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Duality in Hypercomplex Function Theory

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Let $\mathscr{A}$ be the Clifford algebra constructed over a quadratic n-dimensional real vector space with orthogonal basis $\{e_1,..., e_n\}$, and e_0 be the identity of $\mathscr{A}$. Furthermore, let $M_k(\Omega; \mathscr{A})$ be the set of $\mathscr{A}$-valued functions defined in an open subset Ω of $\mathbf{R}^{m+1}$ $(1 \leqslant m \leqslant n)$ which satisfy $D^k f = 0$ in Ω, where D is the generalized Cauchy-Riemann operator $D = \sum_{i=0}^{m} e_i(\partial/\partial x_i)$ and $k \in \mathbf{N}$. The aim of this paper is to characterize the dual and bidual of $M_k(\Omega; \mathscr{A})$. It is proved that, if $M_k(\Omega; \mathscr{A})$ is provided with the topology of uniform compact convergence, then its strong dual is topologically isomorphic to an inductive limit space of Fréchet modules, which in its turn admits $M_k(\Omega; \mathscr{A})$ as its dual. In this way, classical results about the spaces of holomorphic functions and analytic functionals are generalized.

1. INTRODUCTION

In his well known paper [9] Köthe has shown that if $\mathscr{D}$ is an arbitrary proper open subset of the Riemann sphere $\mathscr{O}$ and $\mathscr{H}(\mathscr{D})$ denotes the space of locally holomorphic functions on $\mathscr{D}$ provided with its natural topology, then its dual may be identified with $\mathscr{H}(\mathscr{U})$, the space of locally holomorphic functions on $\mathscr{U} = \mathscr{O}\backslash\mathscr{D}$, and conversely. Let us recall that $\mathscr{H}(\mathscr{U})$ is in fact an inductive limit space and that the respective duals are endowed with the strong topology.

Almost simultaneously Grothendieck has developed in [8] a duality theory for vector valued holomorphic functions defined on a proper open subset of $\mathscr{O}$, generalizing in this way Köthe's result.

Afterwards Tillmann has worked out in [12] and [13] respectively a duality theory for harmonic functions in n-dimensional Euclidean space, $n \geqslant 3$, and for analytic functions on Riemann surfaces.

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As to the case of holomorphic functions of several complex variables, we can cite the work of Lelong [10] and the thesis [2] of Braun, in which a generalization of Aizenberg's result in [1] is given too.

Finally we mention the thesis [3] of Chauveheid, in which he has characterized the dual of the space of strong solutions in an open subset $\Omega \subset \mathbf{R}^n$ of an arbitrary elliptic differential operator with constant coefficients.

In this paper, which is a continuation of [4, 5, 6], we study the dual of the space $M_k(\Omega; \mathscr{A})$ consisting of those functions $f\colon \Omega \to \mathscr{A}$ which satisfy $D^k f = 0$ in Ω, where $k \in \mathbf{N}$, $k \geqslant 1$, Ω is an open subset of $\mathbf{R}^{m+1}$, $\mathscr{A}$ is the Clifford algebra constructed over an n-dimensional *real* quadratic vector space ($1 \leqslant m \leqslant n$) and $D = \sum_{i=0}^{m} e_i(\partial/\partial x_i)$ is a hypercomplex differential operator generalizing the classical Cauchy–Riemann operator. In fact the operator D^k determines a strongly elliptic system of 2^n homogeneous differential equations, each of order k. In the particular case that $m = n = 1$, the solutions of $D^k f = 0$ in Ω are nothing else but the polyanalytic functions in Ω, so that for $k = 1$, the space of holomorphic functions in Ω is obtained. For general m, n and k, $M_k(\Omega; \mathscr{A})$ constitutes a subclass of the set of $\mathbf{R}^{2^n}$-valued polyharmonic functions of order k.

As a main result it is proved that if $M_k(\Omega; \mathscr{A})$ is equipped with the topology of uniform compact convergence, then its strong dual $M_k(\Omega; \mathscr{A})_b^*$ is topologically isomorphic to an inductive limite space $\tilde{M}_k^{(l)}(\text{co}\,\Omega; \mathscr{A})$ of Fréchet modules (Theorem 2.3) and that conversely, the strong dual $\tilde{M}_k^{(l)}(\text{co}\,\Omega; \mathscr{A})_b^*$ is topologically isomorphic to $M_k(\Omega; \mathscr{A})$ (Theorem 5.1).

It should be noted that the proof of these results relies heavily upon two types of Runge approximation theorems obtained in [6], whereas in classical function theory the Runge approximation theorem appears to be a simple corollary to duality (see [11]).

We have thus generalized classical results concerning the spaces $\mathscr{H}(\Omega)$ and $\mathscr{H}'(\Omega)$ of respectively holomorphic functions and analytic functionals in Ω.

2. Preliminaries

In this section we repeat briefly some notions and results from [4]–[6] to which frequent appeal will be made in the sequel. Let $\mathscr{A}$ be the Clifford algebra constructed over a quadratic n-dimensional *real* vector space with orthogonal basis $\{e_1, \ldots, e_n\}$ such that $e_i^2 = -e_0$, for $i = 1, \ldots, n$, e_0 being the identity in $\mathscr{A}$. Furthermore, let an arbitrary basic element of $\mathscr{A}$ be denoted by $e_A = e_{i_1} e_{i_2} \cdots e_{i_h}$, where $A = \{i_1, \ldots, i_h\} \subset N = \{1, 2, \ldots, n\}$ with $i_1 < i_2 < \cdots < i_h$ and put for any $\lambda = \sum_A \lambda_A e_A \in \mathscr{A}$, $|\lambda|_0^2 = 2^n \sum_A \lambda_A^2$. Then $|\cdot|_0$ is a norm on $\mathscr{A}$ (see [4]).

Let $m \leqslant n$, $m \neq 0$, and let Ω be an open non empty subset of $\mathbf{R}^{m+1}$. Then in [4] we have studied properties of the solutions of the equation $D^k f = 0$ ($f D^k = 0$) in Ω, where $k \in \mathbf{N}$, $k \geqslant 1$, $f \in C_k(\Omega; \mathscr{A})$ and $D = \sum_{i=0}^{m} e_i(\partial/\partial x_i)$. These solutions have been called left (right) k-monogenic functions in Ω; their set constitutes a right (left) $\mathscr{A}$-module $M_k(\Omega; \mathscr{A})(M_k^{(l)}(\Omega; \mathscr{A}))$ which becomes a right (left) Fréchet $\mathscr{A}$-module for the topology of uniform compact convergence (see [4, Theorem 3.1]). Moreover, note that, since $\bar{D}^k D^k = D^k \bar{D}^k = e_0 \Delta^k$ where $\Delta = \sum_{i=0}^{m} (\partial^2/\partial x_i^2)$ is the Laplacian in $(m + 1)$-dimensional Euclidean space, $M_k(\Omega; \mathscr{A})$ is a submodule of the $\mathscr{A}$-valued polyharmonic functions of order k in Ω.

If $K \subset \mathbf{R}^{m+1}$ is compact, we have introduced in [5] the right (left) $\mathscr{A}$-module $\tilde{M}_k(\text{co } K; \mathscr{A})(\tilde{M}_k^{(l)}(\text{co } K; \mathscr{A}))$ consisting of those elements in $M_k(\text{co } K; \mathscr{A})$ $(M_k^{(l)}(\text{co } K; \mathscr{A}))$ which are regular at infinity with respect to the fundamental solution E_k of D^k. It has been proved that $\tilde{M}_k(\text{co } K; \Omega)(\tilde{M}_k^{(l)}(\text{co } K; \Omega))$ is a closed submodule of $M_k(\text{co } K; \Omega)(M_k^{(l)}(\text{co } K; \Omega))$ (see [5, Theorem 3.1]).

Finally, three Runge type approximation theorems have been obtained in [6], to wit

(i) If K is a compact subset of $\mathbf{R}^{m+1}$ and $\vec{a}$ is a subset of co K having one point in each bounded component of co K, then the set of "rational" functions $M_k(\mathbf{R}^{m+1}; \mathscr{A}) \oplus \mathscr{R}^*(\vec{a})$ is uniformly dense in $M_k(K; \mathscr{A})$, the latter being the $\mathscr{A}$-module of functions which are (k)-monogenic in some open neighborhood of K (see [6, Lemma 3.3]).

(ii) If α is a subset of co Ω having one point in each component of co Ω, then the set of "rational" functions $M_k(\mathbf{R}^{m+1}; \mathscr{A}) \oplus \mathscr{R}^*(\alpha)$ is dense in $M_k(\Omega; \mathscr{A})$ for the topology of uniform compact convergence (see [6, Theorem 3.1]).

(iii) If K is a compact subset of $\mathbf{R}^{m+1}$ and α is a subset of K having one point in each component of K, then the set of "rational" functions $\mathscr{R}^*(\alpha)$ is dense in $\tilde{M}_k(\text{co } K; \mathscr{A})$ for the topology of uniform compact convergence (see [6, Theorem 4.1]).

3. The Indicatrix of Fantappie

In the sequel we assume that Ω is an open non empty subset of $\mathbf{R}^{m+1}$, $(K_j)_{j=1}^{\infty}$ is the compact exhaustion of Ω given by

$$K_j = \left\{x \in \Omega : |\, x \,| \leqslant j \text{ and } d(x, \text{co } \Omega) \geqslant \frac{1}{j}\right\}$$

and $\{p_{K_j} : j \in \mathbf{N}\}$ is the proper system of seminorms on $M_k(\Omega; \mathscr{A})$ associated to it, i.e. for each $j \in \mathbf{N}$,

$$p_{K_j}(f) = \sup_{x \in K_j} |f(x)|_0, \qquad f \in M_k(\Omega; \mathscr{A}).$$

Now let T be a bounded right $\mathscr{A}$-linear functional on $M_k(\Omega; \mathscr{A})$, that is, there exist $C > 0$ and $j \in \mathbf{N}$ such that for all $f \in M_k(\Omega; \mathscr{A})$

$$| T(f)|_0 \leqslant C p_{K_j}(f).$$

Then by the Hahn–Banach and Riesz representation theorems (see [5]) there exists an $\mathscr{A}$-valued measure μ in $\mathbf{R}^{m+1}$ supported on K_j such that for all $f \in M_k(\Omega; \mathscr{A})$

$$T(f) = \int d\mu(x) f(x).$$

Furthermore, let $\varphi \in \mathscr{D}_\infty(\Omega; \mathbf{R})$ with $\varphi(t) = 1$ on an open neighborhood $\omega_\varphi \subset \Omega$ of K_j. Then by virtue of the representation formula established in [5, Theorem 4.1], for each $x \in K_j$

$$f(x) = \int E_k(x - t)\, D^k(f\varphi)(t)\, dt^{m+1}.$$

Hence, using Fubini's theorem,

$$\begin{aligned} T(f) &= \int d\mu(x) \int E_k(x - t)\, D^k(f\varphi)(t)\, dt^{m+1} \\ &= \int \left[\int d\mu(x)\, E_k(x - t)\right] D^k(f\varphi)(t)\, dt^{m+1}. \end{aligned}$$

Put

$$\begin{aligned} t_k(t) &= \int d\mu(x)\, E_k(x - t) \\ &= (-1)^k \int d\mu(x)\, E_k(t - x) \\ &= (-1)^k \mu * E_k . \end{aligned}$$

Then, up to the constant $(-1)^k$, t_k equals the right Cauchy transform of μ defined in [5]. Let us recall that $\mu * E_k$ is right k-monogenic in co$[\mu]$ and regular at infinity with respect to E_k so that $t_k \in \tilde{M}_k^{(l)}(\text{co } K_j ; \mathscr{A})$.

In analogy with classical function theory, t_k is called the *indicatrix of Fantappié* associated to T in co K_j.

In view of the foregoing considerations we have

PROPOSITION 3.1. *Let T be a right $\mathscr{A}$-linear functional on $M_k(\Omega; \mathscr{A})$ which is bounded by p_{K_j} and let t_k be its associated indicatrix of Fantappié in* co K_j. *Then if $\varphi \in \mathscr{D}_\infty(\Omega; \mathbf{R})$ with $\varphi(t) = 1$ on some open neighborhood ω_φ of K_j,*

$$T(f) = \int t_k(t)\, D^k(f\varphi)(t)\, dt^{m+1}$$

and this for all $f \in M_k(\Omega; \mathscr{A})$.

In the sequel, for $j \in \mathbf{N}$ fixed, Φ_j will stand for the set of functions $\varphi \in \mathscr{D}_\infty(\Omega; \mathbf{R})$ such that $\varphi(t) = 1$ on some open neighborhood ω_φ of K_j which is contained in Ω.

Now take $u \in \tilde{M}_k^{(l)}(\text{co } K_j; \mathscr{A})$, $\varphi \in \Phi_j$ and call for each $f \in M_k(\Omega; \mathscr{A})$,

$$T(f) = \int u(t)\, D^k(f\varphi)(t)\, dt^{m+1}. \tag{3.1}$$

Then T is well defined on $M_k(\Omega; \mathscr{A})$. Moreover, its definition does not depend on the choice of $\varphi \in \Phi_j$. Indeed, if $\varphi_1, \varphi_2 \in \Phi_j$, then $\psi = \varphi_1 - \varphi_2 \in \mathscr{D}_\infty(\Omega \backslash K_j; \mathbf{R})$ so that, using Green's identity (see [4]),

$$\begin{aligned} \int_{[\psi]} u(t)\, D^k(f\psi)(t)\, dt^{m+1} &= (-1)^k \int_{[\psi] \subset \Omega \backslash K_j} uD^k \cdot (f\psi)(t)\, dt^{m+1} \\ &= 0 \end{aligned}$$

We now claim that T, given by (3.1), is a bounded right $\mathscr{A}$-linear functional on $M_k(\Omega; \mathscr{A})$. It is obvious that T is right $\mathscr{A}$-linear on $M_k(\Omega; \mathscr{A})$. To prove that T is bounded, we proceed as follows. As u determines a right $\mathscr{A}$-distribution $\mathscr{T}_u$ in co K_j (see [5]) where

$$\mathscr{T}_u(\psi) = \int_{\text{co}K_j} u(t)\, \psi(t)\, dt^{m+1}, \qquad \psi \in \mathscr{D}_\infty(\text{co } K_j; \mathscr{A}),$$

we have that for any distributional extension $\mathscr{U}^{(r)}$ of u — that is, $\mathscr{U}^{(r)}$ is a right $\mathscr{A}$-distribution in $\mathbf{R}^{m+1}$ with $\mathscr{U}^{(r)} = \mathscr{T}_u$ in co K_j — and each ψ of the form $\psi = D^k(f\varphi)$,

$$\begin{aligned} \langle \mathscr{U}^{(r)}, D^k(f\varphi)\rangle &= \int_{\text{co}K_j} u(t)\, D^k(f\varphi)(t)\, dt^{m+1} \\ &= T(f). \end{aligned}$$

Now let $\epsilon > 0$ be such that $K = \{x \in \Omega : d(x, K_j) \leqslant 2\epsilon\} \subset \omega_\varphi$. Then, as $\mathscr{U}^{(r)}D^k$ is a right $\mathscr{A}$-distribution in $\mathbf{R}^{m+1}$ with compact support contained in K_j, we have that there exist $M \in \mathbf{N}$ and $C > 0$ such that for all $\psi \in \mathscr{D}_\infty(\mathbf{R}^{m+1}; \mathscr{A})$,

$$|\langle \mathscr{U}^{(r)}D^k, \psi\rangle|_0 \leqslant C \sup_{|\alpha| \leqslant M} \sup_{x \in K} |\, \partial^\alpha \psi(x)|_0 .$$

In particular, for $\psi = f\varphi$

$$\begin{aligned} |\langle \mathscr{U}^{(r)}D^k, f\varphi\rangle|_0 &\leqslant C \sup_{|\alpha| \leqslant M} \sup_{x \in K} |\, \partial^\alpha (f\varphi)(x)|_0 \\ &= C \sup_{|\alpha| \leqslant M} \sup_{x \in K} |\, \partial^\alpha f(x)|_0 . \end{aligned}$$

Since the components of f are polyharmonic in Ω, a suitable compact neighborhood K_η of K ($K_\eta \subset \omega_\varphi$) may be found such that

$$|\langle \mathscr{U}^{(r)} D^k, f\varphi \rangle|_0 \leqslant C^* \sup_{x \in K_\eta} |f(x)|_0 .$$

As

$$\begin{aligned}\langle \mathscr{U}^{(r)} D^k, f\varphi \rangle &= (-1)^k \langle \mathscr{U}^{(r)}, D^k(f\varphi) \rangle \\ &= (-1)^k \, T(f),\end{aligned}$$

we get finally that

$$| \, T(f)|_0 \leqslant C^* \sup_{x \in K_\eta} |f(x)|_0 ,$$

which implies that T is bounded on $M_k(\Omega; \mathscr{A})$.

In view of the foregoing considerations we have

PROPOSITION 3.2. *Let $j \in \mathbf{N}$ be fixed, $u \in \tilde{M}_k^{(l)}(\mathrm{co}\, K_j \,; \mathscr{A})$ and $\varphi \in \Phi_j$. Then $T : M_k(\Omega; \mathscr{A}) \to \mathscr{A}$ given by*

$$T(f) = \int u(t) \, D^k(f\varphi)(t) \, dt^{m+1}, \qquad f \in M_k(\Omega; \mathscr{A}),$$

is a bounded right $\mathscr{A}$-linear functional.

Remark. If $u \in \tilde{M}_k^{(l)}(\mathrm{co}\, K_j \,; \mathscr{A})$ and $\varphi \in \Phi_j$ are given, then a compact neighborhood $K_\eta \subset \omega_\varphi$ of K_j may be found such that the right $\mathscr{A}$-linear functional T on $M_k(\Omega; \mathscr{A})$ defined by (3.1) is bounded by p_{K_η}. Note that in the definition of T the domain of integration may be restricted to $[\varphi] \backslash K_\eta$ so that, using Green's identity, we obtain that

$$\begin{aligned}T(f) = (-1)^k &\int_{[\varphi]\backslash K_\eta} u D^k \cdot f\varphi \, dt^{m+1} \\ &+ (-1)^{k+1} \int_{\partial([\varphi])\backslash K_\eta)} \sum_{j=0}^{k-1} (-1)^j u D^{k-1-j} d\sigma D^j (f\varphi).\end{aligned}$$

Since $uD^k = 0$ in $[\varphi] \backslash K_\eta$, $\varphi(t) = 0$ on $\partial[\varphi]$ and $\varphi(t) = 1$ on ∂K_η, we get finally that

$$T(f) = \int_{\partial K_\eta} \sum_{j=0}^{k-1} (-1)^{j+k} u D^{k-1-j} d\sigma D^j f. \tag{3.2}$$

Conversely, let T be a right $\mathscr{A}$-linear functional on $M_k(\Omega; \mathscr{A})$ which is bounded by p_{K_j} and let t_k be its associated indicatrix of Fantappié. Furthermore, choose

$\varphi \in \Phi_j$ and let K be a suitable compact neighborhood of K_j satisfying $K \subset [\varphi] \subset \omega_\varphi$. Then, applying once again Green's identity, we have that

$$T(f) = \int_{\partial K} \sum_{j=0}^{k-1} (-1)^{j+k} t_k D^{k-1-j} d\sigma D^j f.$$

But, since in co K_j, $t_k D^{k-1-j} = t_{j+1}$ (see [4]), we get at last that

$$T(f) = \int_{\partial K} \sum_{j=0}^{k-1} (-1)^{j+k} t_{j+1} d\sigma D^j f. \tag{3.3}$$

The formulas (3.2) and (3.3) should be compared with their analogues in classical duality theory, where the relationship between an analytic functional and its indicatrix of Fantappié is usually given by an integral taken over the boundary of a domain. We describe this relationship in the Propositions 3.1 and 3.2 by an integral taken over the whole of $\mathbf{R}^{m+1}$ and this by using Schwartz's space $\mathscr{D}_\infty(\Omega; \mathbf{R})$. The question arises of course whether or not T is uniquely determined by $u \in \tilde{M}_k^{(l)}(\text{co } K_j; \mathscr{A})$. To this end we introduce.

Definition 3.1 .Let $j \in \mathbf{N}$ be fixed. Then we define the following subset L_j of $\tilde{M}_k^{(l)}(\text{co } K_j; \mathscr{A})$: an element $u \in \tilde{M}_k^{(l)}(\text{co } K_j; \mathscr{A})$ is said to belong to L_j if and only if there exists $\varphi \in \Phi_j$ such that for all $f \in M_k(\Omega; \mathscr{A})$

$$\int u(t)\, D^k(f\varphi)(t)\, dt^{m+1} = 0.$$

It is clear that L_j is a submodule of $\tilde{M}_k^{(l)}(\text{co } K_j; \mathscr{A})$ and that for any $u \in L_j$, its associated bounded right $\mathscr{A}$-linear functional is nothing else but the zero functional.

Moreover, if $u \in L_j$ then we have in fact that for each $\varphi^* \in \Phi_j$,

$$\int u(t)\, D^k(f\varphi^*)(t)\, dt^{m+1} = 0$$

and this for all $f \in M_k(\Omega; \mathscr{A})$.

The meaning of Definition 3.1 will become clear from

Proposition 3.3. *Let $j \in \mathbf{N}$ be fixed. Then $u \in L_j$ if and only if there exists $i_0 > j$ such that $u = 0$ in* co K_{i_0}.

Proof. Let $u \in \tilde{M}_k^{(l)}(\text{co } K_j; \mathscr{A})$ be zero outside some K_{i_0} ($i_0 > j$) and $\varphi \in \mathscr{D}_\infty(\Omega; \mathscr{A})$ be such that $\varphi \in \Phi_{i_0}$. Then obviously $\varphi \in \Phi_j$ and for any $f \in M_k(\Omega; \mathscr{A})$,

$$\int u(t)\, D^k(f\varphi)(t)\, dt^{m+1} = 0$$

which proves that $u \in L_j$.

Conversely, suppose that $u \in L_j$. Then for some $\varphi \in \Phi_j$

$$\int u(t)\, D^k(f\varphi)(t)\, dt^{m+1} = 0$$

and this for all $f \in M_k(\Omega; \mathscr{A})$.

Now let $i_0 > j$ be the least index such that $[\varphi] \subset \mathring{K}_{i_0}$. Then we prove that $u = 0$ in co K_{i_0}.

Indeed, let K be a compact subset of ω_φ such that $K_j \subset \mathring{K}$ and choose $\psi \in C_\infty(\Omega; \mathbf{R})$ such that $\psi(t) = 0$ on an open neighborhood ω_ψ of K_j contained in $\mathring{K}$ and $\psi(t) = 1$ in co K. Then $u\psi \in C_\infty(\mathbf{R}^{m+1}; \mathscr{A})$ and $u\psi(t) = u(t)$ in co K. Hence, if K_η is a suitable compact subset of ω_φ with $K \subset \mathring{K}_\eta$, we have that for all $f \in M_k(\Omega; \mathscr{A})$

$$\int u(t)\, D^k(f\varphi)(t)\, dt^{m+1} = \int_{[\varphi]\setminus K_\eta} (u\psi)(t)\, D^k(f\varphi)(t)\, dt^{m+1}$$

$$= 0.$$

Furthermore, in view of Runge's theorem (see [6, Lemma 3.3]), for any $f^* \in M_k(K_{i_0}; \mathscr{A})$, a sequence $(h_s)_{s\in\mathbf{N}}$ of left (k)-monogenic functions may be found, all of them having their singularities off K_{i_0}, such that $(h_s)_{s\in\mathbf{N}}$ converges uniformly on K_{i_0} to f^*. Hence, for each multiindex $\alpha \in \mathbf{N}^{m+1}$, $(\partial^\alpha h_s)_{s\in\mathbf{N}}$ converges uniformly on any compact subset $H \subset \mathring{K}_{i_0}$ to $\partial^\alpha f^*$. Thus we obtain that the relation

$$\int_{[\varphi]\setminus K_\eta} (u\psi)(t)\, D^k(f^*\varphi)(t)\, dt^{m+1} = 0$$

is valid for all $f^* \in M_k(K_{i_0}; \mathscr{A})$.

Now take $a \in$ co K_{i_0}; then clearly $E_k(t-a) \in M_k(K_{i_0}; \mathscr{A})$ so that

$$\int_{[\varphi]\setminus K_\eta} (u\psi)(t)\, D^k(E_k(t-a)\,\varphi(t))\, dt^{m+1} = 0.$$

Using Green's identity, we find that

$$(-1)^k \int_{[\varphi]\setminus K_\eta} (u\psi) D^k \cdot E_k(t-a)\,\varphi(t)\, dt^{m+1} = 0$$

or

$$(-1)^k \int_{[\varphi]\setminus K_\eta} (u\psi)\, D^k \cdot E_k(t-a)\, dt^{m+1} - (-1)^k \int_{[\varphi]\setminus K_\eta} (u\psi)\, D^k$$

$$E_k(t-a)(1-\varphi(t))\, dt^{m+1} = 0.$$

As in co K $(u\psi)\, D^k = uD^k = 0$ and $1 - \varphi(t) = 0$, the second term in the left member of the last equality vanishes. As to its first term, remark that $u\psi$ is a $C_\infty(\mathbf{R}^{m+1}; \mathscr{A})$-function which extends u so that, u being regular at infinity with respect to E_k,

$$\begin{aligned} 0 &= (-1)^k \int (u\psi)\, D^k \cdot E_k(t-a)\, dt^{m+1} \\ &= ((u\psi)\, D^k * E_k)(a) \\ &= (u\psi)(a) \\ &= u(a). \end{aligned}$$

Since $a \in \operatorname{co} K_{i_0}$ has been taken arbitrarily, we thus have proved that $u = 0$ in co K_{i_0}. ∎

It is clear that if u_1 and u_2, both belonging to $\tilde{M}_k^{(l)}(\operatorname{co} K_j ; \mathscr{A})$ for some fixed $j \in \mathbf{N}$, determine the same bounded $\mathscr{A}$-linear functional, then $u_1 - u_2 \in L_j$. Conversely, let T be a right $\mathscr{A}$-linear functional on $M_k(\Omega; \mathscr{A})$ bounded by p_{K_j} and let $t_k \in \tilde{M}_k^{(l)}(\operatorname{co} K_j ; \mathscr{A})$ be its associated indicatrix of Fantappié. Then in view of the foregoing considerations, any element of the form $t_k + h$, $h \in L_j$, determines the same T.

The aim of the following section is to explore this relationship more deeply.

4. The Dual of $M_k(\Omega; \mathscr{A})$

Let again Ω be an open non empty subset of $\mathbf{R}^{m+1}$, $(K_j)_{j=1}^\infty$ be the compact exhaustion of Ω and $\{p_{K_j} : j \in \mathbf{N}\}$ be its associated system of seminorms on $M_k(\Omega; \mathscr{A})$. Furthermore, let for $j \in \mathbf{N}$ fixed, L_j be the submodule of $\tilde{M}_k^{(l)}$ (co K_j ; $\mathscr{A}$) given in Definition 3.1. Then we assert that L_j is closed in $\tilde{M}_k^{(l)}$ (co K_j ; $\mathscr{A}$), the latter being provided with the topology of uniform compact convergence.

Indeed, suppose that for some sequence $(u_s)_{s\in\mathbf{N}}$ in L_j which converges to u in $\tilde{M}_k^{(l)}(\operatorname{co} K_j ; \mathscr{A})$, $u \notin L_j$. Take $\varphi \in \Phi_j$; then there ought to exist $f \in M_k(\Omega; \mathscr{A})$ such that

$$\int u(t)\, D^k(f\varphi)(t)\, dt^{m+1} \neq 0.$$

Since for each $s \in \mathbf{N}$,

$$\int u_s(t)\, D^k(f\varphi)(t)\, dt^{m+1} = 0,$$

where the integral is taken over the compact set $[\varphi]\backslash\omega \subset \text{co}\, K_j$, we obtain by a classical argument that

$$\int u(t)\, D^k(f\varphi)\, dt^{m+1} = 0,$$

clearly a contradiction.

For each $j \in \mathbf{N}$, we may thus consider the quotient left $\mathscr{A}$-module $\tilde{M}_k^{(l)}(\text{co}\, K_j\,;\, \mathscr{A})/L_j$, consisting of the elements $[u] = u + L_j$, $u \in \tilde{M}_k^{(l)}(\text{co}\, K_j\,;\, \mathscr{A})$. Equipped with the system of seminorms $\tilde{\mathscr{P}}_j = \{\tilde{p}_{\mathscr{K}} : \mathscr{K} \subset \text{co}\, K_j \text{ compact}\}$, where $\tilde{p}_{\mathscr{K}}([u]) = \inf_{h\in[u]} p_{\mathscr{K}}(h)$, $\tilde{M}_k^{(l)}(\text{co}\, K_j\,;\, \mathscr{A})/L_j$ becomes a left Fréchet $\mathscr{A}$-module.

For convenience we put $\mathscr{E}_j = \tilde{M}_k^{(l)}(\text{co}\, K_j\,;\, \mathscr{A})/L_j$, $j \in \mathbf{N}$. Now we claim that $\mathscr{E}_j$ may be continuously embedded in $\mathscr{E}_{j+1}$.

Clearly $L_j \subset L_{j+1}$, $j \in \mathbf{N}$. Furthermore, define $I_j : \mathscr{E}_j \to \mathscr{E}_{j+1}$ by $I_j([u]) = u + L_{j+1}$. Then obviously the definition of $I_j([u])$ is independent of the representative chosen. Moreover, I_j is injective.

Indeed, assume that $I_j([u]) = I_j([v])$. Then $u - v \in L_{j+1}$ so that, if $f \in M_k(\Omega; \mathscr{A})$ and $\varphi \in \Phi_{j+1}$,

$$\int (u - v) \cdot D^k(f\varphi)\, dt^{m+1} = 0.$$

But, as $u - v \in \tilde{M}_k^{(l)}(\text{co}\, K_j\,;\, \mathscr{A})$, this implies that $u - v \in L_j$ or $[u] = [v]$.

Finally I_j is continuous since for each $\mathscr{K} \subset \text{co}\, K_{j+1}$ compact,

$$\tilde{p}_{\mathscr{K}}(I_j([u])) = \inf_{h\in L_{j+1}} p_{\mathscr{K}}(u + h) \leqslant \inf_{h\in L_j} p_{\mathscr{K}}(u + h) = \tilde{p}_{\mathscr{K}}([u]).$$

Thus it makes sense to introduce

Definition 4.1. *Let Ω and $(K_j)_{j=1}^{\infty}$ be as described until now. Then we call $\tilde{M}_k^{(l)}(\text{co}\, \Omega; \mathscr{A}) = \mathscr{L}_j\mathscr{E}_j$ the inductive limit of the left $\mathscr{A}$-modules $\mathscr{E}_j$. Endowed with its inductive limit topology, we denote this left $\mathscr{A}$-module by $\tilde{M}_k^{(l)}(\text{co}\, \Omega; \mathscr{A})_{\text{ind}}$.*

Let us recall that, if $\mathscr{Q}$ denotes the proper system of seminorms defining the inductive limit topology on $\tilde{M}_k^{(l)}(\text{co}\, \Omega; \mathscr{A})$, an arbitrary element $Q \in \mathscr{Q}$ is given by (see [7])

$$Q([u]) = \inf_{[u]=\sum_{(j)}[u_j]} \sum_{(j)} c_j P_j([u_j]),$$

where for each $j \in \mathbf{N}$, $c_j > 0$ and $P_j \in \tilde{\mathscr{P}}_j$.

Now call $M_k(\Omega; \mathscr{A})^*$ the left $\mathscr{A}$-module consisting of all bounded right $\mathscr{A}$-linear functionals on $M_k(\Omega; \mathscr{A})$. Then our main objective is to show that

$M_k(\Omega; \mathscr{A})^*$, provided with the strong topology, is topologically isomorphic to $\tilde{M}_k^{(l)}(\text{co}\,\Omega; \mathscr{A})$, i.e. $M_k(\Omega; \mathscr{A})_b^* \simeq \tilde{M}_k^{(l)}(\text{co}\,\Omega; \mathscr{A})_{\text{ind}}$.

Let us repeat that an arbitrary seminorm p_B from the system defining the strong topology on $M_k(\Omega; \mathscr{A})^*$ is given by

$$p_B(T) = \sup_{f \in B} | T(f)|_0 ,$$

B being a bounded subset of $M_k(\Omega; \mathscr{A})$.

Combining the results of the foregoing section with the previous considerations, we get immediately

THEOREM 4.1. *Let $J : M_k(\Omega; \mathscr{A})^* \to \tilde{M}_k^{(l)}(\text{co}\,\Omega; \mathscr{A})$ be defined by $J(T) = [t_k]$, where t_k is the indicatrix of Fantappié associated to $T \in M_k(\Omega; \mathscr{A})^*$. Then J is an isomorphism between these left $\mathscr{A}$-modules.*

Note that J remains an isomorphism between $M_k(\Omega; \mathscr{A})^*$ and $\tilde{M}_k^{(l)}(\text{co}\,\Omega; \mathscr{A})$ when these spaces are considered as real vector spaces. This property will be used implicitly in the proof of Theorem 4.2 below.

Now we assert that J is a topological isomorphism. In order to prove this assertion we proceed as follows. First of all remark that $M_k(\Omega; \mathscr{A})$, considered as a real vector space, is a Schwartz space.

Indeed, as each component f_A of $f \in M_k(\Omega; \mathscr{A})$ satisfies $\Delta^k f_A = 0$ in Ω, $M_k(\Omega; \mathscr{A})$ is a subspace of $\prod_{A \in \mathscr{P}N} \text{Harm}_k(\Omega; \mathbf{R})$ where

$$\text{Harm}_k(\Omega; \mathbf{R}) = \{g \colon \Omega \to \mathbf{R} : \Delta^k g = 0 \text{ in } \Omega\}.$$

Equipped with the topology of uniform compact convergence, $\text{Harm}_k(\Omega; \mathbf{R})$ becomes a real Schwartz space and so does $\prod_{A \in \mathscr{P}N} \text{Harm}_k(\Omega; \mathbf{R})$. Since on $M_k(\Omega; \mathscr{A})$ the induced product topology is equivalent with the natural topology, we obtain that $M_k(\Omega; \mathscr{A})$ is a real Schwartz space. Consequently, $M_k(\Omega; \mathscr{A})$ being a Fréchet space too, we find that its real dual $M_k(\Omega; \mathscr{A})_b'$—that is the space of bounded real linear functionals on $M_k(\Omega; \mathscr{A})$, endowed with the strong topology—becomes a sequentially complete bornological space (see [7]).

As a second step, note that $M_k(\Omega; \mathscr{A})^*$ may always be provided with an inductive limit topology since $M_k(\Omega; \mathscr{A})^* = \mathscr{L}_j M_k(\Omega; \mathscr{A})^*_{\|\cdot\|_j}$, where for each $j \in \mathbf{N}$, $M_k(\Omega; \mathscr{A})^*_{\|\cdot\|_j}$ is the left Banach $\mathscr{A}$-module of right $\mathscr{A}$-linear functionals T on $M_k(\Omega; \mathscr{A})$ which are bounded by p_{K_j}.

Let us repeat that

$$\| T \|_j = \sup_{\{f : p_{K_j}(f) \leqslant 1\}} | T(f)|_0 .$$

Denote this inductive limit space by $M_k(\Omega; \mathscr{A})^*_{\text{ind}}$.

As is well known, the strong topology is weaker than the inductive limit topology on $M_k(\Omega; \mathscr{A})^*$. In the following proposition, we shall prove that these two topologies are in fact equivalent.

PROPOSITION 4.1. *If considered as real convex spaces, we have that*

(i) $M_k(\Omega, \mathscr{A})^*_{\text{ind}} \simeq M_k(\Omega; \mathscr{A})'_b$

(ii) $M_k(\Omega, \mathscr{A})^*_b \simeq M_k(\Omega; \mathscr{A})'_b$.

Hence

(iii) $M_k(\Omega, \mathscr{A})^*_b \simeq M_k(\Omega; \mathscr{A})^*_{\text{ind}}$.

Proof. To prove these assertions, let us first recall that the function $\theta : M_k(\Omega; \mathscr{A})^* \to M_k(\Omega; \mathscr{A})'$ defined by $\theta(T) = \tau_{e_0} T$, $T \in M_k(\Omega; \mathscr{A})^*$, is an isomorphism between these real vector spaces.

Here $\tau_{e_0} T(f) = 2^n [Tf]_0$, $[Tf]_0$ being the e_0-component of $Tf \in \mathscr{A}$ and this for any $f \in M_k(\Omega; \mathscr{A})$. Moreover, if $\mathscr{T} \in M_k(\Omega; \mathscr{A})'$, then $\theta^{-1}(\mathscr{T}) = T^*$ is completely determined by (see [4])

$$T^*(f) = 2^{-n} \sum_A e_A \mathscr{T}(f\bar{e}_A), \qquad f \in M_k(\Omega; \mathscr{A}).$$

(i) We show that θ is a topological isomorphism between $M_k(\Omega; \mathscr{A})^*_{\text{ind}}$ and $M_k(\Omega; \mathscr{A})'_b$.

Indeed, let B be a bounded subset of $M_k(\Omega; \mathscr{A})$. Then for each $T \in M_k(\Omega; \mathscr{A})^*$,

$$p_B(\theta(T)) = \sup_{f \in B} | \tau_{e_0} T(f)| \leqslant | e_0 |_0 \sup_{f \in B} | Tf |_0 = | e_0 |_0 \, p_B(T).$$

As there exist $C > 0$ and q such that $p_B(T) \leqslant Cq(T)$ for all $T \in M_k(\Omega; \mathscr{A})^*$, q being a seminorm from the system which defines the inductive limit topology, we obtain that

$$p_B(\theta(T)) \leqslant C \, | e_0 |_0 \, q(T), \qquad T \in M_k(\Omega; \mathscr{A})^*.$$

Consequently θ is continuous.

Since $M_k(\Omega; \mathscr{A})^*_{\text{ind}}$ is an inductive limit of real Banach spaces and $M_k(\Omega; \mathscr{A})'_b$ is a real sequentially complete bornological space, a classical corollary to the closed graph theorem yields that θ^{-1} is continuous.

(ii) We prove that θ is a topological isomorphism between $M_k(\Omega; \mathscr{A})^*_b$ and $M_k(\Omega; \mathscr{A})'_b$.

Indeed, from the first part in the proof of (i) it follows already that θ is continuous.

Now let B be a bounded subset of $M_k(\Omega; \mathscr{A})$, $\mathscr{T}$ be an arbitrary element of $M_k(\Omega; \mathscr{A})'$ and $T^* = \theta^{-1}\mathscr{T}$. Then, since $|e_A|_0 = 2^{n/2}$ for each $A \in \mathscr{P}N$, we find that for all $f \in M_k(\Omega; \mathscr{A})$

$$\begin{aligned} |T^*f|_0 &\leqslant 2^{-n} \sum_A |e_A|_0 \, |\mathscr{T}(f\bar{e}_A)| \\ &= 2^{-n/2} \sum_A |\mathscr{T}(f\bar{e}_A)|. \end{aligned}$$

Put $B_A = B\bar{e}_A$, $A \in \mathscr{P}N$, and call $B^* = \bigcup_{A\in\mathscr{P}N} B_A$. Then B^* is a bounded subset of $M_k(\Omega; \mathscr{A})$ and

$$\sup_{f\in B} |T^*f|_0 \leqslant 2^{n/2} \sup_{g\in B^*} |\mathscr{T}(g)|,$$

which implies that θ^{-1} is continuous.

(iii) of course follows immediately from (i) and (ii). ∎

Finally, we arrive at

THEOREM 4.2. *Let $J : M_k(\Omega; \mathscr{A})^* \to \tilde{M}_k^{(l)}(\text{co}\,\Omega; \mathscr{A})$ be defined as in Theorem 4.1. Then J is a topological isomorphism between $M_k(\Omega; \mathscr{A})_b^*$ and $\tilde{M}_k^{(l)}(\text{co}\,\Omega; \mathscr{A})$.*

Proof. From Theorem 4.1 we know that J is an isomorphism. To prove the continuity of J, by Proposition 4.1(iii) it suffices to show that J restricted to each $M_k(\Omega; \mathscr{A})^*_{\|\cdot\|_j}$ is continuous.

Let $\mathscr{Q}$ be an arbitrary element of $\mathscr{Q}$. Then for each $T \in M_k(\Omega; \mathscr{A})^*_{\|\cdot\|_j}$, we have that

$$\begin{aligned} Q(J(T)) &\leqslant c_j P_j(J(T)) \\ &= c_j \inf_{h\in[t_k]} (\sup_{t\in\mathscr{K}} |h(t)|_0) \\ &\leqslant c_j \sup_{y\in\mathscr{K}} |t_k(y)|_0 \\ &\leqslant c_j \sup_{y\in\mathscr{K}} \left| \int_{[\mu]} d\mu(x)\, E_k(y-x) \right|_0 \\ &\leqslant c_j C(K_j) \sup_{w\in K_j-\mathscr{K}} |E_k(w)|_0 \\ &\leqslant C(K_j; \mathscr{K}). \end{aligned}$$

Here $\mathscr{K}$ is the compact subset of co K_j to which the seminorm P_j on $\mathscr{E}_j$ is associated and $C(K_j; \mathscr{K})$ is a positive constant depending on Q.

Hence, for each $T \in M_k(\Omega; \mathscr{A})^*_{\|\cdot\|_j}$

$$\varrho\left(J\left(\frac{T}{\| T \|_j}\right)\right) \leqslant C(K_j; \mathscr{K})$$

or

$$\varrho(J(T)) \leqslant C(K_j; \mathscr{K}) \| T \|_j .$$

Of course this implies that J is continuous.

As $M_k(\Omega; \mathscr{A})^*_{\text{ind}}$ and $\tilde{M}_k^{(l)}(\text{co}\,\Omega; \mathscr{A})_{\text{ind}}$ are both inductive limits of Fréchet spaces, a classical corollary to the closed graph theorem implies that J^{-1} is continuous. ∎

5. The Dual of $\tilde{M}_k^{(l)}(\text{co}\,\Omega; \mathscr{A})$

The main objective of this section is to characterize $\tilde{M}_k^{(l)}(\text{co}\,\Omega; \mathscr{A})^*$, the right $\mathscr{A}$-module consisting of all bounded left $\mathscr{A}$-linear functionals on $\tilde{M}_k^{(l)}$ $(\text{co}\,\Omega; \mathscr{A})_{\text{ind}}$.

We claim that $\tilde{M}_k^{(l)}(\text{co}\,\Omega; \mathscr{A})_b^*$ is topologically isomorphic to $M_k(\Omega; \mathscr{A})$. To this end we proceed as follows. Associate to each $f \in M_k(\Omega; \mathscr{A})$ the $\mathscr{A}$-functional $\mathbf{T}_f : \tilde{M}_k^{(l)}(\text{co}\,\Omega; \mathscr{A}) \to \mathscr{A}$ defined by

$$\mathbf{T}_f([t_k]) = T(f) = \int t_k(t)\, D^k(f\varphi)(t)\, dt^{m+1} \tag{5.1}$$

where $T = J^{-1}([t_k])$, J being the topological isomorphism given in Theorem 4.2.

Recall that the integral representation in (5.1) is independent of the representative chosen in $[t_k]$ and of $\varphi \in \Phi_j$.

Furthermore, note that for f and φ fixed, there exists a positive constant C_j such that

$$| T(f)|_0 \leqslant C_j \sup_{t \in [\varphi] \setminus \omega_\varphi} | t_k(t)|_0 .$$

Obviously $\mathbf{T}_f$ is a left $\mathscr{A}$-linear functional on $\tilde{M}_k^{(l)}(\text{co}\,\Omega; \mathscr{A})$. Moreover $\mathbf{T}_f$ is bounded.

Indeed, let $[t_k] \in \tilde{M}_k^{(l)}(\text{co}\,\Omega; \mathscr{A})$ and let $[t_k] = \sum_{(j)} [t_k]_j$ be an arbitrary decomposition of $[t_k]$, $[t_k]_j \in \mathscr{E}_j$. Taking $u_j \in [t_k]_j$ and putting $u = \sum_{(j)} u_j$ and $T_j = J^{-1}([t_k]_j)$, we obtain that

$$\begin{aligned} | \mathbf{T}_f([t_k])|_0 &= \left| \mathbf{T}_f\left(\sum_{(j)} [t_k]_j\right)\right|_0 \\ &\leqslant \sum_{(j)} | \mathbf{T}_f([t_k]_j)|_0 \\ &= \sum_{(j)} | T_j(f)|_0 . \end{aligned}$$

If for each j occurring in the above decomposition, $\varphi_j \in \Phi_j$ is taken arbitrarily but fixed, we have that there exist positive constants C_j such that

$$| T_j(f)|_0 \leqslant C_j \sup_{t \in [\varphi_j] \setminus \omega_{\varphi_j}} | u_j(t)|_0 ,$$

whence, putting $\mathscr{K}_j = [\varphi_j] \setminus \omega_{\varphi_j}$,

$$| T_j(f)|_0 \leqslant C_j \inf_{u_j \in [t_k]_j} \sup_{t \in \mathscr{K}_j} | u_j(t)|_0 .$$

Consequently

$$| \mathbf{T}_f([t_k])|_0 \leqslant \sum_{(j)} C_j \inf_{u_j \in [t_k]_j} \sup_{t \in \mathscr{K}_j} | u_j(t)|_0$$

so that, since the decomposition of $[t_k]$ has been taken arbitrarily,

$$| \mathbf{T}_f([t_k])|_0 \leqslant \inf_{[t_k] = \Sigma_{(j)} [t_k]_j} \sum_{(j)} C_j P_j([t_k]_j),$$

where for each j, P_j is a seminorm on $\tilde{M}_k^{(l)}(\text{co } K_j ; \mathscr{A})$ determined by

$$P_j([t_k]_j) = \inf_{u_j \in [t_k]_j} \sup_{t \in \mathscr{K}_j} | u_j(t)|_0 .$$

Hence $\mathbf{T}_f \in \tilde{M}_k^{(l)}(\text{co } \Omega; \mathscr{A})^*$.

THEOREM 5.1. *Let $I : M_k(\Omega; \mathscr{A}) \to \tilde{M}_k^{(l)}(\Omega; \mathscr{A})_b^*$ be defined by $I(f) = \mathbf{T}_f$, $\mathbf{T}_f$ being given by* (5.1). *Then I is a topological isomorphism between these right $\mathscr{A}$-modules.*

Proof. Obviously I is right $\mathscr{A}$-linear. Now we show that I is injective or equivalently, that from $T(f) = 0$ for all $T \in M_k(\Omega; \mathscr{A})^*$ it follows that $f = 0$. To this end, choose an arbitrary $a \in \Omega$, take the least index $j \in \mathbf{N}$ such that $a \in K_j$ and consider the function $E_k(a - x)$, $x \in \text{co } K_j$. Then $E_k(a - x) \in \tilde{M}_k^{(l)}(\text{co } K_j ; \mathscr{A})$.

Since we assume that $T(f) = 0$ for all $T \in M_k(\Omega; \mathscr{A})^*$, we have that for $\varphi \in \Phi_j$ fixed

$$\int E_k(a - x) \, D^k(f\varphi)(x) \, dx^{m+1} = 0.$$

Hence, if K_η is a suitable compact neighborhood of K_j such that $K_\eta \subset \omega_\varphi$, we get by Green's identity that

$$\int_{\partial K_\eta} \sum_{j=0}^{k-1} (-1)^{j+k} E_k(a - x) \, D_x^{k-1-j} \, d\sigma_x D^j f(x) = 0$$

or

$$\int_{\partial K_\eta} \sum_{j=0}^{k-1} (-1)^j E_{j+1}(x-a)\, d\sigma_x D^j f(x) = 0.$$

But, in view of Cauchy's formula (see [4]), the left hand side equals $f(a)$ so that $f(a) = 0$. Since $a \in \Omega$ has been taken arbitrarily, $f = 0$ in Ω.

Next we prove that I is surjective.

Let $\mathbf{T} \in M_k^{(l)}(\mathrm{co}\,\Omega; \mathscr{A})^*$ be given. Then we show that there exists $f \in M_k(\Omega; \mathscr{A})$ such that $\mathbf{T} = \mathbf{T}_f$.

For each $a \in \Omega$, call again $j \in \mathbf{N}$ the least index such that $a \in K_j$ and put

$$f(a) = \mathbf{T}([E_k(a-x)]).$$

Then clearly f is well defined in Ω. Moreover, using classical arguments, we may easily check that $f \in C_\infty(\Omega; \mathscr{A})$ and that for each multiindex $\alpha \in \mathbf{N}^{m+1}$

$$\partial^\alpha f(a) = \mathbf{T}(\partial^\alpha[E_k(a-x)]) = \mathbf{T}([\partial^\alpha E_k(a-x)]).$$

Hence by virtue of the left $\mathscr{A}$-linearity of $\mathbf{T}$

$$\begin{aligned} D^k f(a) &= \mathbf{T}([D_a{}^k(E_k(a-x)]) \\ &= 0 \end{aligned}$$

which implies that $f \in M_k(\Omega; \mathscr{A})$.

Now we prove that $\mathbf{T} = \mathbf{T}_f$.

Note that in any case

$$(\mathbf{T}_f - \mathbf{T})([E_k(a-x]) = 0. \tag{5.2}$$

Take $j \in \mathbf{N}$ fixed and consider the restriction of $\mathbf{T}_f - \mathbf{T}$ to $\mathscr{E}_j$. If $[u] \in \mathscr{E}_j$ and $h \in [u]$, by Runge's theorem there exists a sequence $(h_r)_{r\in\mathbf{N}}$ of functions belonging to $\tilde{M}_k^{(l)}(\mathrm{co}\,K_j, \mathscr{A})$ with singularities in K_j, each of the form

$$h_r(x) = \sum_{(i)} \sum_{(l_1,\ldots,l_p)} \sum_{(s)} \frac{\partial^p \partial^s E_k(x-a_i)}{\partial x_{l_1} \cdots \partial x_{l_p}} \lambda_{l_1\cdots l_p}^{(s)(i)},$$

where $a_i \in K_j$ for each i, such that $(h_r)_{r\in\mathbf{N}}$ converges to h in $\tilde{M}_k^{(l)}(\mathrm{co}\,K_j; \mathscr{A})$.

From (5.2) it then follows that for any $r \in \mathbf{N}$, $(\mathbf{T}_f - \mathbf{T})([h_r]) = 0$ so that also $(\mathbf{T}_f - \mathbf{T})([u]) = 0$. Hence $\mathbf{T}_f - \mathbf{T} = 0$ on each $\mathscr{E}_j$ and thus $\mathbf{T}_f - \mathbf{T} = 0$ on $\tilde{M}_k^{(l)}(\mathrm{co}\,\Omega; \mathscr{A})$.

As a final step, we show that I is bicontinuous. On the one side, let $\mathscr{B}$ be bounded in $\tilde{M}_k^{(l)}(\mathrm{co}\,\Omega; \mathscr{A})_{\mathrm{ind}}$ and put $\mathscr{B}^* = J^{-1}\mathscr{B}$. Then $\mathscr{B}^*$ is bounded in $M_k(\Omega; \mathscr{A})_b^*$. Since $M_k(\Omega; \mathscr{A})$ is bornological, $\mathscr{B}^*$ is equicontinuous and hence contained in the polar of a semiball, say $\mathscr{B}^* \subset b_{p_{K_j}}^{\Delta}(r)$ (see [7]).

Consequently, if $f \in M_k(\Omega; \mathscr{A})$ and $I(f) = \mathbf{T}_f$,

$$\begin{aligned} p_{\mathscr{B}}(\mathbf{T}_f) &= \sup_{[u]\in\mathscr{B}} | \mathbf{T}_f([u])|_0 \\ &= \sup_{T\in\mathscr{B}^*} | T(f)|_0 \\ &\leqslant \sup_{T\in b^{\Delta}_{p_{K_j}}(r)} | T(f)|_0 \\ &\leqslant \frac{1}{r} p_{K_j}(f) \end{aligned}$$

or I is continuous.

To prove that I^{-1} is continuous, remark first that the real convex space $M_k(\Omega; \mathscr{A})$ is evaluable and has representable seminorms. Hence its natural system of seminorms is equivalent with the system $\{\pi_{\mathscr{B}} : \mathscr{B}$ bounded in $M_k(\Omega; \mathscr{A})'_b\}$, where for each $\mathscr{B}$, $\pi_{\mathscr{B}}(f) = \sup_{\mathscr{T}\in\mathscr{B}} | \mathscr{T}(f)|$, $f \in M_k(\Omega; \mathscr{A})$ (see [7]).

Consequently, for any natural seminorm p_{K_j} on $M_k(\Omega; \mathscr{A})$, there exist $C > 0$ and $\mathscr{B}$ b-bounded in $M_k(\Omega; \mathscr{A})'$ such that for all $f \in M_k(\Omega; \mathscr{A})$, $p_{K_j}(f) \leqslant C\pi_{\mathscr{B}}(f)$.

Putting $\theta^{-1}\mathscr{B} = \mathscr{B}^*$, where θ is the topological isomorphism from Proposition 4.1 (iii), we obtain that

$$\begin{aligned} p_{K_j}(I^{-1}(\mathbf{T})) &= p_{K_j}(f) \\ &\leqslant C\pi_{\mathscr{B}}(f) \\ &= C \sup_{T\in\mathscr{B}^*} | T(f)|_0 \\ &= C \sup_{[t_k]\in J(\mathscr{B}^*)} | \mathbf{T}([t_k])|_0 \\ &= C p_{J(\mathscr{B}^*)}(\mathbf{T}), \end{aligned}$$

where J is the topological isomorphism given in Theorem 4.2. This proves the continuity of I^{-1}. ∎

REFERENCES

1. L. A. AIZENBERG, The general form of a linear continuous functional in spaces of functions that are holomorphic in convex domains of C^n, *Soviet Math.* **7** (1966), 196–202.
2. R. BRAUN, "Cauchy-Fantappié Formeln und Dualität in der Funktionentheorie," Dissertation, Universität Göttingen, 1972.
3. P. CHAUVEHEID, "Application de l'analyse fonctionnelle à la théorie des fonctions analytiques," Thèse, Université de Liège, 1973.
4. R. DELANGHE AND F. BRACKX, Hypercomplex function theory and Hilbert modules with reproducing kernel, *Proc. London Math. Soc.* (3) **37** (1978), 545–576.

5. R. Delanghe and F. Brackx, Regular solutions at infinity of a generalized Cauchy-Riemann operator, *Simon Stevin* **53** (1979), 13–30.
6. R. Delanghe and F. Brackx, Runge's theorem in hypercomplex function theory, *J. Appr. Theory*, in press.
7. H. G. Garnir, M. De Wilde, and J. Schmets, "Analyse fonctionnelle," T. I., Birkhäuser, Basel, 1968.
8. A. Grothendieck, Sur certains espaces de fonctions holomorphes, I, II, *J. Reine Angew. Math.* **192** (1953), 35–64, 77–95.
9. G. Köthe, Dualität in der Funktionentheorie, *J. Reine Angew. Math.* **191** (1953), 30–49.
10. P. Lelong, "Fonctionnelles analytiques et fonctions entières (n variables)," Les Presses de l'Université de Montréal, 1968.
11. L. A. Rubel and B. A. Taylor, Functional analysis proofs of some theorems in function theory, *Amer. Math. Monthly* **76** (1969), 483–488; Corrections, *Amer. Math. Monthly* **77** (1970), 58.
12. H. G. Tillmann, Dualität in der Potentialtheorie, *Port. Math.* **13** (1954), 55–86.
13. H. G. Tillmann, Dualität in der Funktionentheorie auf Riemannschen Flächen, *J. Reine Angew. Math.* **195** (1956), 76–101.