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Higher secant varieties of the Segre varieties $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$

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Dedicated to Prof. Wolmer Vasconcelos on the occasion of his 65th birthday

Abstract

Let $V_t = \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ (t-copies) embedded in $\mathbb{P}^N$ ($N = 2^t - 1$) via the Segre embedding. Let $(V_t)^s$ be the subvariety of $\mathbb{P}^N$ which is the closure of the union of all the secant $\mathbb{P}^{s-1}$'s to V_t. The *expected dimension* of $(V_t)^s$ is $\min\{st + (s-1), N\}$.

This is not the case for $(V_4)^3$, which we conjecture is the only defective example in this infinite family. We prove (Theorem 2.3): if $e_t = [\frac{2^t}{t+1}] \equiv \delta_t (\mathrm{mod}\, 2)$ and $s_t = e_t - \delta_t$ then $(V_t)^s$ has the expected dimension, except possibly when $s = s_t + 1$. Moreover, whenever $t = 2^k - 1$, $(V_t)^s$ has the expected dimension for every s.

MSC: 14M99; 14M12; 14A05

0. Introduction

The problem of determining the dimensions of the higher secant varieties of the classically studied projective varieties (and to describe the defective ones) is a problem with a long and interesting history (see e.g. [7,8,11,12,14]).

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In the case of the Segre varieties there is much interest in this question, and not only among geometers. In fact, this particular problem is strongly connected to questions in representation theory, coding theory, algebraic complexity theory (see our paper [6] for some recent results as well as a summary of known results, and also [2]) and, surprisingly enough, also in algebraic statistics (e.g. see [9,10]).

We address this problem here; more precisely we will study the higher secant varieties of the product of t-copies of $\mathbb{P}^1$, i.e. of

$$\mathbb{P}^{\mathbf{n}} = \mathbb{P}^1 \times \cdots \times \mathbb{P}^1, \quad \mathbf{n} = (1, \ldots, 1)$$

embedded in the projective space $\mathbb{P}^N (N = 2^t - 1)$ by the complete linear system $\mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})$, where $\mathbf{a} = (a_1, \ldots, a_t) = (1, 1, \ldots, 1)$. We denote this embedding of $\mathbb{P}^{\mathbf{n}}$ by V_t.

In Section 1 we recall some classical results by Terracini regarding such secant varieties and we also introduce one of the fundamental observations (Proposition 1.3) which allows us to convert certain questions about ideals of varieties in multiprojective space to questions about ideals in standard polynomial rings. In this section we also recall some lemmata which are extremely useful in dealing with the postulation of non-reduced zero-dimensional schemes in projective space.

In Section 2 we give our main theorem (Theorem 2.3). We first remark on the cases $t = 2, 3, 4$ separately (showing that for $t = 4$ there is a defective secant variety). Finally, if $t > 4$ and $t = 2^k - 1$ we recall (see [6]) that all the higher secant varieties of V_t have the expected dimension. If $t > 4$ and $t \neq 2^k - 1$, we again show that all the higher secant varieties have the expected dimension—except, possibly, for one higher secant variety for each such t.

Our method is essentially this (see Section 1): we use Terracini's lemma (as in [5,6]) to translate the problem of determining the dimensions of higher secant varieties into that of calculating the value, at $(1, \ldots, 1)$, of the Hilbert function of generic sets of 2-fat points in $\mathbb{P}^{\mathbf{n}}$. Then we show, by passing to an affine chart in $\mathbb{P}^{\mathbf{n}}$ and then homogenizing in order to pass to $\mathbb{P}^t$, that this last calculation amounts to computing the Hilbert function of a very particular subscheme of $\mathbb{P}^t$. Finally, we study the postulation of these special subschemes of $\mathbb{P}^t$ (mainly) by using the "differential Horace method" introduced by Alexander and Hirschowitz [1].

1. Preliminaries, the multiprojective-affine-projective method

Let us recall the notion of higher secant varieties.

Defintion 1.1. Let $X \subseteq \mathbb{P}^N$ be a closed irreducible projective variety of dimension n. The s^{th} *higher secant variety* of X, denoted X^s, is the closure of the union of all linear spaces spanned by s independent points of X.

Recall that, for X as above, there is an inequality involving the dimension of X^s. Namely,

$$\dim\ X^s \leqslant \min\{N, sn + s - 1\}$$

and one "expects" the inequality should, in general, be an equality.

When X^s does not have the expected dimension, X is said to be $(s-1)$-*defective*, and the positive integer

$$\delta_{s-1}(X) = \min\{N, sn+s-1\} - \dim\ X^s$$

is called the $(s-1)$-*defect* of X. Probably the most well-known defective variety is the Veronese surface, X, in $\mathbb{P}^5$ for which $\delta_1(X) = 1$.

A classical result about higher secant varieties is Terracini's lemma (see [13,6]):

Terracini's lemma. *Let X be an integral scheme embedded in* $\mathbb{P}^N$. *Then*

$$T_P(X^s) = \langle T_{P_1}(X), \ldots, T_{P_s}(X)\rangle,$$

where $P_1, \ldots, P_s$ *are s generic points on X, and P is a generic point of* $\langle P_1, \ldots, P_s\rangle$ *(the linear span of* $P_1, \ldots, P_s$ *); here* $T_{P_i}(X)$ *is the projectivized tangent space of X in* $\mathbb{P}^N$.

Let $Z \subset X$ be a scheme of s generic *2-fat points*, that is a scheme defined by the ideal sheaf $\mathscr{I}_Z = \mathscr{I}^2_{P_1} \cap \cdots \cap \mathscr{I}^2_{P_s} \subset \mathscr{O}_X$, where $P_1, \ldots, P_s$ are s generic points. Since there is a bijection between hyperplanes of the space $\mathbb{P}^N$ containing the subspace $\langle T_{P_1}(X), \ldots, T_{P_s}(X)\rangle$ and the elements of $H^0(X, \mathscr{I}_Z(1))$, we have:

Corollary 1.2. *Let X and Z be as above*; *then*

$$\dim\ X^s = \dim\langle T_{P_1}(X), \ldots, T_{P_s}(X)\rangle = N - \dim\ H^0(X, \mathscr{I}_Z(1)).$$

Now, let $X = \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ (t-times) and let $V_t \subset \mathbb{P}^N (N = 2^t - 1)$ be the embedding of X given by $\mathscr{O}_X(1, \ldots, 1)$. By applying the corollary above to our case we get

$$\dim\ V_t^s = H(Z, (1, 1, \ldots, 1)) - 1,$$

where $Z \subset \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ is a set of s generic 2-fat points, and where $\forall \mathbf{j} \in \mathbb{N}^t$, $H(Z, \mathbf{j})$ is the Hilbert function of Z, i.e.

$$H(Z, \mathbf{j}) = \dim\ R_{\mathbf{j}} - \dim\ H^0(\mathbb{P}^1 \times \cdots \times \mathbb{P}^1, \mathscr{I}_Z(\mathbf{j})),$$

where $R = k[x_{0,1}, x_{1,1}, \ldots, x_{0,t}, x_{1,t}]$ is the multi-graded homogeneous coordinate ring of $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$.

Now consider the birational map

$$g : \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 - -- \to \mathbb{A}^t,$$

where

$$((x_{0,1}, x_{1,1}), \ldots, (x_{0,t}, x_{1,t})) \longmapsto \left(\frac{x_{1,1}}{x_{0,1}}, \frac{x_{1,2}}{x_{0,2}}, \ldots, \frac{x_{1,t}}{x_{0,t}}\right).$$

This map is defined in the open subset of $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ given by $\{x_{0,1}x_{0,2} \ldots x_{0,t} \neq 0\}$.

Let $S=k[z_0, z_{1,1}, z_{1,2}, \dots, z_{1,t}]$ be the coordinate ring of $\mathbb{P}^t$ and consider the embedding $\mathbb{A}^t \to \mathbb{P}^t$ whose image is the chart $\mathbb{A}^t_0 = \{z_0 \neq 0\}$. By composing the two maps above we get

$$f : \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 - - - \to \mathbb{P}^t,$$

with

$$\begin{aligned} &((x_{0,1}, x_{1,1}), \dots, (x_{0,t}, x_{1,t})) \\ &\quad \longmapsto \left(1, \frac{x_{1,1}}{x_{0,1}}, \frac{x_{1,2}}{x_{0,2}}, \dots, \frac{x_{1,t}}{x_{0,t}}\right) \\ &\quad = (x_{0,1}x_{0,2}\cdots x_{0,t}, x_{1,1}x_{0,2}\cdots x_{0,t}, \dots, x_{0,1}\cdots x_{0,t-1}x_{1,t}). \end{aligned}$$

Let $Z \subset \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ be a zero-dimensional scheme which is contained in the affine chart $\{x_{0,1}x_{0,2}\dots x_{0,t} \neq 0\}$ and let $Z' = f(Z)$. We want to construct a scheme $W \subset \mathbb{P}^t$ such that $\dim(I_W)_t = \dim(I_Z)_{(1,\dots,1)}$.

Let $Q_0, Q_1, Q_2, \dots, Q_t$ be the coordinate points of $\mathbb{P}^t$. The defining ideal of Q_i, $(1 \leqslant i \leqslant t)$, is

$$I_{Q_i} = (z_0, z_{1,1}, \dots, \widehat{z_{1,i}}, \dots, z_{1,t}).$$

Let W_i be the subscheme of $\mathbb{P}^t$ denoted by $(t-1)Q_i$, i.e. the scheme defined by the ideal $I_{Q_i}^{t-1}$.

Proposition 1.3. *Let Z, Z' be as above and let $W = Z' + W_1 + \cdots + W_t \subset \mathbb{P}^n$. Then we have*

$$\dim(I_W)_t = \dim(I_Z)_{(1,\dots,1)}.$$

Proof. First note that

$$R_{(1,\dots,1)} = \langle (x_{0,1}^{1-s_1} N_1)(x_{0,2}^{1-s_2} N_2) \cdots (x_{0,t}^{1-s_t} N_t) \rangle,$$

where the $N_i = x_{1,i}^{s_i}$ and either $s_i = 0$ or 1.

By dehomogenizing (via f above) and then substituting $z_{i,j}$ for $(x_{i,j}/x_{0,j})$, and finally homogenizing with respect to z_0, we see that

$$R_{(1,\dots,1)} \simeq \langle z_0^{t-s_1-\cdots-s_t} M_1 M_2 \dots M_t \rangle,$$

where $M_i = z_{1,i}^{s_i}$ and either $s_i = 0$ or 1.

Claim. $(I_{(W_1+\cdots+W_t)})_t = (I_{W_1} \cap \cdots \cap I_{W_t})_t = \langle z_0^{t-s_1-\cdots-s_t} M_1 \dots M_t \rangle$, where $M_i = z_{1,i}^{s_i}$ and either $s_i = 0$ or 1.

Proof $(\subseteq)$. Since both vector spaces are generated by monomials, it is enough to show that the monomials of the left-hand side of the equality are contained in the right-hand side of the equality.

Consider $M = z_0^{t-s_1-\cdots-s_t} M_1 M_2 \ldots M_t$ (as above). We now show that this monomial is in I_{W_i} (for each i). Notice that $M_j \in I_{Q_i}^{s_i}$ (for $j \neq i$) and that $z_0^{t-s_1-\cdots-s_t} \in I_{Q_i}^{t-s_1-\cdots-s_t}$. Thus, $M \in I_{Q_i}^{(t-s_1-\cdots-s_t)+(s_1+\cdots+\hat{s}_i+\cdots+s_t)} = I_{Q_i}^{t-s_i}$. Since $s_i \leqslant 1$ we have $t-1 \leqslant t - s_i$ and so $M \in I_{Q_i}^{t-1}$ as well, and that is what we wanted to show.

($\supseteq$). To prove this inclusion, consider an arbitrary monomial $M \in S_t$. Such an M can be written $M = z_0^{\alpha_0} M_1 \cdots M_t$ where $M_i = z_{1,i}^{\alpha_i}$.

Now, $M \in (I_{(W_1+\cdots+W_t)})_t$ means $M \in (I_{W_i})_t$ for each i, hence

$$\alpha_0 + \alpha_1 + \cdots + \hat{\alpha}_i + \cdots + \alpha_t \geqslant t-1$$

for $i = 1, \ldots, t$.

Since

$$\alpha_0 + \alpha_1 + \cdots + \alpha_t = t$$

then $t - \alpha_i \geqslant t-1$ for each such i and so $\alpha_i \leqslant 1$ for each i. That finishes the proof of the claim. □

Now, since Z and Z' are isomorphic (f is an isomorphism between the two affine charts $\{z_0 \neq 0\}$ and $\{x_{0,1}x_{0,2}\ldots x_{0,t} \neq 0\}$), it immediately follows (via the two different dehomogeneizations) that $(I_Z)_{(1,\ldots,1)} \cong (I_W)_t$. □

When Z is given by s generic 2-fat points, we have the obvious corollary:

Corollary 1.4. *Let $Z \subset \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ be a generic set of s 2-fat points and let $W \subset \mathbb{P}^t$ be as in Proposition 1.3, i.e. $W = 2P_1 + \cdots + 2P_s + (t-1)Q_1 + \cdots + (t-1)Q_t$. Then we have*

$$\dim V_t^s = H(Z, (1, \ldots, 1)) - 1 = (2^t - 1) - \dim(I_W)_t.$$

Now we give some preliminary lemmata and observations (for notation and proofs we refer the reader to [1, Section 2 and Corollary 9.3]).

Lemma 1.5 (*Castelnuovo's inequality*). *Let $\mathcal{D} \subseteq \mathbb{P}^n$ be a smooth hypersurface of degree d, and let $Z \subseteq \mathbb{P}^n$ be a zero-dimensional scheme. The scheme Z' defined by the ideal $(I_Z : I_{\mathcal{D}})$ is called the residual of Z with respect to $\mathcal{D}$, and denoted by $Res_{\mathcal{D}} Z$; the schematic intersection $Z'' = Z \cap \mathcal{D}$ is called the trace of Z on $\mathcal{D}$, and denoted by $Tr_{\mathcal{D}} Z$. Then for $t \geqslant d$*

$$\dim(I_{Z,\mathbb{P}^n})_t \leqslant \dim(I_{Z',\mathbb{P}^n})_{t-d} + \dim(I_{Z'',\mathcal{D}})_t.$$

Lemma 1.6 (*Horace's differential lemma*). *Let $H \subseteq \mathbb{P}^n$ be a hyperplane, and let $P_1, \ldots, P_r$ be generic points in $\mathbb{P}^n$. Let $Z = \check{Z} + 2P_1 + \cdots + 2P_r \subseteq \mathbb{P}^n$ be a (zero-dimensional) scheme, let $\check{Z}' = Res_H \check{Z}$, and $\check{Z}'' = Tr_H \check{Z}$. Let $P_1', \ldots, P_r'$ be generic points in H. Let $D_{2,H}(P_i') = 2P_i' \cap H$, and $Z' = \check{Z}' + D_{2,H}(P_1') + \cdots + D_{2,H}(P_r')$ (Z' a subscheme of $\mathbb{P}^n$), $Z'' = \check{Z}'' + P_1' + \cdots + P_r'$($Z''$ a subscheme of $\mathbb{P}^{n-1} \simeq H$). Then $\dim(I_Z)_t = 0$ if the* *following two conditions are satisfied*:

$$Degue \quad \dim(I_{Z'})_{t-1} = \dim(I_{\check{Z}'+D_{2,H}(P'_1)+\cdots+D_{2,H}(P'_r)})_{t-1} = 0$$

and

$$Dime \quad \dim(I_{Z''})_t = \dim(I_{\check{Z}''+P'_1+\cdots+P'_r})_t = 0.$$

The following (obvious) remark is very useful.

Remark 1.7. Let $Z, Z' \subseteq \mathbb{P}^n$, be zero-dimensional schemes such that $Z' \subseteq Z$. Then

(i) if Z imposes independent conditions to the hypersurfaces of I_t, then the same is true for the scheme Z';
(ii) if $Z' \neq Z$ and $\dim(I_{Z'})_t = 0, 1$, then $\dim(I_Z)_t = 0$.

2. Secant varieties of the Segre embeddings of $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$

We write $(\mathbb{P}^1)^t$ for $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$, ($t$ times). By Corollary 1.4, we know that dim $V_t^s = 2^t - 1 - \dim(I_W)_t$, where W is the subscheme of $\mathbb{P}^t$ defined by the ideal

$$I = \wp_1^2 \cap \cdots \cap \wp_s^2 \cap q_1^{t-1} \cap \cdots \cap q_t^{t-1},$$

where $\wp_i \leftrightarrow P_i$, $q_i \leftrightarrow Q_i$ and $\{P_1, \ldots, P_s\}$ is a set of generic points in $\mathbb{P}^t$ and Q_i is the i^{th} coordinate point of $\mathbb{P}^t$ lying on the hyperplane $\{z_0 = 0\}$.

Let us start with an example.

Example 2.1. Since the case $t = 2$ is trivial let us begin with $t = 3$, and $V_3 \subset \mathbb{P}^7$ the Segre embedding of $(\mathbb{P}^1)^3$. In this case it is well known (e.g. see [6, Example 2.4]) that dim $V_3^2 = 7$. But let us check it with our method.

We have dim $V_3^2 = 7 - \dim(I_W)_3$, where $W \subset \mathbb{P}^3$ is the scheme defined by the ideal

$$I = \wp_1^2 \cap \wp_2^2 \cap q_1^2 \cap q_2^2 \cap q_3^2.$$

It suffices to show that $I_3 = 0$. But this is well known (see [1]).

Example 2.2. Now let $t = 4$. In this case $V_4 \subset \mathbb{P}^{15}$. From [6] (Proposition 2.3 and Example 3.2) we have dim $V_4^2 = 9$ and $V_4^4 = \mathbb{P}^{15}$.

Let us now consider V_4^3. Its expected dimension is 14 but we will show that this variety is defective, and has dimension 13, i.e. $\delta_2(V_4) = 1$.

To see why this is so, recall that

$$\dim\ V_4^3 = 15 - \dim_k(I_W)_4,$$

where W is the subscheme of $\mathbb{P}^4$ defined by

$$I_W = \wp_1^2 \cap \wp_2^2 \cap \wp_3^2 \cap q_1^3 \cap q_2^3 \cap q_3^3 \cap q_4^3.$$

In this case, the 7 points in the support of W lie on a rational normal curve in $\mathbb{P}^4$ and by results in [4] we obtain that $\dim_k(I_W)_4 = 2$ and hence $\dim V_4^3 = 13$ (and not 14, as expected).

We now come to the main theorem of this section (and of the paper). We consider $V_t = (\mathbb{P}^1)^t \subset \mathbb{P}^N$, $N = 2^t - 1$ and its higher secant varieties. We show that for each such t all the higher secant varieties V_t^s, have the expected dimension (except for *at most* one s for each such t).

Theorem 2.3. *Let V_t be the Segre embedding of $(\mathbb{P}^1)^t$ in $\mathbb{P}^N$, $N=2^t-1$, and let $e_t = [\frac{2^t}{t+1}] \equiv \delta_t \pmod 2$, $\delta_t \in \{0, 1\}$; $s_t = e_t - \delta_t$. Then we have the following*:

(1) *If $s \leqslant s_t$, then* $\dim V_t^s = s(t+1) - 1$;
(2) *If $s \geqslant s_t + 2$, then* $\dim V_t^s = N$.
That is, for $s \neq s_t + 1$,

$$\dim V_t^s = \min\{s(t+1) - 1; N\}.$$

Proof. First notice that, for every t, s_t (and hence $s_t + 2$) is an even integer.

By Corollary 1.4, to find $\dim V_t^s$ we have to compute $\dim(I_{W_{(s,t)}})_t$, where $W_{(s,t)}$ is the subscheme of $\mathbb{P}^t$

$$W_{(s,t)} = (t-1)Q_1 + \cdots + (t-1)Q_t + 2P_1 + \cdots + 2P_s \subset \mathbb{P}^t,$$

where the Q_i are the coordinate points of $\mathbb{P}^t$ in the hyperplane $z_0 = 0$ and $P_1, \ldots, P_s$ are s generic points in $\mathbb{P}^t$, and show it has the expected dimension in the two cases considered by this theorem, i.e., for $s \neq s_t + 1$:

$$\dim(I_{W_{s,t}})_t = \max\{2^t - s(t+1); 0\} = \begin{cases} 2^t - s(t+1) & \text{for } s \leqslant s_t; \\ 0 & \text{for } s \geqslant s_t + 2. \end{cases}$$

As far as Case (1) is concerned, it suffices (by Remark 1.7) to prove the theorem only for $s = s_t$. As far as Case (2) is concerned, the claim is that for $s \geqslant s_t + 2$ we have $V_t^s = \mathbb{P}^N$. Thus it suffices, again by Remark 1.7, to prove the theorem in this case only for $s = s_t + 2$.

Since we need to consider schemes like $W_{(s,t)}$ for s_t and $s_t + 2$ (both even integers) we will proceed, for a while, by considering schemes like $W_{(s,t)}$ where s is ANY even integer.

We start by letting H be a hyperplane of $\mathbb{P}^t$ which contains $\{Q_2, \ldots, Q_t\}$ but does not contain Q_1. We place $P'_1, \ldots, P'_{\frac{s}{2}}$ on that hyperplane (generically) and $P_{\frac{s}{2}+1}, \ldots, P_s$ off that hyperplane (generically).

Now consider the scheme (which is a specialization of a scheme like $W_{(s,t)}$)

$$Z_{(s,t)} = \left((t-1)Q_1 + \cdots + (t-1)Q_t + 2P'_1 + \cdots + 2P'_{\frac{s}{2}}\right) + 2P_{\frac{s}{2}+1} + \cdots + 2P_s.$$

Our goal will be to show that for s (even), as in the theorem, the ideal of the scheme $Z_{(s,t)}$ (in degree t) has the same dimension as the ideal of the scheme $W_{(s,t)}$, in degree t.

We now perform the processes of *Degue* and *Dime* on this scheme, with our fixed scheme being $(t-1)Q_1 + \cdots + (t-1)Q_t + 2P'_1 + \cdots + 2P'_{\frac{s}{2}}$, and obtain:

$$(Degue)\ Z' = Z'_{(s,t)} = (t-1)Q_1 + (t-2)Q_2 + \cdots + (t-2)Q_t + P'_1 + \cdots + P'_{\frac{s}{2}} + D_{2,H}(P'_{\frac{s}{2}+1}) + \cdots + D_{2,H}(P'_s) \subset \mathbb{P}^t,$$

where $P'_{\frac{s}{2}+1}, \ldots, P'_s$ are generic points in H, and

$$(Dime)\ Z'' = Z''_{(s,t)} = (t-1)Q_2 + \cdots + (t-1)Q_t + 2P'_1 + \cdots + 2P'_{\frac{s}{2}} + P'_{\frac{s}{2}+1} + \cdots + P'_s \subset H \simeq \mathbb{P}^{t-1}.$$

Lemma 2.4. *Let Z' and Z'' be as above. If we let*

$$Z''' = Z'''_{(\frac{s}{2},t-1)} = (t-2)Q_2 + \cdots + (t-2)Q_t + 2R_1 + \cdots + 2R_{\frac{s}{2}} + R_{\frac{s}{2}+1} + \cdots + R_s$$

be the subscheme of $H \simeq \mathbb{P}^{t-1}$ where $R_1, \ldots, R_s$ are generic points of H, then

$$\dim(I_{Z'})_{t-1} = \dim(I_{Z''})_t = \dim(I_{Z'''})_{t-1}.$$

Proof. We first consider $(I_{Z'})_{t-1}$. Notice that any form of degree $t-1$ containing $(t-1)Q_1$ has to be a cone with vertex at Q_1. Since $Q_1 \notin H$ we have

$$\dim_k(I_{Z'})_{t-1} = \dim_k(I_{Z'\cap H,H})_{t-1}.$$

But $Z' \cap H$, considered as a subscheme of $H \simeq \mathbb{P}^{t-1}$, is the scheme

$$(t-2)Q_2 + \cdots + (t-2)Q_t + P'_1 + \cdots + P'_{\frac{s}{2}} + 2P'_{\frac{s}{2}+1} + \cdots + 2P'_s$$

and this has the form of Z''' above. That shows one of the equalities of the lemma.

As for the other equality, one first notes that every form in $(I_{Z''})_t$ has to have the linear form H' (which describes the hyperplane of $\mathbb{P}^{t-1}$ containing $Q_2, \ldots, Q_t$) as a factor.

Thus, $\dim(I_{Z''})_t = \dim(I_{Res_{H'}(Z'')})_{t-1}$. But,

$$Res_{H'}(Z'') = (t-2)Q_2 + \cdots + (t-2)Q_t + 2P'_1 + \cdots + 2P'_{\frac{s}{2}} + P'_{\frac{s}{2}+1} + \cdots + P'_s$$

and this also has the form of Z''' above. □

Since the scheme Z''' of the lemma is of the type $W_{(\frac{s}{2},t-1)} + R_{\frac{s}{2}+1} + \cdots + R_s$ and $W_{(\frac{s}{2},t-1)}$ is precisely the scheme we have to consider if we wish to find the dimension of $(V_{t-1})^{\frac{s}{2}}$, the stage is set for an induction argument on t.

In order to start the induction argument we need to establish some base cases. The cases $t = 2$ and $t = 3$ are trivial.

$t=4$: In this case $s_t=s_4=2$ and we have already seen (Example 2.2) that the theorem is true for $s\leqslant 2$ and $s\geqslant 4$. Notice that (in this case) the missing value ($s=3$) actually gives a defective secant variety.

$t=5$: We will do this case in some detail as it shows the general method of the proof.

We have $s_t=s_5=4$ and $s_t+2=s_5+2=6$. So, it will be enough to show that: dim $V_5^4=23$ and that $V_5^6=\mathbb{P}^{31}$.

For V_5^4 we need to consider the scheme

$$W=W_{(4,5)}=4Q_1+\cdots+4Q_5+2P_1+\cdots+2P_4,$$

where $P_1,\ldots,P_4$ are general points of $\mathbb{P}^5$ and show that

$$\dim(I_W)_5=(2^5-1)-23=8.$$

To do that, it would be enough to show that a scheme like

$$Z=Z_{(4,5)}=4Q_1+\cdots+4Q_5+2P_1'+2P_2'+2P_3+2P_4$$

(where P_1', P_2' are general points in a hyperplane $H\supset\langle Q_2,\ldots,Q_5\rangle$ such that $Q_1\notin H$, and P_3, P_4 are generic points in $\mathbb{P}^5$) satisfies $\dim(I_Z)_5=8$. But, to prove that, it is enough to show that by adding 8 points to Z we get a scheme Z^+ for which $\dim(I_{Z^+})_5=0$.

Choose 8 points $\{T_1,\ldots,T_8\}$ so that the first four are generically chosen on H and the last four are generically chosen in $\mathbb{P}^5$. If we perform *Degue* and *Dime* on Z^+ we get (see Lemma 2.4 above)

$$\begin{aligned}(Z^+)'&=4Q_1+3Q_2+\cdots+3Q_5+P_1'+P_2'+D_{2,H}(P_3')+D_{2,H}(P_4')\\&\quad+T_5+\cdots+T_8\\&=Z'+T_5+\cdots+T_8\end{aligned}$$

and

$$\begin{aligned}(Z^+)''&=4Q_2+\cdots+4Q_5+2P_1'+2P_2'+P_3'+P_4'+T_1+\cdots+T_4\\&=Z''+T_1+\cdots+T_4.\end{aligned}$$

By Lemma 2.4 we have

$$\dim(I_{Z'})_4=\dim(I_{Z''})_5=\dim(I_{Z'''})_4,$$

where $Z'=Z'_{(4,5)}$, $Z''=Z''_{(4,5)}$, and Z''' is the subscheme of $\mathbb{P}^4$ given by $3Q_2+\cdots+3Q_5+2R_1+2R_2+R_3+R_4$ where $R_1,\ldots,R_4$ are 4 generic points in $\mathbb{P}^4$.

But, we already know that for $W_{(2,4)}=3Q_2+\cdots+3Q_5+2R_1+2R_2$ we have $\dim(I_{W_{(2,4)}})_4=6$.

Thus, $\dim(I_{Z'})_4=4$ and $\dim(I_{Z''})_5=4$. It follows that

$$\dim(I_{(Z^+)'})_4=0\quad\text{and}\quad\dim(I_{(Z^+)''})_5=0.$$

By Lemma 1.6 we thus have $\dim(I_{Z^+})_5=0$ as we wanted to show. That finishes this calculation.

We now need to show that $V_5^6 = \mathbb{P}^{31}$. As before, we consider a scheme Z of the type

$$Z = Z_{(6,5)} = 4Q_1 + \cdots + 4Q_5 + 2P_1' + \cdots + 2P_3' + 2P_4 + \cdots + 2P_6,$$

where P_1', P_2', P_3' are generically chosen on the hyperplane H, and P_4, P_5, P_6 are generically chosen in $\mathbb{P}^5$. By Lemma 2.4, (and also using Lemma 1.6) we will be done if, considered as a subscheme of $\mathbb{P}^4$,

$$Z'''_{(3,4)} = Z''' = 3Q_2 + \cdots + 3Q_5 + 2R_1 + 2R_2 + 2R_3 + R_4 + R_5 + R_6$$

satisfies $(I_{Z'''})_4 = 0$.

But, for $W = W_{(3,4)}$ we know that $\dim(I_W)_4 = 2$ (see Example 2.2). It then follows that $\dim(I_{Z'''})_4 = 0$, which is what we wanted to show.

It is worth mentioning that although the theorem does not cover the case $\{t = 5,\ s = 5\}$ we were able to show, using the computer algebra system *CoCoA* [3], that $\dim V_5^5 = 29$, the expected dimension.

Now we prove the theorem by induction on t. We proceed to the general case of $t \geqslant 6$, noting that the theorem has been proved for $t = 2, 3, 4, 5$. In fact, and we will use this, for $t = 5$ we have the expected dimension even for V_5^5, something not mentioned in the theorem.

To study $V_t^{s_t}$ $(t \geqslant 6)$ we need to consider the schemes

$$W = W_{(s_t,t)}) = (t-1)Q_1 + \cdots + (t-1)Q_t + 2P_1 + \cdots + 2P_{s_t} \subset \mathbb{P}^t,$$

where the P_i are generic points of $\mathbb{P}^t$ and the Q_i are the coordinate points of $\mathbb{P}^t$ on the hyperplane $z_0 = 0$.

We want to show that

$$\dim(I_W)_t = 2^t - s_t(t+1).$$

We write $2^t - s_t(t+1) = 2r$.

We follow the procedure outlined in the case $t = 5$. We first form the scheme

$$Z = Z_{(s_t,t)} = (t-1)Q_1 + \cdots + (t-1)Q_t + 2P_1' + \cdots + 2P'_{\frac{s_t}{2}} + 2P_{\frac{s_t}{2}+1} + \cdots + 2P_{s_t} \subset \mathbb{P}^t,$$

where $P_1', \ldots, P'_{\frac{s_t}{2}}$ are generically chosen on $H \simeq \mathbb{P}^{t-1}$, $H \supset \langle Q_2, \ldots, Q_t \rangle$, $Q_1 \notin H$ and $P_{\frac{s_t}{2}+1}, \ldots, P_{s_t}$ are generically chosen points in $\mathbb{P}^t$.

It will be enough to show that $\dim(I_Z)_t = 2r$. In order to do that it will be enough to show that adding $2r$ simple points of $\mathbb{P}^t$ to Z gives a scheme whose defining ideal, in degree t, is 0.

We choose $2r$ simple points of $\mathbb{P}^t$ and call them $\{T_1, \ldots, T_r, T_{r+1}, \ldots, T_{2r}\}$ where $T_1, \ldots, T_r$ are chosen generically in H and $T_{r+1}, \ldots, T_{2r}$ are chosen generically in $\mathbb{P}^t$.

Form the scheme

$$Z^+ = Z + T_1 + \cdots + T_{2r} \subset \mathbb{P}^t.$$

If we let $(\ )'$ denote the operation of *Degue* and $(\ )''$ the operation of *Dime*, we get

$$(Z^+)' = Z' + T_{r+1} + \cdots + T_{2r} \quad \text{and} \quad (Z^+)'' = Z'' + T_1 + \cdots + T_r. \qquad (\dagger)$$

By Lemma 2.4 we have

$$\dim(I_{Z'})_{t-1} = \dim(I_{Z''})_t = \dim(I_{Z'''})_{t-1},$$

where

$$Z''' = (t-2)Q_2 + \cdots + (t-2)Q_t + 2R_1 + \cdots + 2R_{\frac{s_t}{2}} + R_{\frac{s_t}{2}+1} + \cdots + R_{s_i} \subset \mathbb{P}^{t-1}$$

for $R_1, \ldots, R_{s_t}$ generic points of $\mathbb{P}^{t-1}$.

If we let $\tilde{W} = W_{(\frac{s_t}{2}, t-1)}$ then we can rewrite Z''' as

$$Z''' = \tilde{W} + R_{\frac{s_t}{2}+1} + \cdots + R_{s_t}.$$

Suppose, for the moment, that the dimension of $(I_{\tilde{W}})_{t-1}$ is as expected, i.e.

$$\dim(I_{\tilde{W}})_{t-1} = 2^{t-1} - \frac{s_t}{2}\,(t).$$

It follows that

$$\dim(I_{Z'''})_{t-1} = 2^{t-1} - \frac{s_t}{2}\,(t) - \frac{s_t}{2}.$$

Using (†) we conclude that

$$\dim(I_{(Z^+)'})_{t-1} = \dim(I_{(Z^+)''})_t = \left[2^{t-1} - \frac{s_t}{2}\,(t) - \frac{s_t}{2}\right] - r = 0.$$

Hence we can apply Lemma 1.6 to Z^+, which finishes the theorem in this case.

As far as Case (2) is concerned, to show that

$$V_t^{s_t+2} = \mathbb{P}^N \quad (N = 2^t - 1)$$

we need to consider the schemes

$$W = W_{(s_t+2,t)} = (t-1)Q_1 + \cdots + (t-1)Q_t + 2P_1 + \cdots + 2P_{s_t+2} \subset \mathbb{P}^t,$$

where the P_i are generic points of $\mathbb{P}^t$ and the Q_i the coordinate points on the hypersurface $z_0 = 0$. We want to show that

$$\dim(I_W)_t = 0.$$

We follow the procedure we described above (slightly simpler in this case) by forming

$$Z = Z_{(s_t+2,t)} = (t-1)Q_1 + \cdots + (t-1)Q_t + 2P'_1 + \cdots + 2P'_{\frac{s_t+2}{2}} + 2P_{\frac{s_t+2}{2}+1} + \cdots + 2P_{s_t+2},$$

where $P'_1, \ldots, P'_{\frac{s_t+2}{2}}$ are generic points on $H \simeq \mathbb{P}^{t-1},\ H \supset \langle Q_2, \ldots, Q_t \rangle,\ Q_1 \notin H$ and

$$P_{\frac{s_t+2}{2}+1}, \ldots, P_{s_t+2}$$

are generic points in $\mathbb{P}^t$.

Using the same procedure as above, we seek to apply Lemma 1.6. To do that it is enough (by Lemma 2.4) to prove that if

$$Z''' = (t-2)Q_2 + \cdots + (t-2)Q_t + 2R_1 + \cdots + 2R_{\frac{s_t+2}{2}} + R_{\frac{s_t+2}{2}+1} + \cdots + R_{s_t+2}$$

(where $R_1, \ldots, R_{s_t+2}$ are generic points of $\mathbb{P}^{t-1}$), then $\dim(I_{Z'''})_{t-1} = 0$.

But, if we let $\tilde{\tilde{W}} = W_{(\frac{s_t+2}{2}, t-1)}$ then we can rewrite Z''' as

$$Z''' = \tilde{\tilde{W}} + R_{\frac{s_t+2}{2}+1} + \cdots + R_{s_t+2}.$$

If we assume for the moment that $\dim(I_{\tilde{\tilde{W}}})_{t-1}$ is as expected, i.e.

$$\dim(I_{\tilde{\tilde{W}}})_{t-1} = 2^{t-1} - \frac{s_t+2}{2}\,(t)$$

we then have

$$\dim(I_{Z'''})_{t-1} = \max\left\{0, \left[2^{t-1} - \frac{s_t+2}{2}\,(t)\right] - \frac{s_t+2}{2}\right\} = 0$$

and we are done.

We now deal with the final step of the induction argument. We saw above that to prove that $V_t^{s_t}$ and $V_t^{s_t+2}$ have the correct dimensions it is enough to show that:

$(V_{t-1})^{s_t/2}$ and $(V_{t-1})^{(s_t+2)/2}$ have, respectively, the expected dimensions.

$t = 6$: In this case $s_6 = 8$ and for $V_5 = V_{t-1}$ we have already observed that $(V_5)^4$ and $(V_5)^5$ have the expected dimensions. So, the theorem is proved for $t = 6$.

$t = 7$: In this case $s_7 = 16$ and so $\frac{s_7}{2} = 8 = s_6$. Since V_6^8 has the expected dimension then Case (1) holds for $t = 7$ and we have

$$\dim\ V_7^{s_7} = 16(7+1) - 1 = 2^7 - 1 = N.$$

It follows that $\dim\ V_7^s = N$ for all $s \geqslant 16$. In particular, Case (2) holds also for $t = 7$.

$t \geqslant 8$: To use the induction hypothesis it would be enough to have

$$\frac{s_t}{2} \leqslant \frac{s_t+2}{2} \leqslant s_{t-1}.$$

Since $s_7 = 16$, $s_8 = 28$, and $s_9 = 50$, we have these inequalities for $t = 8, 9$.

For $t \geqslant 10$ notice that

$$s_t = \frac{2^t}{t+1} - \varepsilon_1\ (0 \leqslant \varepsilon_1 < 2) \quad \text{and} \quad s_{t-1} = \frac{2^{t-1}}{t} - \varepsilon_2\ (0 \leqslant \varepsilon_2 < 2).$$

Thus

$$s_{t-1} - \frac{s_t + 2}{2} = \frac{1}{2t(t+1)} 2^t - \varepsilon_2 + \frac{\varepsilon_1}{2} - 1 \geqslant \frac{2^t}{2t(t+1)} - 3.$$

This last is $\geqslant 0$ if and only if

$$2^t \geqslant 6t(t+1)$$

and this last happens always when $t \geqslant 10$.

This computation finishes the proof of Theorem 2.3. □

Remark 2.5. In analogy with the case $t = 7$, if $t = 2^k - 1$ then $V_t^{s_t} = \mathbb{P}^N (N = 2^t - 1)$, hence $V_t^s = \mathbb{P}^N$ for all $s \geqslant s_t$. So, for $t = 2^k - 1$ ALL the s-secant varieties to V_t have the expected dimension (see also our paper [6]).

Given the results we have seen above, it makes sense to conjecture:

Conjecture. *For $V_t = (\mathbb{P}^1)^t \subset \mathbb{P}^N$, $(N = 2^t - 1)$, except for the case $\{t = 4,\ s = 3\}$, the dimension of V_t^s is always that expected.*

From our computations above, the first open cases of the conjecture are for $\{t = 6,\ s = 9\}$, $\{t = 8,\ s = 29\}$, and $\{t = 9,\ s = 51\}$.

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