Further Bounds for the Smallest Singular Value and the Spectral Condition Number

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Abstract—We derive monotonic sequences of bounds for the extreme singular values. In particular, we find further lower bounds for the smallest singular value which improve the bounds of Yu and Gun. Also, we give new upper bounds for the spectral condition number. © 1999 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Let $A$ be an $n \times n$ complex matrix. Let

$$\sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_n(A)$$

be the singular values of $A$. It is well known that

$$\sigma_1^2(A) + \sigma_2^2(A) + \cdots + \sigma_n^2(A) = \|A\|_F^2$$

and

$$\sigma_1(A) \sigma_2(A) \cdots \sigma_n(A) = |\det A|,$$

where $\|A\|_F$ and $\det A$ denote the Frobenius norm of $A$ and the determinant of $A$, respectively.

Estimating the extreme singular values has a theoretical and practical interest. For example, the spectral condition number $\kappa_2(A) = \sigma_1(A)/\sigma_n(A)$ measures the sensibility of the solution of $Ax = b$ to errors in the data or to round-off errors. One can estimate $\kappa_2(A)$ using a lower bound for $\sigma_n(A)$ and an upper bound for $\sigma_1(A)$.

To minimize the numerical round-off errors in solving the system $Ax = b$, it is normally convenient that the rows of $A$ be properly scaled before the solution procedure begins. One way is to premultiply by the diagonal matrix

$$D = \text{diag} \left\{ \frac{\alpha_1}{r_1(A)}, \frac{\alpha_2}{r_2(A)}, \cdots, \frac{\alpha_n}{r_n(A)} \right\},$$

where $\alpha_i$ and $r_i(A)$ denote the $i$-th singular value and the $i$-th row of $A$, respectively.

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where $r_i(A)$ is the Euclidean norm of the $i$th row of $A$ and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are positive real numbers such that
\[ \alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2 = n. \]
Clearly, the Frobenius norm of the coefficient matrix $B = DA$ of the scaled system is equal to $\sqrt{n}$ and if $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 1$, then each row of $B$ is a unit vector in the Euclidean norm. Also, we can define $B = AD$,
\[
D = \text{diag}\left\{ \frac{\alpha_1}{c_1(A)}, \frac{\alpha_2}{c_2(A)}, \ldots, \frac{\alpha_n}{c_n(A)} \right\},
\]
where $c_i(A)$ is the Euclidean norm of the $i$th column of $A$. Again, $\|B\|_F = \sqrt{n}$ and if $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 1$, then each column of $B$ is a unit vector in the Euclidean norm.

In this work, among other results, we develop an increasing sequence of lower bounds for the smallest singular value. If $B$ is an $n \times n$ complex matrix and $\|B\|_F = \sqrt{n}$, then the first term of the sequence is the bound of Hong and Pan (in [1], this bound is a consequence of Lemmas 1 and 2):
\[
\sigma_n(B) > \left( \frac{n-1}{n} \right)^{(n-1)/2} |\det B| \quad (3)
\]
and the second term improves the result of Yu and Gu [2, Lemma 1]:
\[
\sigma_n(B) > \left( \frac{n-1}{n} \right)^{(n-1)/2} |\det B| \left[ 1 + \frac{1}{2} \left( \frac{n-1}{n} \right)^n |\det B|^2 \right]. \quad (4)
\]
Consequently, as we will see later, other results of these authors are also improved.

In [3], the best possible upper bound for the spectral condition number using only $\|A\|_F$, $\det A$, and $n$ is obtained. In particular, if $B$ is an $n \times n$ complex matrix and $\|B\|_F = \sqrt{n}$, then the best upper bound for $\kappa_2(B)$ using only $\det B$ is
\[
\kappa_2(B) \leq 1 + \sqrt{1 - |\det B|^2}. \quad (5)
\]
In this paper, by scaling the matrix $A$ and then using (5), we derive new upper bounds for $\kappa_2(A)$.

## 2. MONOTONIC SEQUENCES OF BOUNDS FOR THE EXTREME SINGULAR VALUES

Let $A$ be an $n \times n$ nonsingular complex matrix. Let $\sigma_k(A)$ be fixed. Making use of the fact that the geometric mean of positive numbers does not exceed their arithmetic mean, we have
\[
\prod_{j \neq k} \sigma_j^2(A) \leq \left( \frac{\sum_{j \neq k} \sigma_j^2(A)}{n-1} \right)^{n-1}.
\]
Multiplying both sides of the inequality by $\sigma_k^2(A)$ we obtain
\[
\sigma_k^2(A) \sigma_1^2(A) \sigma_2^2(A) \cdots \sigma_n^2(A) \leq \left( \frac{\sum_{j \neq k} \sigma_j^2(A)}{n-1} \right)^{n-1} \sigma_k^2(A). \quad (6)
\]
Now, using (1) and (2), inequality (6) becomes
\[
|\det A|^2 \leq \left( \frac{\|A\|_F^2 - \sigma_k^2(A)}{n-1} \right)^{n-1} \sigma_k^2(A). \quad (7)
\]
Finally, from (7), we get Lemma 1.
LEMMA 1. Let \( A \) be an \( n \times n \) nonsingular complex matrix. Then, each singular value \( \sigma_k(A) \) satisfies
\[
\left( \sigma_k^{2/(n-1)}(A) \right)^n - \|A\|_F^2 \sigma_k^{2/(n-1)}(A) + (n-1) |\text{det} \ A|^2/(n-1) \leq 0. \tag{8}
\]

Inequality (8) suggests to study the function
\[
f(x) = x^n - \|A\|_F^2 x + (n-1) |\text{det} \ A|^2/(n-1). \tag{9}
\]

Some properties of the function \( f \) are given in the next lemma.

LEMMA 2. Let
\[
r = \left( \frac{\|A\|_F^2}{n} \right)^{1/(n-1)}.
\tag{10}
\]

Then,
\[
1. \quad f(r) \leq 0.
\tag{11}
\]
The equality holds if and only if
\[
|\text{det} \ A|^{1/n} = \frac{\|A\|_F}{\sqrt{n}}. \tag{12}
\]

2. \[ \frac{df}{dx}(r) = 0. \]

3. \[ \frac{df}{dx}(x) < 0, \quad \text{for } 0 < x < r, \]
\[ \frac{df}{dx}(x) > 0, \quad \text{for } x > r. \]

4. \( f \) is a convex function for \( x \geq 0 \).

PROOF. From the geometric-arithmetic-mean inequality, we obtain
\[
|\text{det} \ A|^2 = \prod_{i=1}^n \sigma_i^2(A) \leq \left( \frac{\|A\|_F^2}{n} \right)^n.
\]

Then,
\[
|\text{det} \ A|^{2/(n-1)} - \left( \frac{\|A\|_F^2}{n} \right)^{n/(n-1)} \leq 0.
\]

Therefore,
\[
f(r) = \left( \frac{\|A\|_F^2}{n} \right)^{n/(n-1)} - n \left( \frac{\|A\|_F^2}{n} \right)^{1/(n-1)} + (n-1)|\text{det} \ A|^{2/(n-1)}
\]
\[
= (n-1) \left( |\text{det} \ A|^{2/(n-1)} - \left( \frac{\|A\|_F^2}{n} \right)^{n/(n-1)} \right) \leq 0.
\]

We note that the equality takes place if and only if \( \sigma_1(A) = \sigma_2(A) = \cdots = \sigma_n(A) \). Clearly, this condition is equivalent to \( |\text{det} \ A|^{1/n} = \|A\|_F/\sqrt{n} \). Thus, we have proved (11) and (12). The rest of the proof is straightforward. \( \blacksquare \)

The following corollary is immediate.
COROLLARY 3. Let $A$ be an $n \times n$ nonsingular complex matrix. Let $f$ and $r$ as defined in (9) and (10), respectively. Then,

1. the function $f$ is strictly decreasing in the interval $(0, r)$, it is strictly increasing for $x > r$ and, for $x > 0$, it has a global minimum at $r$;
2. there exists a unique $\alpha \in (0, r]$ and there exists a unique $\beta \geq r$ such that $f(\alpha) = 0$ and $f(\beta) = 0$;
3. for $x > 0$, $f(x) \leq 0$ if and only if $x \in [\alpha, \beta]$.

THEOREM 4. Let $A$ be an $n \times n$ nonsingular complex matrix. Then, the singular values $\sigma_k(A)$ lie in the interval

$$[\alpha^{(n-1)/2}, \beta^{(n-1)/2}],$$

where $\alpha$ and $\beta$, $\alpha \leq \beta$, are the positive roots of the equation

$$x^n - \|A\|_F^2 x + (n - 1) |\det A|^{2/(n-1)} = 0.$$ 

In particular,

$$\alpha^{(n-1)/2} \leq \sigma_n(A) \quad \text{and} \quad \sigma_1(A) \leq \beta^{(n-1)/2}.$$ 

PROOF. From Lemma 1 we have

$$f(\sigma_k^{2/(n-1)}(A)) \leq 0.$$ 

Making use of Point 3 in Corollary 3, we conclude that

$$\sigma_k^{2/(n-1)}(A) \in [\alpha, \beta], \quad k = 1, 2, \ldots, n.$$ 

That is,

$$\alpha \leq \sigma_k^{2/(n-1)}(A) \leq \beta.$$ 

Raising both sides of the last inequality to the power $(n - 1)/2$ we find that the singular values lie in the interval given by (13).

The positive roots $\alpha$ and $\beta$ of equation (14) can be found by any standard numerical method for solving nonlinear equations. Taking into consideration the convexity of $f$ for $x > 0$, it is particularly convenient to use the Newton-Raphson's method.

The Newton-Raphson iterates for equation (14) are

$$x_{k+1} = \frac{n - 1}{\|A\|_F^2} \left( 1 - \frac{\det A}{\|A\|_F^2} \right) \frac{x_k^n}{x_k^{n-1}}.$$ 

The convexity of the function $f$, for $x \geq 0$, guarantees that if $x_0 = 0$, then the sequence $(x_k)$ is increasing and convergent to $\alpha$. Now, we want to find an initial guess $x_0$ in order to obtain a sequence converging to $\beta$. We have

$$f \left( \|A\|_F^{2/(n-1)} \right) = (n - 1) |\det(A)|^{2/(n-1)} > 0 \quad \text{and} \quad r < \|A\|_F^{2/(n-1)}.$$ 

Hence, $\beta < \|A\|_F^{2/(n-1)}$. Thus, again by the convexity of the function $f$, if $x_0 = \|A\|_F^{2/(n-1)}$, then the sequence $(x_k)$ is decreasing and convergent to $\beta$.

Therefore, taking into account (15), we have Theorem 5.
THEOREM 5. Let \( A \) be an \( n \times n \) nonsingular complex matrix. Let \((x_k)\) be the sequence defined by (16). Then,

1. if \( x_0 = 0 \), \((x_k^{(n-1)/2})\) is an increasing sequence of lower bounds for \( \sigma_n(A) \);
2. if \( x_0 = \|A\|_F^{2/(n-1)} \), \((x_k^{(n-1)/2})\) is a decreasing sequence of upper bounds for \( \sigma_1(A) \).

EXAMPLE 6. Let

\[
A = \begin{bmatrix}
-1 & 2 & 1 \\
1 & 0 & 2 \\
3 & 2 & 1 \\
0 & 3 & 2
\end{bmatrix}
\]

For this matrix equation (14) is \( x^4 - 48x + 55.2339 = 0 \). Then we have the following.

<table>
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<th>( x_k )</th>
<th>( x_k^{(n-1)/2} )</th>
<th>( k )</th>
<th>( x_k )</th>
<th>( x_k^{(n-1)/2} )</th>
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<td>0</td>
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<td>6.9282</td>
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<tr>
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<td>1.2344</td>
<td>1</td>
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<td>5.8608</td>
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<tr>
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<td>1.3023</td>
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</tr>
<tr>
<td>3</td>
<td>1.1929</td>
<td>1.3029</td>
<td>3</td>
<td>3.1169</td>
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<tr>
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<td>1.3029</td>
<td>4</td>
<td>3.1168</td>
<td>5.5026</td>
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<tr>
<td></td>
<td>5</td>
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<td></td>
<td>5.5026</td>
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</tr>
</tbody>
</table>

The singular values of \( A \) are 5.1234, 3.3655, 2.7743, 1.6514.

The following corollary is immediate.

COROLLARY 7. Let \( B \) be an \( n \times n \) nonsingular complex matrix such that \( \|B\|_F = \sqrt{n} \). Let \((x_k)\) be the sequence defined by

\[
x_{k+1} = \frac{n - 1}{n} \frac{\det B^{2/(n-1)} - x_k^n}{1 - x_k^{n-1}}.
\]

Then,

1. if \( x_0 = 0 \), \((x_k^{(n-1)/2})\) is an increasing sequence of lower bounds for \( \sigma_n(B) \);
2. if \( x_0 = n^{1/(n-1)} \), \((x_k^{(n-1)/2})\) is a decreasing sequence of upper bounds for \( \sigma_1(B) \).

3. FURTHER LOWER BOUNDS FOR THE SMALLEST SINGULAR VALUE

Let \( B \) be a complex matrix of order \( n \times n \) with \( \|B\|_F = \sqrt{n} \). Let \( x_0 = 0 \). Then, from (17),

\[
x_1 = \frac{n - 1}{n} \frac{\det B^{2/(n-1)}}{1 - x_1^{n-1}}.
\]

Hence,

\[
x_1^{(n-1)/2} = \frac{n - 1}{n} \frac{(n-1)^{1/2}}{\det B} < \sigma_n(B).
\]

This is the lower bound of Hong and Pan for \( \sigma_n(B) \) given in (3). We already observed that in [1] this result is a consequence of two previous lemmas. Here, it follows immediately. Now, we want to relate \( x_2^{(n-1)/2} \), \( x_0 = 0 \), with the lower bound of Yu and Gu given in (4). From (17),

\[
x_2 = \frac{n - 1}{n} \frac{\det B^{2/(n-1)} \frac{1 - ((n - 1)/n)^n}{\det B^2}}{1 - ((n - 1)/n)^{n-1} \det B^2} = \frac{n - 1}{n} \frac{\det B^{2/(n-1)} \left(1 + \left(\frac{n - 1}{n}\right)^n\right)^{-1} \det B^2}{1 - ((n - 1)/n)^{n-1} \det B^2}.
\]
We know that $x_2^{(n-1)/2}$ is a lower bound for $\sigma_n(B)$. Then,

$$\sigma_n(A) > x_2^{(n-1)/2}$$

$$= \left( \frac{n-1}{n} \right)^{(n-1)/2} \left| \det B \right| \left[ 1 + \left( \frac{n-1}{n} \right)^{n-1} \left| \det B \right|^2 \right]^{(n-1)/2}$$

$$> \left( \frac{n-1}{n} \right)^{(n-1)/2} \left| \det B \right| \left[ 1 + \frac{1}{2} \frac{(n-1)/n)^n}{\det B^2} \left| \det B \right|^2 \right]^{(n-1)/2},$$

for $n \geq 3$.

Thus, we have derived a new lower bound for $\sigma_n(B)$.

**Lemma 8.** Let $B$ be an $n \times n$ nonsingular complex matrix, $n \geq 3$, with $\|B\|_F = \sqrt{n}$. Then,

$$\sigma_n(B) > \left( \frac{n-1}{n} \right)^{(n-1)/2} \left| \det B \right| \left[ 1 + \frac{1}{2} \theta(B) \left( \frac{n-1}{n} \right)^n \left| \det B \right|^2 \right],$$

where

$$\theta(B) = \frac{1}{1 - ((n-1)/n)^{n-1} \left| \det B \right|^2}.$$  

(19)

**Remark 1.** We have $|\det(B)| \leq 1$ for any matrix $B$ with $\|B\|_F = \sqrt{n}$. Hence,

$$\left( \frac{n-1}{n} \right)^{n-1} \left| \det B \right|^2 < 1.$$

Therefore, $\theta(B) > 1$ and thus, we can conclude that (18) gives a better lower bound than the lower bound of Yu and Gu given in (4). Consequently, the lower bounds for $\sigma_n(A)$ derived in [2] by the use of (4) can be also improved.

Since the matrices $PA$, $AP$, and $A$ have the same singular values for any permutation matrix $P$, we assume for the rest of this section, without loss of generality, that the rows and columns of $A$ are such that

$$r_1(A) \geq r_2(A) \geq \cdots \geq r_n(A),$$

$$c_1(A) \geq c_2(A) \geq \cdots \geq c_n(A).$$

Let

$$A = \{ \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \mid 0 < \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \text{ and } \alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2 = n \}. \quad (20)$$

For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in A$, we define

$$D = \text{diag}\left\{ \frac{\alpha_1}{r_1(A)}, \frac{\alpha_2}{r_2(A)}, \ldots, \frac{\alpha_n}{r_n(A)} \right\}$$

and

$$B = DA.$$

Then, $\|B\|_F^2 = n$. We recall that

$$\sigma_n(ST) \geq \sigma_n(S) \sigma_n(T)$$

and

$$\sigma_n(ST) \geq \sigma_n(S) \sigma_n(T).$$
for any matrices $S$ and $T$. Using this fact and Lemma 8,

\[
\sigma_n(A) = \sigma_n(D^{-1}B) \\
\geq \sigma_n(D^{-1}) \sigma_n(B) \\
> \frac{r_n(A)}{\alpha_n} \left( \frac{n - 1}{n} \right)^{(n-1)/2} \left| \det B \right| \left[ 1 + \frac{1}{2} \theta_r \left( \frac{n - 1}{n} \right)^n \prod_{i=1}^{n} \alpha_i \right] \\
= \left( \frac{n - 1}{n} \right)^{(n-1)/2} r_n(A) \prod_{i=1}^{n-1} \alpha_i \left[ 1 + \frac{1}{2} \theta_r \left( \frac{n - 1}{n} \right)^n \prod_{i=1}^{n} \frac{\alpha_i^2}{r_i^2(A)} \left| \det A \right| \right],
\]

where

\[
S_r = \frac{\left| \det A \right|}{\prod_{i=1}^{n} r_i(A)}
\]

and

\[
\theta_r = \frac{1}{1 - ((n - 1)/n)^{n-1} \sum_{i=1}^{n} \alpha_i^2}.
\]

The corresponding result for columns can be obtained by considering

\[
D = \text{diag} \left\{ \frac{\alpha_1}{c_1(A)}, \frac{\alpha_2}{c_2(A)}, \ldots, \frac{\alpha_n}{c_n(A)} \right\},
\]

\[
B = AD,
\]

\[
S_c = \frac{\left| \det A \right|}{\prod_{i=1}^{n} c_i(A)},
\]

and

\[
\theta_c = \frac{1}{1 - ((n - 1)/n)^{n-1} \sum_{i=1}^{n} \alpha_i^2}.
\]

We have obtained Theorem 9.

**Theorem 9.** Let $A$ be an $n \times n$ nonsingular complex matrix, $n \geq 3$. Let $A$ as in (20). If $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in A$, then

\[
\sigma_n(A) > \left( \frac{n - 1}{n} \right)^{(n-1)/2} \min \left\{ r_{\min}(A) S_r \prod_{i=1}^{n-1} \alpha_i \left[ 1 + \frac{1}{2} \theta_r \left( \frac{n - 1}{n} \right)^n \prod_{i=1}^{n} \frac{\alpha_i^2}{r_i^2(A)} \right], \right. \\
\left. c_{\min}(A) S_c \prod_{i=1}^{n-1} \alpha_i \left[ 1 + \frac{1}{2} \theta_c \left( \frac{n - 1}{n} \right)^n \prod_{i=1}^{n} \frac{\alpha_i^2}{c_i^2(A)} \right]\right\}. \quad (21)
\]

Putting $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 1$ in (21) we get the following.

**Corollary 10.** Let $A$ be an $n \times n$ nonsingular complex matrix, $n \geq 3$. Then,

\[
\sigma_n(A) > \left( \frac{n - 1}{n} \right)^{(n-1)/2} \max \left\{ r_{\min}(A) S_r C_r, c_{\min}(A) S_c C_c \right\}, \quad (22)
\]
The bound in (22) improves the main result of Hong and Pan [1, Theorem 1]. In this point, we look for the best bound in (21). For this purpose, we consider the function

$$g(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left(\frac{n-1}{n}\right)^{(n-1)/2} \rho_r(A) \prod_{i=1}^{n-1} \alpha_i \left[1 + \frac{1}{2} \theta_r \left(\frac{n-1}{n}\right)^n \rho_r^{n-1} \prod_{i=1}^{n} \alpha_i^2\right],$$

$$\theta_r = \frac{1}{1 - ((n-1)/n)^{n-1} \rho_r \prod_{i=1}^{n} \alpha_i^2}.$$  

The function $g$ is continuous in the closure $\overline{A}$ of the set $A$ defined in (20). Then, there exists $\alpha^* \in \overline{A}$ such that $g$ achieves its maximum value over $\overline{A}$ at $\alpha^*$. Let $\alpha^* = (\alpha_1^*, \alpha_2^*, \ldots, \alpha_n^*)$. There exists a sequence $(\alpha_1^{(k)}, \alpha_2^{(k)}, \ldots, \alpha_n^{(k)}), (\alpha_1^{(k)}, \alpha_2^{(k)}, \ldots, \alpha_n^{(k)}) \in A$, such that $\lim_{k \to \infty} (\alpha_1^{(k)}, \alpha_2^{(k)}, \ldots, \alpha_n^{(k)}) = (\alpha_1^*, \alpha_2^*, \ldots, \alpha_n^*)$. We claim that $\alpha^*$ lies in $A$. Since $\alpha_1^2 \leq \alpha_2^2 \leq \cdots \leq \alpha_n^2$ and $(\alpha_1^2)^2 + (\alpha_2^2)^2 + \cdots + (\alpha_n^2)^2 = n$, for all $k$, we obtain that $\alpha_1^* \leq \alpha_2^* \leq \cdots \leq \alpha_n^*$ and that $(\alpha_1^*)^2 + (\alpha_2^*)^2 + \cdots + (\alpha_n^*)^2 = n$. Now it is clear that $\alpha_n^* > 0$. If $\alpha_i^* = 0$ for some $i$, $1 \leq i \leq n-1$, then $g(\alpha^*) = 0$ which is not the maximum value of $g$. Hence, all the components of $\alpha^*$ are positive and thus, $\alpha^* \in \overline{A}$. Let

$$D = \{\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \mid \alpha_i \geq 0, \text{ for } 1 \leq i \leq n \text{ and } \alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2 = n\}.$$  

Since $\alpha_i^* > 0$ for all $i$, $\alpha^*$ lies in the interior of $D$. We point out that the function $g$ and the constraint $\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2 = n$ are both symmetric in the variables $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$. From this, we can conclude that $\alpha_1^* = \alpha_2^* = \cdots = \alpha_{n-1}^*$. Finally, taking into account that fact $\alpha_{n-1} \leq (\rho_{n-1}(A)/\rho_n(A))\alpha_n$ for points in $A$, we obtain that the function $g$ achieves its maximum over $A$ at

$$\alpha_i^* = \alpha_2^* = \cdots = \alpha_{n-1}^* = \frac{\rho_{n-1}(A)}{\rho_n(A)}\alpha_n,$$

$$\alpha_n^* = \left(\frac{n}{1 + (n-1)(\rho_{n-1}(A)/\rho_n(A))^2}\right)^{1/2}.\quad (23)$$

Therefore, the values given in (23) lead to the best bound in (21), which we give in the following corollary.

**Corollary 11.** Let $A$ be an $n \times n$ nonsingular complex matrix, $n \geq 3$. Then,

$$\sigma_n(A) > \max \{\rho_{\min}(A) S_r D_r, c_{\min}(A) S_r D_c\},\quad (24)$$

where

$$D_r = \left(\frac{n-1}{n-1+p}\right)^{(n-1)/2} \left[1 + \frac{1}{2} \frac{((n-1)/(n-1+p))^n S_r^2 p}{1 - ((n-1)/(n-1+p))^n S_r^2 p}\right],$$

$$p = \left(\frac{\rho_n(A)}{\rho_{n-1}(A)}\right)^2,$$

$$D_c = \left(\frac{n-1}{n-1+q}\right)^{(n-1)/2} \left[1 + \frac{1}{2} \frac{((n-1)/(n-1+q))^n S_c^2 q}{1 - ((n-1)/(n-1+q))^n S_c^2 q}\right],$$

$$q = \left(\frac{c_n(A)}{c_{n-1}(A)}\right)^2.$$
The bound given in (24) improves the bound of Yu and Gu [2, Corollary 1], that we write in the form:

\[ \sigma_n(A) > \max \left\{ \min_r(B_r, C_c) \right\} \]

where

\[ B_r = \left( \frac{n-1}{n-1+p} \right)^{n-1/2} \left[ 1 + \frac{1}{2} \left( \frac{n-1}{n-1+p} \right)^n S_r^2 p \right], \]

\[ B_c = \left( \frac{n-1}{n-1+q} \right)^{n-1/2} \left[ 1 + \frac{1}{2} \left( \frac{n-1}{n-1+q} \right)^n S_c^2 q \right]. \]

The above-mentioned authors arrived at (25), proving that the factor

\[ \left( \frac{n-1}{n} \right)^{n-1/2} r_n(A) S_r \prod_{i=1}^{n-1} \alpha_i, \]

in the lower bound

\[ \sigma_n(A) > \left( \frac{n-1}{n} \right)^{n-1/2} r_n(A) S_r \prod_{i=1}^{n-1} \alpha_i \left[ 1 + \frac{1}{2} \left( \frac{n-1}{n} \right)^n S_r^2 \prod_{i=1}^{n} \alpha_i^2 \right], \]

achieves its maximum value at \( \alpha^* = (\alpha_1^*, \alpha_2^*, \ldots, \alpha_n^*) \) given by (23).

**Example 12.** Let

\[ A = \begin{bmatrix} -1 & 2 & 1 & 2 \\ 1 & 0 & 2 & 1 \\ 3 & 2 & 1 & 1 \\ 0 & 3 & 2 & -2 \end{bmatrix}. \]

For this matrix, (22), (24), and (25) give the following lower bounds for the smallest singular value.

\[
\begin{array}{ccc}
(22) & (24) & (25) \\
1.2595 & 1.2734 & 1.2603
\end{array}
\]

**4. New Upper Bounds for the Spectral Condition Number**

We begin reviewing some recent upper bounds for the spectral condition number of a nonsingular matrix \( A \) of order \( n \times n \). As in Section 3, we assume without loss of generality that

\[ r_1(A) \geq r_2(A) \geq \cdots \geq r_n(A) \]

and

\[ c_1(A) \geq c_2(A) \geq \cdots \geq c_n(A). \]

Guggenheimer, Edelman and Johnson in [4] derived the bound:

\[ \kappa_2(A) < \frac{2}{\det A} \left( \frac{\|A\|_F^2}{n} \right)^{n/2}. \]  

(26)

Let \( B \) a nonsingular matrix of order \( n \times n \) such that \( \|B\|_F = \sqrt{n} \). For the matrix \( B \), (26) takes the very simple form

\[ \kappa_2(B) < \frac{2}{\det B}. \]  

(27)
Bound (26) was improved by Merikoski, Urpala, Virtanen, Tam and Uhling. In fact, they proved in [3] that

\[ \kappa_2(A) \leq \frac{1 + \sqrt{1 - (n/\|A\|_F^2)^n |\det A|^2}}{1 - \sqrt{1 - (n/\|A\|_F^2)^n |\det A|^2}} \]  

is the best possible upper bound for \( \kappa_2(A) \) in terms only of \( \det A, \|A\|_F, \) and \( n \).

For the matrix \( B \), (28) becomes

\[ \kappa_2(B) \leq \frac{1 + \sqrt{1 - |\det B|^2}}{|\det B|}. \]  

For \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n), \alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2 = n, \) and \( \alpha_i > 0, \) for all \( i \), we define

\[ D = \text{diag} \left\{ \frac{\alpha_1}{r_1(A)}, \frac{\alpha_2}{r_2(A)}, \ldots, \frac{\alpha_n}{r_n(A)} \right\} \]

and

\[ B = DA. \]

Thus, \( \|B\|_F = \sqrt{n} \). Now, we remember that \( \kappa_2(ST) \leq \kappa_2(S)\kappa_2(T) \) for nonsingular matrices \( S \) and \( T \). Hence, \( \kappa_2(A) \leq \kappa_2(D^{-1})\kappa_2(B) \). By applying (29) to \( B \), we have

\[ \kappa_2(A) \leq \frac{\max_i \left( \frac{r_i(A)}{\alpha_i} \right)}{\min_i \left( \frac{r_i(A)}{\alpha_i} \right)} \frac{1 + \sqrt{1 - \left( \frac{n}{\prod_{i=1}^n \alpha_i} \right)^2 |\det A|^2}}{\prod_{i=1}^n \frac{n}{\prod_{i=1}^n \alpha_i} |\det A|}. \]  

(30)

Suppose that the components of \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \) are chosen such that

\[ \max_i \left( \frac{r_i(A)}{\alpha_i} \right) = \frac{r_1(A)}{\alpha_1} \quad \text{and} \quad \min_i \left( \frac{r_i(A)}{\alpha_i} \right) = \frac{r_n(A)}{\alpha_n}. \]

Thus, (30) becomes

\[ \kappa_2(A) \leq \frac{r_1(A)}{\alpha_1} \frac{\alpha_n}{r_n(A)} \frac{1 + \sqrt{1 - \left( \frac{n}{\prod_{i=1}^n \alpha_i} \right)^2 S_r^2}}{\left( \prod_{i=1}^n \frac{n}{\prod_{i=1}^n \alpha_i} \right) S_r}. \]

The corresponding result for columns also holds. Both bounds are given in the next theorem.

**THEOREM 13.** Let \( A \) be an \( n \times n \) nonsingular complex matrix. Let \( \alpha_1, \alpha_2, \ldots, \alpha_n \) be positive real numbers such that \( \alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2 = n \).

1. If \( \alpha_1, \alpha_2, \ldots, \alpha_n \) satisfy the condition

\[ \max_i \left( \frac{r_i(A)}{\alpha_i} \right) = \frac{r_{\max}(A)}{\alpha_1} \quad \text{and} \quad \min_i \left( \frac{r_i(A)}{\alpha_i} \right) = \frac{r_{\min}(A)}{\alpha_n}, \]  

(31)

then

\[ \kappa_2(A) \leq \frac{r_{\max}(A)}{\alpha_1} \frac{\alpha_n}{r_{\min}(A)} \frac{1 + \sqrt{1 - \left( \frac{n}{\prod_{i=1}^n \alpha_i} \right)^2 S_r^2}}{\left( \prod_{i=1}^n \frac{n}{\prod_{i=1}^n \alpha_i} \right) S_r}. \]  

(32)
2. If $\alpha_1, \alpha_2, \ldots, \alpha_n$ satisfy the condition
\[
\max_i \left( \frac{c_i(A)}{\alpha_i} \right) = \frac{c_{\max}(A)}{\alpha_1} \quad \text{and} \quad \min_i \left( \frac{c_i(A)}{\alpha_i} \right) = \frac{c_{\min}(A)}{\alpha_n},
\]
then
\[
\kappa_2(A) \leq \frac{c_{\max}(A) \alpha_n}{\alpha_1 c_{\min}(A)} \frac{1 + \sqrt{1 - \left( \prod_{i=1}^{n} \alpha_i \right)^2 S_r^2}}{\left( \prod_{i=1}^{n} \alpha_i \right) S_c}.
\]  

3. If $\alpha_1, \alpha_2, \ldots, \alpha_n$ satisfy conditions (31) and (33), then
\[
\kappa_2(A) \leq \min \left\{ \frac{r_{\max}(A) \alpha_n}{r_{\min}(A) \alpha_1} \frac{1 + \sqrt{1 - \left( \prod_{i=1}^{n} \alpha_i \right)^2 S_r^2}}{\left( \prod_{i=1}^{n} \alpha_i \right) S_r}, \frac{c_1(A) \alpha_n}{c_{\max}(A) c_{\min}(A)} \frac{1 + \sqrt{1 - \left( \prod_{i=1}^{n} \alpha_i \right)^2 S_c^2}}{\left( \prod_{i=1}^{n} \alpha_i \right) S_c} \right\}.
\]

We observe that
\[
\alpha_1 = \alpha_2 = \cdots = \alpha_n = 1
\]
satisfies conditions (31) and (33). For these particular values, the corresponding bounds for the spectral condition number are given in the next corollary.

**Corollary 14.** Let $A$ be an $n \times n$ nonsingular complex matrix. Then,
\[
1. \quad \kappa_2(A) \leq \frac{r_{\max}(A) 1 + \sqrt{1 - S_r^2}}{r_{\min}(A)},
\]
\[
2. \quad \kappa_2(A) \leq \frac{c_{\max}(A) 1 + \sqrt{1 - S_c^2}}{S_c}.
\]

Then,
\[
3. \quad \kappa_2(A) \leq \min \left\{ \frac{r_{\max}(A) 1 + \sqrt{1 - S_r^2}}{r_{\min}(A)} \frac{c_{\max}(A) 1 + \sqrt{1 - S_c^2}}{S_c} \right\}.
\]

Now, we prove that, under an additional condition, (36) and (37) are the best bounds in (32) and (34), respectively.

**Theorem 15.** Let $A$ be an $n \times n$ nonsingular complex matrix. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be positive real numbers such that $\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2 = n$.

1. If $\alpha_1, \alpha_2, \ldots, \alpha_n$ satisfy (31) and if $\alpha_1 \leq \alpha_n$, then (36) is the best possible upper bound for $\kappa_2(A)$ in (32).
2. If $\alpha_1, \alpha_2, \ldots, \alpha_n$ satisfy (33) and if $\alpha_1 \leq \alpha_n$, then (37) is the best possible upper bound for $\kappa_2(A)$ in (34).
PROOF. Assume that $\alpha_1 \leq \alpha_n$. We have

$$\left( \prod_{i=1}^{n} \alpha_i^2 \right)^{1/n} \leq \frac{\sum_{i=1}^{n} \alpha_i^2}{n} = 1.$$  

Then,

$$\left( \frac{\alpha_n}{\alpha_1} \right)^2 - \left( \prod_{i=1}^{n} \alpha_i \right)^2 \geq \left( \frac{\alpha_n}{\alpha_1} \right)^2 - 1 \geq 0 \quad (39)$$

and

$$1 \geq S_r \left( \prod_{i=1}^{n} \alpha_i \right)^2. \quad (40)$$

From inequalities (39) and (40) it follows that

$$\left( \frac{\alpha_n}{\alpha_1} \right)^2 - \left( \prod_{i=1}^{n} \alpha_i \right)^2 \geq S_r^2 \left( \prod_{i=1}^{n} \alpha_i \right)^2 \left( \left( \frac{\alpha_n}{\alpha_1} \right)^2 - 1 \right). \quad (41)$$

Hence,

$$\left( \frac{\alpha_n}{\alpha_1} \right)^2 \left( 1 - S_r^2 \left( \prod_{i=1}^{n} \alpha_i \right)^2 \right) \geq \left( 1 - S_r^2 \right) \left( \prod_{i=1}^{n} \alpha_i \right)^2. \quad (42)$$

Finally, from (41),

$$\frac{\alpha_n}{\alpha_1} \sqrt{1 - \left( \prod_{i=1}^{n} \alpha_i \right)^2} \geq \sqrt{1 - S_r^2}.$$

Moreover,

$$\left( \frac{\alpha_1}{\alpha_n} \right)^2 \left( \prod_{i=1}^{n} \alpha_i \right)^2 \leq \left( \frac{\alpha_1}{\alpha_n} \right)^2 \leq 1.$$

Then,

$$\frac{\alpha_n}{\alpha_1} \frac{1}{\prod_{i=1}^{n} \alpha_i} \geq 1. \quad (43)$$

Now, adding inequalities (42) and (43), we have

$$\frac{\alpha_n}{\alpha_1} \frac{1}{\prod_{i=1}^{n} \alpha_i} \geq 1 + \sqrt{1 - S_r^2}.$$

Finally,

$$\frac{r_1 \alpha_n}{r_2 \alpha_1} \frac{1}{S_r} \left( \prod_{i=1}^{n} \alpha_i \right) S_r \geq \frac{r_1}{r_2} \frac{1 + \sqrt{1 - S_r^2}}{S_r}. \quad (44)$$

This shows that (36) gives the best bound in (32), if $\alpha_1 \leq \alpha_n$. The corresponding inequality for the columns can be obtained in a similar way.
EXAMPLE 16. Let

\[
A = \begin{bmatrix}
1 & 1 & 3 & 3 \\
2 & 4 & 0 & 2 \\
2 & 45 & 0 & 1 \\
3 & 2 & 1 & 1
\end{bmatrix}.
\]

Bound (38) gives \( \kappa_2(A) \leq 97.06 \). Let \( \alpha_1 = 0.9, \alpha_2 = \alpha_3 = 1, \) and \( \alpha_4 = \sqrt{4 - 2 - (0.9)^2} \). These values for the components of \( \alpha \) satisfy conditions (31) and (33). Also, \( \alpha_1 < \alpha_4 \). For these values, bound (35) gives \( \kappa_2(A) \leq 119.93 \). This example illustrates Theorem 15.

Next, we give another corollary of Theorem 13.

Taking into consideration the symmetry of the second right-hand side of (32) in the variables \( \alpha_2, \ldots, \alpha_n \) and, from condition (33), that \( \alpha_1 \leq (r_1(A)/r_2(A))\alpha_2 \) and \( \alpha_{n-1} \leq (r_{n-1}(A)/r_n(A))\alpha_n \), we choose

\[
\alpha_2 = \cdots = \alpha_{n-1}, \quad \alpha_1 = \frac{r_1(A)}{r_2(A)}\alpha_2, \quad \alpha_n = \frac{r_n(A)}{r_{n-1}(A)}\alpha_2, \quad \text{and} \quad \alpha_2 = \left(\frac{r_1(A)/r_2(A)}{2} + (n-2) + (r_n(A)/r_{n-1}(A))^2\right)^{1/2}.
\]

We can see that this, another election for the components of \( \alpha \), also satisfies condition (33) and leads to the following corollary, in which we include also the corresponding result for the columns of \( A \).

COROLLARY 17. Let \( A \) be an \( n \times n \) nonsingular complex matrix. Then,

1. \[
\kappa_2(A) \leq \frac{r_2(A)}{r_{n-1}(A)} \frac{1 + \sqrt{1 - \frac{S_r^2 E_r^2}{c_2(A) c_{n-1}(A)}}}{S_r E_r},
\]

where

\[
E_r = \frac{r_{\max}(A) r_{\min}(A)}{r_2(A) r_{n-1}(A)} \left(\frac{n}{(r_{\max}(A)/r_2(A))^2 + (n-2) + (r_{\min}(A)/r_{n-1}(A))^2}\right)^{n/2},
\]

2. \[
\kappa_2(A) \leq \frac{c_2(A)}{c_{n-1}(A)} \frac{1 + \sqrt{1 - \frac{S_c^2 E_c^2}{c_2(A) c_{n-1}(A)}}}{S_c E_c},
\]

where

\[
E_r = \frac{c_{\max}(A) c_{\min}(A)}{c_2(A) c_{n-1}(A)} \left(\frac{n}{(c_{\max}(A)/c_2(A))^2 + (n-2) + (c_{\min}(A)/c_{n-1}(A))^2}\right)^{n/2},
\]

Then,

3. \[
\kappa_2(A) \leq \min \left\{ \frac{r_2(A)}{r_{n-1}(A)} \frac{1 + \sqrt{1 - \frac{S_r^2 E_r^2}{c_2(A) c_{n-1}(A)}}}{S_r E_r}, \frac{c_2(A)}{c_{n-1}(A)} \frac{1 + \sqrt{1 - \frac{S_c^2 E_c^2}{c_2(A) c_{n-1}(A)}}}{S_c E_c} \right\}.
\]

REMARK 2. We observe that in Corollary 17

1. \( r_1(A) \geq r_2(A) \geq r_i(A), c_1(A) \geq c_2(A) \geq c_i(A), \) for \( i = 3, \ldots, n \) and \( r_1(A) \geq r_{n-1}(A) \geq r_n(A), c_1(A) \geq c_{n-1}(A) \geq c_n(A), \) for all \( i = 1, \ldots, n-3; \)

2. \( \alpha_1 = (r_1(A)/r_2(A))\alpha_2 \) and \( \alpha_n = (r_n(A)/r_{n-1}(A))\alpha_n, \) then, \( \alpha_1 = (r_1(A)/r_2(A)) (r_{n-1}(A)/r_n(A))\alpha_n \) (this implies that \( \alpha_n < \alpha_1, \) if \( r_1(A)/r_n(A) > 1 \) or \( r_{n-1}(A)/r_n(A) > 1 \).
We finish observing that bounds (28), (38), and (45) are not comparable, as we see in the following example.

**Example 18.** Let

\[
A_1 = \begin{bmatrix}
1 & 1 & 3 & 3 \\
2 & 4 & 0 & 2 \\
2 & 4.5 & 0 & 1 \\
3 & 2 & 1 & 1
\end{bmatrix},
\]

\[
A_2 = \begin{bmatrix}
1 & 1 & 3 & 3 \\
2 & 4 & 0 & 2 \\
2 & 45 & 0 & 1 \\
3 & 2 & 1 & 1
\end{bmatrix},
\]

\[
A_3 = \begin{bmatrix}
1 & 10 & 1 & 5 \\
10 & 2 & 0 & 20 \\
2 & 0.5 & 0 & 1 \\
3 & 2 & 1 & 1
\end{bmatrix}.
\]

Then,

\[
\begin{array}{cccc}
\text{(28)} & \text{(38)} & \text{(45)} & \kappa_2 \\
A_1 & 29.54 & 36.83 & 30.26 & 14.03 \\
A_2 & 804.56 & 97.06 & 737.98 & 39.48 \\
A_3 & 203.90 & 168.64 & 106.31 & 57.49
\end{array}
\]

In this example, we see bound (28), which is the best upper bound for the spectral condition number in terms only of \(\|A\|_F\), \(\det A\), and \(n\), is strongly improved by (38) for the matrix \(A_2\) and by (45) for the matrix \(A_3\).

**REFERENCES**