) + 1$$
$$= \#A - \#V + 1.$$

If for instance $k > 1$ then one can look at the subquiver of Q obtained by deleting the $k - 1$ edges, if this quiver is reduced then we are in the previous situation. If this is not the case Q contains a subquiver of the form
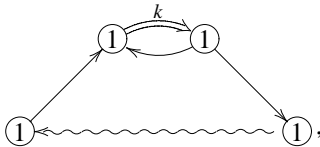
which is not coregular because it is reducible to ($*$).

- If $\#V > 2$ and there are no subquivers of the form ($*$), we can consider an arbitrary vertex v. Construct a new quiver Q' by performing the following substitution for v:

Q' is again reduced without loops and has the same number of primitive cycles, so by induction

$$\text{Dim}\,\mathbb{C}[\text{iss}_1\, Q] = \text{Dim}\,\mathbb{C}[\text{iss}_1\, Q'] > \#A' - \#V' + 1$$
$$= \#A + (kl - k - l) - \#V + 1 + 1 > \#A - \#V + 1. \quad \square$$

All this leads to the proof of our main theorem.

Proof. Theorem 1.1 follows immediately from Lemmas 5.1 and 5.3 and the fact that, as proven in [6], the quiver settings that are listed in the theorem are coregular. $\square$

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